Talk 6:
Determining the β-function in $N_f = 3$ QCD

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Overview

Determining the fundamental parameters of the SM.

A theoretical problem in strongly coupled QFT

$$S_{\rm QCD}[A_\mu,\psi,\overline{\psi}] = \int d^4x\, \frac{1}{4g^2} F^a_{\mu\nu}F^a_{\mu\nu} + \sum_f \overline{\psi_f}\gamma_\mu(\partial_\mu + m_q + \imath A_\mu)\psi_f$$

- Fundamental parameters ($m_{\overline{\rm MS}}, \alpha_{\overline{\rm MS}}$) naturally defined at high energies (M_Z).
- Well measured and clean QCD quantities naturally defined at low energies (M_p, M_π).
- In principle one-to-one correspondence, but…
- Relating fundamental parameters with low energy hadronic quantities requires non-perturbative formulation of QFT $\Longrightarrow$ Lattice QCD

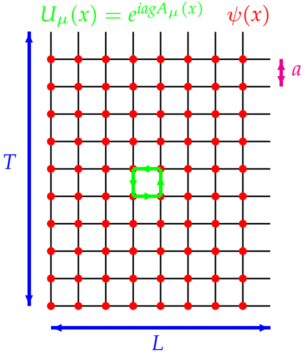
$$\Lambda = \mu \times \left[b_0 g^2(\mu)\right]^{-b_1/2b_0^2} e^{-\frac{1}{2b_0 g^2(\mu)}} \exp\left\{-\int_0^{g^2(\mu)} du \left[\frac{1}{\beta(u)} + \frac{1}{b_0 u^3} - \frac{b_1}{b_0 u}\right]\right\}$$

$$M = \bar m(\mu)[2b_0\bar g^2(\mu)]^{-d_0/2b_0} \exp\left\{-\int_0^{\bar g(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\},$$

Overview

Lattice YM in one slide

Lattice field theory $\longrightarrow$ Non Perturbative definition of QFT.



$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U]\, O(U)_{\text{Wick}} e^{-S_G[U]} \det(D)$$

- Compute the integral numerically $\rightarrow$ Monte Carlo sampling.
- Observable computed averaging over samples

$$\langle O \rangle = \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} O(U_i) + \mathcal{O}(1/\sqrt{N_{\text{conf}}})$$

- One to one relation between a and β.

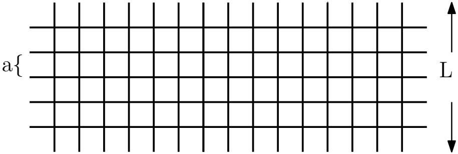
$$S_G[U] = \frac{\beta}{6} \sum_{p \in \text{Plaquettes}} Tr(1 - U_p - U_p^+) \xrightarrow[a \to 0]{} -\frac{1}{2} \int d^4x\, \text{Tr}(F_{\mu\nu} F_{\mu\nu})$$

FINITE SIZE SCALING AND STEP SCALING FUNCTION [LÜSCHER, WEISZ, WOLFF '91]

$$\alpha(\mu) = \frac{3r^2}{4} F(r)\Big|_{\mu=1/r}$$

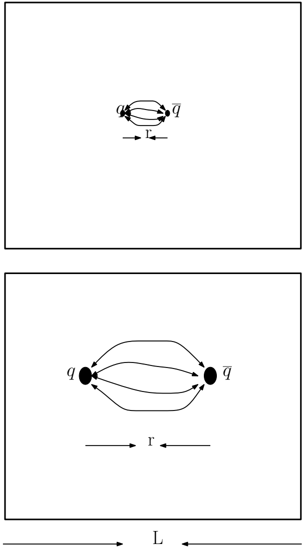
Huge computer resources

$L/a \sim 100 - 1000.$



$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

Finite size scaling and step scaling function [Lüscher, Weisz, Wolff '91]



Conditions:

- Small cutoff effects:

$$r/a \gg 1 \quad (\sim 10)$$

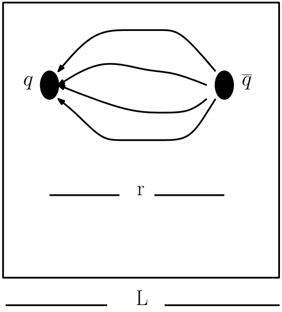
- I want to change μ from perturbative to non-perturbative: Change r by a factor 10.
- FV effects small:

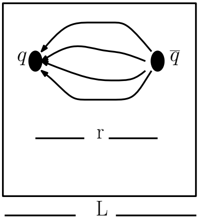
$$L/a \gg r/a \quad (\sim 10)$$

- Huge lattices

$$L/a > 1000$$

Finite size scaling and step scaling function [Lüscher, Weisz, Wolff '91]





Finite volume renorm. scheme:

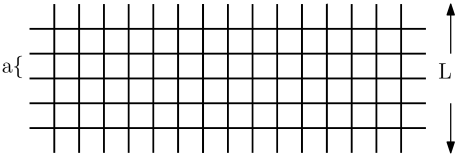
- Fix
$$\mu L = \text{constant}$$
- No FV corrections.
- Only condition
$$L/a \gg 1 \quad (\sim 10)$$
- Coupling depends on one scale: L
$$g^2(\mu) \text{ notation} : g^2(L), g^2(1/L)$$
- Step scaling function: How much changes the coupling when we change the renormalization scale:
$$\sigma_s(u) = g^2(sL)\Big|_{g^2(L)=u}$$
achieved by simple changing $L/a \to sL/a$!

FINITE SIZE SCALING AND STEP SCALING FUNCTION [LÜSCHER, WEISZ, WOLFF '91]

$$\alpha(\mu) = \frac{3r^2}{4} F(r)\Big|_{\mu=1/r}$$

Huge computer resources

$L/a \sim 100 - 1000.$



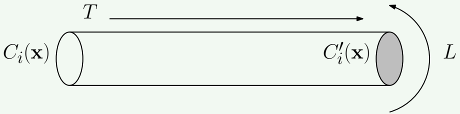
$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

Finite volume renormalization schemes

- Finite volume as renormalization scale $\mu L =$ constant.
- Coupling $\alpha(\mu)$ depends on no other scale but L (Notation: $\alpha(L), \alpha(1/L)$).
- Finite Volume effects part of the scheme **[Lüscher, Weisz, Wolff. 1991]**.
- $a \ll 1/\mu$ easily achieved: $L/a \sim 10 - 40$
- Boundary conditions become relevant:
  - Periodic bc. bad for matching with perturbation theory **[Gonzalez-Arroyo et al '81]**.
  - Schrödinger Functional **[Lüscher et al. '92]**
  - Twisted **[de Divitiies '94]**

Schrödinger Functional couplings

Schrödinger Functional: Dirichlet bc at $x_0 = 0, T$, periodic in $\mathbf{x}$



Background field choice

$$C_i(\mathbf{x}) = \frac{\imath}{L}\left[\mathrm{diag}(-\pi/3, 0, \pi/3) + \eta(\lambda_8 + \nu\lambda_3)\right]$$

$$C_i'(\mathbf{x}) = \frac{\imath}{L}\left[\mathrm{diag}(-\pi, \pi/3, 2\pi/3) - \eta(\lambda_8 - \nu\lambda_3)\right]$$

$$\frac{12\pi}{g_\nu^2(L)} = \left\langle \frac{\partial S}{\partial \eta}\Big|_{\eta=0} \right\rangle = \frac{12\pi}{g^2(L)} - 12\pi\nu\bar{v}$$

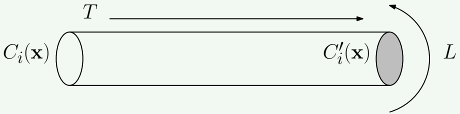
No background field

$$C_i(\mathbf{x}) = C_i'(\mathbf{x}) = 0$$

Gradient flow coupling g^2_{GF}

Schrödinger Functional couplings

Schrödinger Functional: Dirichlet bc at $x_0 = 0, T$, periodic in $\mathbf{x}$



Nice properties of g^2_{SF}

- $\delta_{\text{stat}} g^2_{\text{SF}} \sim \mathcal{O}(g^4) \Longrightarrow$ high precision for small g^2_{SF}.
- Known 3-loop $\beta-$ function $\Longrightarrow \mathcal{O}(\alpha^2)$ corrections in the determination of Λ
- Finite volume $\Longrightarrow$ No IR renormalons. Known NP contribution $\mathcal{O}(e^{-2.6/\alpha})$

Nice properties of g^2_{GF}

- $\delta_{\text{stat}} g^2_{\text{GF}} \sim \mathcal{O}(g^2)$ but with a continuum limit $\Longrightarrow$ high precision for large g^2_{GF}.
- Statistical precision largely independent of a or g^2.

Overview

Yang-Mills gradient flow: basics [Narayanan, Neuberger '06; Lüscher '10]

- Add "extra" (flow) time coordinate t ($\neq x_0$). Define gauge field $B_\mu(x,t)$

$$\begin{aligned} G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\ \frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,t=0) = A_\mu(x)\,. \end{aligned}$$

- Since

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)$$

  $\lim_{t\to\infty} B_\mu(t,x) = A_\mu^{\rm classical}(x)$.
- Correlation functions of the "smooth" field $B_\mu(x,t)$

$$G(x_1, x_2, \dots) = \langle B(x_1,t)B(x_2,t)\cdots\rangle$$

  are finite after the usual bare parameter renormalization **[Lüscher, Weisz. '11]**.
- For example, in pure YM

$$E(x,t) = \frac{1}{4}\langle G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\rangle$$

  is finite (for $t > 0$) after the usual coupling renormalization.

Gradient Flow: How it works

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad B_\mu(x,0) = A_\mu(x)$$

Expand the flow field in powers of g_0.

$$B_\mu(x,t) = \sum_{n=1}^{\infty} B_{\mu,n}(x,t) g_0^n$$

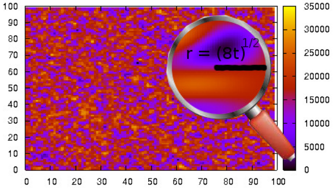
$GF \equiv$ Heat equation (+ gauge terms)

$$\frac{dB_{\mu,1}(x,t)}{dt} = \partial_\nu^2 B_{\mu,1}(x,t)$$

that has solution

$$B_{\mu,1}(x,t) = \sum_p e^{-p^2 t} e^{\imath px} \tilde{A}_\mu(p)$$

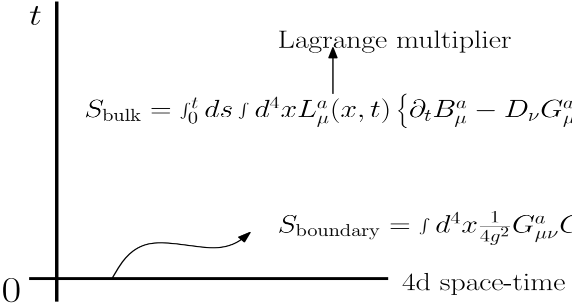
$$B_{\mu,1}(x,t) = \frac{1}{4\pi t} \int d^4y\, e^{-\frac{(x-y)^2}{4t}} A_\mu(y)$$



We are “looking” at world with a resolution $\sim \sqrt{8t}$.

5d Local formulation [Lüscher, Weisz '11]

We can see the theory as a 5d local field theory [Zinn-Justin '86, Zinn-Justin, Zwanziger '88]



$$S_{\text{bulk}} = \int_0^t ds \int d^4x \, L_\mu^a(x,t) \left\{ \partial_t B_\mu^a - D_\nu G_{\mu\nu}^a \right\}$$

$$S_{\text{boundary}} = \int d^4x \, \frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a$$

$$S_{\text{Total}} = S_{\text{bulk}} + S_{\text{boundary}}$$

Theory finite to all orders of perturbation theory

- Power counting
- Theory has BRS invariance
- No loops on the bulk ⇒ No extra counterterms ⇒ No operator mixing for $t > 0$.

GRADIENT FLOW: COUPLING [LÜSCHER '10]

Take the Energy density as a candidate observable

$$\langle E(t)\rangle = \frac{1}{4}\int \mathcal{D}A_\mu G^a_{\mu\nu}(x,t)G^a_{\mu\nu}(x,t)e^{-S[A]}$$

In perturbation theory we have:

$$\langle E(t)\rangle = \frac{3g^2_{\overline{MS}}}{16\pi^2 t^2}(1 + c_1 g^2_{\overline{MS}} + \mathcal{O}(g^4_{\overline{MS}}))$$

and in terms of the running coupling $\alpha(\mu)$ at scale $\mu = 1/\sqrt{8t}$.

$$t^2\langle E(t)\rangle = \frac{3}{4\pi}\alpha_{\overline{MS}}(\mu)\left[1 + c_1'\alpha_{\overline{MS}}(\mu) + \mathcal{O}(\alpha^2_{\overline{MS}})\right]$$

Therefore one can define the strong coupling at a scale $\mu = 1/\sqrt{8t} = 1/cL$

$$\alpha(\mu) = \frac{4\pi}{3}t^2\langle E(t)\rangle = \alpha_{\overline{MS}}(\mu) + \ldots$$

- Non-perturbative definition.
- Easy to evaluate on the lattice.
- precise (smooth observable).

- Fits well with finite size scaling

$$\mu = 1/cL$$

WHY IS A GOOD CHOICE? $N_f = 2$ AND $SU(3)$ SIMULATIONS [P. FRITZSCH, A.R. '13]

L/a	6	8	10	12	16
β	5.2638	5.4689	5.6190	5.7580	5.9631
$\kappa_{\rm sea}$	0.135985	0.136700	0.136785	0.136623	0.136422
$N_{\rm meas}$	12160	8320	8192	8280	8460
$\bar{g}^2_{\rm SF}(L_1)$	4.423(75)	4.473(83)	4.49(10)	4.501(91)	4.40(10)
$\bar{g}^2_{\rm GF}(\mu)$ ($c = 0.3$)	4.8178(46)	4.7278(46)	4.6269(47)	4.5176(47)	4.4410(53)
$\bar{g}^2_{\rm GF}(\mu)$ ($c = 0.4$)	6.0090(86)	5.6985(86)	5.5976(97)	5.4837(97)	5.410(12)
$\bar{g}^2_{\rm GF}(\mu)$ ($c = 0.5$)	7.106(14)	6.817(15)	6.761(19)	6.658(19)	6.602(24)

Advantages of GF coupling definition

- $\mathcal{O}(10^3)$ less expensive at $g^2 \sim 4$ (1 CPU day $\to$ some CPU years).
- Finite variance when $a \to 0$ (i.e. $\mathcal{V} \sim \langle E^2(t)\rangle - \langle E(t)\rangle^2$).
- Statistical precision independent of coupling value $\delta g^2/g^2 \sim$ constant.
- Smaller $c \implies$ Larger cutoff effects, more precision. ($c \in [0.3, 0.5]$)

Ideal for matching with hadronic regime of QCD

SOLVING THE FLOW EQUATION ON THE LATTICE

The continuum equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad \left(\sim -g_0^2 \frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)$$

How do the links $V_\mu(x,t) = \exp[B_\mu(t,x)]$ change with the t?

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S^{\rm latt}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

- Is this the best option?
- Which lattice action $S^{\rm latt}$?

The Zeuthen flow

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \left(1 + \frac{a^2}{12} D_\mu D_\mu^*\right) \frac{\delta S^{\rm LW}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

This equation is **the result of a computation**.

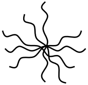
Lattice people hate discovering "new physics"

S = Standard model + Quantum Gravity

At low energies ($\ll M_{\rm pl}$)
$\langle O \rangle = \langle O \rangle_{SM} + \mathcal{O}(1/M_{\rm pl})$

We simulate

$S = \sum_{x,\mu\neq\nu} \mathrm{Tr}(1 - U_\mu(x)U_\nu(x+\mu)\cdots)$



Multi gluon interactions:
6,8,10,12,... gluon vertices.
at energy scales $1/a$

⟹ UNIVERSALITY
(Symmetries, dimensions, ...)

We obtain QCD

$S = -\frac{1}{2}\int \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \ldots$

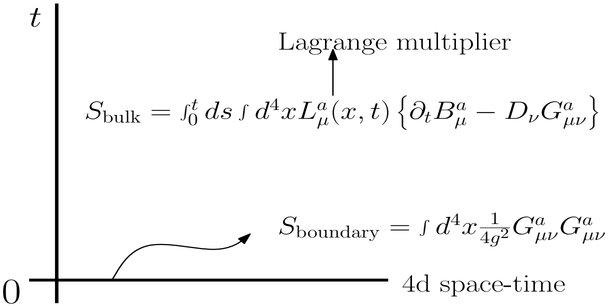
At low energies ($\ll 1/a$)
$\langle O \rangle_{\rm latt} = \langle O \rangle_{\rm QCD} + \mathcal{O}(a^2)$

Symanzik improvement program

Fine tune (i.e. cook) a lattice action $S^{\rm latt}$ such that the efective theory at energy scales much smaller than the cutoff looks as close as possible to the continuum.

5d Local formulation [Lüscher, Weisz '11]

We can see the theory as a 5d local field theory [Zinn-Justin '86, Zinn-Justin, Zwanziger '88]



$$S_{\text{Total}} = S_{\text{bulk}} + S_{\text{boundary}}$$

Theory finite to all orders of perturbation theory

- Power counting
- Theory has BRS invariance
- No loops on the bulk $\Rightarrow$ No extra counterterms $\Rightarrow$ No operator mixing for $t > 0$.

The Symanzik effective action for the Gradient flow

Action composed of bulk part and boundary part

$$S_{\text{bndry}} = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_i \alpha_i \int d^4x\, O_i^{\mathrm{d}=6}(x)$$

$$S_{\text{bulk}} = -2\int_0^\infty dt \int d^4x\,\mathrm{Tr}\left\{L_\mu(x,t)[\partial_t B_\mu(x,t) - D_\mu G_{\mu\nu}]\right\} + \sum_i \int_0^\infty dt \int d^4x\, O_i^{\mathrm{d}=8}(x,t)$$

Possible bulk counterterms

- Remember: No loops in the bulk $\Rightarrow$ No new counterterms are generated.
- Classical improvement in the bulk is equivalent to **non-perturbative** improvement.

Classical expansion of the flow equation

$$\left(a^2\partial_t V_\mu\right) V_\mu^{-1} = a^3 \partial_t B_\mu + \frac{1}{2}a^4 D_\mu \partial_t B_\mu + \frac{1}{6}a^5 D_\mu^2 \partial_t B_\mu + \mathcal{O}(a^6)$$

$$-\partial_{x,\mu}\left[g_0^2 S_{\rm lat}(V)\right] = \sum_{\nu=0}^{3}\Big\{a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu} + \frac{1}{12}a^5\Big[(1+12(c_1-c_2))\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu + 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Big]G_{\nu\mu}\Big\} + \mathcal{O}(a^6)$$

Some conclusions

- Correct continuum flow equation

$$\partial_t B_\mu = D_\nu G_{\nu\mu}$$

- $\mathcal{O}(a)$ corrections cancel.
- No value of c_1, c_2 for which the $\mathcal{O}(a^2)$ corrections cancel!

CLASSICAL EXPANSION OF THE FLOW EQUATION

$$
\begin{aligned}
\left(a^2\partial_t V_\mu\right) V_\mu^{-1} &= a^3\partial_t B_\mu + \frac{1}{2}a^4 D_\mu \partial_t B_\mu + \frac{1}{6}a^5 D_\mu^2 \partial_t B_\mu + \mathcal{O}(a^6) \\
-\partial_{x,\mu}\left[g_0^2 S_{\mathrm{lat}}(V)\right] &= \sum_{\nu=0}^{3}\Big\{a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu} \\
&\quad + \frac{1}{12}a^5\Big[(1+12(c_1-c_2))\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu \\
&\quad + 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Big] G_{\nu\mu}\Big\} + \mathcal{O}(a^6)
\end{aligned}
$$

the Symanzik/LW flow ($c_1 = -1/12$, $c_2 = 0$), is "almost" $\mathcal{O}(a^2)$ improved

$$
\partial_t B_\mu = \sum_{\nu=0}^{3}\left\{D_\nu G_{\nu\mu}(x,t) - \frac{1}{12}a^2 D_\mu^2 D_\nu G_{\nu\mu} + \mathcal{O}(a^3)\right\}
$$

The Zeuthen flow

$$
\begin{aligned}
\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} &= -g_0^2\left(1 + \frac{1}{12}a^2 D_\mu^* D_\mu\right)\partial_{x,\mu}\left[g_0^2 S_{\mathrm{LW}}(V)\right] \\
aD_\mu F(x) &= V_\mu(x,t)F(x+a\hat{\mu})V_\mu(x,t)^\dagger - F(x), \ldots
\end{aligned}
$$

Continuum limit of flow quantities

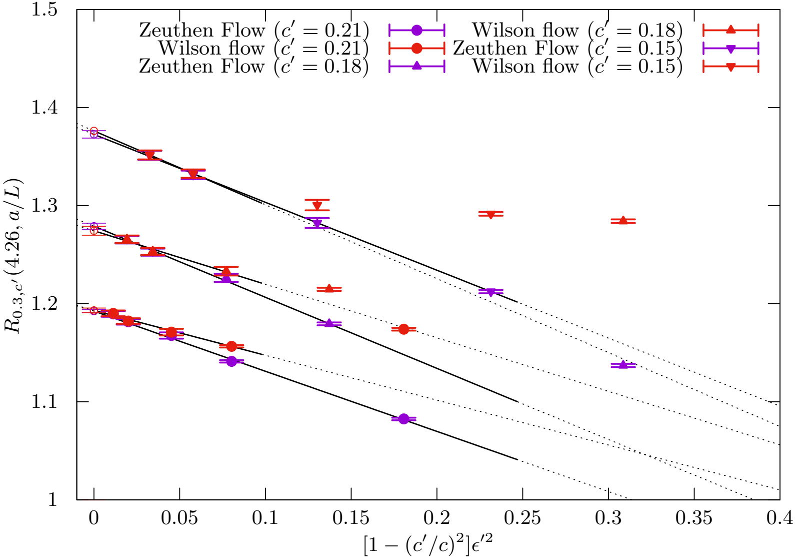
Study the general quantity

$$R_{c,c'}(u, a/L, s) = \left.\frac{g_c^2(L)}{g_{c'}^2(sL)}\right|_{g_c^2(L)=u} = R_{c,c'}(u, 0, s)\left\{1 + A_{c,c'}(u)[\epsilon^2 - \epsilon'^2] + \ldots\right\},$$

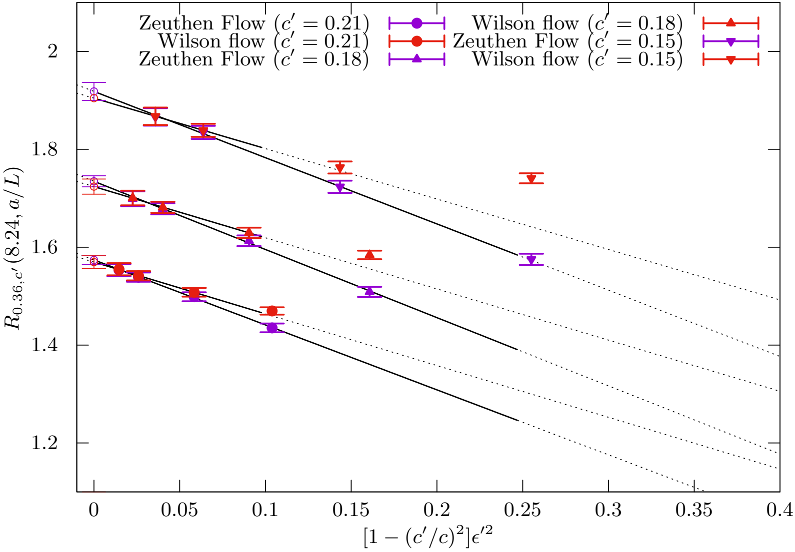
with $\epsilon = a/(cL)$ and $\epsilon' = a/(c'sL)$.

- $R_{c,c'}(u, a/L, s)$ is mainly a function of sc'.
- Step scaling function $\Longrightarrow R_{c,c}(u, a/L, 2)$.
- Instead study $R_{c,c'}(u, a/L, 1) \Longrightarrow$ we can use $L/a = 8, 12, 16, 24, 32$

Scaling of the ratios $R_{c,c'}(u, a/L, 1)$



Scaling of the ratios $R_{c,c'}(u, a/L, 1)$



SCALING OF THE RATIOS $R_{c,c'}(u, a/L, 1)$

Three sources of cutoff effects in flow quantities

- Quantum effects at $t = 0$. Very complicated dependence on g_0^2
- Integrating the flow equation
- Evaluating an observable

Scaling of the ratios $R_{c,c'}(u, a/L, 1)$

Three sources of cutoff effects in flow quantities

- Quantum effects at $t = 0$. Very complicated dependence on g_0^2
- ~~Integrating the flow equation~~ Zeuthen flow
- ~~Evaluating an observable~~ Classically improved discretization

Conclusions: Still lot to understand!

- In our data:
  - Wilson Flow: Breaking of scaling at $(a/cL)^2 = 0.15$
  - Zeuthen Flow: Breaking of scaling at $(a/cL)^2 = 0.3$
  - We use $L/a = 8, c = 0.3 \Longrightarrow (a/cL)^2 = 0.17$
- Zeuthen flow **not** cooked for this!
- $\mathcal{O}(a^2)$ effects still significant!
- Main suspect: The "extra" boundary counterterm:

$$O_4(x) = \mathrm{tr}\{L_\mu(0,x) D_\nu F_{\nu\mu}\}$$

  complicated (i.e. receives quantum corrections).

CONNECTING WITH THE HADRONIC REGIME OF QCD

We need

$$L_{\rm had}\Lambda = ??$$

$L_{\rm had}$ is a hadronic scale: One must be able to compute $F_k L_{\rm had}$ in "usual" large volume lattice simulations

$$g^2_{\rm GF}(L_{\rm had}) = 11.31$$

But the result of the high energy SF running gives at $g^2_{\rm SF}(L_0) = 2.012$ the result

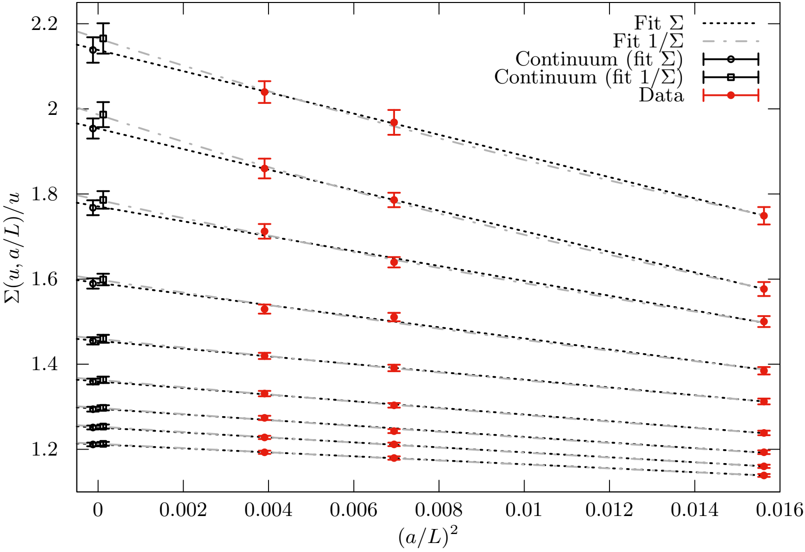
$$L_0\Lambda = 0.0308(8)$$

Strategy

1. Determine the β-function, and compute

$$\frac{L_{\rm had}}{L_0} = \exp\left\{\int_{g(L_0)}^{g(L_{\rm had})} \frac{dx}{\beta(x)}\right\}$$

Step scaling function $L/a = 8, 12, 16 \to 16, 24, 32$



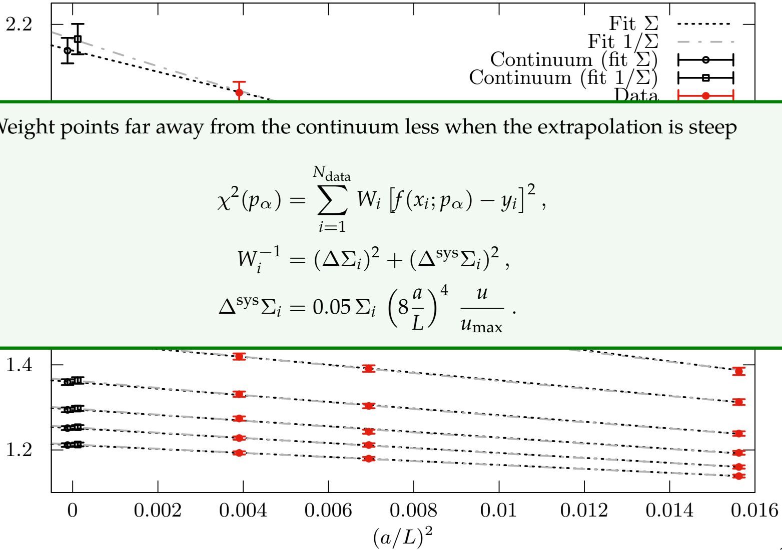
Step scaling function $L/a = 8, 12, 16 \to 16, 24, 32$

Systematic difference between ansatz for continuum extrapolation

$$\Sigma(u_i, a/L) = \sigma_i + r(a/L)^2 \qquad \text{and} \qquad \frac{1}{\Sigma(u_i, a/L)} = \frac{1}{\sigma_i} + \tilde{r}(a/L)^2$$

u_i	σ_i		$(1/\sigma_i - 1/u_i) \times 10^2$	
6.5489	14.005(175)	14.184(197)	−8.13(10)	−8.22(12)
5.8673	11.464(123)	11.654(146)	−8.32(10)	−8.46(13)
5.3013	9.371(79)	9.468(89)	−8.19(11)	−8.30(12)
4.4901	7.139(47)	7.181(51)	−8.26(11)	−8.34(12)
3.8643	5.622(28)	5.641(30)	−8.09(10)	−8.15(14)
3.2029	4.354(19)	4.367(21)	−8.25(12)	−8.32(13)
2.7359	3.541(14)	3.550(15)	−8.31(12)	−8.38(13)
2.3900	2.991(10)	2.996(10)	−8.40(12)	−8.46(13)
2.1257	2.575(9)	2.578(9)	−8.21(14)	−8.26(14)
Constant fit:			−8.233(37)	−8.316(42)

Step scalling function $L/a = 8, 12, 16 \to 16, 24, 32$



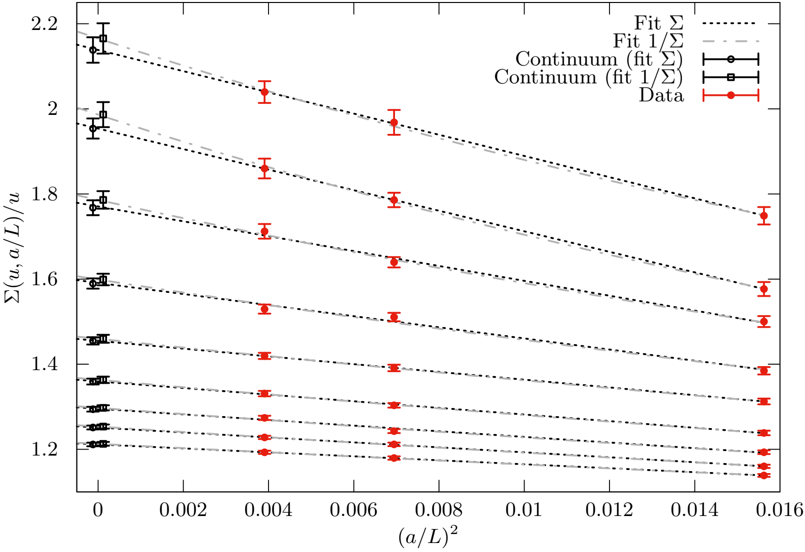
Weight points far away from the continuum less when the extrapolation is steep

$$\chi^2(p_\alpha) = \sum_{i=1}^{N_{\text{data}}} W_i \left[f(x_i;p_\alpha) - y_i\right]^2,$$

$$W_i^{-1} = (\Delta\Sigma_i)^2 + (\Delta^{\text{sys}}\Sigma_i)^2,$$

$$\Delta^{\text{sys}}\Sigma_i = 0.05\,\Sigma_i \left(8\frac{a}{L}\right)^4 \frac{u}{u_{\max}}.$$

Step scaling function $L/a = 8, 12, 16 \to 16, 24, 32$



Determination of $\sigma(u)$

Combined analysis

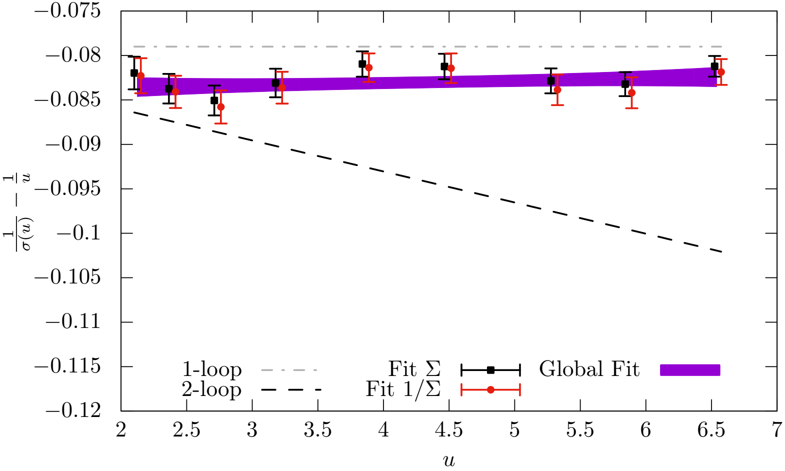
Combine continuum extrapolation with parametrization of $\sigma(u)$

$$\frac{1}{\Sigma(u,a/L)} - \frac{1}{u} = \tilde{P}(u) + \rho(u)\left(\frac{a}{L}\right)^2$$

$$\tilde{P}(x) = \sum_{k=0}^{n_p} c_k x^k; \qquad \rho(x) = \sum_{k=0}^{n_p} r_k x^k$$

Flexible: No need to tune, no need to fit: just simulations at L/a and $2L/a$ at matching g_0.

Determination of $\sigma(u)$



DETERMINATION OF $\sigma(u)$

What have we learned?

- Even seing "perfect $\mathcal{O}(a^2)$ scaling", we need to add systematic uncertainty due to "large" cutoff effects
- Non-perturbative step scaling function is just a shifted 1-loop in $g^2 \in [2 - 6.5]$!
- PT completely broken. Probably large 3-loop coefficient in this scheme.
- $\alpha = 0.2$ ($\sim$ 4GeV) far from applicability of PT at this level of precision.
- Consistents with our conclusions in the SF scheme **[arXiv:1604.06193]**

Determination of the β-function

Use the exact relation

$$\log 2 = \int_{g(L)}^{g(2L)} \frac{dx}{\beta(x)}$$

with the ansatz

$$\beta(x;p) = -\frac{x^3}{P(x)}; \qquad P(x) = \sum p_k x^{2k}$$

Fit your data using

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log 2 - \int_{\sqrt{u_i}}^{\sqrt{\sigma(u_i)}} \frac{dx}{\beta(x;p)} \right]^2$$

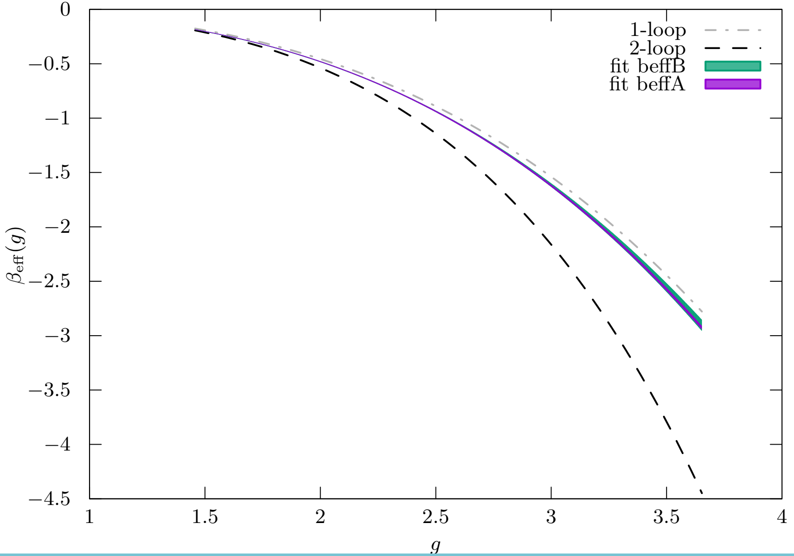
NOTE: Very flexible: No tunning, no fitting: just simulations at different L/a and matching g_0

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log s - \int_{\sqrt{u_i}}^{\sqrt{\Sigma_s(u_i,a/L)+\rho(u,s)\left(\frac{a}{L}\right)^2}} \frac{dx}{\beta(x;p)} \right]^2$$

$$\chi^2(p) = \sum_{\text{data}} \frac{1}{\delta I^2} \left[\log s - \int_{\sqrt{u_i}}^{\sqrt{\Sigma_s(u_i,a/L)}} \frac{dx}{\beta(x;p)} + \tilde{\rho}(u,s) \left(\frac{a}{L}\right)^2 \right]^2$$

The jogging coupling in $N_f = 3$ QCD

–Mattia dalla Brida



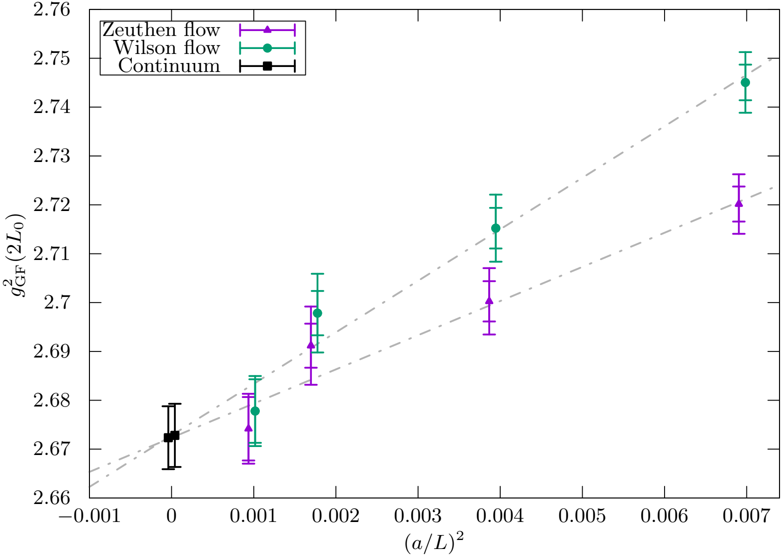
Matching with L_0 scale

L_0 defined via $g^2_{\rm SF}(L_0) = 2.012$

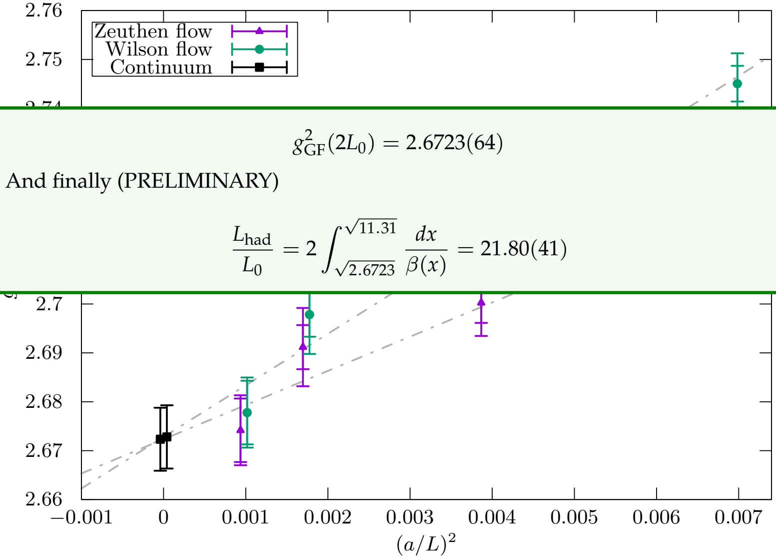
How to relate with $g^2_{\rm GF}$?

- Take a few of $\beta, L/a$ s.t. $g^2_{SF}(L) = 2.012$
- Compute in $\beta, 2L/a$ s.t. $g^2_{GF}(L) =?$

Matching with L_0 scale



Matching with L_0 scale



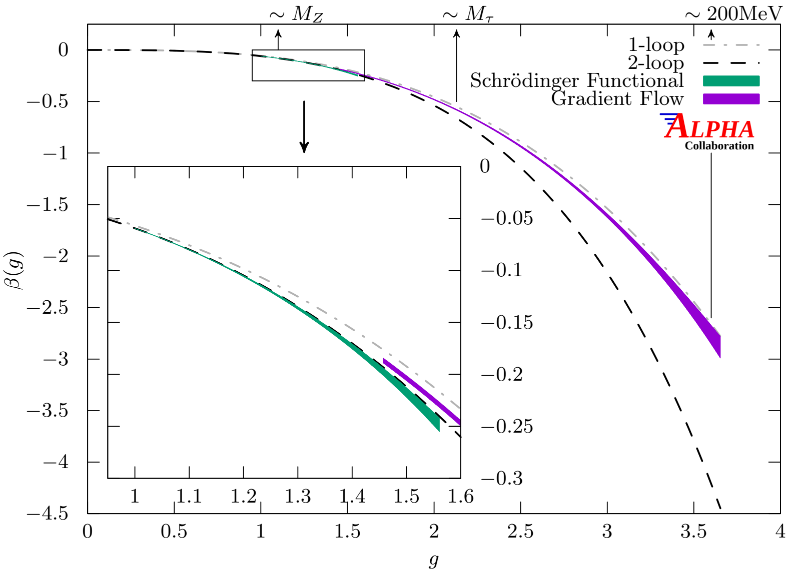
$$g^2_{\mathrm{GF}}(2L_0) = 2.6723(64)$$

And finally (PRELIMINARY)

$$\frac{L_{\mathrm{had}}}{L_0} = 2\int_{\sqrt{2.6723}}^{\sqrt{11.31}} \frac{dx}{\beta(x)} = 21.80(41)$$

Overview

Determination of β-function in QCD



Conclusions

- Finite size scaling key for a first principle determination of the fundamental parameters of the SM.
- Matching with PT requires care when precision increases
- Gradient Flow ideal for matching non-perturbative regimes of strongly coupled QFT
- Determination of β- function allows more flexibility than $\sigma(u)$.
- $\mathcal{O}(a^2)$ cutoff effects still large! Better understanding.
- 3-loop coefficient?