Karlsruher Institut für Technologie

Advances in the nested soft-collinear subtraction scheme

Chiara Signorile-Signorile

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In collaboration with: Fabrizio Caola, Federica Devoto, Kirill Melnikov, Raoul Röntsch, Davide Maria Tagliabue

Take-home message

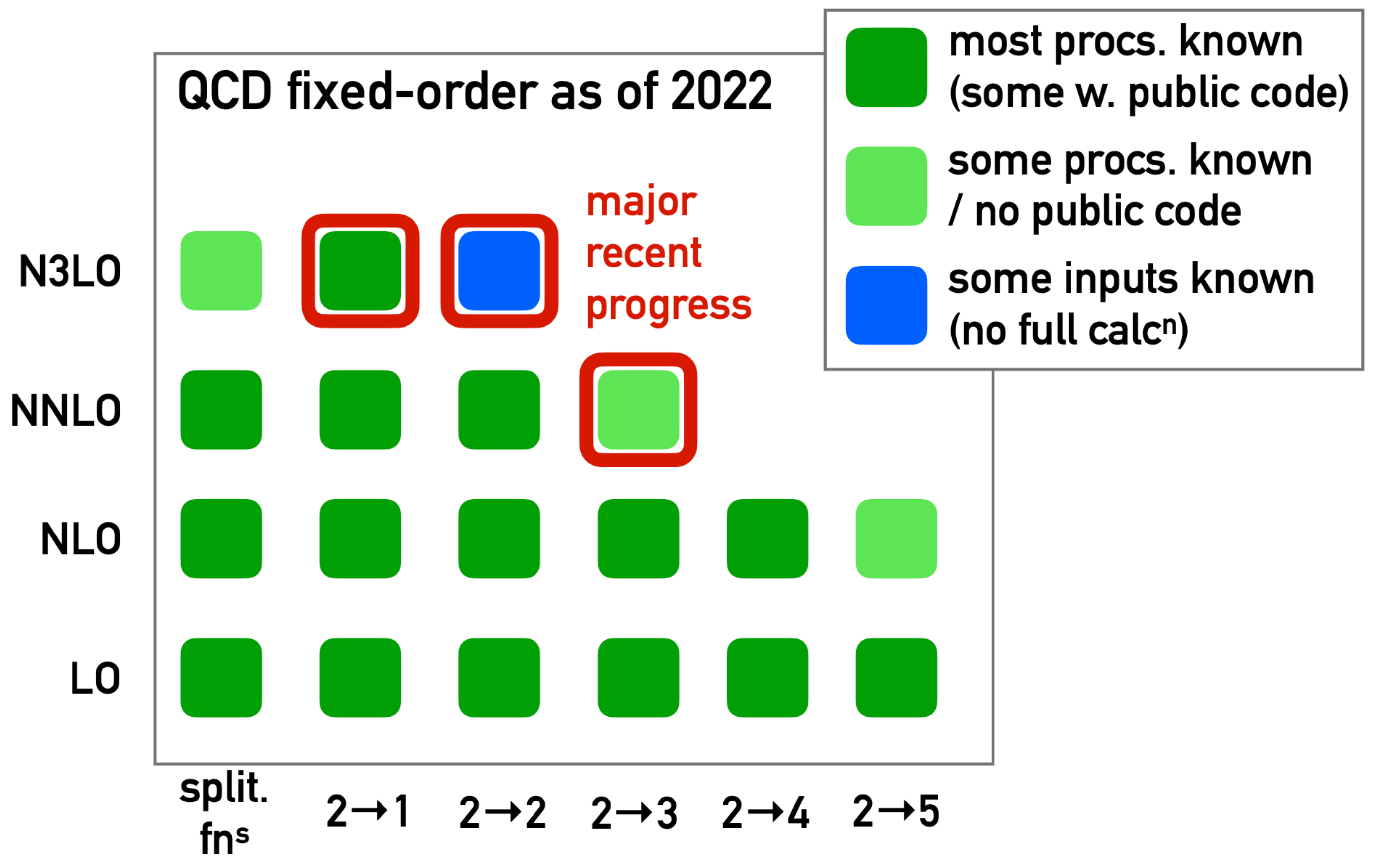
**When the complexity of the problem increases,
look at simple, recurring structures!**

The problem

Hard collisions at the LHC are described in terms of quark and gluon cross sections

$$d\sigma = \int dx_1\, dx_2\, f_i(x_1) f_j(x_2)\, d\sigma_{ij}\, \mathcal{F}\left(1 + \mathcal{O}\left(\frac{\Lambda^n_{\mathrm{QCD}}}{Q^n}\right)\right), \quad n \geq 1$$

$$d\sigma_{ij} = d\sigma_{ij,\mathrm{LO}}\left(1 + \alpha_s\, \Delta^{QCD}_{ij,\mathrm{NLO}} + \alpha_{ew}\, \Delta^{EW}_{ij,\mathrm{NLO}} + \alpha_s^2\, \Delta^{QCD}_{ij,\mathrm{NNLO}} + \alpha_s\, \alpha_{ew}\, \Delta^{QCD\otimes EW}_{ij,\mathrm{NNLO}} + \ldots\right)$$



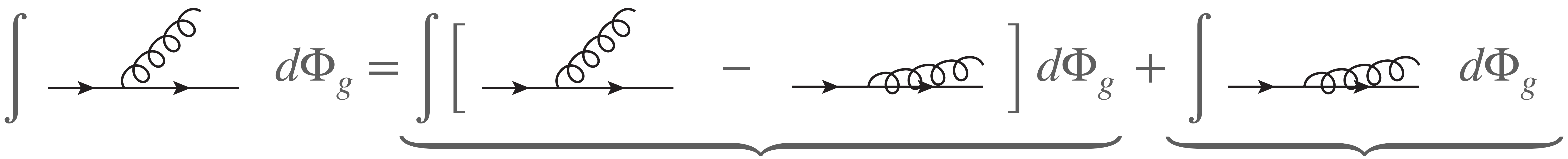
[Salam @ICHEP2023]

Problem: extract infrared $1/\epsilon$ poles in d-dimension **without integrating over the resolved phase space**

→ **fully differential** predictions for IR-safe observables

- **Phase space singularities** of the real radiation
- Explicit **poles from virtual contributions**

Slicing
Subtraction ← This talk



Finite in d=4, integrable numerically

exposes the same $1/\epsilon$ poles as the virtual correction

Why is NNLO so difficult?

Common starting point, **common problems**:

- clear understanding of which **singular configurations** do actually contribute,
- Understanding how to deal with multiple radiators and overlapping singularities,
- **Integrate the subtraction terms** in d-dimensions.

Subtraction

$$\int |\mathcal{M}|^2 F_J \, \mathrm{d}\phi_d = \int \left(|\mathcal{M}|^2 F_J - K\right) \mathrm{d}\phi_4 + \int K \, \mathrm{d}\phi_d$$

Many schemes are available:

Antenna *[Gehermann-De Ridder et al. 0505111]*

ColorfullNNLO *[Del Duca et al. 1603.08927]*

Nested-soft-collinear subtraction *[Caola et al. 1702.01352]*

STRIPPER subtraction *[Czakon 1005.0274]*

Analytic Sector Subtraction *[Magnea et al. 1806.09570]*

Geometric IR subtraction *[Herzog 1804.07949]*

Unsubtraction *[Sborlini et al. 1608.01584]*

FDR *[Pittau, 1208.5457]*

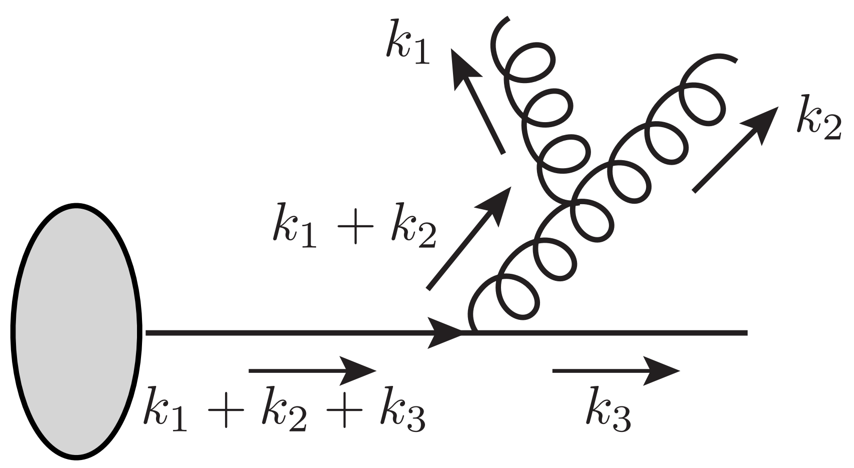
Universal Factorisation *[Sterman et al. 2008.12293]*

Despite the common problem a **variety of different strategies** have been designed.

Most of them feature a **relevant degree of complexity**, which might **hide simplifications and recurring patterns**.

Nested soft-collinear subtraction at NNLO: generalities

Extension of **FKS subtraction to NNLO** *[Caola, Melnikov, Röntsch 1702.01352]*



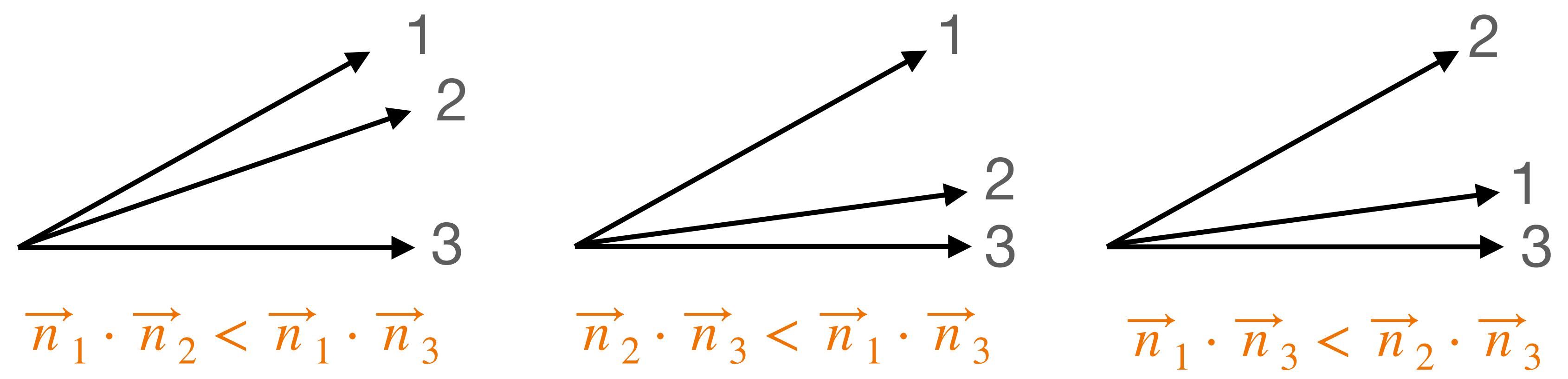
$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

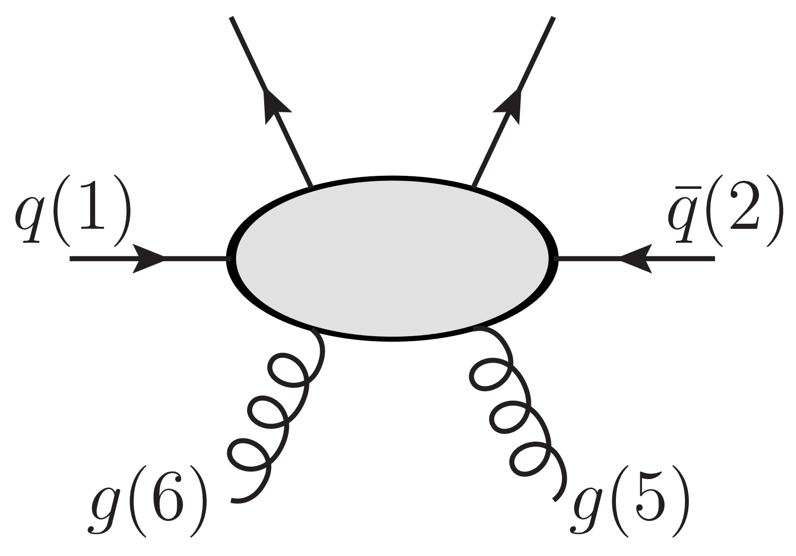
$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

$\vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$\vec{n}_1 \parallel \vec{n}_2$

Strongly-ordered configurations have also to be included: $E_1 \ll E_2\,, \quad E_2 \ll E_1$

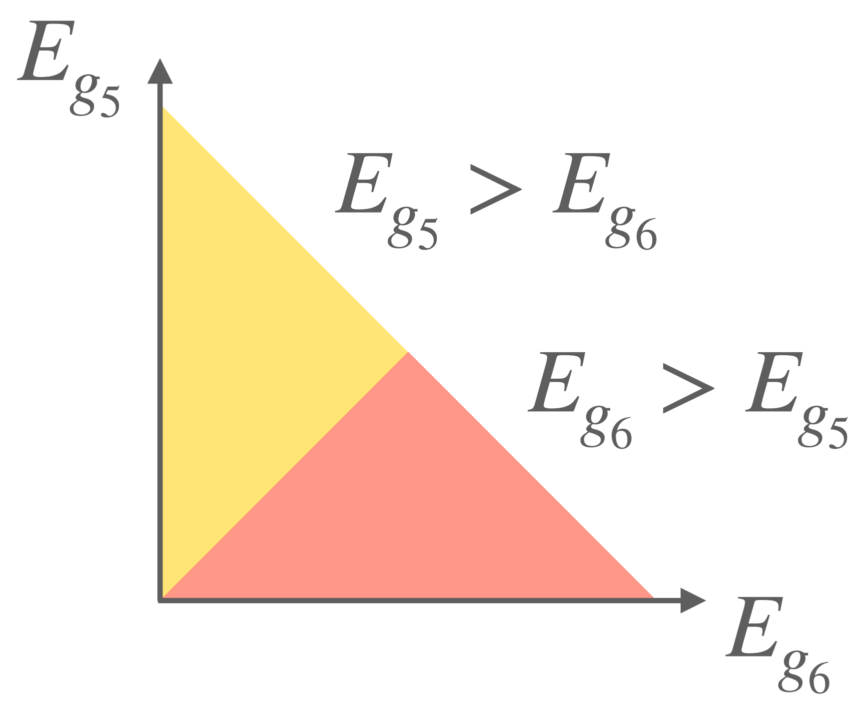




Soft limits:

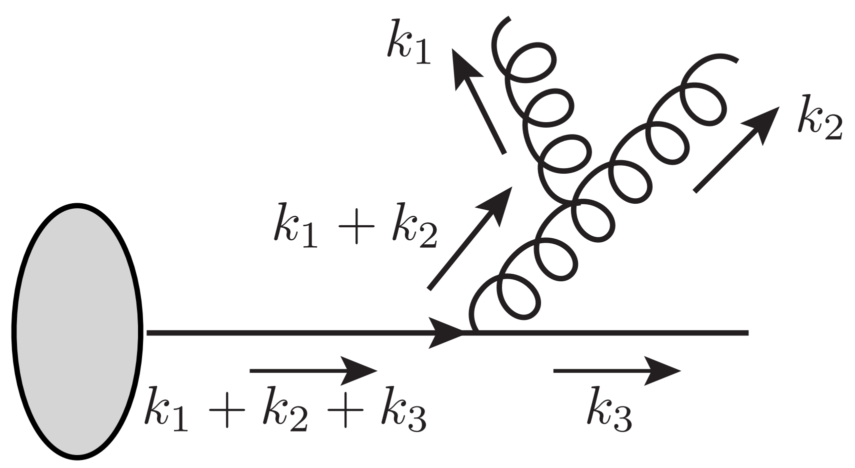
- Non-trivial structure of double-soft eikonal
- Strongly-ordered limits to disentangle

$$1 = \theta\left(E_{g_5} - E_{g_6}\right) + \theta\left(E_{g_6} - E_{g_5}\right)$$



Nested soft-collinear subtraction at NNLO: generalities

Extension of **FKS subtraction to NNLO** *[Caola, Melnikov, Röntsch 1702.01352]*



$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

$E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

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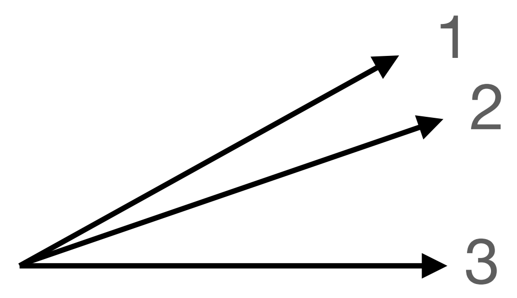
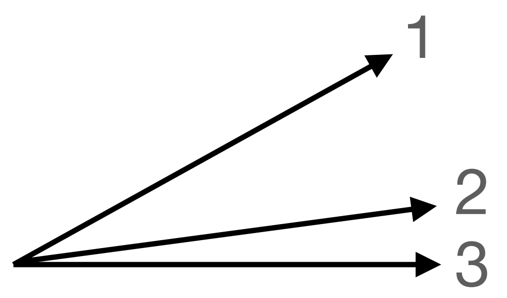
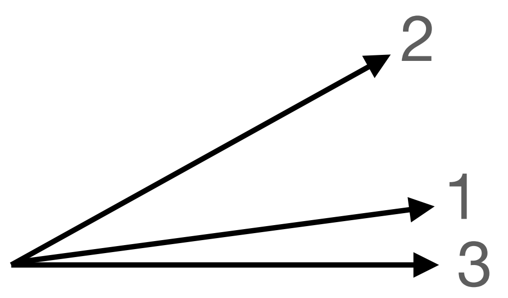


$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$



$\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$

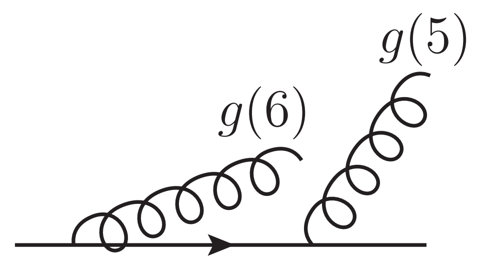


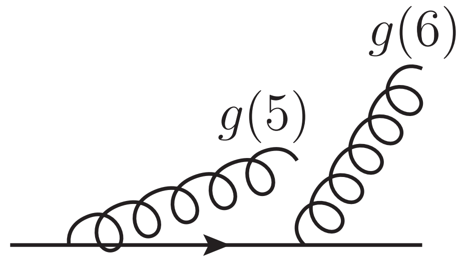
$\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

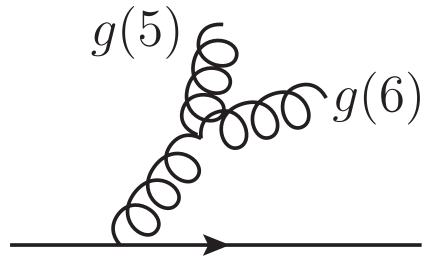
Collinear limits:

- Single, double and triple collinear limits to disentangle
- Strongly-ordered limits to disentangle in triple collinear sectors

$$1 = \sum_{i,j} \omega^{i5,j6}$$

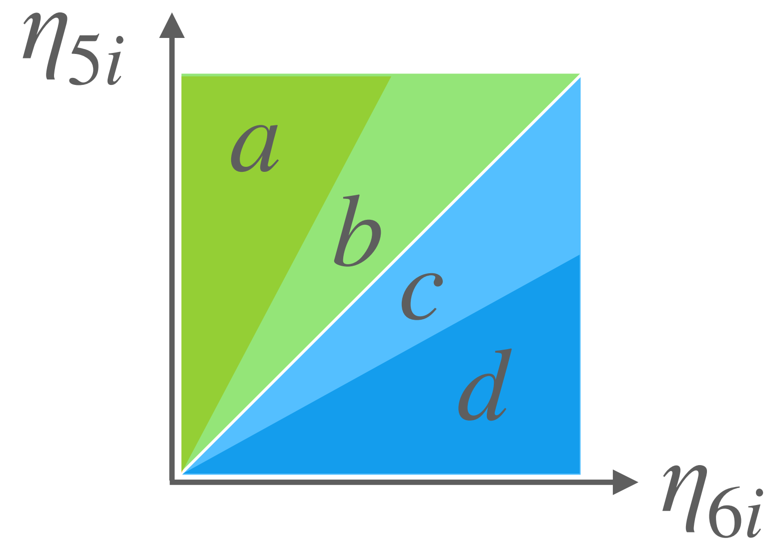






$$\eta_{ab} = \frac{1 - \cos\vartheta_{ab}}{2}$$

$$\omega^{5i,6i} = \omega^{5i,6i} \left(\theta_a + \theta_b + \theta_c + \theta_d \right)$$



Non-trivial structures to integrate → double-soft and triple-collinear kernels *[Caola, Delto, Frellesvig, Melnikov '18, Delto, Melnikov '19]*

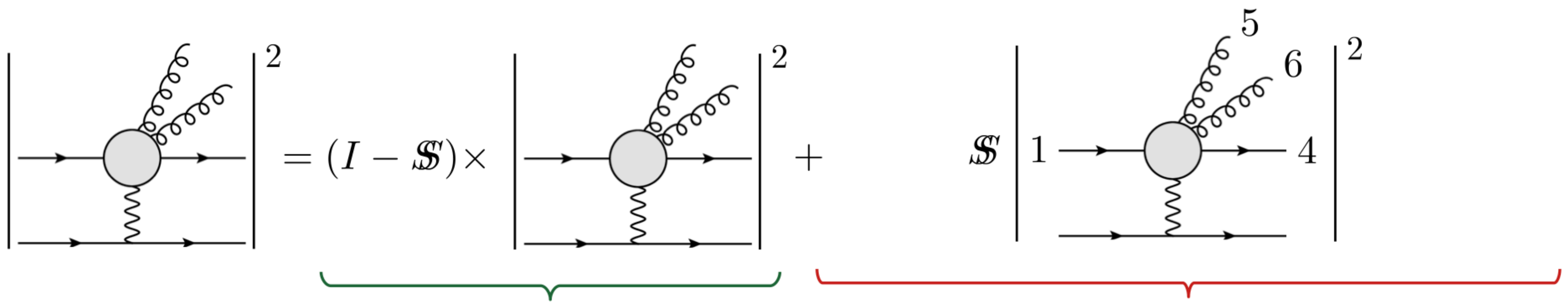
Nested soft-collinear subtraction at NNLO: generalities

[thanks to Konstantin Asteriadis for the nice pictures in this slide]

Example: DIS *[Asteriadis, Caola, Melnikov, Röntsch ’19]*

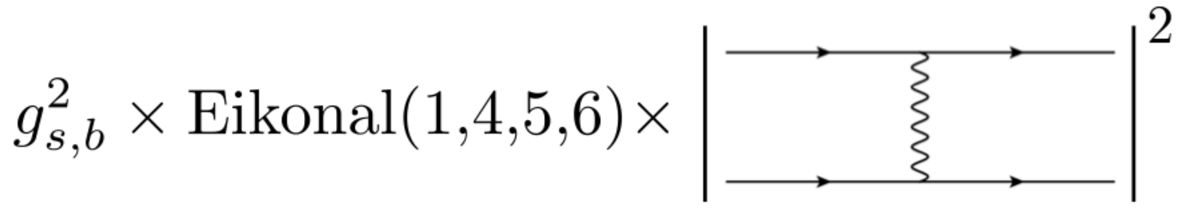
- Extract double soft singularities first ($E_5 \sim E_6 \to 0$) inserting the identity

$$I = (I - \mathbb{S}) + \mathbb{S}$$



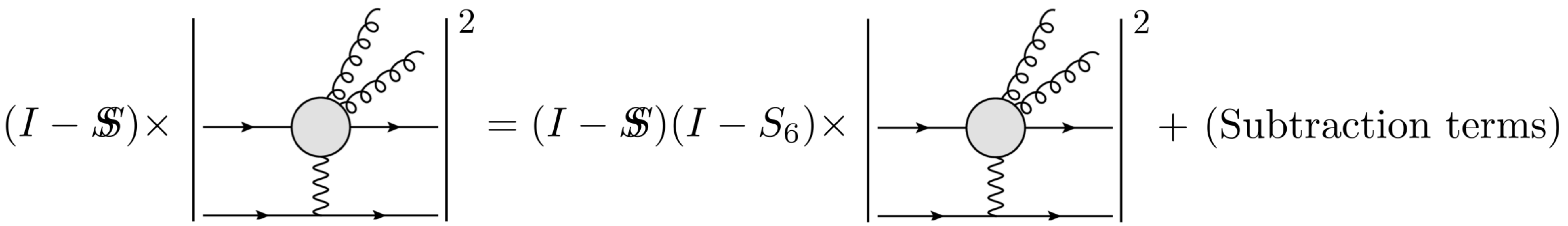
Double-soft singularity regularized but still contains single soft and collinear singularities.

Subtraction term; soft gluons decouple; integrate analytically over phase space of gluons 5 and 6

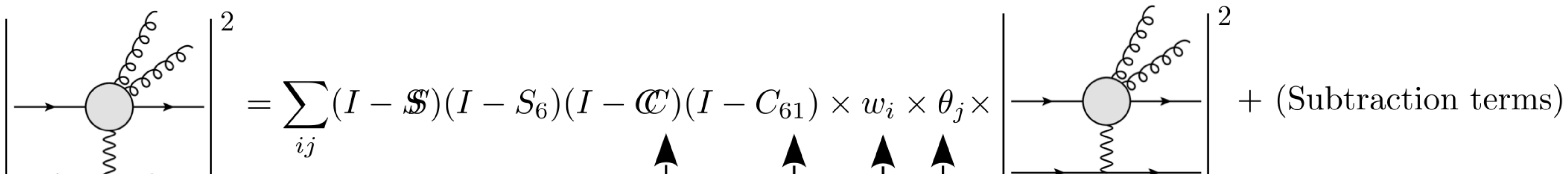


- Gluons ordered in energy -> only one single soft singularity. Insert the identity

$$I = (I - S_6) + S_6$$



- Collinear singularities: partition function *[Frixione, Kunszt, Signer ’96]* and sectoring *[Czakon ’10,11, Czakon, Heymes ’14]* to separate overlapping singularities.



triple collinear singularity

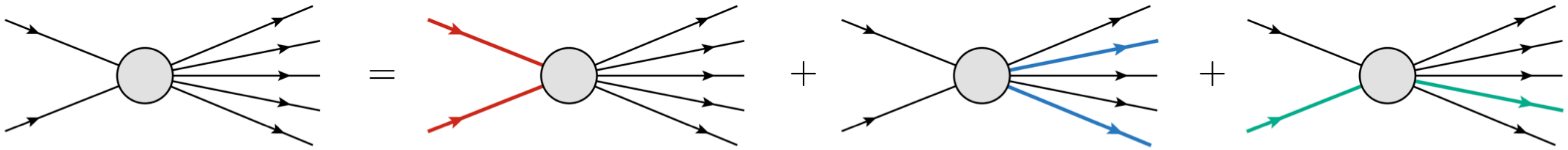
double collinear singularity; e.g. (6||1)

sector; angular ordering

partition function

State of the art:

Separation of complex $pp \to N$ processes into simpler building blocks



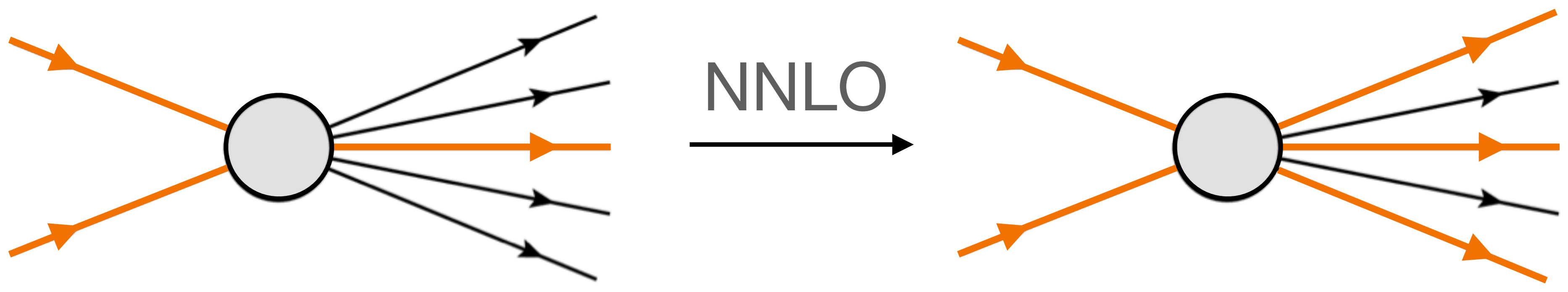
QCD corrections to Drell-Yan
Both **initial state momenta**
[Caola, Melnikov, Röntsch '19]

Higgs decay
Both **final state momenta**
[Caola, Melnikov, Röntsch '19]

Deep Inelastic Scattering
One initial and **one final** state momenta
[Asteriadis, Calola, Melnikov Röntsch '19]

Focus on simple processes→ full control of the procedure, check against analytic results sometime possible.

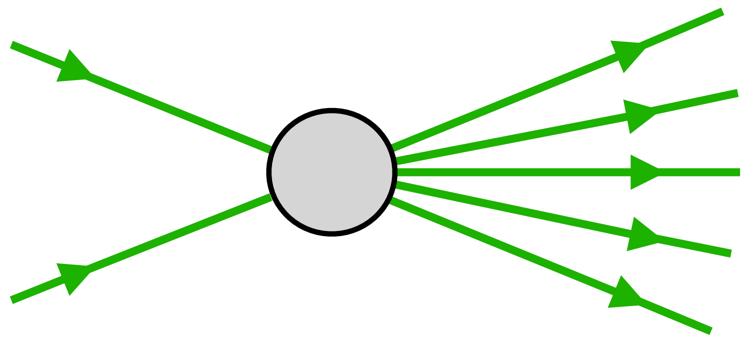
Application to Z+j production



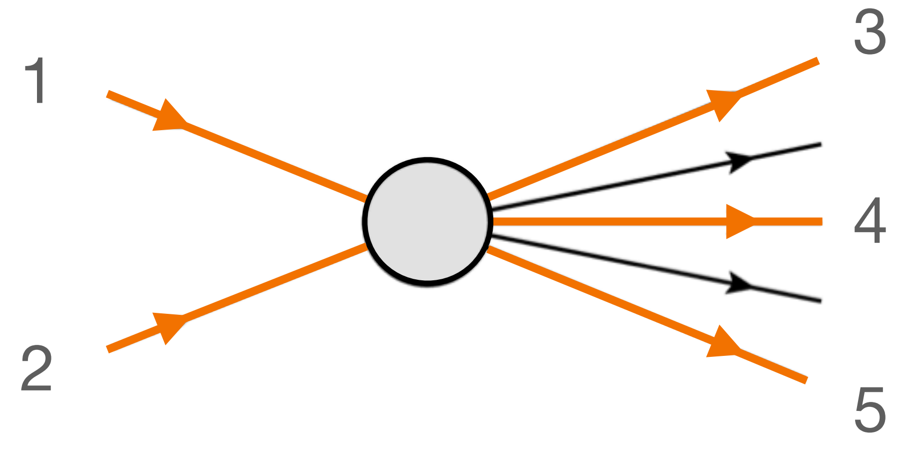
Identify potentially unresolved partons → extra partitioning:

$$\Delta^{(km)} = \frac{p_{\perp,i\neq k,m}}{\sum_{i=3}^{5} p_{\perp,i}}.$$

→ **Prototype for**



Application to Z+j production



$$
\begin{aligned}
\frac{1}{3!}\Big\langle F_{\mathrm{LM}}(1_q,2_{\bar{q}};3_g,4_g,5_g)\Big\rangle = {} & \langle S_{45}\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\rangle + \langle (I-S_4)S_5\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\rangle \\
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i}) + \Theta^{(b)}C_{45,i}(I-C_{45}) \\
& \quad + \Theta^{(c)}C_{45,i}(I-C_{4i}) + \Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\Big\rangle \\
& - \Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in\mathrm{DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\Big\rangle \\
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\big[\Theta^{(a)}C_{5i} + \Theta^{(b)}C_{45} + \Theta^{(c)}C_{4i} + \Theta^{(d)}C_{45}\big]\,\omega_{4i5i} \\
& \quad + \sum_{(ij)\in\mathrm{DC}}\big[C_{4i}+C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\Big\rangle
\end{aligned}
$$

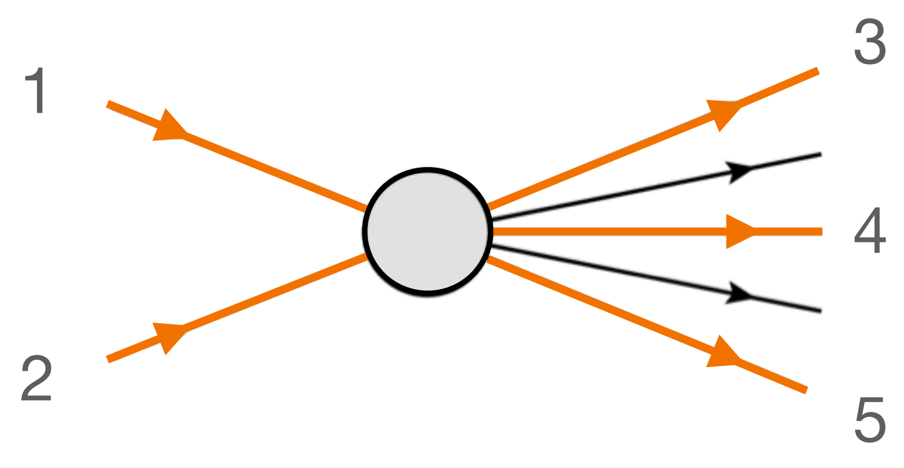
Subtraction terms

$$
\begin{aligned}
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i}) + \Theta^{(b)}(I-C_{45,i})(I-C_{45}) \\
& \quad + \Theta^{(c)}(I-C_{45,i})(I-C_{4i}) + \Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i} \\
& \quad + \sum_{(ij)\in\mathrm{DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)}F_{\mathrm{LM}}^{4>5}\Big\rangle
\end{aligned}
$$

Fully regulated contribution

$$
\begin{aligned}
(ij)\in\mathrm{DC} &\longrightarrow (ij)\in\{(12),(13),(21),(23),(31),(32)\} \\
i\in\mathrm{TC} &\longrightarrow i\in\{1,2,3\}\,.
\end{aligned}
$$

Application to Z+j production



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\end{aligned}
$$

In principle generalisable to n-partons

Subtraction terms

Implemented numerically → no issues in increasing the number of partons

Fully regulated contribution

$$
\begin{aligned}
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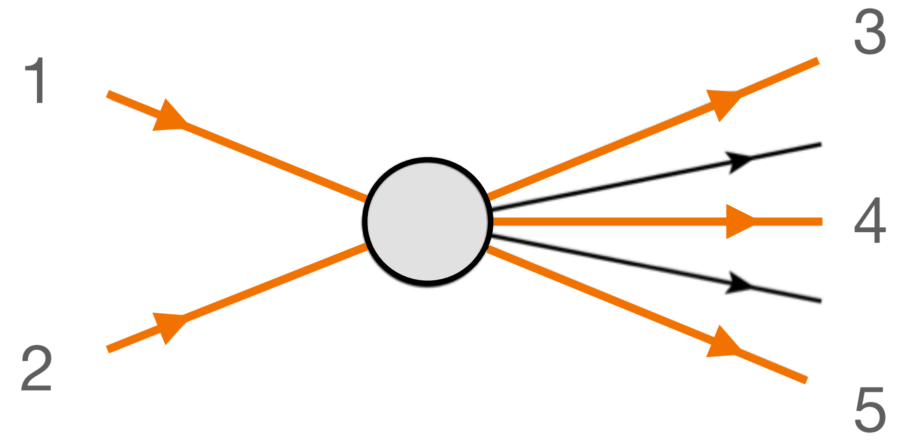
Application to Z+j production



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$$-\Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in{\rm DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)}F_{\rm LM}^{4>5}\Big\rangle$$
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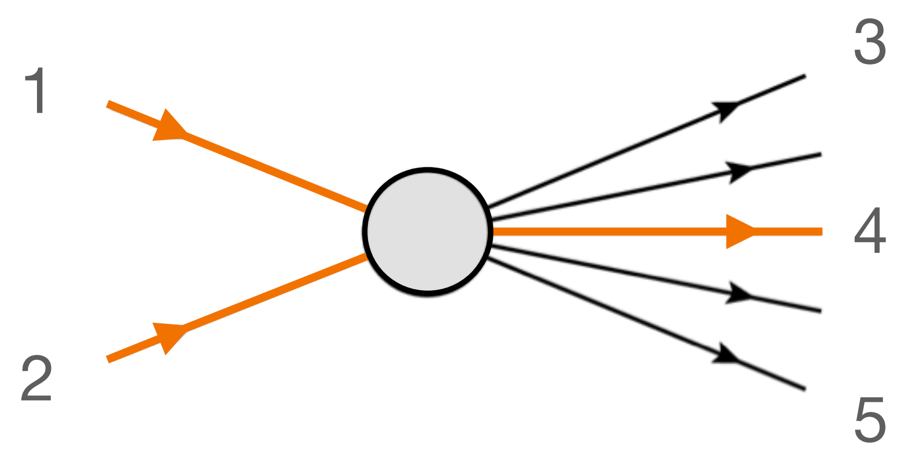
$$(ij)\in{\rm DC}\longrightarrow(ij)\in\{(12),(13),(21),(23),(31),(32)\}$$
$$i\in{\rm TC}\longrightarrow i\in\{1,2,3\}\,.$$

Drawbacks identified with Z+j

- The **bookkeeping** becomes immediately **cumbersome** → large number of subtraction terms.
- Calculating all **subtraction terms separately** may **hide a number of simplifications** that can occur **before explicit evaluation**.
- Writing color-correlations as Casimir operators in the **intermediate steps facilitates the calculation**, **BUT** leads to **non-trivial generalisation** to n-partons.

Summary of the talk



- A **subtraction scheme** based of **FKS** was proposed.
- Singular **kernels** for initial- and final-state emission are **known**. **Integration** of the most complicated **double-unresolved limits performed for arbitrary kinematics**.
- **Application to simple processes** worked out straightforwardly.
- **In principle**, **general formulas for subtraction terms and fully-resolved components** for an **arbitrary number of partons are available**.
- This can be done because we know how to deal with **multiple radiators** [**partitioning**, **energy ordering**]
- However, for **non-trivial processes** (e.g. V+j) **several difficulties** arise: partitioning, energy ordering and Casimir operators **obscure simplifications that are suggested by the simple structure of Catani's operator**.
- This suggests that we may need to take **some steps back**.

Lesson from NLO: $q\bar{q} \to V + ng\,(+g_k)$

Virtual corrections: color-correlations, elastic terms

$$\left\langle F_{\mathrm{LV}}(1\ldots n)\right\rangle = \frac{\alpha_s}{2\pi}\left\langle 2\Re\big(\mathcal{I}_1(\epsilon)\big)\, F_{\mathrm{LM}}\right\rangle$$

$$\boxed{\mathcal{I}_1(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i \frac{1}{\mathbf{T}_i^2}\left(\mathbf{T}_i^2\frac{1}{\epsilon^2} + \gamma_i\frac{1}{\epsilon}\right)\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\lambda_{ij}\pi\epsilon}}$$

Soft real: color-correlations, elastic terms

$$\left\langle S_k\,\Delta^{(k)}\,F_{\mathrm{LM}}(1\ldots n|k)\right\rangle = \left\langle I_{1,R}(\epsilon)\,F_{\mathrm{LM}}\right\rangle$$

$$\boxed{I_{1,R}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{n}\eta_{ij}^{-\epsilon}K_{ij}\mathbf{T}_i\cdot\mathbf{T}_j}$$

$$\bar{P}_{qq}^{\mathrm{AP},0}(z) = 2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)$$

Lesson from NLO: $q\bar{q} \to V + ng\,(+g_k)$

Virtual corrections: color-correlations, elastic terms

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Soft real: color-correlations, elastic terms

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Hard-collinear IS: no color-correlations, boosted and elastic terms

$$\sum_{i=1}^{2}\left\langle (I-S_k)\,C_{ik}\,\omega_{ik}\,\Delta^{(k)}\,F_{\mathrm{LM}}(1\ldots n|k)\right\rangle = [\alpha_s]\sum_{i=1}^{2}\left\langle -\frac{1}{\epsilon}\bar{P}_{qq}^{\mathrm{AP},0}(z)\otimes F_{\mathrm{LM}}^{(i)}(z) + P_{\mathrm{fin,qq}}\otimes F_{\mathrm{LM}}^{(i)}(z) + \hat{\Gamma}_{q_i}\,F_{\mathrm{LM}}\right\rangle$$

Hard-collinear FS: no color-correlations, elastic terms

$$\sum_{i=3}^{n}\left\langle (I-S_k)\,C_{ik}\,\omega_{ik}\,\Delta^{(k)}\,F_{\mathrm{LM}}(1\ldots n|k)\right\rangle = [\alpha_s]\sum_{i=3}^{n}\left\langle\hat{\Gamma}_{g_i}F_{\mathrm{LM}}\right\rangle$$

Define a collinear operator

$$\boxed{\sum_{i=1}^{n}\left\langle\hat{\Gamma}_{f[i]}F_{\mathrm{LM}}\right\rangle \equiv \left\langle I_C(\epsilon)\,F_{\mathrm{LM}}\right\rangle}$$

PDFs renormalisation: no color-correlations, boosted terms

$$\mathrm{d}\sigma_{\mathrm{nlo}}^{\mathrm{PDF}} = \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}\sum_{i=1}^{2}\left\langle\bar{P}_{qq}^{\mathrm{AP},0}(z)\otimes F_{\mathrm{LM}}^{(i)}(z)\right\rangle$$

$$\bar{P}_{qq}^{\mathrm{AP},0}(z) = 2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\delta(1-z)$$

Lesson from NLO: $q\bar{q} \to V + ng\,(+g_k)$

Combining everything together

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{NLO}} &= \mathrm{d}\sigma_{\mathrm{R}} + \mathrm{d}\sigma_{\mathrm{V}} + \mathrm{d}\sigma_{\mathrm{PDF}} \\ &= \langle F_{\mathrm{LM}}(1\ldots n|k)\rangle + \langle F_{\mathrm{LV}}(1\ldots n)\rangle + \mathrm{d}\sigma_{\mathrm{PDF}} \end{aligned}$$

where the subtraction for the real contribution is done iteratively starting with the soft singularities, we get

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{NLO}} = \; & \left\langle \left[[\alpha_s] I_{1,R}(\epsilon) + \frac{\alpha_s}{2\pi} 2\Re\left(\mathcal{I}_1(\epsilon)\right) + I_C(\epsilon) \right] F_{\mathrm{LM}} \right\rangle \\ & + [\alpha_s] \sum_{i=1}^{2} \left\langle -\frac{1}{\epsilon} \bar{P}_{qq}^{\mathrm{AP},0}(z) \otimes F_{\mathrm{LM}}^{(i)}(z) + P_{\mathrm{fin,qq}} \otimes F_{\mathrm{LM}}^{(i)}(z) \right\rangle + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon} \sum_{i=1}^{2} \left\langle \bar{P}_{qq}^{\mathrm{AP},0}(z) \otimes F_{\mathrm{LM}}^{(i)}(z) \right\rangle \\ & + \sum_{i=1}^{n} \langle (I - S_k)(I - C_{ik})\, \Delta^{(k)}\, \omega^{ik}\, F_{\mathrm{LM}}(1\ldots n|k) \rangle \end{aligned}$$

Lesson from NLO: $q\bar{q} \to V + ng\,(+g_k)$

Combining everything together

$$
\begin{aligned}
d\sigma_{\rm NLO} &= d\sigma_{\rm R} + d\sigma_{\rm V} + d\sigma_{\rm PDF} \\
&= \big\langle F_{\rm LM}(1\ldots n|k)\big\rangle + \big\langle F_{\rm LV}(1\ldots n)\big\rangle + d\sigma_{\rm PDF}
\end{aligned}
$$

where the subtraction for the real contribution is done iteratively starting with the soft singularities, we get

$$
\begin{aligned}
d\sigma_{\rm NLO} = &\left\langle \boxed{\left[[\alpha_s] I_{1,R}(\epsilon) + \frac{\alpha_s}{2\pi} 2\Re\left(\mathcal{I}_1(\epsilon)\right) + I_C(\epsilon)\right]} F_{\rm LM}\right\rangle \\
&\left.\begin{aligned}
&+ [\alpha_s] \sum_{i=1}^{2} \left\langle -\frac{1}{\epsilon} \bar{P}_{qq}^{\rm AP,0}(z) \otimes F_{\rm LM}^{(i)}(z) + P_{\rm fin,qq} \otimes F_{\rm LM}^{(i)}(z)\right\rangle + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}\sum_{i=1}^{2}\left\langle \bar{P}_{qq}^{\rm AP,0}(z) \otimes F_{\rm LM}^{(i)}(z)\right\rangle \\
&+ \sum_{i=1}^{n} \langle (I - S_k)(I - C_{ik})\,\Delta^{(k)}\,\omega^{ik}\,F_{\rm LM}(1\ldots n|k)\rangle
\end{aligned}\right\} \text{manifestly finite}
\end{aligned}
$$

$$
\Longrightarrow \boxed{[\alpha_s] I_{1,T}(\epsilon) \equiv \left(\frac{\alpha_s}{2\pi}\right) 2{\rm Re}\left(\mathcal{I}_1(\epsilon)\right) + [\alpha_s]\, I_{1,R}(\epsilon) + [\alpha_s]\, I_C(\epsilon) \qquad \text{FINITE}}
$$

Simple interplay between $\left[V + S_i R + (I - S_i) C_{ij} R\right]_{\rm elastic}$ **and** $\left[(1 - S_i) C_{ij} R\right]_{\rm boost} + {\rm PDFs}$

Lesson from NLO

Simple interplay between $\underbrace{\left[V + S_i R + (I - S_i) C_{ij} R\right]_{\text{elastic}}}$ **and** $\left[(1 - S_i) C_{ij} R\right]_{\text{boost}} + \text{PDFs}$ **should arise also at NNLO.**

$$[\alpha_s] I_{1,T}(\epsilon) \equiv \left(\frac{\alpha_s}{2\pi}\right) 2\mathrm{Re}\big(\mathcal{I}_1(\epsilon)\big) + [\alpha_s]\, I_{1,R}(\epsilon) + [\alpha_s]\, I_C(\epsilon)$$

What we are going to see

- Starting from **IR poles of double-virtual**, we want to find **subtraction terms** that can "**complete**" it:
  - identify structures similar to those encountered at NLO,
  - **get rid of color-correlations and reduce the rest to a sum over external-leg contributions.**
- ***Elementary, my dear Watson!***
- Well… **yes and no**:
  - Clearly the poles have to cancel, thus a **relations between different contributions must exist**.
  - However, finding such relations is **not easy** because of **partitioning** and **energy ordering**, that are **crucial to fully define** the **singular configurations**.

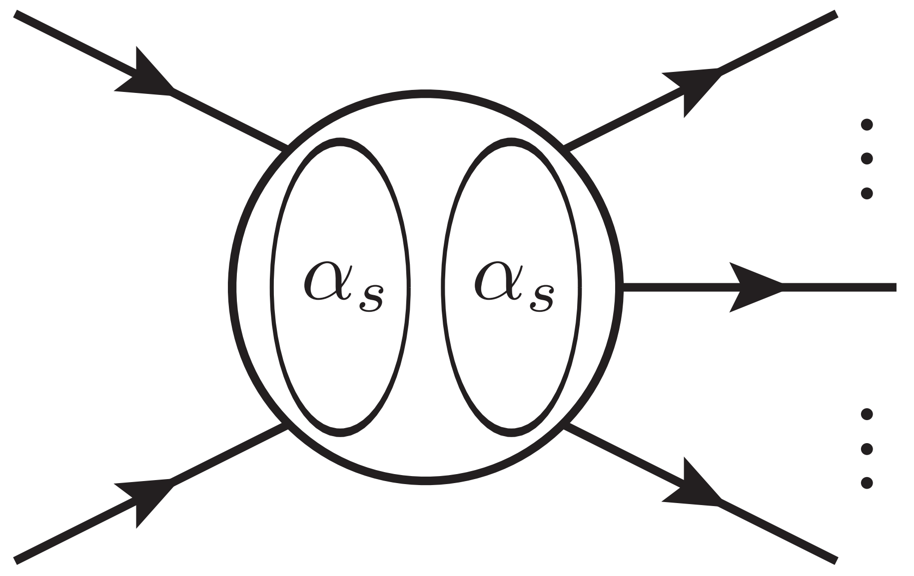
Double virtual contribution

Universal structure, regulated by Catani's operator, valid for any number of external coloured partons *[Catani '98]* . Features a single structure with color-correlations

Color-correlations inside
$\mathcal{I}_1(\epsilon)$
(already encountered at NLO)

$$
\begin{aligned}
\langle F_{\mathrm{LVV}} \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\langle & \frac{1}{2}\Big[2\Re\big(\mathcal{I}_1(\epsilon)\big)\Big]^2 F_{\mathrm{LM}} - \frac{\beta_0}{\epsilon} 2\Re\big(\mathcal{I}_1(\epsilon)\big) F_{\mathrm{LM}} \\
& + \frac{e^{-\epsilon\gamma_{\mathrm{E}}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \frac{\beta_0}{\epsilon} 2\Re\big(\mathcal{I}_1(2\epsilon)\big) F_{\mathrm{LM}} + \frac{e^{-\epsilon\gamma_{\mathrm{E}}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} K\, 2\Re\big(\mathcal{I}_1(2\epsilon)\big) F_{\mathrm{LM}} \\
& + 2\, \underbrace{\frac{e^{\epsilon\gamma_{\mathrm{E}}}}{4\epsilon\,\Gamma(1-\epsilon)} \mathcal{H}_2(\epsilon)}_{\text{Process-dependent}} F_{\mathrm{LM}} + 2\Re\big(\mathcal{I}_1(\epsilon)\big) F_{\mathrm{LV}}^{\mathrm{fin}} + \underbrace{F_{\mathrm{LVV}}^{\mathrm{fin}} + F_{\mathrm{LV^2}}^{\mathrm{fin}} \Bigg\rangle}_{\text{Finite remainders from 2-loop and (1-loop)}^2\text{ amplitudes}},
\end{aligned}
$$

$$
K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \,.
$$



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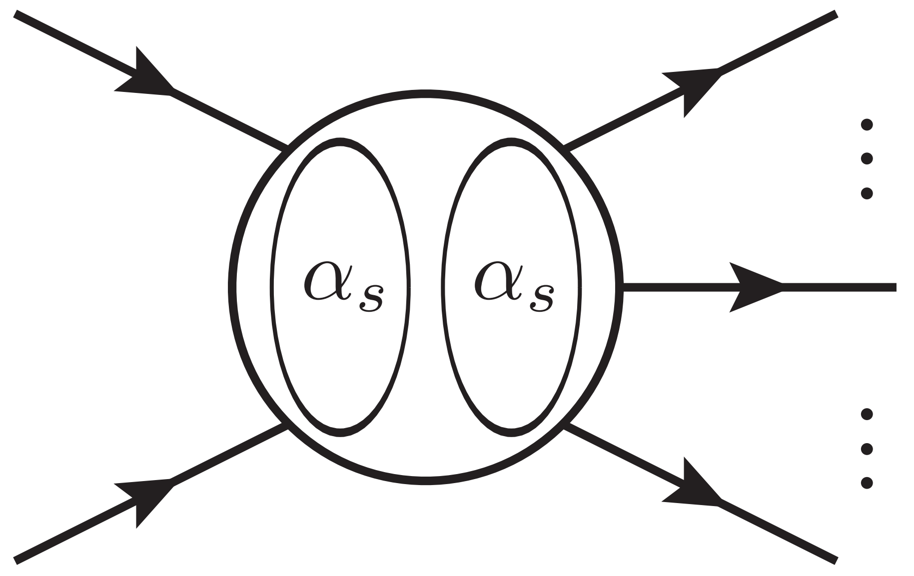
$$\langle F_{\mathrm{LVV}} \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \Big\langle \frac{1}{2}\left[2\Re(\mathcal{I}_1(\epsilon))\right]^2 F_{\mathrm{LM}} - \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(\epsilon)) F_{\mathrm{LM}}$$
$$+ \frac{e^{-\epsilon\gamma_{\mathrm{E}}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \frac{\beta_0}{\epsilon} 2\Re(\mathcal{I}_1(2\epsilon)) F_{\mathrm{LM}} + \frac{e^{-\epsilon\gamma_{\mathrm{E}}}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} K\, 2\Re(\mathcal{I}_1(2\epsilon)) F_{\mathrm{LM}}$$
$$+ \underbrace{2\,\frac{e^{\epsilon\gamma_{\mathrm{E}}}}{4\epsilon\,\Gamma(1-\epsilon)}\mathcal{H}_2(\epsilon)}_{\text{Process-dependent}}\, F_{\mathrm{LM}} + 2\Re(\mathcal{I}_1(\epsilon)) F^{\mathrm{fin}}_{\mathrm{LV}} + \underbrace{F^{\mathrm{fin}}_{\mathrm{LVV}} + F^{\mathrm{fin}}_{\mathrm{LV^2}}}_{\text{Finite remainders from 2-loop and (1-loop)}^2\text{ amplitudes}} \Big\rangle ,$$

Color-correlations inside
$\mathcal{I}_1(\epsilon)$
(already encountered at NLO)

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f \,.$$

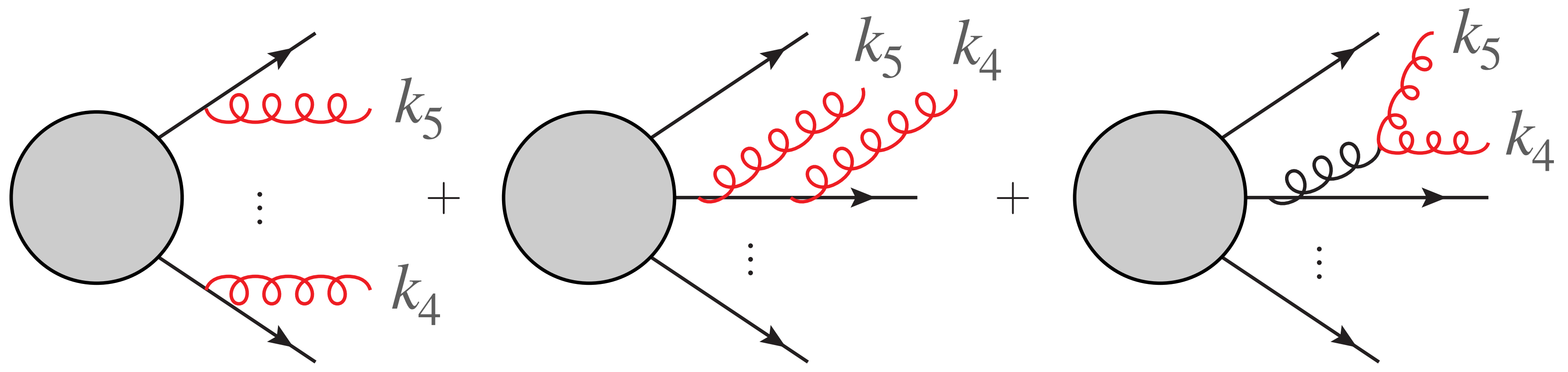
However:

- different **arguments**
- different **powers** → suggests a **specific patter of cancellation.**
- different **prefactors**



Double soft

Different color structure: single-correlated (T^2) and double-correlated (T^4)
[Catani, Grazzini '99]



$$\langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5)\rangle = \langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5)\rangle_{T^4} + \langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5)\rangle_{T^2}.$$

$$\langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5)\rangle_{T^4} =$$

$$= \frac{1}{2}\Big\langle \int [dp_4][dp_5]\Theta(E_4 - E_5) \sum_{i\neq j}^{n} S_{ij}(p_4) \sum_{k\neq m}^{n} S_{km}(p_5)\{\mathbf{T}_i\cdot\mathbf{T}_j, \mathbf{T}_k\cdot\mathbf{T}_l\}\, F_{\mathrm{LM}}\Big\rangle$$

$$= \frac{[\alpha_s]^2}{2}\Big\langle I_{1,R}^2(\epsilon) F_{\mathrm{LM}}\Big\rangle,$$

Same structure as NLO with color-correlations

$$\boxed{I_{1,R}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{n} \eta_{ij}^{-\epsilon} K_{ij}\mathbf{T}_i\cdot\mathbf{T}_j}$$

Double soft

Different color structure: single-correlated (T^2) and double-correlated (T^4)

[Catani, Grazzini '99]

Valid for arbitrary angle between the emitters

$$
\begin{aligned}
\langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5)\rangle_{T^2} = {} & (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2\left\{\frac{1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left[\frac{11}{12}-\ln(s^2)\right]\right.\\
& +\frac{1}{\epsilon^2}\left[2\mathrm{Li}_2(c^2)+\ln^2(s^2)-\frac{11}{6}\ln(s^2)+\frac{11}{3}\ln 2-\frac{\pi^2}{4}-\frac{16}{9}\right]\\
& +\frac{1}{\epsilon}\left[6\mathrm{Li}_3(s^2)+2\mathrm{Li}_3(c^2)+\left(2\ln(s^2)+\frac{11}{3}\right)\mathrm{Li}_2(c^2)-\frac{2}{3}\ln^3(s^2)\right.\\
& \quad+\left(3\ln(c^2)+\frac{11}{6}\right)\ln^2(s^2)-\left(\frac{22}{3}\ln 2+\frac{\pi^2}{2}-\frac{32}{9}\right)\ln(s^2)\\
& \quad\left.-\frac{45}{4}\zeta_3-\frac{11}{3}\ln^2 2-\frac{11}{36}\pi^2-\frac{137}{18}\ln 2+\frac{217}{54}\right]\\
& +4\mathrm{G}_{-1,0,0,1}(s^2)-7\mathrm{G}_{0,1,0,1}(s^2)+\frac{22}{3}\mathrm{Ci}_3(2\delta)+\frac{1}{3\tan(\delta)}\mathrm{Si}_2(2\delta)\\
& +2\mathrm{Li}_4(c^2)-14\mathrm{Li}_4(s^2)+4\mathrm{Li}_4\left(\frac{1}{1+s^2}\right)-2\mathrm{Li}_4\left(\frac{1-s^2}{1+s^2}\right)\\
& +2\mathrm{Li}_4\left(\frac{s^2-1}{1+s^2}\right)+\mathrm{Li}_4(1-s^4)+\left[10\ln(s^2)-4\ln\left(1+s^2\right)\right.\\
& \left.+\frac{11}{3}\right]\mathrm{Li}_3(c^2)+\left[14\ln(c^2)+2\ln(s^2)+4\ln\left(1+s^2\right)+\frac{22}{3}\right]\mathrm{Li}_3(s^2)\\
& +4\ln(c^2)\mathrm{Li}_3(-s^2)+\frac{9}{2}\mathrm{Li}_2^2(c^2)-4\mathrm{Li}_2(c^2)\mathrm{Li}_2(-s^2)+\left[7\ln(c^2)\ln(s^2)\right.\\
& \left.-\ln^2(s^2)-\frac{5}{2}\pi^2+\frac{22}{3}\ln 2-\frac{131}{18}\right]\mathrm{Li}_2(c^2)+\left[\frac{2}{3}\pi^2-4\ln(c^2)\ln(s^2)\right]\times\\
& \mathrm{Li}_2(-s^2)+\frac{\ln^4(s^2)}{3}+\frac{\ln^4\left(1+s^2\right)}{6}-\ln^3(s^2)\left[\frac{4}{3}\ln(c^2)+\frac{11}{9}\right]\\
& +\ln^2(s^2)\left[7\ln^2(c^2)+\frac{11}{3}\ln(c^2)+\frac{\pi^2}{3}+\frac{22}{3}\ln 2-\frac{32}{9}\right]-\frac{\pi^2}{6}\ln^2\left(1+s^2\right)\\
& +\zeta_3\left[\frac{17}{2}\ln(s^2)-11\ln(c^2)+\frac{7}{2}\ln\left(1+s^2\right)-\frac{21}{2}\ln 2-\frac{99}{4}\right]+\ln(s^2)\times\\
& \left[-\frac{7\pi^2}{2}\ln(c^2)+\frac{22}{3}\ln^2 2-\frac{11}{18}\pi^2+\frac{137}{9}\ln 2-\frac{208}{27}\right]-12\mathrm{Li}_4\left(\frac{1}{2}\right)\\
& +\frac{143}{720}\pi^4-\frac{\ln^4 2}{2}+\frac{\pi^2}{2}\ln^2 2-\frac{11}{6}\pi^2\ln 2+\frac{125}{216}\pi^2+\frac{22}{9}\ln^3 2\\
& \left.+\frac{137}{18}\ln^2 2+\frac{434}{27}\ln 2-\frac{649}{81}+\mathcal{O}(\epsilon)\right\},
\end{aligned}
$$

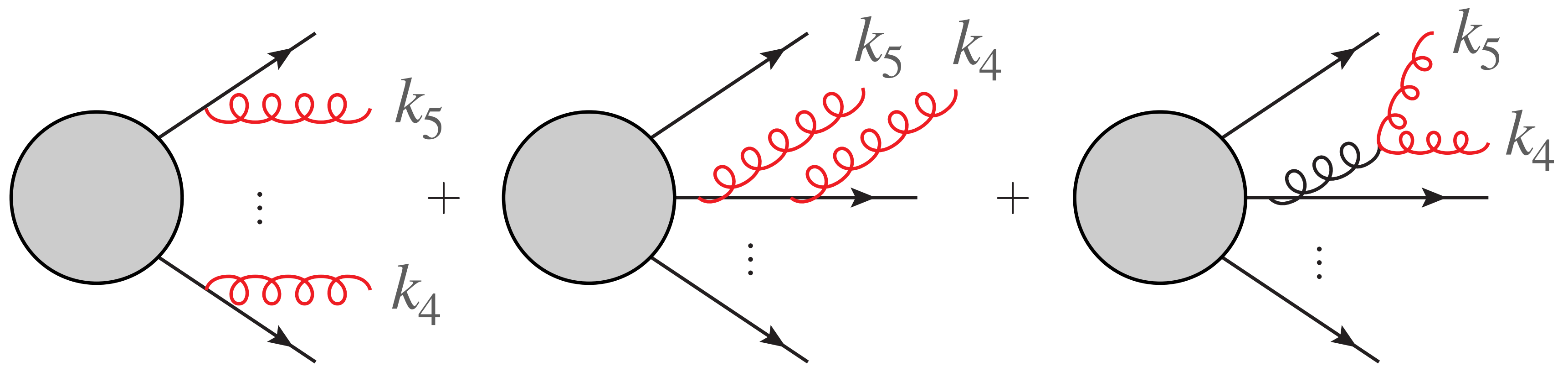
[Caola, Delto, Frellesvig, Melnikov 1807.05835]

$$\delta=\frac{\delta_{12}}{2},\ s=\sin\frac{\delta_{12}}{2},\ c=\cos\frac{\delta_{12}}{2} \qquad \mathrm{Ci}_n(z)=\frac{\mathrm{Li}_n(e^{iz})+\mathrm{Li}_n(e^{-iz})}{2},\ \mathrm{Si}_n(z)=\frac{\mathrm{Li}_n(e^{iz})-\mathrm{Li}_n(e^{-iz})}{2i}$$

Quite involved expression… however…

Double soft

Different color structure: single-correlated (T^2) and double-correlated (T^4)
[Catani, Grazzini '99]



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$$\langle S_{45} F_{\rm LM}^{4>5}(4,5)\rangle_{T^4} =$$
$$= \frac{1}{2}\left\langle \int [dp_4][dp_5]\Theta(E_4 - E_5) \sum_{i\neq j}^{n} S_{ij}(p_4) \sum_{k\neq m}^{n} S_{km}(p_5) \{\mathbf{T}_i\cdot\mathbf{T}_j, \mathbf{T}_k\cdot\mathbf{T}_l\}\, F_{\rm LM}\right\rangle$$
$$= \frac{[\alpha_s]^2}{2}\left\langle I_{1,R}^2(\epsilon)\, F_{\rm LM}\right\rangle,$$

Same structure as NLO with color-correlations

$$\boxed{I_{1,R}(\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{n} \eta_{ij}^{-\epsilon} K_{ij}\,\mathbf{T}_i\cdot\mathbf{T}_j}$$

However the pole content can be expressed in a compact way

Different structures, but clearly related

$$\langle S_{45} F_{\rm LM}^{4>5}(4,5)\rangle_{T^2} = \left\langle \int [dp_4][dp_5]\Theta(E_4 - E_5) \sum_{i\neq j}^{n} S_{ij}(p_4,p_5)\, \mathbf{T}_i\cdot\mathbf{T}_j\, F_{\rm LM}\right\rangle$$
$$= [\alpha_s]^2 \left[\frac{C_A}{\epsilon^2} c_1(\epsilon) + \frac{\beta_0}{\epsilon} c_2(\epsilon) + \beta_0\, c_3(\epsilon)\right] \langle \tilde{I}_{1,R}(2\epsilon) F_{\rm LM}\rangle + \langle S_{45} F_{\rm LM}^{4>5}(4,5)\rangle_{T^2}|_{\rm fin},$$

Another structure with color-correlations, but different wrt NLO

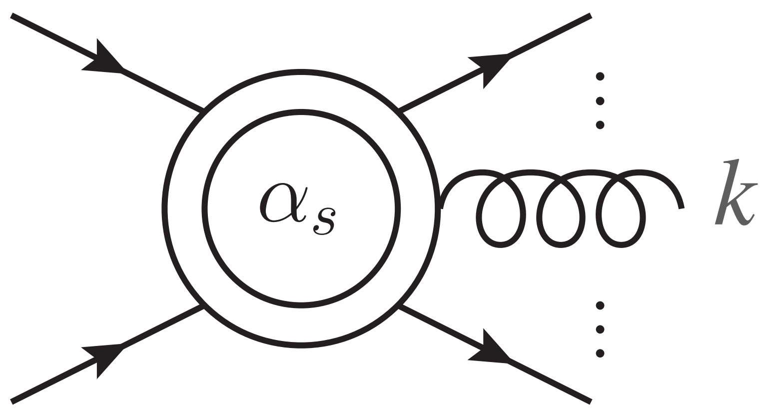
$$\boxed{\tilde{I}_{1,R}(2\epsilon) = -\frac{(2E_{\rm max}/\mu)^{-4\epsilon}}{(2\epsilon)^2}\sum_{i\neq j}^{n} \eta_{ij}^{-2\epsilon} \tilde{K}_{ij}\,\mathbf{T}_i\cdot\mathbf{T}_j}$$

$$c_1(\epsilon) = 1 + \left(\frac{\pi^2}{6} - \frac{32}{9}\right)\epsilon^2 + \left(\frac{217}{27} - \frac{137}{9}\log 2 - 22\log^2 2 + \frac{11\zeta_3}{2}\right)\epsilon^3$$
$$c_2(\epsilon) = 1 + \frac{\pi^2}{3}\epsilon^2, \qquad c_3(\epsilon) = 4\log 2 + 8\epsilon \log^2 2\,.$$

$$\boxed{\tilde{I}_{1,R}(2\epsilon) = I_{1,R}(2\epsilon) + \mathcal{O}(\epsilon)}$$

Soft real-virtual

Universal IR structure of RV-corrections under soft limit *[Catani, Grazzini 0007142]*



$$S_k F_{\mathrm{LRV}}(1\dots n|k) = -g_{s,b}^2 \sum_{l,m=1}^{n} \Bigg\{ \mathcal{S}_{lm}(k)\, F_{\mathrm{LV}}^{(lm)} - \frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\,\mathcal{S}_{lm}(k)\,F_{\mathrm{LM}}^{(lm)} - [\alpha_s]\,C_A\,A_K\,2^{-\epsilon}\Big(\mathcal{S}_{lm}(k)\Big)^{1+\epsilon} F_{\mathrm{LM}}^{(lm)}$$

$$+ [\alpha_s]\frac{2^{-\epsilon}\pi\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)} \sum_{\substack{p=1\\ p\neq l,m,k}}^{n} \mathcal{S}_{lm}(k)\Big(\mathcal{S}_{mp}(k)\Big)^{\epsilon} F_{\mathrm{LM}}^{(lmp)} \Bigg\},$$

$$A_K = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)}$$

$$\mathcal{S}_{ab}(k) = \frac{p_a\cdot p_b}{p_a\cdot k\; p_b\cdot k}$$

$$F_{\mathrm{LV}}^{(lm)} = 2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j|\mathcal{M}_1\rangle$$
$$= \frac{\alpha(\mu)}{2\pi}\,2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j\,\mathcal{I}_1(\epsilon)|\mathcal{M}_0\rangle + \frac{\alpha(\mu)}{2\pi}\,2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j|\mathcal{M}_{1,\mathrm{fin}}\rangle$$

$$F_{\mathrm{LM}}^{(lmp)} \sim \langle\mathcal{M}^{(0)}|\sum_{a,b,c} f_{abc}\,\mathbf{T}_l^a\,\mathbf{T}_m^b\,\mathbf{T}_p^c\,|\mathcal{M}^{(0)}\rangle \longrightarrow$$

New color structure -> finite after integration over unresolved PS,

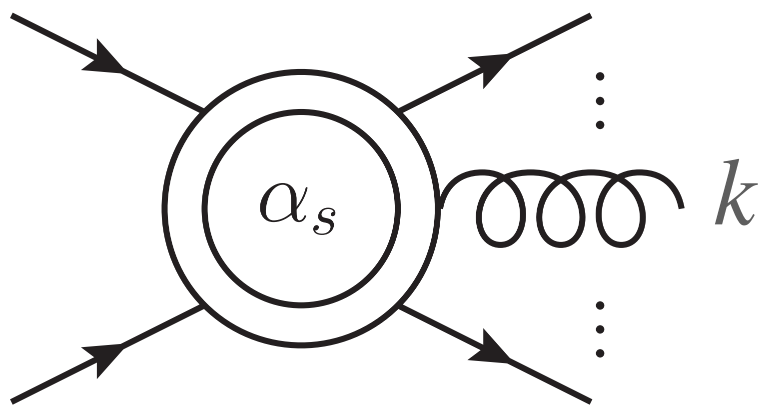
Vanishes for 3 partons at Born-level

After performing the integration over the unresolved parton phase space we get a compact expression:

$$\big\langle S_4\,F_{\mathrm{LRV}}(4)\big\rangle = \frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle I_{1,R}(\epsilon)\Big[2\Re(\mathcal{I}_1(\epsilon))F_{\mathrm{LM}} + F_{\mathrm{LV}}^{\mathrm{fin}}\Big]\Big\rangle$$
$$-[\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle I_{1,R}(\epsilon)F_{\mathrm{LM}}\Big\rangle - [\alpha_s]^2\,C_A\,A_K\,\Big\langle\widetilde{I}_{1,R}(2\epsilon)F_{\mathrm{LM}}\Big\rangle$$

Soft real-virtual

Universal IR structure of RV-corrections under soft limit *[Catani, Grazzini 0007142]*



$$S_k F_{\mathrm{LRV}}(1\ldots n|k) = -\,g_{s,b}^2 \sum_{l,m=1}^{n} \left\{ \mathcal{S}_{lm}(k)\, F_{\mathrm{LV}}^{(lm)} - \frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\, \mathcal{S}_{lm}(k)\, F_{\mathrm{LM}}^{(lm)} - [\alpha_s]\, C_A\, A_K\, 2^{-\epsilon} \Big(\mathcal{S}_{lm}(k)\Big)^{1+\epsilon} F_{\mathrm{LM}}^{(lm)} \right.$$

$$\left. + [\alpha_s] \frac{2^{-\epsilon}\pi\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)} \sum_{\substack{p=1\\ p\neq l,m,k}}^{n} \mathcal{S}_{lm}(k)\Big(\mathcal{S}_{mp}(k)\Big)^{\epsilon} F_{\mathrm{LM}}^{(lmp)} \right\},$$

$$A_K = \frac{\Gamma^3(1+\epsilon)\,\Gamma^5(1-\epsilon)}{\epsilon^2\,\Gamma(1+2\epsilon)\,\Gamma^2(1-2\epsilon)}$$

$$\mathcal{S}_{ab}(k) = \frac{p_a\cdot p_b}{p_a\cdot k\; p_b\cdot k}$$

$$F_{\mathrm{LV}}^{(lm)} = 2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j|\mathcal{M}_1\rangle$$

$$= \frac{\alpha(\mu)}{2\pi}\, 2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j\,\mathcal{I}_1(\epsilon)|\mathcal{M}_0\rangle + \frac{\alpha(\mu)}{2\pi}\, 2\Re\langle\mathcal{M}_0|\mathbf{T}_i\cdot\mathbf{T}_j|\mathcal{M}_{1,\mathrm{fin}}\rangle$$

$$F_{\mathrm{LM}}^{(lmp)} \sim \langle\mathcal{M}^{(0)}|\sum_{a,b,c} f_{abc}\,\mathbf{T}_l^a\,\mathbf{T}_m^b\,\mathbf{T}_p^c\,|\mathcal{M}^{(0)}\rangle \longrightarrow$$

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$$- [\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle I_{1,R}(\epsilon)F_{\mathrm{LM}}\Big\rangle - [\alpha_s]^2 C_A\, A_K\,\Big\langle \widetilde{I}_{1,R}(2\epsilon)F_{\mathrm{LM}}\Big\rangle$$

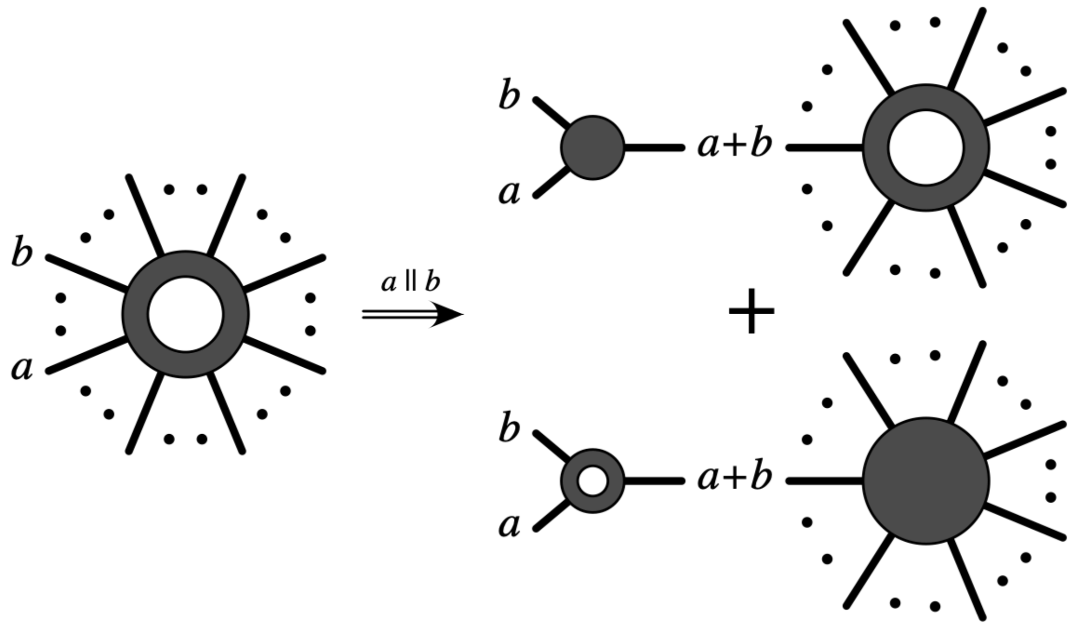
Structures and **color coefficients** already encountered in **double-virtual** and **double-soft**.

A pattern begins to arise…

Hard-collinear real-virtual and single soft RR

Also in this case the IR structure is know in full generality *[Kosower '99, Bern, Del Duca et al. '99]*.

For $q\bar{q} \to V + ggg$ the integrated contribution reads



$$
\begin{aligned}
\sum_{i=1}^{3} \langle (I-S_4) C_{4i}\, \Delta^{(4)}\, F_{\rm LV}(4) \rangle = {} & [\alpha_s]^2 \left\langle I_C(\epsilon)\, 2\Re\big(\bar{\mathcal{I}}_1(\epsilon)\big) F_{\rm LM} \right\rangle + \frac{\alpha_s}{2\pi}[\alpha_s] \left\langle I_C(\epsilon) F_{\rm LV}^{\rm fin} \right\rangle \\
& - [\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon} \left\langle I_C(\epsilon) F_{\rm LM} + \sum_{k=1}^{2} P_{qq}^{\rm gen}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle \\
& + [\alpha_s]^2 \left\langle \Gamma_g^{\rm 1loop}\, F_{\rm LM} \right\rangle + \frac{[\alpha_s]^2}{\epsilon} \sum_{i=k}^{2} \left\langle P_{qq}^{\rm 1loop} \otimes F_{\rm LM}^{(k)}(z) \right\rangle \\
& + [\alpha_s]^2 \sum_{k=1}^{2} \left\langle P_{qq}^{\rm gen}(z) \otimes 2{\rm Re}\big(\bar{\mathcal{I}}_1(z,\epsilon)\big) F_{\rm LM}^{(k)}(z) \right\rangle + [\alpha_s]\frac{\alpha_s}{2\pi} \sum_{k=1}^{2} \left\langle P_{qq}^{\rm gen}(z) \otimes F_{\rm LV}^{{\rm fin},(k)}(z) \right\rangle
\end{aligned}
$$

Same structure as NLO

One-loop splitting functions, known analytically

Single soft: different subtraction terms combined → careful with the limits order

$$
\begin{aligned}
\sum_{i=1}^{3} \Big\langle (I-S_4) C_{4i} & \left[\left\langle S_5\, \Delta^{(45)}\, F_{\rm LM}^{4>5}(4,5) \right\rangle \right] + S_5\, (I-S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\rm LM}^{5>4}(4,5) \Big\rangle = \\
& + [\alpha_s]^2 \sum_{k=1}^{2} \left\langle I_{1R}(\epsilon) P_{qq}^{\rm gen}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle + [\alpha_s]^2 \left\langle I_{1R}(\epsilon)\, I_C(\epsilon)\, F_{\rm LM} \right\rangle \\
& + \frac{[\alpha_s]^2}{\epsilon^2} N_s\, C_A \left[\sum_{k=1}^{2} \left\langle \left(\frac{2E_k}{\mu}\right)^{-2\epsilon} \tilde{P}_{qq}^{\rm gen,}(z) \otimes F_{\rm LM}^{(k)}(z) \right\rangle + \sum_{k=1}^{3} \left\langle \left(\frac{2E_k}{\mu}\right)^{-2\epsilon} \hat{\Gamma}^{(k)\,{\rm e.o.}}\, F_{\rm LM} \right\rangle \right]
\end{aligned}
$$

Status so far

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R n_f\,.$$

$\langle F_{\mathrm{LVV}} \rangle$	$\frac{1}{2}\left[2\Re(\mathcal{I}_1(\epsilon))\right]^2$	$\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(\epsilon))$	$K\, 2\Re(\mathcal{I}_1(2\epsilon))$	$\frac{\beta_0}{\epsilon}\, 2\Re(\mathcal{I}_1(2\epsilon))$	
$\langle S_{45} F_{\mathrm{LM}}^{4>5}(4,5) \rangle$	$\frac{1}{2}\, I_{1,R}^2(\epsilon)$		$\frac{C_A}{\epsilon^2}\, \widetilde{I}_{1,R}(2\epsilon)$	$\frac{\beta_0}{\epsilon}\, \widetilde{I}_{1,R}(2\epsilon)$	$\beta_0\, \widetilde{I}_{1,R}(2\epsilon)$
$\langle S_4\, F_{\mathrm{LRV}}(4) \rangle$	$I_{1,R}(\epsilon)\, 2\Re(\mathcal{I}_1(\epsilon))$	$\frac{\beta_0}{\epsilon}\, I_{1,R}(\epsilon)$	$C_A\, A_K\, \widetilde{I}_{1,R}(2\epsilon)$		
$\langle (I - S_4) C_{4i}\, \Delta^{(4)}\, F_{\mathrm{LV}}(4) \rangle$	$I_C(\epsilon)\, 2\Re(\bar{\mathcal{I}}_1(\epsilon))$	$\frac{\beta_0}{\epsilon}\, I_C(\epsilon)$			
$\Big\langle (I - S_4) C_{4i} \left[\langle S_5\, \Delta^{(45)}\, F_{\mathrm{LM}}^{4>5}(4,5) \rangle \right] + S_5\, (I - S_4)\, C_{4i}\, \Delta^{(45)}\, F_{\mathrm{LM}}^{5>4}(4,5) \Big\rangle$	$I_{1R}(\epsilon)\, I_C(\epsilon)$				

A term $I_C^2(\epsilon)$ needed to reconstruct $(I_1 + I_{1,R} + I_C)^2$

$\rightarrow$ look at double-collinear

reconstruct $I_1(\epsilon) + I_{1,R}(\epsilon) + I_C(\epsilon)$ but with extra $1/\epsilon$

Clear interplay $\rightarrow$ $C_A, 2\epsilon$

non-transparent cancellation

Suggest $I_1(2\epsilon) + I_{1,R}(2\epsilon) + I_C(2\epsilon)$ but with extra $1/\epsilon$

Hard-collinear real-virtual and single soft RR

Manipulations required to reconstruct recurring structures and match, for instance, PDFs-like corrections

$$\frac{1}{2}\Big\langle \sum_{i,j}(I-S_4)\,(I-S_5)\,C_{4i}C_{5j}\,\Delta^{(45)}F_{\rm LM}(4,5)\Big\rangle = \Big\langle \frac{1}{2}[\alpha_s]^2\Big(I_C(\epsilon)\Big)^2 F_{\rm LM} + \sum_{k=1}^{2} G^{(k)}(z)\,F_{\rm LM}^{(k)}(z) + G^{(3)}\,F_{\rm LM}$$
$$+\frac{1}{2}\,[\alpha_s]^2\sum_{k=1}^{2}\big[P_{qq}^{\rm gen}\otimes P_{qq}^{\rm gen}(z)\big]_{\rm pdf}\,F_{\rm LM}^{(k)}(z) + [\alpha_s]^2\sum_{k=1}^{2}P_{qq}^{\rm gen}\otimes I_C(z,\epsilon)F_{\rm LM}^{(k)}(z)$$
$$+\,[\alpha_s]^2\,P_{qq}^{\rm gen}(z_1)\otimes F_{\rm LM}(z_1,z_2)\otimes P_{qq}^{\rm gen}(z_2)\Big\rangle$$

Cancellation of the double-color-correlated contributions

$$\frac{1}{2}\Big\langle\Big(\frac{\alpha_s}{2\pi}\,2\Re\big(\mathcal{I}_1(\epsilon)\big)+[\alpha_s]I_{1,R}(\epsilon)+[\alpha_s]I_C(\epsilon)\Big)^2\,F_{\rm LM}\Big\rangle = \frac{1}{2}[\alpha_s]^2\Big\langle I_{1,T}^2(\epsilon)\,F_{\rm LM}\Big\rangle$$
$$\longrightarrow \text{ finite}$$

Same combination encountered at NLO:
finite, and easy to be computed.

Conclusions

1. Subtraction schemes are necessary ingredients to obtain precise theoretical predictions.

2. **Nested-soft collinear subtraction** provides an efficient method to deal with n-parton processes:

   I. combine different subtraction terms **to get rid of color-correlations** (and boosted contributions),

   II. reduce the subtraction terms to **few, recurring structures**.

3. Pole cancellation proven analytically for the case-study $q\bar{q} \to V + ggg$.

Finite remainders in agreement with the “old-fashion approach”

Work in progress

1. Generalisation to $q\bar{q} \to V + ng$

2. Generalisation to arbitrary final- and initial-state partons.

Thank you!

Backup

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\boxed{\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]}F_{\rm LM}\right\rangle$$

$$\boxed{\frac{\beta_0}{\epsilon}[\alpha_s]I_{1,T}(\epsilon)}$$

$$+\boxed{\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle}+[\alpha_s]^2\,\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\boxed{\frac{\beta_0}{\epsilon}\left\langle I_{1,T}(2\epsilon)\,F_{\rm LM}\right\rangle}\underbrace{-\frac{\alpha_s}{2\pi}[\alpha_s]\frac{\beta_0}{\epsilon}\left\langle I_C(2\epsilon)\,F_{\rm LM}\right\rangle+\Sigma^{(1)}_{T_i\cdot T_j,\,\rm fin}}_{\text{No singular, color-correlated contributions}}$$

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\boxed{\frac{\beta_0}{\epsilon}\Big\langle\Big[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\Big]}F_{\rm LM}\Big\rangle$$

$$+\boxed{\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle}+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\Big\langle\Big[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\boxed{\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\Big]F_{\rm LM}}\Big\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle\underset{\text{finite}}{\boxed{c_\epsilon\,K\,I_{1,T}(2\epsilon)}}\,F_{\rm LM}\Big\rangle-\underbrace{\frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle c_\epsilon\,K\,I_{1,R}(2\epsilon)\,F_{\rm LM}\Big\rangle}_{\text{Singular and color-correlated}}-\underbrace{\frac{\alpha_s}{2\pi}[\alpha_s]\Big\langle c_\epsilon\,K\,I_C(2\epsilon)\,F_{\rm LM}\Big\rangle}_{\text{color-uncorrelated}}$$

Cancellation of single-color-correlated contributions

$$\widetilde{I}_{1,R}(2\epsilon) \longrightarrow I_{1,R}(2\epsilon)$$

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\left\langle\left[[\alpha_s] I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\right]F_{\rm LM}\right\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+\underbrace{[\alpha_s]^2\,\beta_0\,c_3(\epsilon)}_{\text{finite}}\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\left\langle\left[-[\alpha_s]^2 C_A A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\boxed{\left(\frac{\alpha_s}{2\pi}\right)^2 c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\right]F_{\rm LM}}\right\rangle$$

$$\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle\underbrace{\boxed{c_\epsilon\,K\,I_{1,T}(2\epsilon)}}_{\text{finite}}F_{\rm LM}\right\rangle-\underbrace{\frac{\alpha_s}{2\pi}[\alpha_s]\left\langle c_\epsilon\,K\,I_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle}_{\text{Singular and color-correlated}}-\frac{\alpha_s}{2\pi}\underbrace{[\alpha_s]\left\langle c_\epsilon\,K\,I_C(2\epsilon)\,F_{\rm LM}\right\rangle}_{\text{color-uncorrelated}}$$

$$-C_A A_K+\frac{C_A}{\epsilon^2}c_1 \quad \text{finite}$$

$$\widetilde{I}_{1,R}(2\epsilon) \longrightarrow I_{1,R}(2\epsilon)$$

Cancellation of single-color-correlated contributions

$$-\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\Big\langle\Big[[\alpha_s]I_{1,R}(\epsilon)+\frac{\alpha_s}{2\pi}2\Re\left(\mathcal{I}_1(\epsilon)\right)+I_C(\epsilon)\Big]F_{\rm LM}\Big\rangle$$

$$+\left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{\epsilon}c_\epsilon\left\langle 2\Re(\mathcal{I}_1(2\epsilon))\,F_{\rm LM}\right\rangle+[\alpha_s]^2\frac{\beta_0}{\epsilon}c_2(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle+[\alpha_s]^2\beta_0\,c_3(\epsilon)\left\langle\widetilde{I}_{1,R}(2\epsilon)\,F_{\rm LM}\right\rangle$$

$$+\Big\langle\Big[-[\alpha_s]^2C_A\,A_K\,\widetilde{I}_{1,R}(2\epsilon)+[\alpha_s]^2\frac{C_A}{\epsilon^2}c_1(\epsilon)\,\widetilde{I}_{1,R}(2\epsilon)+\left(\frac{\alpha_s}{2\pi}\right)^2c_\epsilon\,K\,2\Re\left(\mathcal{I}_1(2\epsilon)\right)\Big]F_{\rm LM}\Big\rangle$$

$$[\alpha_s]\frac{\alpha_s}{2\pi}\frac{\beta_0}{\epsilon}\overbrace{\Big\langle\Big(I_{1,T}(2\epsilon)-I_{1,T}(\epsilon)\Big)F_{\rm LM}\Big\rangle}^{1/\epsilon\text{ color-uncorrelated}}$$

$$+\overbrace{\Big\langle\Big[[\alpha_s]^2\Big(-C_A\,A_K+\frac{C_A}{\epsilon^2}c_1(\epsilon)+\beta_0\,c_3(\epsilon)\Big)-\frac{\alpha_s}{2\pi}[\alpha_s]\,c_\epsilon\,K\Big]I_{1,R}(2\epsilon)\,F_{\rm LM}\Big\rangle}^{\propto\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\frac{1}{\epsilon}[\text{color}-\text{correlations}]}$$

$$-\frac{\alpha_s}{2\pi}[\alpha_s]\underbrace{\Big\langle\Big(c_\epsilon\,K+\frac{\beta_0}{\epsilon}\Big)I_C(2\epsilon)\,F_{\rm LM}\Big\rangle}_{1/\epsilon^2\text{ color-uncorrelated}}$$

Peculiar dependence in the color-correlations, that fits perfectly a contribution from triple-collinear sectors $\Theta^{(b)}$

$$\Big\langle\sum_{i\in{\rm TC}}(I-S_{45})\,C_{45}\Theta^{(b)}(F_{\rm LM}-2S_5F_{\rm LM}^{4>5})\,\omega_{4i5i}\,\Delta^{(45)}\Big\rangle$$

$$\longrightarrow\;-4[\alpha_s]^2C_A2^{-2\epsilon}\delta_g(\epsilon)\Big\langle I_{1,R}(2\epsilon)F_{\rm LM}\Big\rangle+\Sigma^{(2)}_{T_i\cdot T_j,\,{\rm fin}}\;\propto-\frac{C_A(C_A+2C_F)}{\epsilon^2}\left(-\frac{131}{72}+\frac{\pi^2}{6}+\frac{11}{6}\log 2\right)+\mathcal{O}(\epsilon^{-1})$$

Useful relations:

$$I_{1,R}(\epsilon) = -\frac{(2E_{\max}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{n}\eta_{ij}^{-\epsilon}K_{ij}\mathbf{T}_i\cdot\mathbf{T}_j\ ,$$

$$\begin{aligned} K_{ij} &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) \\ &= \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-\eta_{ij}) \end{aligned}$$

$$\widetilde{I}_{1,R}(2\epsilon) = -\frac{(2E_{\max}/\mu)^{-4\epsilon}}{(2\epsilon)^2}\sum_{i\neq j}^{n}\eta_{ij}^{-2\epsilon}\widetilde{K}_{ij}\mathbf{T}_i\cdot\mathbf{T}_j$$

$$\begin{aligned} \widetilde{K}_{ij} &= \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\,\eta_{ij}^{1+3\epsilon}{}_2F_1(1+\epsilon,1+\epsilon,1-\epsilon,1-\eta_{ij}) \\ &= \frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\,{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})\ . \end{aligned}$$

$$\widetilde{K}_{ij}(\epsilon) = K_{ij}(2\epsilon)\left[\frac{{}_2F_1(-2\epsilon,-2\epsilon;1-\epsilon,1-\eta_{ij})}{{}_2F_1(-2\epsilon,-2\epsilon,1-2\epsilon,1-\eta_{ij})}\right] = K_{ij}(2\epsilon)\left[1+\mathcal{O}(\epsilon^3)\right]$$

$$\tilde{I}_{1,R}(2\epsilon) = I_{1,R}(2\epsilon) + \mathcal{O}(\epsilon)$$

Useful definitions:

$$\hat{\Gamma}_q = \frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{2E_1}{\mu}\right)^{-2\epsilon}\left[\gamma_q + \frac{C_F}{\epsilon}(1-e^{-2\epsilon L_1})\right]F_{\rm LM}(1\ldots N) \sim \frac{1}{\epsilon}(\gamma_q + 2C_F\, L_1) + \mathcal{O}(\epsilon^0)$$

$$\hat{\Gamma}_g = \frac{1}{\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\gamma^{22}_{z,g\to gg} + \frac{1}{\epsilon}(1-e^{-2\epsilon L_n})\right] \qquad \gamma^{22}_{z,g\to gg} \sim \frac{11}{6} + \frac{1}{9}\left(67-6\pi^2\right)\epsilon + \ldots$$

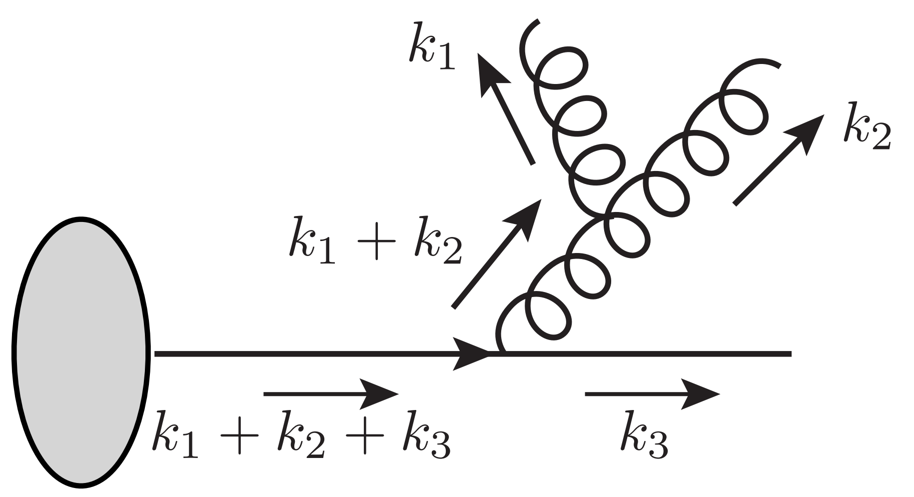
$$\hat{\Gamma}_g(2\epsilon) = \frac{1}{2\epsilon}C_A\left(\frac{2E_n}{\mu}\right)^{-4\epsilon}\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-4\epsilon)}\left[\gamma^{44}_{z,g\to gg} + \frac{1}{2\epsilon}(1-e^{-4\epsilon L_n})\right]$$

$$P^{\rm gen}_{qq}(z) = -\frac{1}{\epsilon}\hat{P}^{\rm AP,0}_{qq}(z) + P'_{\rm fin,qq}(z)$$

$$G^{(1)}(z)\,F^{(1)}_{\rm LM} = \frac{1}{2}[\alpha_s]^2\left[-P^{\rm gen}_{qq}\otimes\Gamma^{(1)}_q(z)F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z) + \Gamma^{(1)}_q P^{\rm gen}_{qq}\otimes F^{(1)}_{\rm LM}(1_q,2_{\bar{q}};3_g|z)\right]$$

$$G^{(3)}(L_3) = \frac{1}{2}\frac{[\alpha_s]^2}{\epsilon^2}C_A^2\left(\frac{2E_3}{\mu}\right)^{-4\epsilon}\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)^2\left(\gamma^{22}_{z,g\to gg}+\frac{1}{\epsilon}\right)\left(\gamma^{42}_{z,g\to gg}-\gamma^{22}_{z,g\to gg}\right)$$

1. Clear understanding of which singular configurations do actually contribute



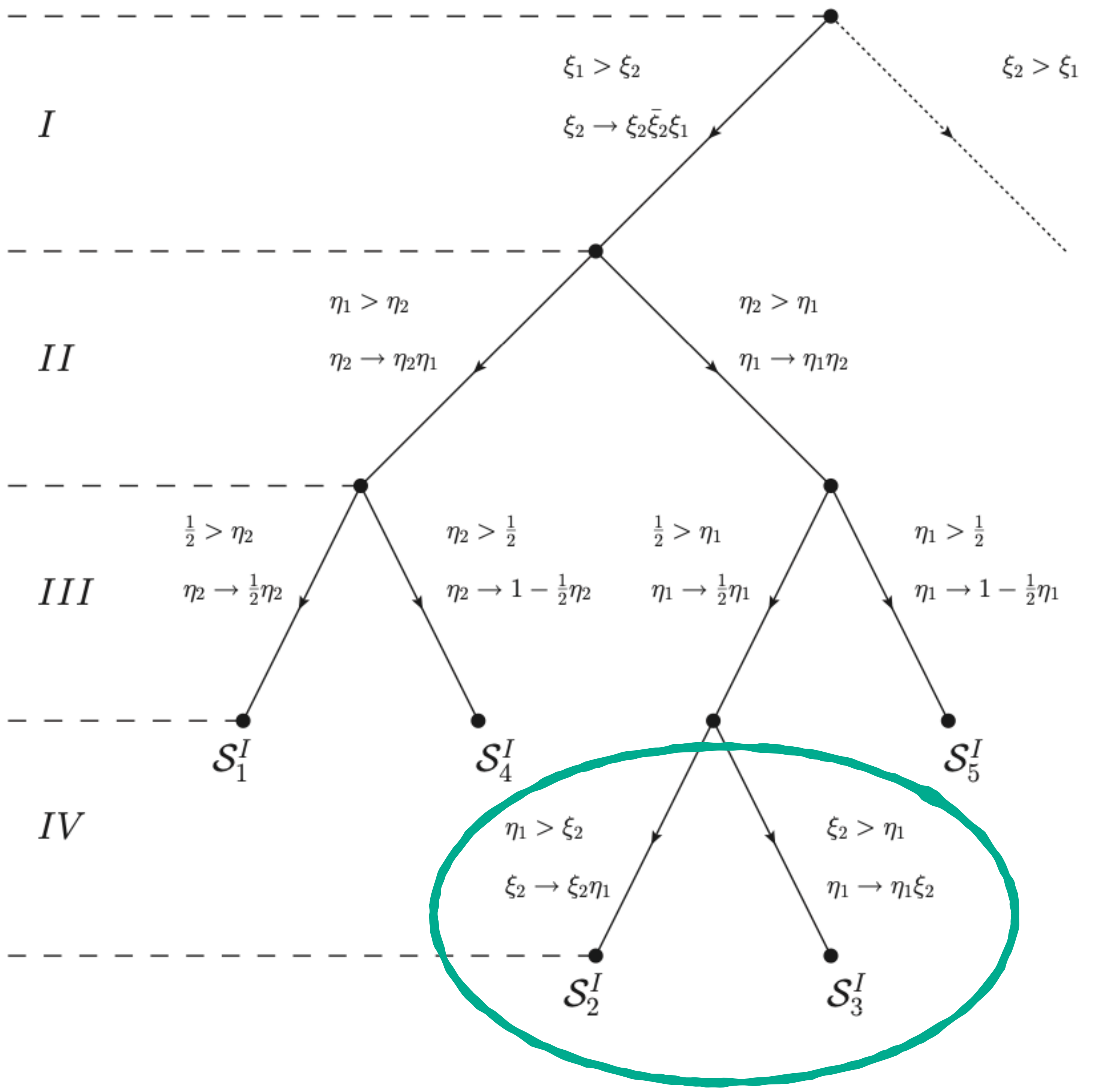
$$\sim \frac{1}{(k_1+k_2)^2}\,\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\,\frac{1}{2k_1\cdot k_2+2k_1\cdot k_3+2k_2\cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.
Do non-commutative limits actually contribute?

STRIPPER was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included

Gauge invariant amplitudes are free of entangled singularities thanks to color coherence: soft parton does not resolve angles of the collinear partons

Soft-collinear limits can be described by taking the known soft and collinear limits sequentially

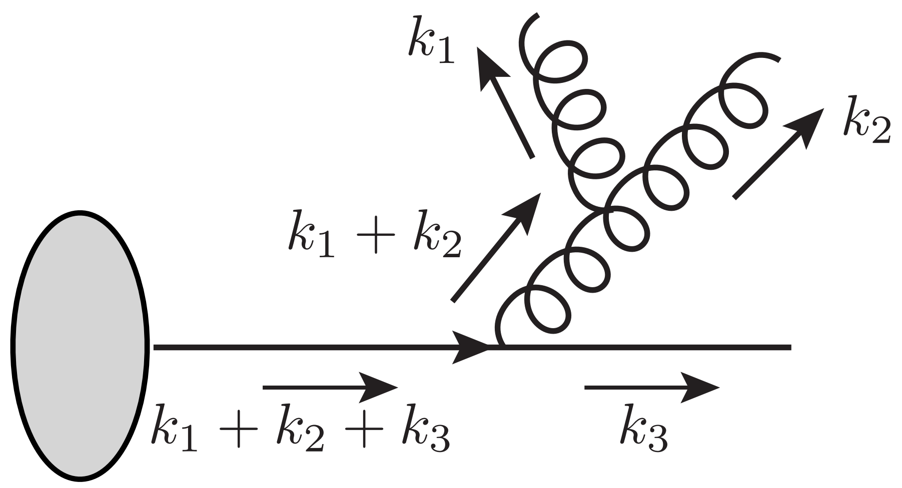


[Czakon 1005.0274]

2. Get to the point where the problem is well defined

a) Identify the overlapping singularities

b) Regulate them

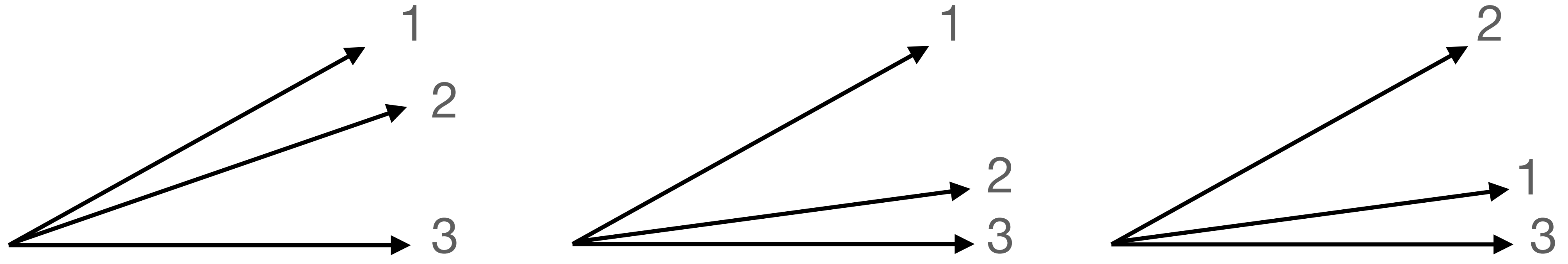


$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

$$\overbrace{E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0}^{\text{Soft origin}} \qquad \overbrace{\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3}^{\text{Collinear origin}}$$

$$E_1 \ll E_2 \,, \quad E_2 \ll E_1$$

Includes **strongly ordered** configurations



$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated. Strongly ordered configurations have to be properly taken into account.

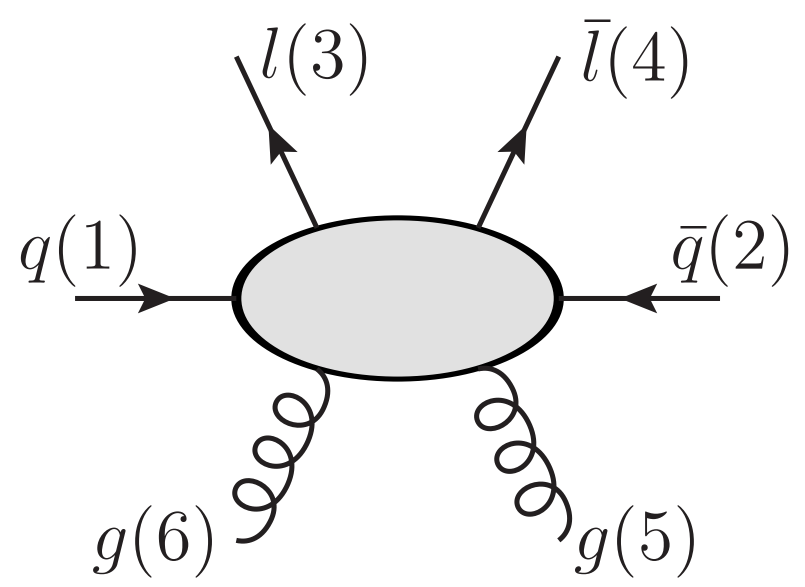
Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Melnikov, Röntsch 1702.01352]*



$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}}$$

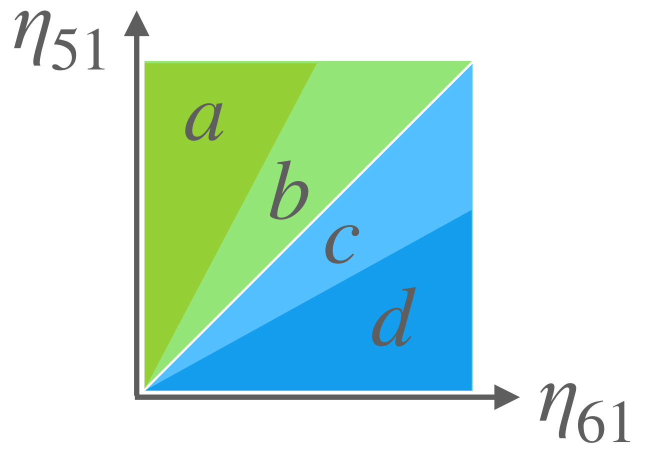
$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$d_{i=5,6} = \rho_{1i} + \rho_{2i} = 2$$

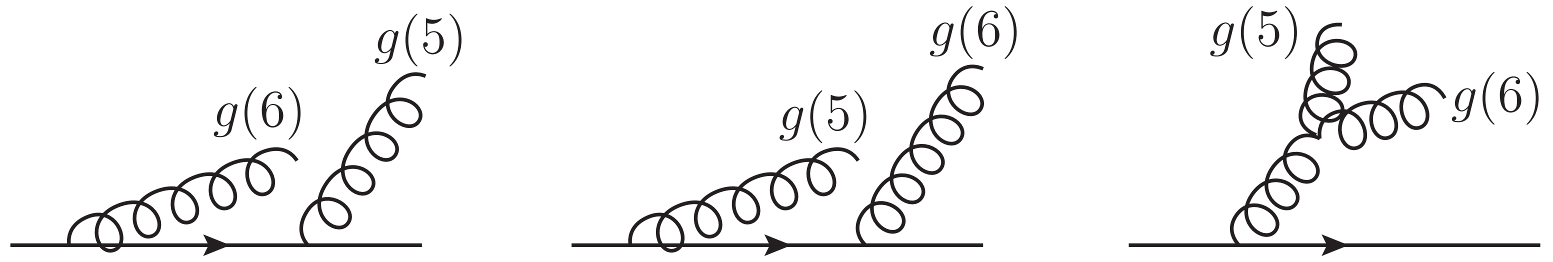
$$d_{5621} = \rho_{56} + \rho_{52} + \rho_{61}$$

$$d_{5612} = \rho_{56} + \rho_{51} + \rho_{62}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$



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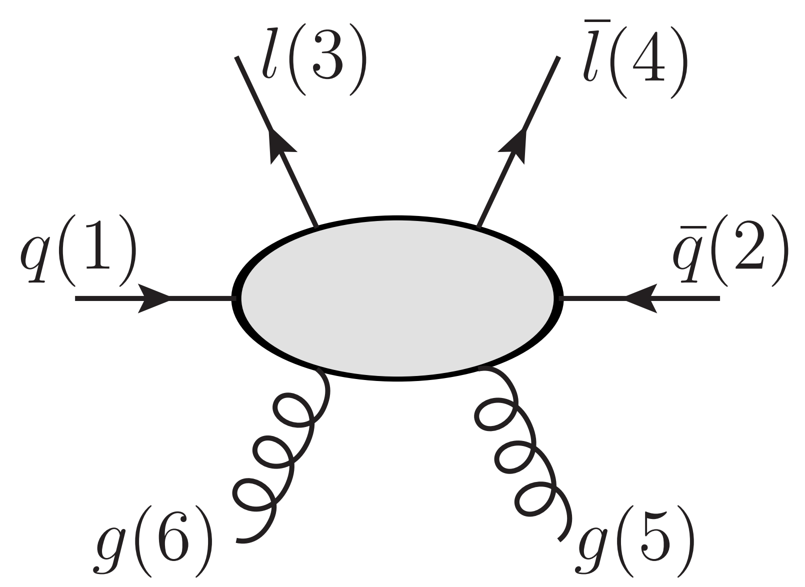

Advantages:

1. Simple definition
2. Structure of collinear singularities fully defined
3. Same strategy holds for NNLO mixed QCDxEW processes
4. **Minimum number of sector**

Disadvantages:

1. Partition based on angular ordering -> Lorentz invariance not preserved
   -> angles defined in a given reference frame
2. Theta function

3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2} \left[RR_{n+2} - K_{n+2}\right] + \int d\Phi_{n+2}\, K_{n+2} \qquad K_{n+2} \supset C_{ij},\, C_{kl},\, S_i,\, S_{ij},\, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\mathrm{unres.}} \otimes \mathrm{Limit} \otimes \mathrm{Constraints}$$

The 'Limit' component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

Different approximations and techniques can be applied: the results assume different form depending on the adopted strategy

Two main structure are the most complicated ones and affect most of the physical processes:

- **Double soft**
- **Triple collinear**

Kernels integration

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Delto, Frellesvig, Melnikov 1807.05835, Delto, Melnikov 1901.05213]*

Two soft parton (5,6) and two hard massless radiator (1,2): arbitrary relative angle between the three-momenta of the radiators

$$I_{12}^{(gg)(56)} = \frac{(1-\epsilon)(s_{51}s_{62} + s_{52}s_{61}) - 2s_{56}s_{12}}{s_{56}^2(s_{51} + s_{61})(s_{52} + s_{62})} + s_{12}\,\frac{s_{51}s_{62} + s_{52}s_{61} - s_{56}s_{12}}{s_{56}s_{51}s_{62}s_{52}s_{61}}\left[1 - \frac{1}{2}\,\frac{s_{51}s_{62} + s_{52}s_{61}}{(s_{51} + s_{61})(s_{52} + s_{62})}\right]$$

$$I_{S_{56}}^{(gg)} = \int [\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\theta(E_{\max} - E_5)\,\theta(E_5 - E_6)\,I_{12}^{(gg)(56)}\big(k_1, k_2, k_5, k_6\big) \qquad [\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}\,(2\pi)\,\delta_+(k_i^2)$$

$$E_5 = E_{\max}\,\xi \qquad E_6 = E_{\max}\,\xi\,z \qquad 0 < \xi < 1\,,\, 0 < z < 1$$

Reverse unitarity: **map phase space integrals onto loop integrals** *[Anastasiou, Melnikov 0207004]*
after defining integral families, integration-by-part identities. Differential equations w.r.t. the ratio of energies of emitted gluons at fixed angle.
Boundary conditions for z=0, and arbitrary angle