# Four-dimensional integration methods for Feynman integrals 

## Johannes M. Henn

Based on 22II.I3967 [hep-th] with R. Ma, K. Yan and Y. Zhang, and 23xx.xxxx with C. Milloy and K. Yan<br>RADCOR conference 2023, May 28 - June 2



## Motivation: physically relevant quantities are often much simpler than intermediate steps

Dimensional regularization often leads to unnecessarily complicated spurious terms that drop out in physically meaningful, scheme invariant quantities, when the regulator is removed.

For example, the infrared- and UV-renormalized two-loop five-particle amplitudes depends on fewer alphabet betters.

There is much progress in decomposing amplitudes in terms of finite and divergent contributions.

## Recent ideas for (infrared) divergences

We understand well the structure of divergences in perturbative QFT.As a consequence, often one can organize calculations in terms of finite quantities.

- Use of integration-by-parts/dimension shift relations [von Manteuffel, Panzer, Schabinger, 2014]
- Tropical ideas for sector decomposition / subtraction [Schultka, 20I8;Arkani-Hamed, Hillmann, Mizera, 2022; talk by Ma]
- Classification of integrands according to degree of divergence; novel approaches to subtraction methods.
[Arkani-Hamed, 2010; JMH, Peraro, Stahlhofen, Wasser, 2019]
[Anastasiou, Sterman, 2018,2022; +Haindl, Yang, Zeng, 2020] [Gluza, Kajda, Kosower, 20IO] [talks by Bertolotti, Karlen Novichkov, Page, Signorile-Signorile, ...]
- Finite loop integrands from Amplituhedron geometry [Arkani-Hamed, JMH, Trnka, 2020]


## Outline

I. Integrals for the three-loop soft anomalous dimension matrix (JMH, Milloy, Yan, in preparation)
2. Proof-of-concept for method of computing leading divergences of Feynman integrals, with application to the three-loop cusp anomalous dimension (JMH, Ma, Yan, Zhang)

## Part I: Soft anomalous dimension matrix.

 <br> \title{Soft anomalous dimension describes infrared <br> \title{
Soft anomalous dimension describes infrared divergences in scattering of massive particles
} divergences in scattering of massive particles
}

Sample two-loop diagrams


Most interesting color structure $\quad f^{a b c} \mathbf{T}_{1}^{a} \mathbf{T}_{2}^{b} \mathbf{T}_{3}^{c} F\left(x_{12}, x_{23}, x_{13}\right)$
Cross-ratios $\frac{\beta_{i} \cdot \beta_{j}}{\sqrt{\beta_{i}^{2} \beta_{j}^{2}}}=-\frac{1}{2}\left(x_{i j}+\frac{1}{x_{i j}}\right)$
[Related talk by Liu: three-loop singularities of amplitudes with one massive particle.]

## Two-loop soft anomalous dimension matrix

 is simple, but this is not obvious.
[Ferroglia, Neubert, Pecjak, Yang (2009)]

[Mitov, Sterman, Sung (2009+20I0)]
Position space techniques
[Chien, Schwartz, Simmons-Duffin, Stewart (201I)] Special gauge choice
Two-loop answer just depends on logarithms and dilogarithms!

$$
\sim f^{a b c} \mathbf{T}_{1}^{a} \mathbf{T}_{2}^{b} \mathbf{T}_{3}^{c} F\left(x_{12}, x_{23}, x_{13}\right)
$$

$$
F=\sum_{i, j, k} \epsilon_{i j k}\left\{r\left(x_{k i}\right) r\left(x_{i j}\right) \log \left(x_{k i}\right)\left[-\operatorname{Li}_{2}\left(1-x_{i j}^{2}\right)+\log ^{2}\left(x_{i j}\right)\right]-r\left(x_{k i}\right) \log \left(x_{k i}\right) \log ^{2}\left(x_{i j}\right)\right\}
$$

# Higher orders in dimensional regularization even contain elliptic functions! 



One might expect to find the simple result from differential equations with a simple polylogarithmic alphabet.

However, dimensionally regulated differential equations are very complicated [JMH, unpublished; Milloy, PhD thesis, 2020]!

But the physical result (leading pole in eps) is simple. Can we obtain this in a better way?

## Try a different regulator

We suspect that the complexity has to do with the choice of regulator. The eikonal integrals are scaleless. One conventionally uses dimensional regularization to regulate the UV, and puts an IR cutoff on all eikonal lines,

$$
\frac{1}{k \cdot v} \longrightarrow \frac{1}{k \cdot v+\delta}
$$

Our idea: only shift one of the three lines in this way. E.g.


Subtlety: Unregulated lines lead to IR divergences. Use known two-line cusp anomalous dimension as a counterterm.
$\longrightarrow$ Test if this setup simplifies the two-loop problem.

## Two-loop from differential equations

With an IR regulator only on one line, the differential equations are significantly simplified.

We find canonical differential equations (16 MI, four in top sector), with the following alphabet:


$$
\begin{array}{r}
\mathcal{A}=\left\{x_{12}-1, x_{12}, x_{12}+1, x_{13}-1, x_{13}, x_{13}+1, x_{23}-1, x_{23}, x_{23}+1,\right. \\
\left.x_{12} x_{13}+x_{23}, x_{12} x_{23}+x_{13}, x_{12}+x_{13} x_{23}, x_{12} x_{13} x_{23}+1\right\}
\end{array}
$$

Not as simple as the final result, but at least polylogarithmic.
Motivated by this, we do a proof-of-principle calculation at three loops.

## Three-loop application

Here we find 66 master integrals. To find a canonical basis, we use a combination of methods, using the programs CANONICA, INITIAL, and DlogBasis. We use absence of unphysical behavior to fix the boundary
 constants. The result is
$w_{122}^{(3)}=\frac{1}{\epsilon}\left[r\left(x_{12}\right)^{2} f_{A}\left(x_{12}, x_{13}, x_{23}\right)+r\left(x_{12}\right) r\left(x_{13}\right) f_{B}\left(x_{12}, x_{13}, x_{23}\right)+r\left(x_{12}\right) r\left(x_{23}\right) f_{C}\left(x_{12}, x_{13}, x_{23}\right]\right.$
We find that the pure functions $f_{A}, f_{B}, f_{C}$ can be written in terms of classical polylogarithms $\operatorname{Li}_{n}(-s)$ with $n \leq 5$.

$$
s \in\left\{x_{12},-x_{12}, x_{13},-x_{13}, \frac{1}{x_{23}}, \frac{1}{x_{13} x_{23}}, \frac{x_{12}}{x_{13} x_{23}}, \frac{x_{13}}{x_{23}}, \frac{x_{12} x_{13}}{x_{23}}, x_{23}, \frac{x_{23}}{x_{13}}, \frac{x_{12} x_{23}}{x_{13}}\right\}
$$

## Our analytic result agrees with numerical evaluations from pySecDec



Absolute value of relative difference between numerical and analytical result at $\mathrm{O}(500)$ different points.

## Conclusion of part I

- We found a 'friendly' regulator for the soft anomalous dimension matrix.
- A three-loop proof-of-concept calculation seems promising.
- Can we find a way of calculating where intermediate steps do not depend on the precise form of the regulator?


## Part 2: Four-dimensional differential equations for leading divergences

[JMH, Ma, Yan, Zhang, 2022]

# Quick reminder: simplified differential equations 

 method for four-dimensional Feynman integrals [JMH, Caron-Huot, 2014]- Operate in space of finite Feynman integrals
- Simpler, four-dimensional integration-by-parts relations
- Fewer master integrals
- 'block-triangular’ structure of differential equations, canonical basis found algorithmically
- Bonus relations that relate loop orders


## Example massive box integral

[JMH, Caron-Huot, 2014]
transcendental weight
$d\left(\begin{array}{l}g_{1} \\ g_{2} \\ g_{3} \\ g_{6}\end{array}\right)=d\left(\begin{array}{cccc}0 & 0 & 0 & 0 \\ \log \left(\frac{\beta_{u}-1}{\beta_{u}+1}\right) & 0 & 0 & 0 \\ \log \left(\frac{\beta_{v}-1}{\beta_{v}+1}\right) & 0 & 0 & 0 \\ 0 & \log \left(\frac{\beta_{u v}-\beta_{u}}{\left.\beta_{u v}+\beta_{u}\right)} \log \left(\frac{\beta_{u v}-\beta_{v}}{\beta_{u v}+\beta_{v}}\right)\right. & 0\end{array}\right)\left(\begin{array}{l}g_{1} \\ g_{2} \\ g_{3} \\ g_{6}\end{array}\right)$

[Cf. Ekta's talk for applications to light-by-light scattering]

Can we use similar techniques for poles of divergent Feynman integrals?

## Angle-dependent cusp anomalous dimension

Given by divergence of eikonal (HQET) integrals:


$$
\sim \frac{1}{\epsilon} \frac{1+x^{2}}{1-x^{2}} \log x, \quad x=e^{i \phi}
$$

$$
\cos \phi=\frac{v_{1} \cdot v_{2}}{\sqrt{v_{1}^{2} v_{2}^{2}}}
$$

Thanks to non-Abelian exponentiation, higher-loop contributions can be organized in terms of graphs that have overall divergences only (up to beta function terms).


$$
\sim \frac{f(x)}{L \epsilon}+\mathcal{O}\left(\epsilon^{0}\right)
$$

We argue that the leading divergence can be computed using four-dimensional methods.
$J^{(L)}(x)=\lim _{\epsilon \rightarrow 0}\left[\operatorname{L\epsilon } I^{(L)}(x)\right]$

## Procedure:

- classify all 'admissible' integrals (that have the same divergence structure)
- write down IBP and DE operators that stay within the space of admissible integrals (using syzygy relations)
- bonus: restrict to integrals with same scaling dimension (property of eikonal integrals)


## Proof-of-concept application



$$
\frac{d}{d x} \vec{J}=B \vec{J}, \quad B \equiv\left(\begin{array}{cccccccc}
0-\frac{2}{(-1+x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{x} & \frac{2 x}{(-1+x)(1+x)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

We initially obtain al $3 x 13$ matrix. Studying the PicardFuchs equation, it can be reduced to an $8 \times 8$ matrix.

$$
\begin{aligned}
J_{1} & =\frac{8}{3} H_{-1,-2,0,0}-\frac{8}{3} H_{-1,2,0,0}+\frac{8}{3} H_{1,-2,0,0}-\frac{8}{3} H_{1,2,0,0}-\frac{8}{3} H_{-1,0,0,0,0}-\frac{8}{3} H_{1,0,0,0,0} \\
& -\zeta_{4} \ln (1-x)+\zeta_{4} \ln (1+x)+\frac{4}{3} \zeta_{3} \ln (1-x) \ln x-\frac{4}{3} \zeta_{3} \ln x \ln (1+x) \\
& +\frac{8}{3} \zeta_{3} \operatorname{Li}_{2}(x)-\frac{2}{3} \zeta_{3} \operatorname{Li}_{2}\left(x^{2}\right) .
\end{aligned}
$$

Result agrees with dim-reg calculation, where 39 MI were needed. [Grozin, JMH, Korchemsky, Marquardt, 2014+2015]

## Discussion

- We developed four-dimensional methods that benefit from the simplicity of the physical results.
- Regularization may obscure this simplicity. For the soft anomalous dimension matrix, we proposed a 'friendly' regularization scheme, where intermediate steps are not overly complicated.
- We presented a four-dimensional method to compute leading divergences of loop integrals. It requires fewer master integrals and is more economical compared to dimensional regularization.

Thank you!


