$B_c \to \eta_c$ Form Factors in the Double-Logarithmic Approximation

Philipp Böer

based on 2205.06021 with Guido Bell and Thorsten Feldmann
& "in progress" with Guido Bell, Thorsten Feldmann, Dennis Horstmann and Vladyslav Shtabovenko

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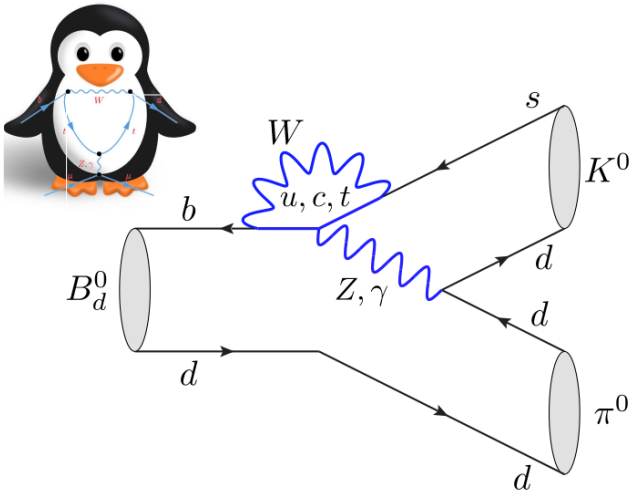
Mainz Institute for Theoretical Physics
JOHANNES GUTENBERG UNIVERSITÄT MAINZ

B physics in the precision era

B physics is a precision laboratory for tests of the Standard Model.



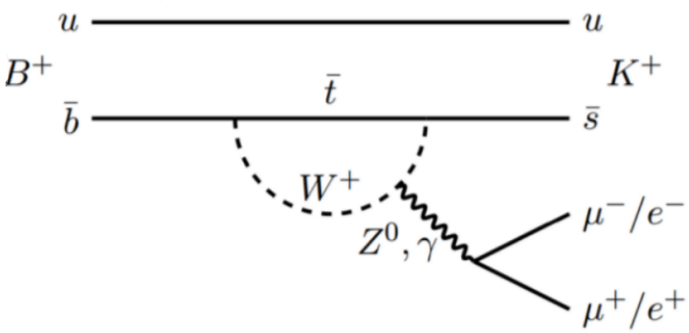


Fig. taken from 1609.02015

- Decays of B hadrons provide insights into flavour-changing interactions
  - $\rightarrow$ extraction of SM parameters
  - $\rightarrow$ probe of new physics through small quantum fluctuations
  - $\rightarrow$ observed anomalies in specific decay channels, e.g. $B \to \pi K$, $B \to K^* \ell\ell$, ...

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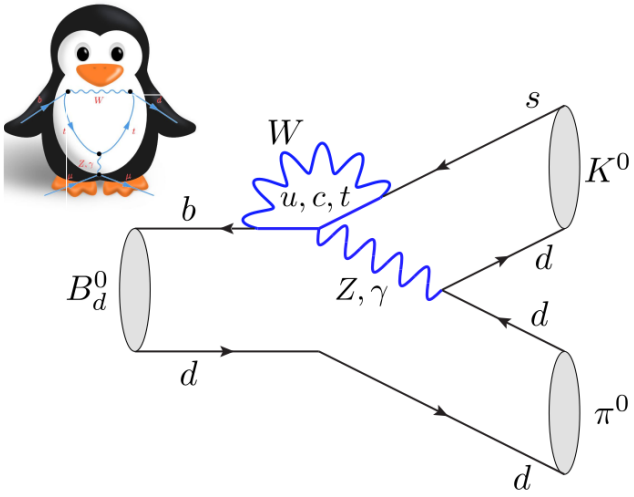

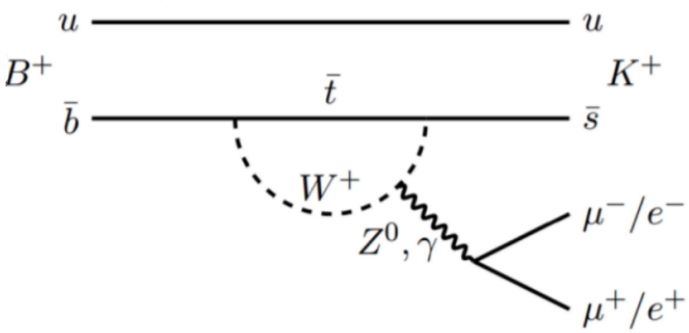

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  - $\rightarrow$ observed anomalies in specific decay channels, e.g. $B \to \pi K$, $B \to K^* \ell\ell$, ...

- Challenging QCD dynamics, in particular for charmless exclusive decays (at large recoil)
  - $\rightarrow$ factorization of decay amplitudes in the heavy-quark limit $m_b \sim E_{\pi,K} \to \infty$ [BBNS '99]
  - $\rightarrow$ clear separation of perturbative and non-perturbative physics
  - $\rightarrow$ **open conceptual problem**: endpoint-divergent convolution integrals can spoil factorization

This work

...is not (yet) about phenomenologically relevant precision calculations. We rather aim at:

- improving the theoretical methods used to describe hard-exclusive processes
- consistent treatment of endpoint-singularities in SCET
- understanding soft-collinear factorization at sub-leading power in $\Lambda_{\rm had}/m_b \simeq 0.2$

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Active field of current research

- $\rightarrow$ heavy-to-light form factors [PB '18] + [Bell,PB,Feldmann,Horstmann, to appear]
- $\rightarrow$ bottom induced $h \rightarrow \gamma\gamma(gg)$ decay [Neubert et al. '19-22]
- $\rightarrow$ off-diagonal gluon thrust [Beneke et al. '22]
- $\rightarrow$ μ-e backscattering [Bell,PB,Feldmann '22]
- $\rightarrow$ QED corrections in leptonic B decays [Feldmann et al. '22, Cornella et al. '23]
- $\rightarrow$ power-corrections in inclusive $\bar{B} \rightarrow X_s\gamma$ [Hurth, Szafron '23]

Recent progress has been made using refactorization ideas [PB '18] and additive rearrangements of the endpoint contributions [Liu/Neubert '19]. However, the problem is considerably more complicated in hadronic hard-exclusive processes [PB '18; Bell/PB/Feldmann '22]. A systematic treatment requires novel theoretical tools!

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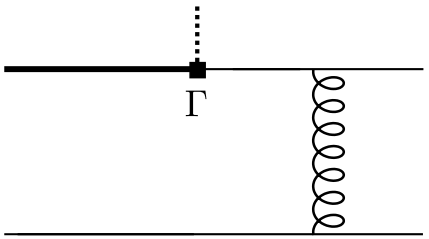
Idea: resum logarithms in $B_c \to \eta_c$ form factor at large recoil for non-relativistic bound states

- ✓ quark mass $m_c \gg \Lambda_{\rm QCD}$ provides physical IR cut-off
- ✓ relativistic dynamics at scales $\mu \gtrsim m_c$ perturbative
- ✓ provides one of the simplest setups to study a sub-leading power hard-exclusive reaction

Heavy-to-light form factors for NR bound states



Form factors parametrize hadronic matrix elements of weak currents, e.g.

$$\langle \eta_c(p)| \bar{c}\gamma^\mu b |B_c(p_B)\rangle = F_+(q^2)(p_B^\mu + p^\mu) + F_-(q^2)q^\mu$$

At large recoil energies $E_{\eta_c} \sim m_{B_c} \to$ soft-collinear factorization (SCET) [Beneke/Feldmann '00]

- ✓ works well for spin-symmetry violating terms
- × fails for spin-symmetric (universal) contribution due to endpoint singularities ("soft-overlap")

$$F_+^{\text{(non-fac)}} = \frac{1}{2E_{\eta_c}} \langle \eta_c(p)| \bar{c}\frac{\not{\bar{n}}\not{n}}{4} b |B_c(p_B)\rangle \qquad (p \simeq En \text{ with } n^2 = \bar{n}^2 = 0, n\cdot\bar{n} = 2)$$

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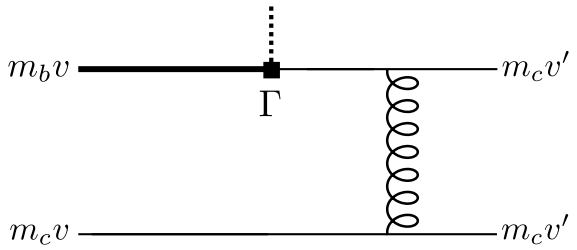

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In the NR limit $m_b \gg m_c \gg \Lambda$ the mesons are entirely dominated by 2-particle Fock states.

- $\to$ $2 \to 2$ scattering process of on-shell massive quarks with ext. current e.g. [Bell/Feldmann '05+'08, Bell '06]
- $\to$ radiative cor. induce double-logs $\sim \alpha_s \ln^2(2\gamma)$ ($v \cdot v' \equiv \gamma \gg 1$ is the large boost between the mesons)

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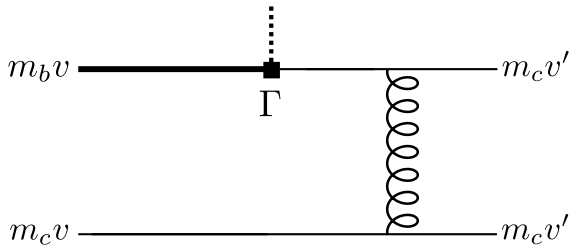

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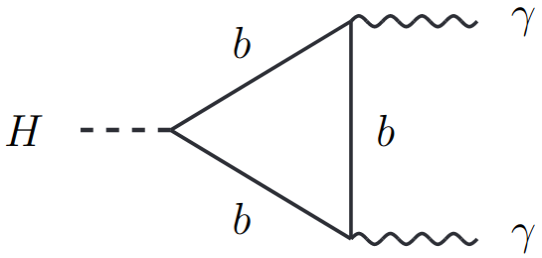
Goal of this work:

- "diagrammatic resummation" of the leading double-logs in the "soft-overlap" $\to$ non-standard!

Prelude I: $h \to \gamma\gamma$ via b-quark loop



- scale hierarchy $M_H \gg m_b$
- sub-leading power due to helicity suppression

The LO graph develops a double-log when the b-quark propagator between the two γ's becomes soft:

→ soft quark effectively goes on-shell: $1/(k^2 - m_b^2) \to -2\pi i\, \delta(k^2 - m_b^2)$

→ other quark propagators become eikonal (largely off-shell) $\Rightarrow$ $\int d^2k_\perp \sim \theta(k_+k_- - m_b^2)$

→ hard cut-offs through external photon momenta

$$i\mathcal{M}^{(0)} \sim \int_0^{m_H} \frac{dk_+}{k_+} \int_0^{m_H} \frac{dk_-}{k_-} \theta(k_+k_- - m_b^2) = \frac{L^2}{2} \qquad \text{with} \qquad L \equiv \ln m_b^2/m_H^2$$

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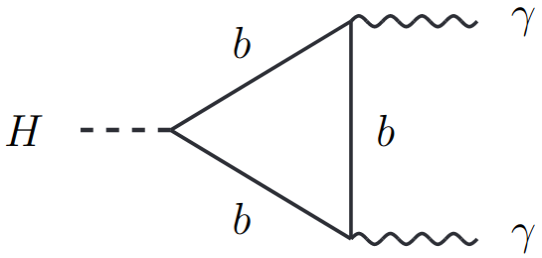

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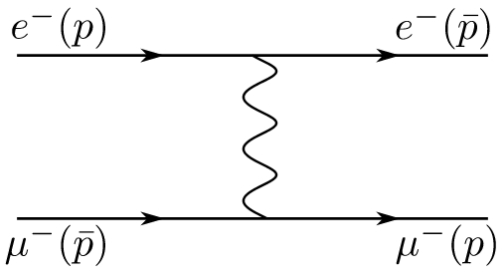
All-order result from soft-gluon exponentiation at hard $H \to (b\bar{b})^*$ vertex: [Kotsky/Yakovlev '98; Liu/Penin '18], SCET analysis in [Liu/Neubert/Mecaj/Wang 19-22]

$$i\mathcal{M}^{(\mathrm{DL})} \sim \int_0^{m_H} \frac{dk_+}{k_+} \int_0^{m_H} \frac{dk_-}{k_-} \theta(k_+k_- - m_b^2) \exp\left\{ -\frac{\alpha_s C_F}{2\pi} \ln \frac{k_+}{m_H} \ln \frac{k_-}{m_H} \right\}$$
$$= \frac{L^2}{2}\, {}_2F_2\left(1,1;\frac{3}{2},2;-\frac{\alpha_s C_F}{4\pi}\frac{L^2}{2}\right)$$

Prelude II: muon-electron backscattering



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- leading-power process

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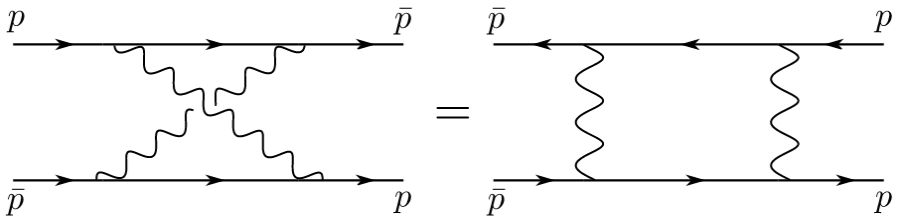

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In the sum of all NLO graphs, double-logarithms from soft-photon configurations cancel

$\rightarrow$ The NLO double log arises from a soft-lepton exchange (two soft lepton propagators due to special kinematics)

$$i\mathcal{M}^{(1)} \simeq i\mathcal{M}^{(0)} \int_0^{\sqrt{s}} \frac{dk_+}{k_+} \int_0^{\sqrt{s}} \frac{dk_-}{k_-} \theta(k_+k_- - m^2) = \frac{L^2}{2} \qquad \text{with} \qquad L \equiv \ln \lambda^2 \equiv \ln m^2/s$$

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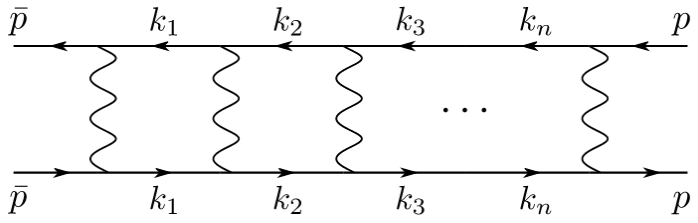

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Configuration is not helicity suppressed $\Rightarrow$ ladder diagrams iterate the one-loop double log

$\rightarrow$ all lepton propagators on shell. Longitudinal momenta are strongly ordered

$$\sqrt{s} \gg k_{n,-} \gg k_{n-1,-} \gg \cdots \gg k_{1,-} \gg m^2/\sqrt{s}$$
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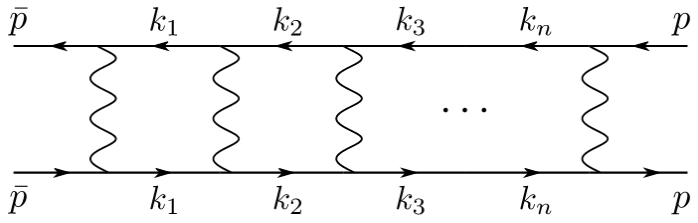

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$$\begin{aligned} i\mathcal{M}^{(n)} &\simeq i\mathcal{M}^{(0)} \int_{\lambda^2}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2}{x_2} \cdots \int_{x_{n-1}}^1 \frac{dx_n}{x_n} \int_{\lambda^2/x_1}^1 \frac{dy_1}{y_1} \int_{\lambda^2/x_2}^{y_1} \frac{dy_2}{y_2} \cdots \int_{\lambda^2/x_n}^{y_{n-1}} \frac{dy_n}{y_n} \\ &= i\mathcal{M}^{(0)} \frac{\ln^{2n} \lambda^2}{n!(n+1)!} \end{aligned}$$

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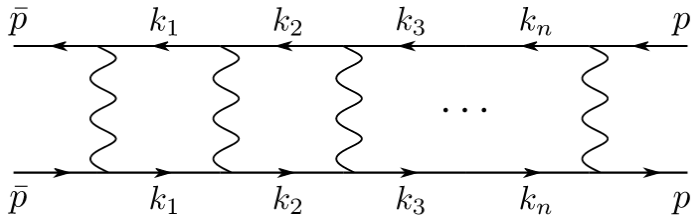

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resums to modified Bessel function [Gorshkov/ Gribov/Lipatov/Frolov 66], SCET analysis in [Bell/PB/Feldmann '22]

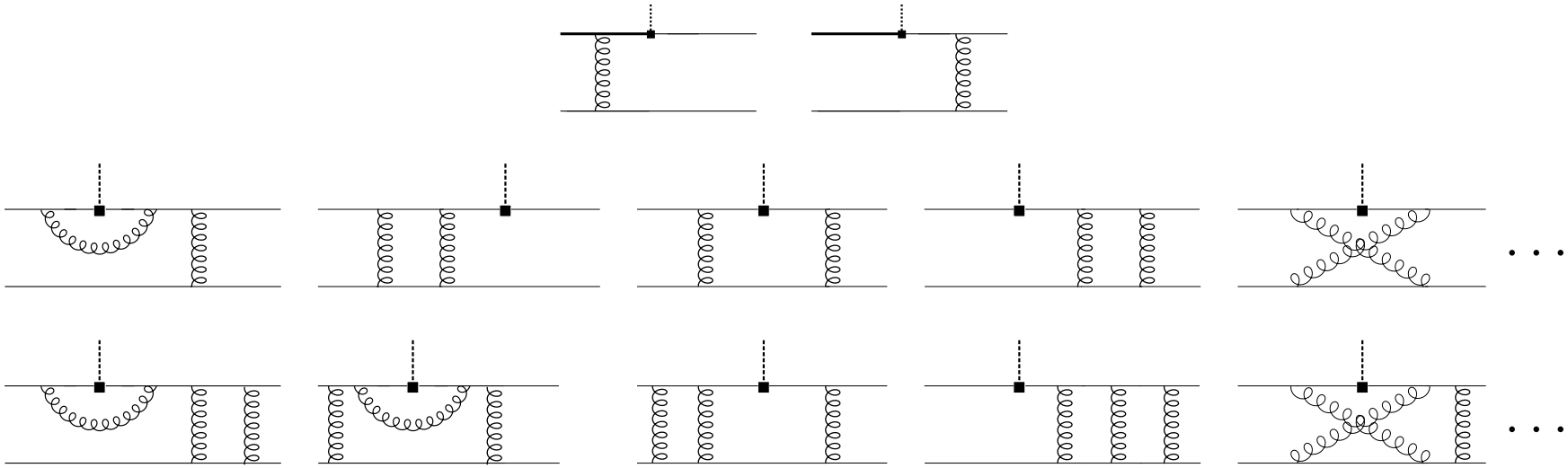
$$i\mathcal{M}^{(\mathrm{DL})} = i\mathcal{M}^{(0)} \frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}}, \qquad \text{with} \qquad z = \frac{\alpha_{\mathrm{em}}}{2\pi} \ln^2 \lambda^2$$

Back to heavy-to-light form factors

The non-relativistic form factors combine features of both cases!

→ endpoint logarithms from rapidity ordered spectator-quark propagators [PB '18] ← soft quarks

→ non-trivial interplay with additional "cusp-logarithms" ← soft gluons



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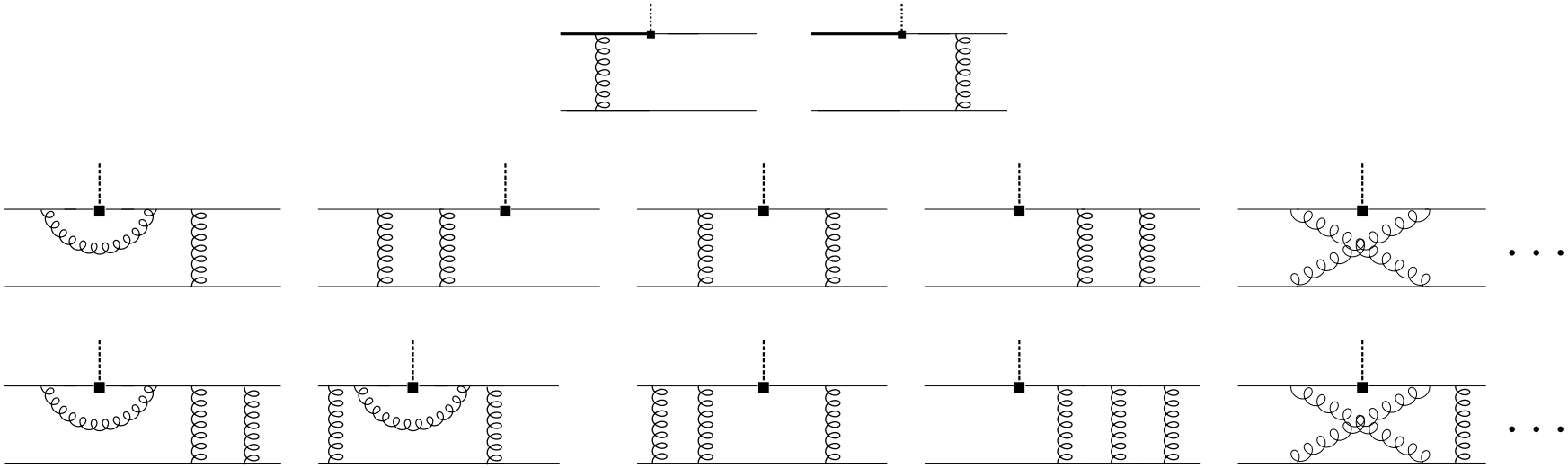


How to analyze the problem systematically?

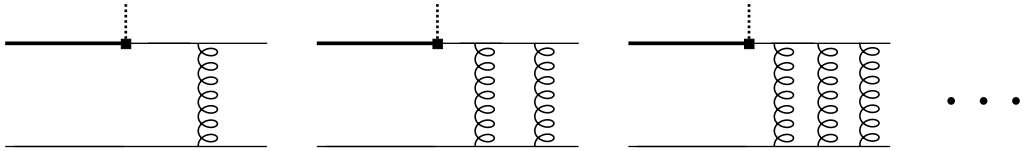
→ so far: abelian limit (QED)

→ first study "pure" endpoint double logarithms

→ include exponentiated soft-gluon contributions

Rapidity-ordered ladder diagrams

Observation I: In light-cone gauge $\bar{n} \cdot A = 0$ the leading pure endpoint logarithms arise only from light-quark ladder diagrams!



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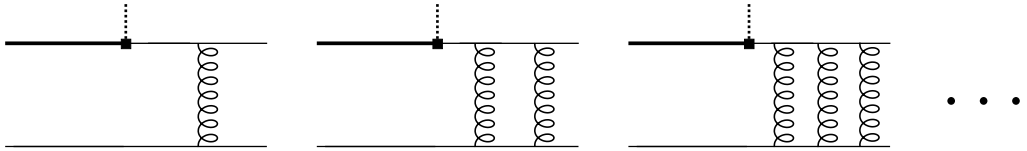

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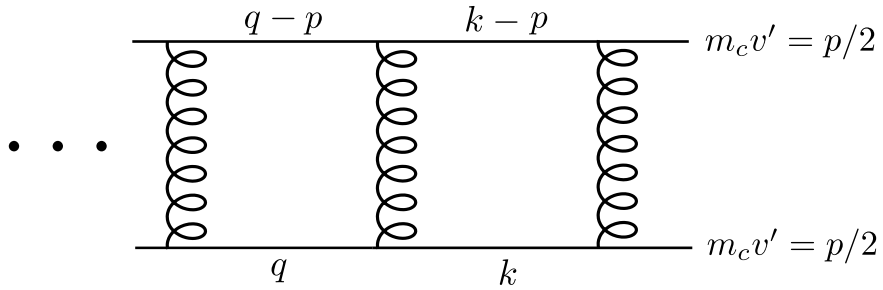
Observation II: Coupling to active (upper) quark line eikonal except for the two rightmost gluons!

$\rightarrow$ more non-eikonal couplings (sub-leading interactions) forbidden by power-counting in $m_b \rightarrow \infty$ limit

$\rightarrow$ similar to μ-e scattering, but more complicated Dirac structure

Integral equations for pure endpoint logarithms



Double-logarithmic series governed by implicit integral equations:

$$f(q_+,q_-) = f^{(0)} + \frac{\alpha}{2\pi}\int_{q_-}^{p_-}\frac{dk_-}{k_-}\int_{m_c^2/k_-}^{q_+}\frac{dk_+}{k_+}\left(f(k_+,k_-) - m_c\, f_m(k_+,k_-)\right)$$

$$f_m(q_+,q_-) = f_m^{(0)} + \frac{\alpha}{2\pi}\int_{q_-}^{p_-}\frac{dk_-}{k_-}\int_{m_c^2/k_-}^{q_+}\frac{dk_+}{k_+}\, f_m(k_+,k_-)$$

→ physical form factor $F \propto f(q_+ = q_- = m_c)$

→ translate into system of PDEs which can be solved in Laplace space

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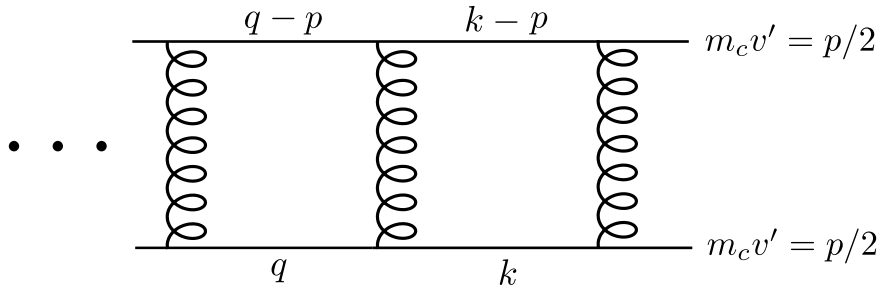

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→ translate into system of PDEs which can be solved in Laplace space

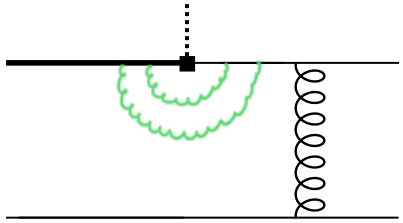
$$F_+^{(\text{non-fac})}(\gamma)\big|_{\text{pure endpoint}} \propto \frac{I_1\left(2\sqrt{z}\right)}{\sqrt{z}} + I_0\left(2\sqrt{z}\right) - \frac{1}{3} \qquad \text{with} \qquad z = \frac{\alpha}{2\pi}\ln^2(2\gamma)$$

- $z \to \infty$ asymptotics: $I_{0,1}(2\sqrt{z})$ grow exponentially

✓ all-order consistency check from analysis of endpoint singularities of B_c and η_c LCDAs in [PB '18]

Including cusp logarithms

In contrast to μ-e scattering, process has a hard interaction vertex



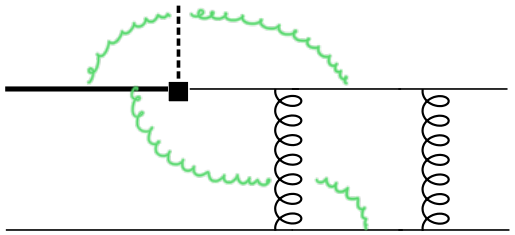
$\rightarrow$ soft gluons exponentiate to global Sudakov suppression factor

$$F_+^{(\text{non-fac})}(\gamma) \equiv \exp\left\{-\frac{\alpha}{4\pi}\ln^2(2\gamma)\right\} \cdot \xi(\gamma)$$

$\rightarrow$ $\xi(\gamma)$ contains all non-factorizable endpoint double-logarithms

Including cusp logarithms

In contrast to μ-e scattering, process has a hard interaction vertex



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In addition, soft-gluon couplings modify each rung in the ladder (similar to $h \to \gamma\gamma$)

→ modified integral equations with off-shell Sudakov factor:

$$f(q_+,q_-) = f^{(0)} + \frac{\alpha}{2\pi}\int_{q_-}^{p_-}\frac{dk_-}{k_-}\int_{m_c^2/k_-}^{q_+}\frac{dk_+}{k_+}\exp\left\{-\frac{\alpha}{2\pi}\ln\frac{p_-}{k_-}\ln\frac{q_+}{k_+}\right\}(f - m_c\, f_m)(k_+,k_-)$$

$$f_m(q_+,q_-) = f_m^{(0)} + \frac{\alpha}{2\pi}\int_{q_-}^{p_-}\frac{dk_-}{k_-}\int_{m_c^2/k_-}^{q_+}\frac{dk_+}{k_+}\exp\left\{-\frac{\alpha}{2\pi}\ln\frac{p_-}{k_-}\ln\frac{q_+}{k_+}\right\} f_m(k_+,k_-)$$

→ $\xi(\gamma) \propto f(q_+ = q_- = m_c)$

→ structure verified up to NNLO, but exponentiation remains conjecture

→ recover μ-e scattering and $h \to \gamma\gamma$ in certain limits

Towards a solution

It is more transparent to work with logarithmic variables $\rho = \ln \frac{q_+ p_-}{m_c^2}$ and $\eta = \ln \frac{p_-}{q_-}$, e.g.

$$f_m(\rho,\eta) = f_m^{(0)} + \hat{\alpha} \int_0^\eta d\eta' \int_{\eta'}^\rho d\rho' \, f_m(\rho',\eta') \, e^{-\hat{\alpha}\eta'(\rho-\rho')} \qquad (\hat{\alpha} \equiv \alpha/2\pi)$$

This can be translated into a PDE

$$(\partial_\rho \partial_\eta + \hat{\alpha}\eta\partial_\eta - \hat{\alpha}) f_m(\rho,\eta) = 0$$

$\rightarrow$ look for solution evaluated at $\rho = \eta = \ln(2\gamma)$

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So far could find closed form for the Laplace transform expressed through the error function

$$f_m(z) = e^{z/2} \mathcal{L}^{-1} \left\{ \frac{2}{\sqrt{\pi}} \frac{e^{-s^2}}{\mathrm{erfc}(s)} \right\} \left(\sqrt{2z} \right) \qquad \text{with} \qquad z = \frac{\alpha}{2\pi} \ln^2(2\gamma)$$

The perturbative series is

$$f_m(z) = \sum_{n=0}^{\infty} c_n (-z)^n = 1 + \frac{z}{2} + \frac{z^2}{24} - \frac{z^3}{720} - \frac{z^4}{40320} + \frac{17 z^5}{3628800} - \frac{107 z^6}{479001600} + \dots$$

with recursively defined coefficients

$$c_0 = 1 \,, \qquad c_n = \frac{-1}{(2n)!} \sum_{k=0}^{n-1} (n+k)! \, c_k$$

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work in progress!

Conclusion

Fundamental open problems:

- $\rightarrow$ even the leading double-logs of very basic (QED) amplitudes defy our current EFT/RGE machinery
- $\rightarrow$ (How) Does soft-collinear factorization at sub-leading power work?
- $\rightarrow$ very active and phenomenologically relevant field of research

Recent progress:

- $\rightarrow$ refactorization and rearrangements can cure endpoint singularities in certain cases
- $\rightarrow$ more complicated in $2 \rightarrow 2$ scattering processes and beyond (μ-e scattering, B decays, ...)
- $\rightarrow$ nested pattern of endpoint-singularities

$B_c \rightarrow \eta_c$ form factors as a perturbative playground:

- $\rightarrow$ non-factorizable endpoint logarithms modify Sudakov suppression
- $\rightarrow$ double-logarithmic series governed by integral equations
- $\rightarrow$ NNLO cross-check soon available (someone inspired to do N^{3}LO??)
- $\rightarrow$ result constraints the all-order IR singularities of the massless scattering amplitude (matching coefficient)

Thank you!

Backup-Slides