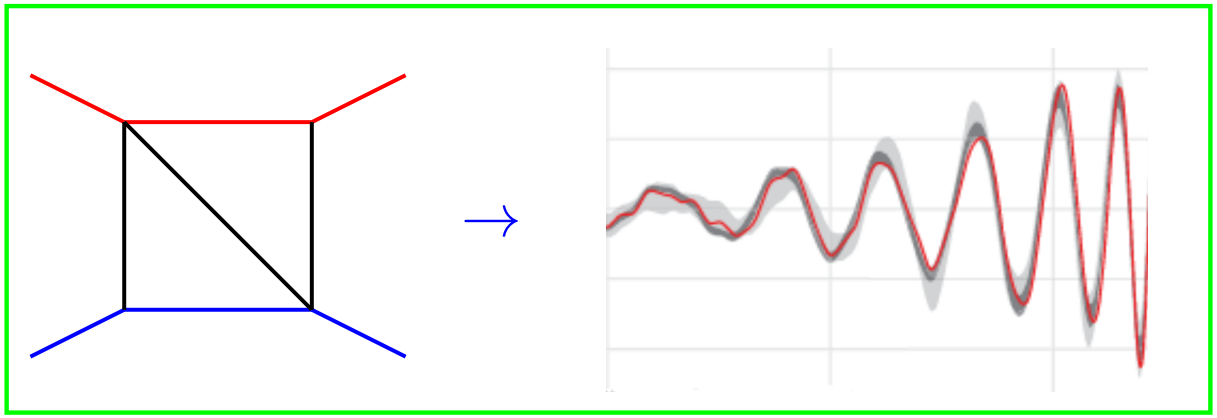
Numerical Unitarity for Binary Dynamics



Fernando Febres Cordero
Department of Physics, Florida State University

RADCOR 2023, Crieff, Scotland, 7/1/2023

Based on [arXiv:2205.07357]
with Manfred Kraus, Guanda Lin, Michael Ruf and Mao Zeng
and [*in progress*]

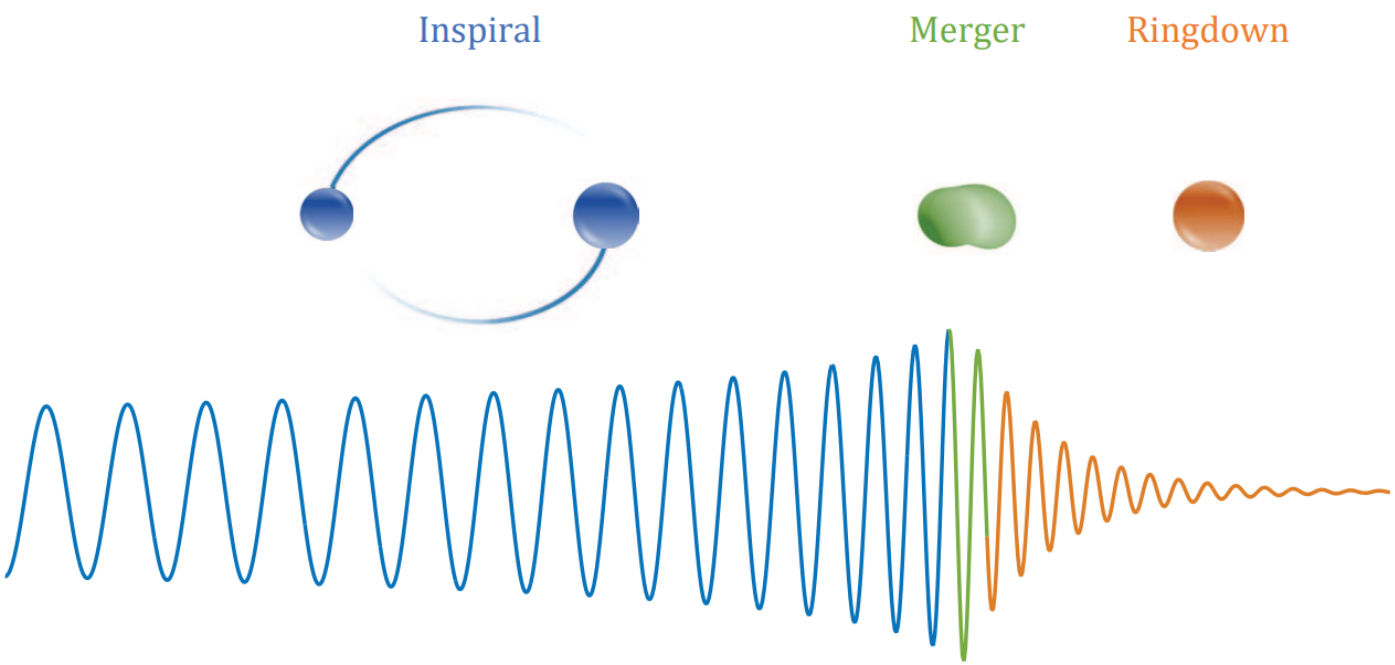
Outline

Precision in Gravitational Waveforms

Conservative Dynamics with Spinning Black Hole at $\mathcal{O}(G^3)$

Multi-Loop Numerical Unitarity for GW Astronomy

Binary Mergers and Gravitational Waves



- A precise theoretical description of the merger process is necessary to exploit the physics potential of measurements at gravitational wave observatories

- The inspiral can be described through perturbative methods (in the *Post-Newtonian* and *Post-Minkowskian* frameworks), the merger with numerical relativity, and the ringdown with black hole pertubation theory

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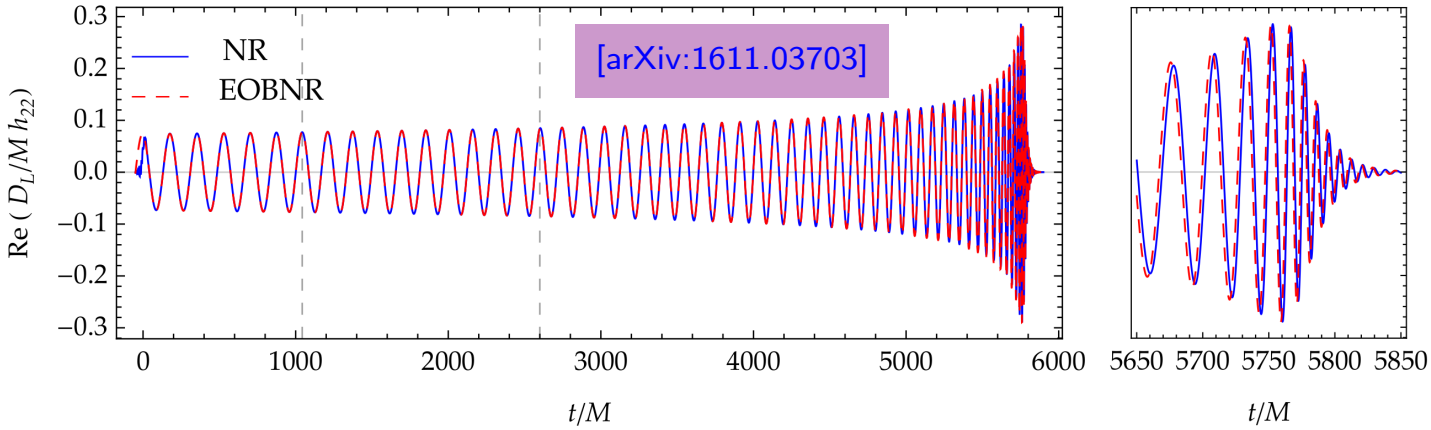

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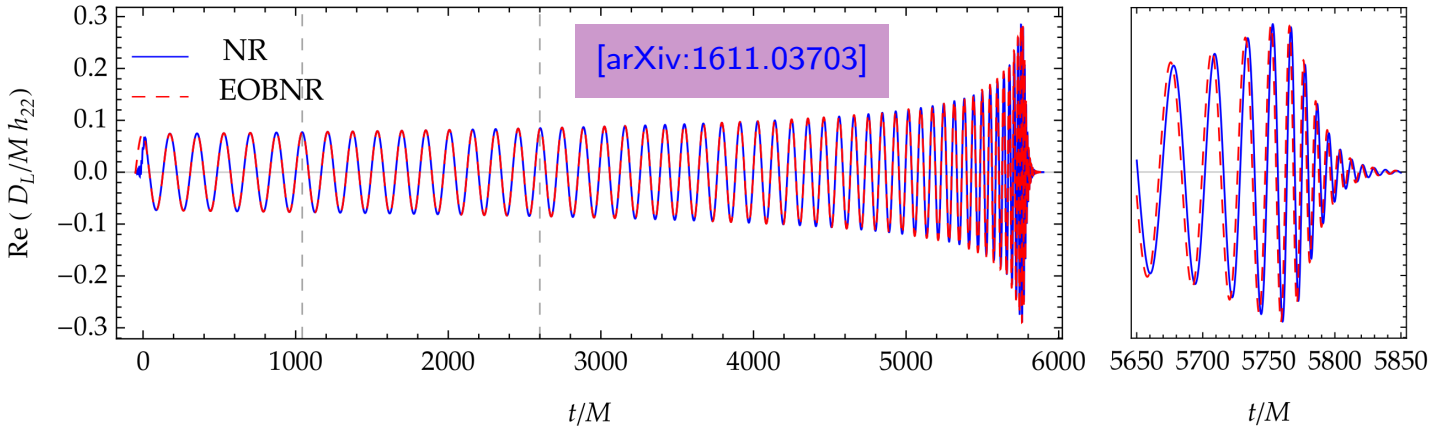

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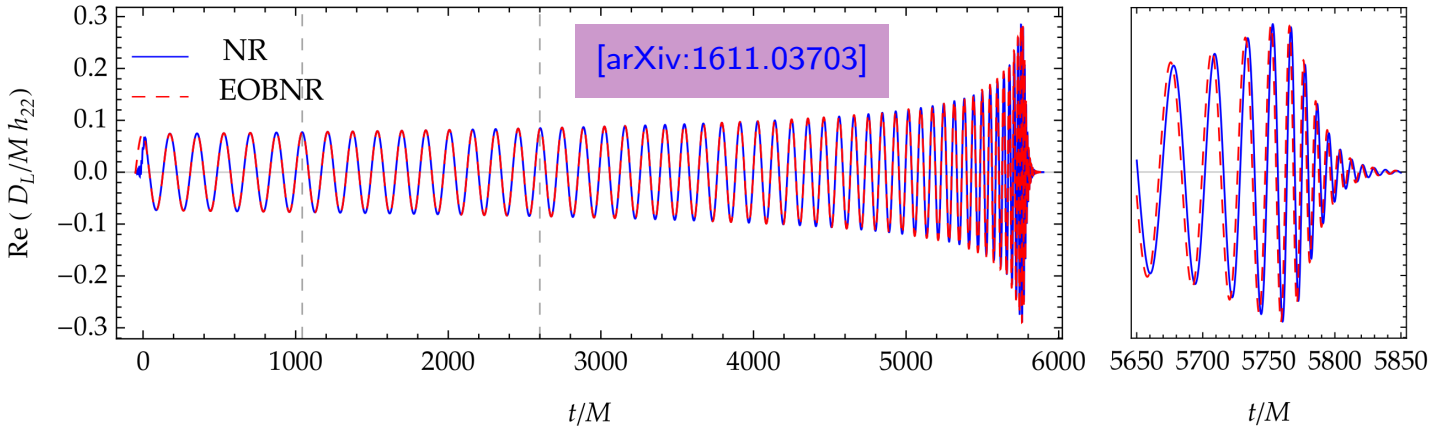

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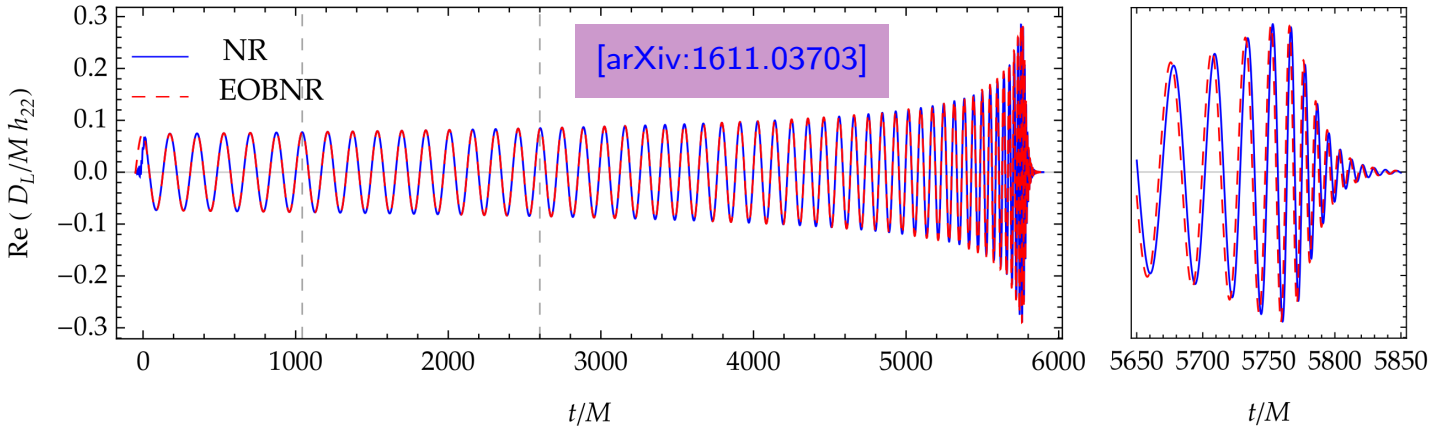

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$\longrightarrow$ scattering amplitudes can provide critical input for theoretical descriptions in the form of conservative potentials and scattering observables!

Frontier in Perturbative Corrections to Newton's Potential

Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng [arXiv:2204.05194]

$$
\begin{aligned}
&G(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)\\
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- New results recently produced with QFT tools
- State-of-the-art PN results
- Expected terms given sensitivity of future GW observatories

Progress for Spinless Binary Merger

Slide by Manfred Kraus

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + \boxed{V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2)} + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$

$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{p}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{r}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^4} + \cdots$$

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Newton	Einstein, Infeld Hoffmann	Ohta, Okamura, Kimura, Hiida	Damour et al Blanchet, Faye	Damour et al	Mastrolia et al Blümlein et al	Blümlein et al

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$$\frac{v^2}{c^2} \sim \frac{GM}{r} \ll 1$$

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→ See Michael Ruf's talk next for *5PL* related calculation

Progress for Binary Merger with a Spinning Black Hole

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Spin Hamilitonian much less known!

Post Newtonian

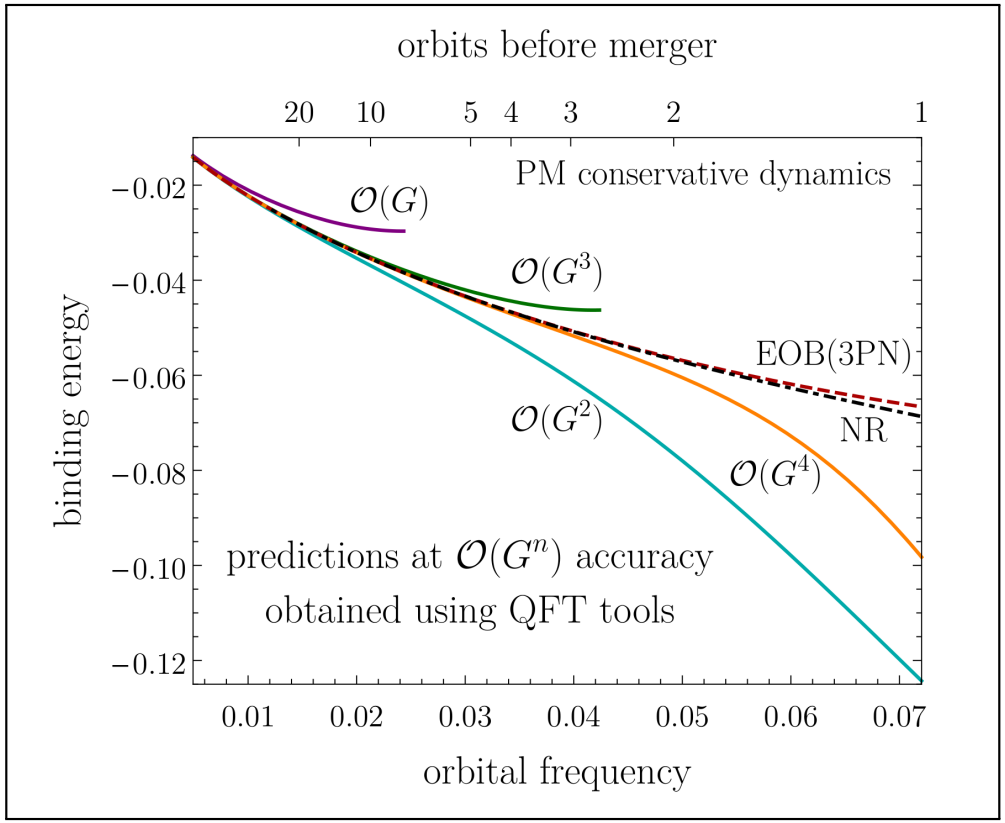
- $\boldsymbol{S}$: 3PN $\mathcal{O}(G^4)$ [Levi et al arXiv:2208.14949, Mastrolia et al arXiv:2209.00611]
- $\boldsymbol{S}^2$: 5PN $\mathcal{O}(G^4)$ [Kim,Levi,Yin arXiv:2112.01509]
- $\boldsymbol{S}^3$: 4PN $\mathcal{O}(G^2)$ [Levi,Mougiakakos,Viera arXiv:1912.06276]
- $\boldsymbol{S}^4$: 5PN $\mathcal{O}(G^2)$ [Levi,Teng arXiv:2008.12280]

Post Minkowskian

- $\boldsymbol{S}^2$: 2PM [Bern et.al arXiv:2005.03071], [Kosmopoulos,Luna arXiv:2102.10137]
  3PM [Febres Cordero, MK, Lin, Ruf, Zeng arXiv:2205.07357]
- $\boldsymbol{S}^4$: 2PM [Chen,Chung,Huang arXiv:2111.13639]
- $\boldsymbol{S}^5$: 2PM [Bern et al arXiv:2203.06202]
- $\boldsymbol{S}^\infty$: 2PM [Aoude,Haddad,Helset arXiv:2203.06197,arXiv:2205.02809]

Phenomenological Impact

Antonelli, Buonanno, Steinhoff, van de Meent, Vines [arXiv:1901.07102]



- Exploration of impact of higher-order corrections
- Benchmark quantities: bounding energy and scattering angle
- NR results serve as reference. 4PM results give excellent results
- Both bound and unbound observables studied

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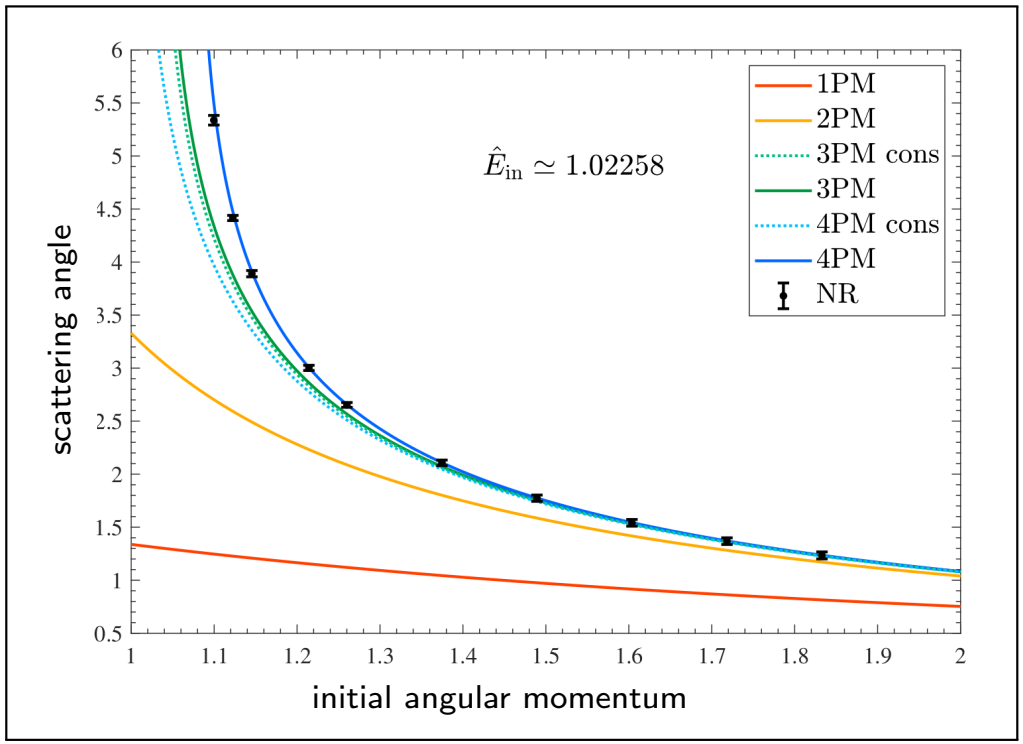
Damour, Rettegno [arXiv:2211.01399]



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- Techniques based on scattering amplitudes can produce information for classical observables
  - $\rightarrow$ Effective field theory approaches
    Goldberger, Rothstein; Cheung, Rothstein, Solon; ...
  - $\rightarrow$ 'in-in' formalism for computing (inclusive) observables
    Kosower, Maybee, O'Connell; Maybee, O'Connell, Vines
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  - → Here: EFT matching in dimensional regularization, modular arithmetics and numerical unitarity

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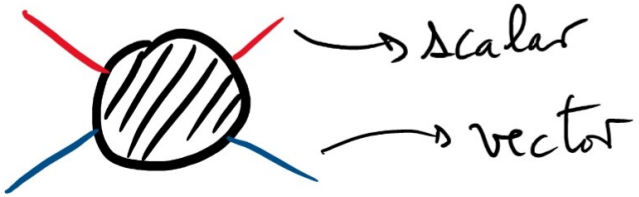

Scattering Amplitude Calculation

Parra-Martinez, Ruf, Zeng [arXiv:2005.04236]

Kinematics:

$$V(p_1) + s(p_2) \to s(p_3) + V(p_4)$$

$$q^\mu = p_2^\mu - p_3^\mu = (0, \boldsymbol{q})$$

$$\bar{p}_i \cdot q = 0$$

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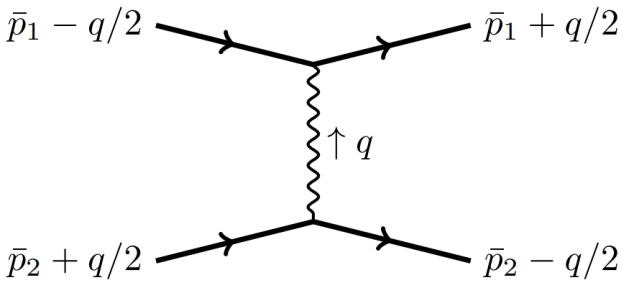

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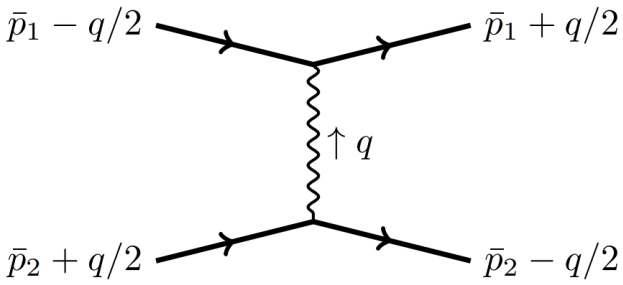

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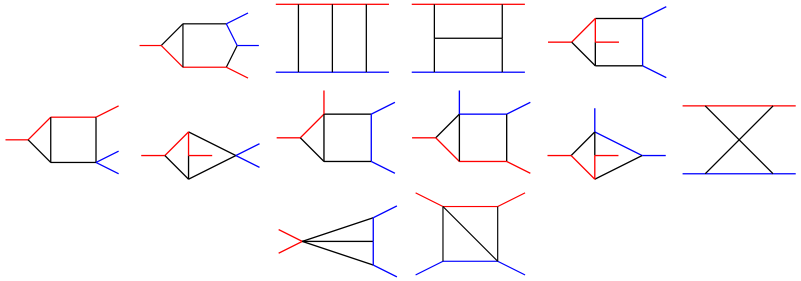
Multi-loop numerical untarity for amplitudes:



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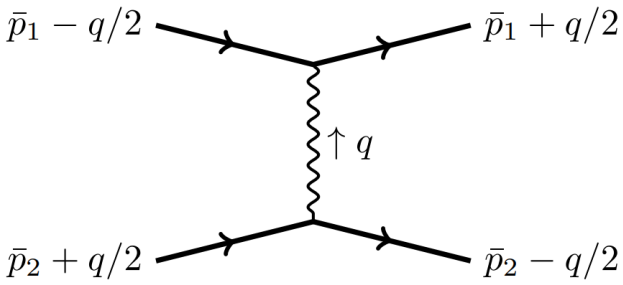

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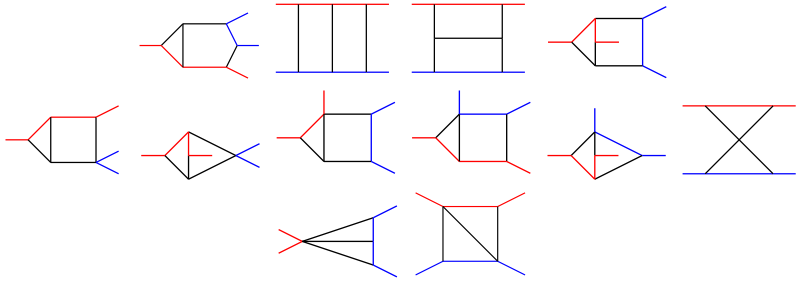

$$\boxed{\{T_1^{\mu\nu}, \ldots, T_5^{\mu\nu}\} = \{\eta^{\mu\nu}, q^\mu q^\nu, q^2 \bar{p}_2^\mu \bar{p}_2^\nu, \bar{p}_2^{[\mu} q^{\nu]}, \bar{p}_2^{(\mu} q^{\nu)}\}}$$

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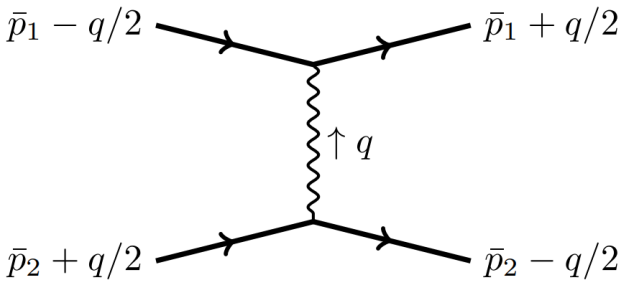
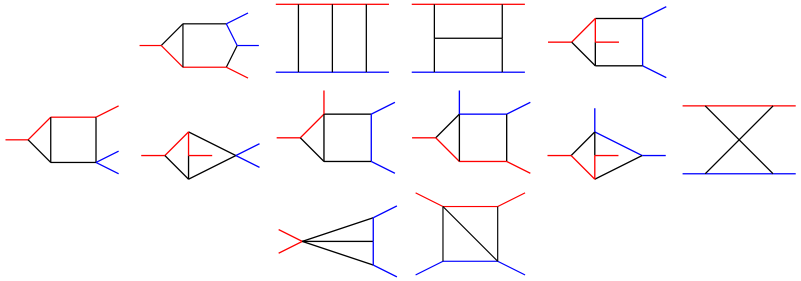
$$q^\mu = p_2^\mu - p_3^\mu = (0, \boldsymbol{q})$$

$$\bar{p}_i \cdot q = 0 \qquad \boxed{\sigma = \frac{p_1 \cdot p_2}{m_A m_\phi}}$$



Bern, Cheung, Roiban, Shen, Solon, Zeng [arXiv:1908.01493]

Multi-loop numerical untarity for amplitudes:



$$\boxed{\{T_1^{\mu\nu}, \ldots, T_5^{\mu\nu}\} = \{\eta^{\mu\nu}, q^\mu q^\nu, q^2 \bar{p}_2^\mu \bar{p}_2^\nu, \bar{p}_2^{[\mu} q^{\nu]}, \bar{p}_2^{(\mu} q^{\nu)}\}}$$

Caravel

Abreu, **FFC**, Ita, Kraus, Page, Pascual, Ruf, Sotnikov [arXiv:2009.11957]

Scattering Amplitude Calculation

IBPs, expansion and integration:

Smirnov, Chuharev [arXiv:1901.07808]

FIRE6, using finite fields

By the method of regions obtain expansion in q^2 to $\mathcal{O}((q^2)^0)$ and ϵ

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Construct appropriate ansatz in the variables m_ϕ, m_A and σ to obtain analytic result

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The amplitudes (an example):

$$
\begin{aligned}
M_4^{(2,2)} = {} & \frac{1}{\epsilon^2}\left[-\frac{(m_A^3 m_\phi + m_A m_\phi^3)\sigma(1-2\sigma^2)^2}{2(8\pi)^2(\sigma^2-1)^2} - \frac{m_A^2 m_\phi^2(1-2\sigma^2)(1-8\sigma^4)}{2(16\pi)^2(\sigma^2-1)^2}\right] \\
& + \frac{1}{\epsilon}\left[-\frac{m_A^2 m_\phi^2 \sigma(-6+\sigma^2)(1+2\sigma^2)}{2(4\pi)^2(\sigma^2-1)^{3/2}}\operatorname{arccosh}(\sigma) + \frac{m_A m_\phi^3 \sigma(3-16\sigma^2+16\sigma^4)}{2(8\pi)^2(\sigma^2-1)^2}\right. \\
& \left. + \frac{m_A^3 m_\phi(\sigma-5\sigma^3+5\sigma^5)}{2(4\pi)^2(\sigma^2-1)^2} + \frac{m_A^2 m_\phi^2(77+189\sigma^2-360\sigma^4+136\sigma^6)}{3(16\pi)^2(\sigma^2-1)^2}\right] + \mathcal{O}(\epsilon^0)
\end{aligned}
$$

...

FFC, Kraus, Lin, Ruf, Zeng [arXiv:2205.07357]

Effective Field Theory Amplitude

- We employ the non-relativistic, non-local, $3D$ EFT:

$$L_{\mathrm{EFT}} = \int_{\boldsymbol{k}} \hat{\phi}^\dagger(-\boldsymbol{k}) \left(\mathrm{i}\partial_t - \sqrt{\boldsymbol{k}^2 + m_\phi^2} \right) \hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k}) \left(\mathrm{i}\partial_t - \sqrt{\boldsymbol{k}^2 + m_A^2} \right) \hat{A}^i(\boldsymbol{k})$$
$$- \int_{\boldsymbol{k},\boldsymbol{k}'} \tilde{V}_{ij}(\boldsymbol{k},\boldsymbol{k}') \hat{A}^{\dagger,i}(\boldsymbol{k}') \hat{A}^j(\boldsymbol{k}) \hat{\phi}^\dagger(-\boldsymbol{k}') \hat{\phi}(-\boldsymbol{k})$$

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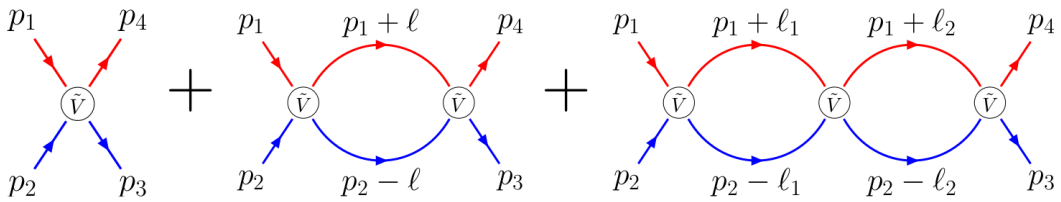
- As described by Cheung, Rothstein, Solon [arXiv:1808.02489] the EFT amplitude is given by iterated bubbles:



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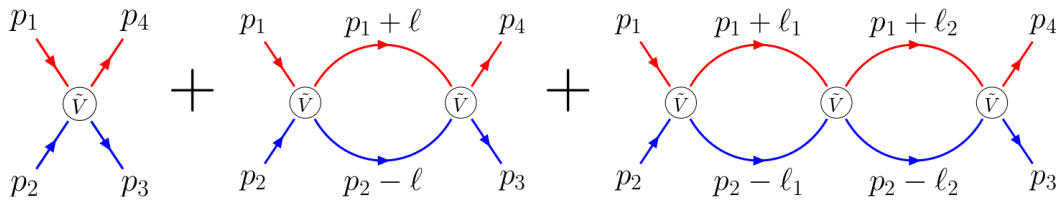

- The potential $\tilde{V}_{ij}$ is decomposed in terms of $\{\delta^{ij}, \bar{\boldsymbol{p}}^{[i}\boldsymbol{q}^{j]}, \dots\}$ $\sim \{\mathbb{1}, -\mathrm{i}(\boldsymbol{q} \times \bar{\boldsymbol{p}}) \cdot \boldsymbol{S}\}$. From the amplitudes' matching, we can then fix the potential!

Example Result: Spin-Orbit Interaction

- The $(\boldsymbol{q} \times \bar{\boldsymbol{p}}) \cdot \boldsymbol{S}$ term in the conservative potential is a linear combination of the coefficients:

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$$+\frac{(4m_A+3m_\phi)E}{4}\sigma(2\sigma^2-1)(5\sigma^2-3)+4m_A m_\phi\sigma(\sigma^2-6)(2\sigma^2+1)\sqrt{\sigma^2-1}\,\mathrm{arccosh}(\sigma)\Big]\,,$$

$$c_{1,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = 0\,, \qquad c_{2,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = E\xi c_1^{(2)}\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} + c_1^{(1)}\left(E\xi\frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + \frac{c_1^{(2)}\left(\frac{2E^2\xi}{\boldsymbol{k}^2}+\frac{1}{\xi}-3\right)}{2E}\right)\,,$$

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- Other terms in the potential more involved, but still relatively simple

Outline

Precision in Gravitational Waveforms

Conservative Dynamics with Spinning Black Hole at $\mathcal{O}(G^3)$

Multi-Loop Numerical Unitarity for GW Astronomy

Two-Loop Numerical Unitarity

Decompose $\mathcal{A}$ in terms of *master* integrals:

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- **Tensor Basis**
- **Scattering Plane Tensor Basis**
- **Master-Surface Basis**

Integrand Parametrizations

- Given diagram Γ make an *adaptive* momentum parametrization

$$\ell_l = \sum_{j \in B_l^p} v_l^j r^{lj} + \underbrace{\sum_{j \in B_l^t} v_l^j \alpha^{lj}}_{\text{ISP's}} + \underbrace{\sum_{i \in B^{ct}} \frac{n^i}{(n^i)^2} \alpha^{li}}_{\text{Common transverse}} + \sum_{i \in B^{\epsilon}} n^i \mu_l^i$$

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These integrand parametrizations have been automated in Caravel. Tipically less than $\mathcal{O}(5\%)$ (IBP-)reducible monomials (with scattering-plane variables) remain!

Master/Surface Decompositions

[Ita '15; Abreu, **FFC**, Ita, Page, Zeng '17]

Consider the integration by parts (IBP) relation on Γ

$$0 = \int \prod_i d^D \ell_i \, \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right]$$

making it *unitarity compatible* (controlling the propagator structure) [Gluza, Kadja, Kosower '10; Schabinger '11]

$$u_j^\nu \frac{\partial}{\partial \ell_j^\nu} \rho_k = f_k \rho_k$$

Write ansatz for u_j^ν expanded in external and loop momenta, and solve polynomial equations using algebraic geometry techniques

Build a full set of surface terms and fill the rest of the space with master integrands

Related [Boehm, Georgoudis, Larsen, Schulze, Zhang '16 - '19]
[Agarwal, von Manteuffel '19]

Surface Terms Factory

[Abreu, **FFC**, Ita, Jaquier, Page, Ruf, Sotnikov [arXiv:2002.12374]]

Solutions to u_j^ν are power-counting independent. When parametrizing a given numerator of a $\Gamma \in \Delta$ we need to consider the required power-counting for the theory at hand.

But we can *industrially* produce surface terms by considering polynomials $t_r(\ell_l)$, and then considering the vector $t_r(\ell_l)u_j^\nu$:

$$m_{\Gamma,(r,s)} = u_j^\nu \frac{\partial t_r(\ell_l)}{\partial \ell_i^\nu} + t_r(\ell_l)\left(\frac{\partial u_j^\nu}{\partial \ell_i^\nu} - \sum_{k\in P_\Gamma} f_k^s\right)$$

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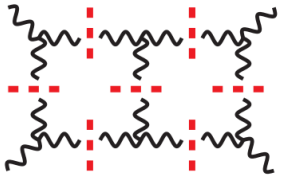
A four-graviton amplitude calculation in **Einstein gravity** structurally the same as a four-gluon amplitude calculation in **QCD**!

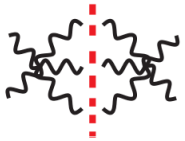
Unitarity Approach to Computing Integrand Coefficients

[Bern, Dixon, Dunbar, Kosower] [Britto, Cachazo, Feng]

- In on-shell configurations of ℓ_l, the integrand factorizes and produces a *cut equation*:



$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma \\ k \in \bar{Q}_{\Gamma'}}} \frac{c_{\Gamma',k}\ m_{\Gamma',k}(\ell_l^\Gamma)}{\prod_{j \in (P_{\Gamma'}/P_\Gamma)} \rho_j(\ell_l^\Gamma)}$$



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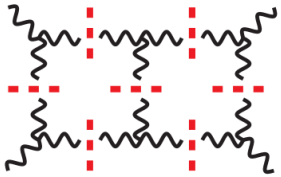

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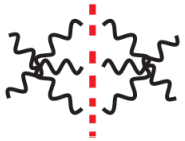

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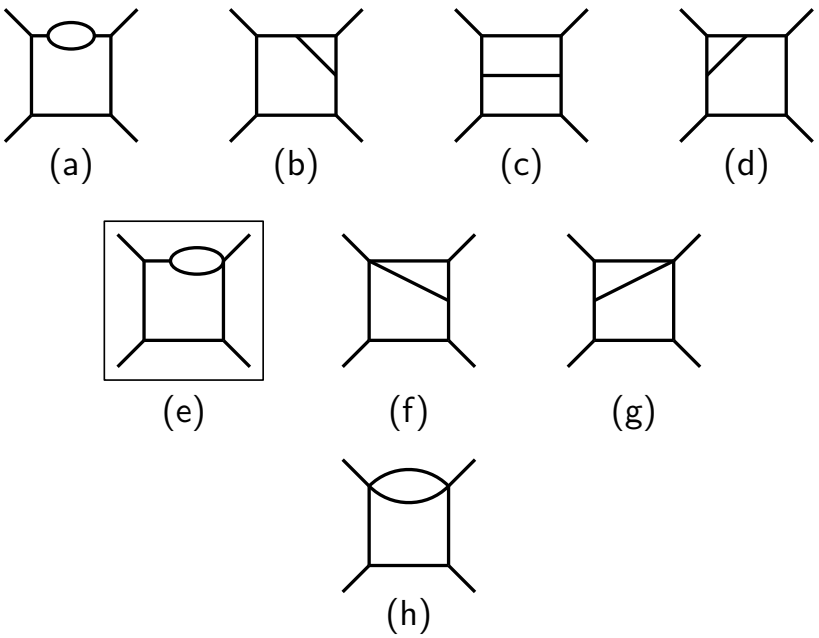
Never construct analytic integrand, numerics for every phase-space point!

Solving Cut Equations

[Abreu, **FFC**, Ita, Jaquier, Page [arXiv:1703.05255]]



Solving Cut Equations

[Abreu, **FFC**, Ita, Jaquier, Page [arXiv:1703.05255]]

For a maximal:

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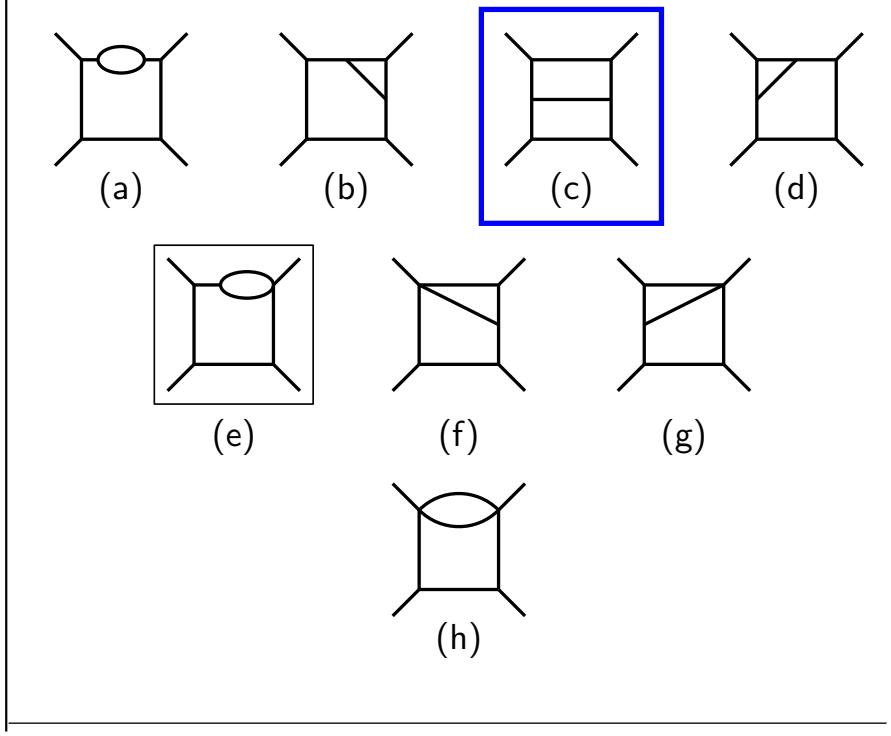

Solving Cut Equations

[Abreu, **FFC**, Ita, Jaquier, Page [arXiv:1703.05255]]

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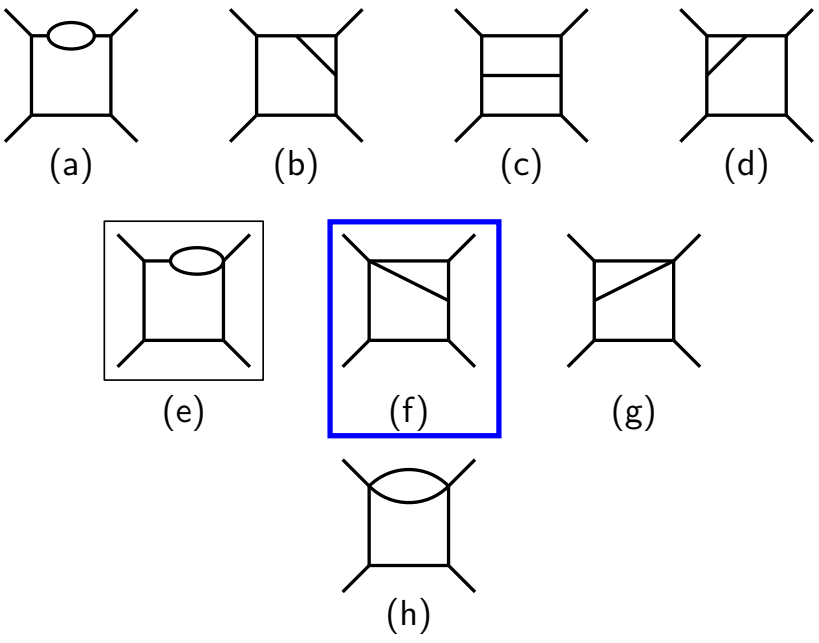
For a next-to-maximal:

$N\left(\,,\ell_l^{\mathrm{f}}\right) = R\left(\,,\ell_l^{\mathrm{f}}\right)$

$-\frac{1}{\rho_{\mathrm{fb}}} N\left(\,,\ell_1^{\mathrm{f}}\right) - \frac{1}{\rho_{\mathrm{fc}}} N\left(\,,\ell_l^{\mathrm{f}}\right)$



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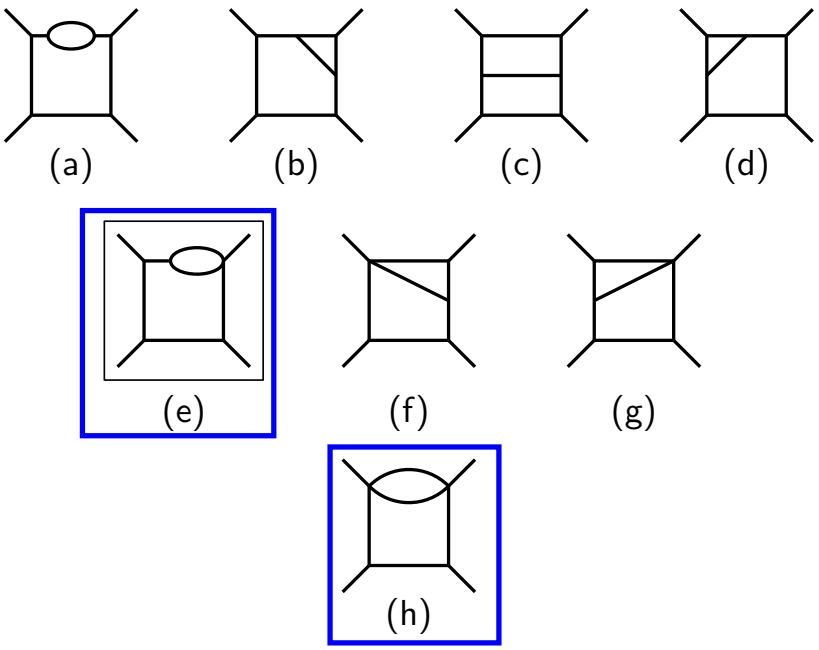


And for the combined single-pole diagram an bubble-box:

$N\left(\,,\ell_l^{\mathrm{h}}\right) + \frac{1}{\rho_{\mathrm{he}}} N\left(\,,\ell_l^{\mathrm{h}}\right) = R\left(\,,\ell_l^{\mathrm{h}}\right)$

$- \frac{1}{\rho_{\mathrm{hf}}} N\left(\,,\ell_l^{\mathrm{h}}\right) - \frac{1}{\rho_{\mathrm{hg}}} N\left(\,,\ell_l^{\mathrm{h}}\right) - \frac{1}{(\rho_{\mathrm{he}})^2} N\left(\,,\ell_l^{\mathrm{h}}\right)$

$- \frac{1}{\rho_{\mathrm{hf}}\rho_{\mathrm{fb}}} N\left(\,,\ell_l^{\mathrm{h}}\right) - \frac{1}{\rho_{\mathrm{hf}}\rho_{\mathrm{fc}}} N\left(\,,\ell_l^{\mathrm{h}}\right) - \frac{1}{\rho_{\mathrm{hg}}\rho_{\mathrm{gd}}} N\left(\,,\ell_l^{\mathrm{h}}\right)$

The Caravel Framework

A framework to explore multi-loop multi-leg scattering amplitudes in the SM and beyond

Caravel

The CARAVEL Framework

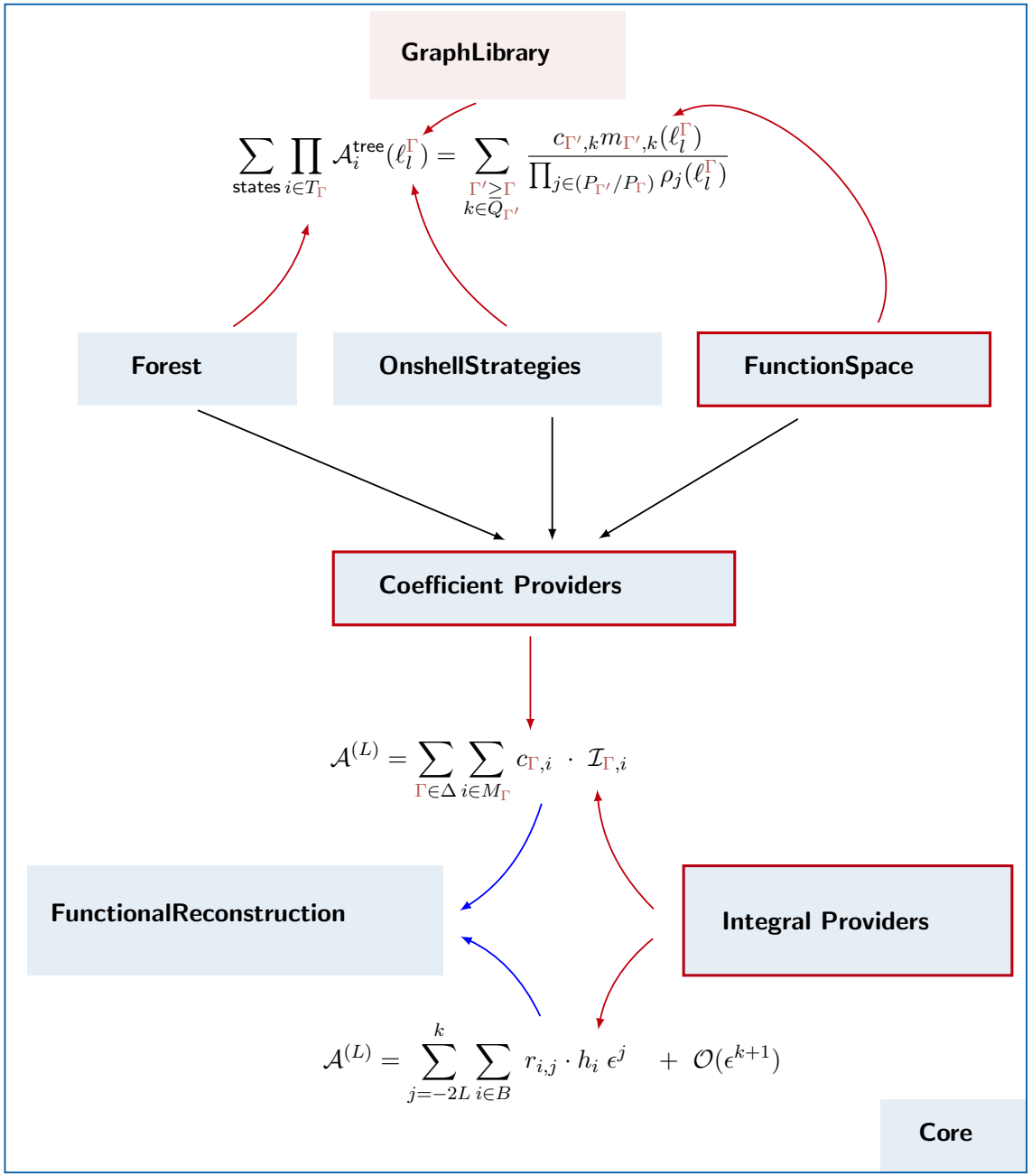
A framework to explore multi-loop multi-leg scattering amplitudes in the SM and beyond

A modular C++17 library implementing the multi-loop numerical unitarity method

[Abreu, Dormans, **FFC**, Ita, Kraus, Page, Pascual, Ruf, Sotnikov, arXiv:2009.11957]

- Numerics in (high-precision) floating-point, rational and modular arithmetic
- Generic design for calculations in QFT, e.g. in the SM, gravity theories, and more
- Algebraic tools for semi-analytical calculations in C++
- Publicly available @ GitLab!

The CARAVEL Framework



Includes general tools for:

- D-dimensional kinematics
- graph isomorphism techniques
- tree-level and multi-loop cut calculations
- Generic scattering-plane integrand parametrizations
- Selected master-surface decompositions
- on-shell phase-space parametrizations
- Feynman integral handling
- Algebraic tools
- ...

Caravel @ GitLab:

`https://gitlab.com/caravel-public/caravel`

Outlook

- QFT techniques developed for collider phenomenology playing a key role in improving the description of compact binary mergers
- We presented the conservative potential for compact binary systems with a spinning black hole at $\mathcal{O}(G^3)$ and to all orders in velocity and including up-to S^2 terms
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Thanks!