Instituto de
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Volume independence beyond large N

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In collaboration with

Antonio González-Arroyo, Liam Keegan,
Matt Koren, Masanori Okawa

Holography, conformal field theories, and lattice

✦ Large N Eguchi-Kawai reduction - Volume independence

Previous talk by A. González-Arroyo

Volume $\longleftrightarrow$ SU(N)

✦ Finite N ? Volume dependence encoded in

Effective size involving both volume and group degrees of freedom

Some relation to the idea of volume independence by

Kotvun, Unsal & Yaffe
Amber, Basar, Cherman, Dorigoni, Hanada, Koren, Poppitz, Sharpe,…

Eguchi-Kawai reduction

Large N observable

$$b = \frac{\beta}{2N^2} = \lambda_L^{-1} \quad \text{fixed}$$

$$O_\infty(b) = \lim_{N\to\infty} \lim_{L\to\infty} O(b, N, L)$$

L^4 lattice

Eguchi-Kawai reduction

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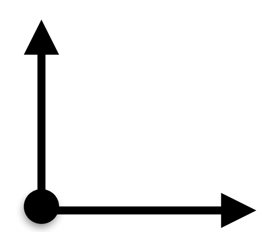
L^4 lattice

Eguchi-Kawai reduction

$$O_\infty(b) = \lim_{N\to\infty} O(b, N, L = 1)$$



$U_\mu \in SU(N)$

one-point lattice

Eguchi-Kawai reduction

$b = \dfrac{\beta}{2N^2} = \lambda_L^{-1}$ fixed

Large N observable

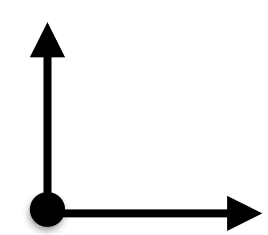
$$O_\infty(b) = \lim_{N\to\infty} \cancel{\lim_{L\to\infty}} O(b, N, L)$$

L^4 lattice

Eguchi-Kawai reduction

$$O_\infty(b) = \lim_{N\to\infty} O(b, N, L = 1)$$

Thermodynamic limit irrespective of L



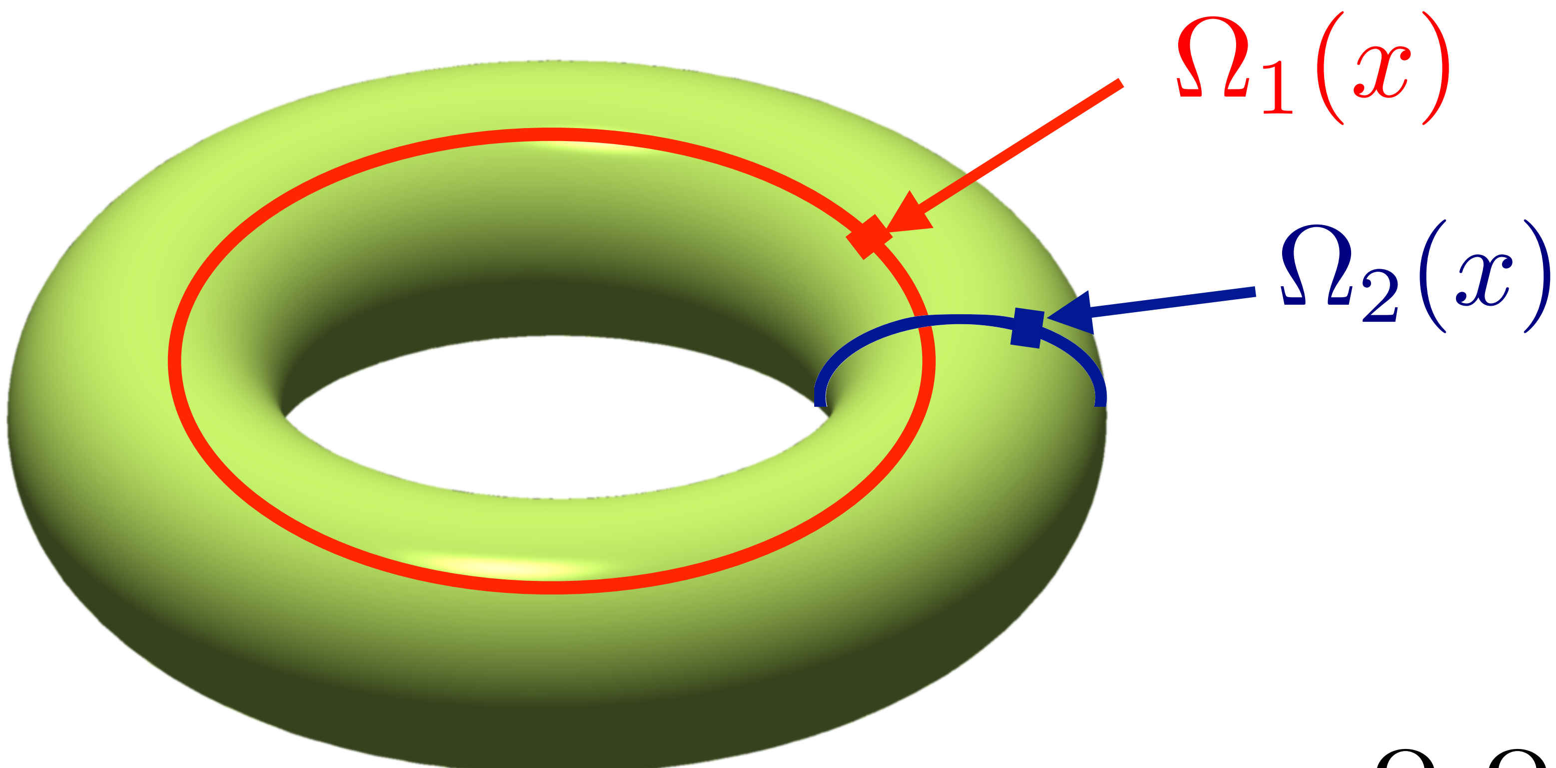
$U_\mu \in SU(N)$

one-point lattice

A different limit - Physics on a finite volume

't Hooft, Luscher, Gonzalez-Arroyo e.a., van Baal

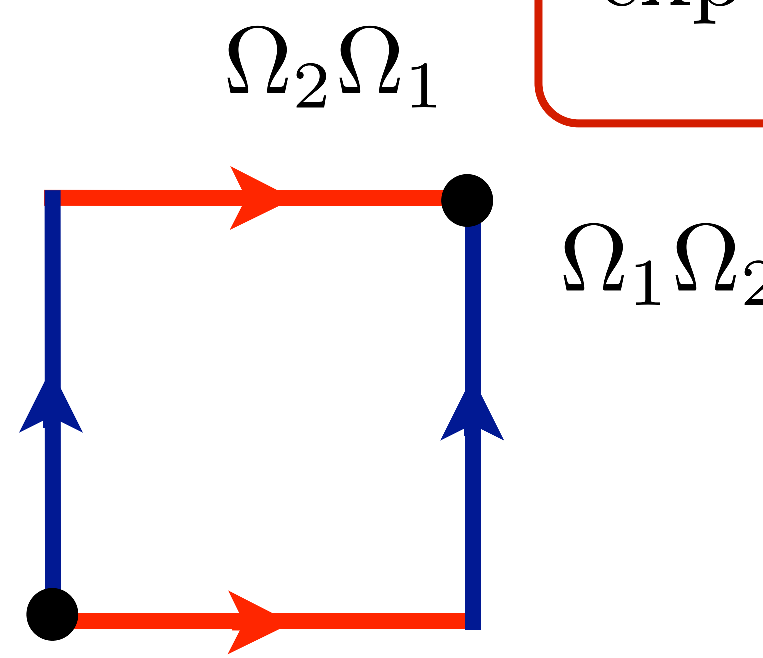
- Formulate the problem in the continuum on a d-torus of Size l



Twist matrices

Twist tensor

$$\boxed{\exp\left\{2i\pi\frac{n_{\mu\nu}}{N}\right\}}$$



G. 't Hooft

For orthogonal twists $\Omega_\mu(x) = \Gamma_\mu$

$$\frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = 0 \,(\mathrm{mod}\, N)$$

consistency $\Gamma_\mu \Gamma_\nu = e^{\frac{2\pi i n_{\mu\nu}}{N}} \Gamma_\nu \Gamma_\mu$

boundary conditions $A_\mu(x + l\,\hat{\nu}) = \Gamma_\nu A_\mu(x) \Gamma_\nu^\dagger$

Twist $\boxed{n_{\mu\nu}}$ $\boxed{\Gamma_\mu}$ Twist Eaters

To implement boundary conditions

$$A_\nu(x) = \mathcal{N} \sum_p{}' e^{ip\cdot x} \, \hat{A}_\nu(p) \, \hat{\Gamma}(p)$$

$$A^a_\mu(p) T_a$$

momentum dependent
basis for the SU(N)
Lie algebra

$$\hat{\Gamma}(p) \propto \Gamma_1^{s_1} \Gamma_2^{s_2} \cdots \Gamma_d^{s_d}$$

$$\boxed{p_\mu = \frac{2\pi n_\mu}{l_{\rm eff}}}$$

To satisfy b.c. momentum is quantised in units of

Effective box - size

$l_{\rm eff} = lN$ for d=2

$l_{\rm eff} = l\sqrt{N}$ for d=4

Conjecture

Volume and N dependence controlled by $l_{\rm eff}$

By varying $l_{\rm eff}$ we can transit from perturbation theory to confinement

$l_{\rm eff}$ determines the dynamics

Link to Eguchi-Kawai reduction

$$l_{\text{eff}} = l\sqrt{N}$$

On a lattice $\quad l = La$

For the one-point lattice in EK reduction - thermodynamic limit

$$l_{\text{eff}} = a\sqrt{N} \xrightarrow[N \to \infty, \ a \text{ fixed}]{} l_{\text{eff}} = \infty$$

We will instead pursue

$$\boxed{l_{\text{eff}} \ \text{fixed} \qquad N \to \infty \ \ \& \ \ l \to 0}$$

Perturbation theory

- Momentum quantized in units of $l_{\rm eff}$
- Free propagator identical that on a finite volume $l_{\rm eff}$
- Group structure constants $\Gamma(p)$

$$F(p,q,-p-q) = -\sqrt{\frac{2}{N}}\sin\left(\frac{\theta_{\mu\nu}}{2}p_\mu q_\nu\right)$$

Momentum dependent phases in the vertices

$$\boxed{\theta_{\mu\nu} = \left(\frac{l_{\rm eff}}{2\pi}\right)^2 \times \tilde{\epsilon}_{\mu\nu}\,\tilde{\theta}}$$

Links to non-commutative gauge theories

González-Arroyo, Korthals Altes, Okawa

$$l_{\text{eff}} = lN$$

$$l_{\text{eff}} = l\sqrt{N}$$

in 2-d

$$\tilde{\theta} = \frac{2\pi\bar{k}}{N}$$

$$n_{ij} = \epsilon_{ij} k$$

$$k\bar{k} = 1 \,(\text{mod}\, N)$$

k and N co-prime

in 4-d

$$\tilde{\theta} = \frac{2\pi\bar{k}}{\sqrt{N}}$$

$$n_{\mu\nu} = \epsilon_{\mu\nu}\, k\, \sqrt{N}$$

$$k\bar{k} = 1 \,(\text{mod}\, \sqrt{N})$$

k and $\sqrt{N}$ co-prime

4d with a 2-d twist Ramos & Keegan

Volume independence

Vertices α $\sqrt{\frac{2\lambda}{V_{\rm eff}}}\sin\left(\frac{\theta_{\mu\nu}}{2}p_\mu q_\nu\right)$

In perturbation theory, physics depends on

$$\boxed{\tilde{\theta}\,,\ \lambda\,,\ l_{\rm eff}}$$

For fixed $\tilde{\theta}$, volume and N dependence encoded in the effective size

The perturbative expansion implies an equivalence between different SU(N) gauge theories

Morita Duality

Links to non-commutative gauge theories

The SU(N) twisted theory is physically equivalent to a non-commutative U(1) gauge theory defined on a periodic torus with periods $l_{\rm eff}$ and non-commutativity parameter θμν.

$$A_\nu(x) = \mathcal{N} \sum_p{}' e^{ip\cdot x}\, \hat{A}_\nu(p)\, \hat{\Gamma}(p)$$

in U(1)

Non-commutativity in the vertices

$$\sqrt{\frac{2\lambda}{V_{\rm eff}}} \sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right)$$

Ambjorn, Makeenko, Nishimura & Szabo

Comment 1

✦ Certain momenta excluded by the twist in SU(N)

$$\mathrm{Tr}\,\hat{\Gamma}(p) = 0$$

$$A_\nu(x) = \mathcal{N} \sum_p{}' e^{ip\cdot x}\, \hat{A}_\nu(p)\, \hat{\Gamma}(p)$$

$$p_\mu = \frac{2\pi n_\mu}{l_{\mathrm{eff}}}$$

for d=4, exclude

$$n_\mu = 0\,(\mathrm{mod}\,\sqrt{N})\,\forall\mu$$

Reintroduces N dependence

Example - SU(N) Wilson loops on a L^4 lattice

$$\log W(b,N,L) = -W_1(N,L)\lambda - W_2(N,L)\lambda^2$$

For TBC

$$W_1(N,L) = F_1(L\sqrt{N}) - \frac{1}{N^2}F_1(L) \xrightarrow[N\to\infty]{} \boxed{F_1(\infty)}$$

Finite N

$$W_1(N,L) = F_1(L\sqrt{N}) - \frac{1}{N^2}F_1(L) \xrightarrow[L\to\infty]{} \boxed{\frac{N^2-1}{N^2}} F_1(\infty)$$

Reintroduces N dependence

Retrieves the correct number of colour degrees of freedom

MGP, González-Arroyo & Okawa

For PBC

$$W_1(N,L) = F_1(L)\,\frac{N^2-1}{N^2} \xrightarrow[N\to\infty]{} \boxed{F_1(L)}$$

retains L dependence

Comment II

✦ Fixed $\tilde{\theta}$

$$\tilde{\theta} = \frac{2\pi\bar{k}}{\sqrt{N}}$$

k and $\sqrt{N}$ co-prime

Not possible to keep it exactly fixed when changing N

The conjecture relies on assuming a smooth dependence on $\tilde{\theta}$

Possible caveats

✦ Non-perturbative effects ?

TEK Symmetry breaking Ishikawa&Okawa, Teper&Vairinhos e.a., Azeyanagi e.a.

✦ Perturbative instabilities in the large N limit

Negative self-enegy $\longrightarrow$ Tachyonic instabilities

Hayakawa, Guralnik e.a., Armoni e.a., Bietenholz e.a., ...

Avoided if $\boxed{k \text{ and } \bar{k} \propto N \quad \text{as} \quad N \to \infty}$

González-Arroyo & Okawa, MGP & González-Arroyo & Okawa

Some applications

- Running of the SU(∞) coupling in 4-dimensions
- Spectrum of 2+1 Yang-Mills theory

SU(N) running coupling

Yang-Mills gradient flow

Flow of gauge potentials

$$\partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t)$$

$$B_\mu(x,t=0) = A_\mu \qquad\qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]$$

Gauge invariant composite fields at positive flow time $\longrightarrow$

Renormalized observables defined at a scale

$$\mu = 1/\sqrt{8t}$$

Lüscher, Lüscher & Weisz

Twisted gradient flow coupling

$$\frac{1}{N}\langle E(t)\rangle = \frac{1}{2N}\langle \mathrm{Tr}\{G_{\mu\nu}(t)G_{\mu\nu}(t)\}\rangle \propto \lambda_{\overline{MS}} + \mathcal{O}(\lambda^2_{\overline{MS}})$$

Action density

Define a renormalized coupling in terms of E(t) Luscher, Fodor et al., Fritszch&Ramos

- Use twisted boundary conditions Ramos
- Set the scale in terms of the torus size

Scale dependence

$$\boxed{\lambda_{\mathrm{TGF}}(l) = \mathcal{N}^{-1}(c)\,\frac{t^2\langle E(t)\rangle}{N}\bigg|_{t=c^2l^2/8}}$$

Scheme dependence

Our proposal

MGP, González-Arroyo, Keegan, Okawa

For SU(N) set the scale in terms of the effective size $\lambda(\tilde{l})$

$$\boxed{\tilde{l} = l\sqrt{N}}$$

Non-perturbative lattice determination $\tilde{l} = La\sqrt{N}$

For TEK

$L = 1$

$$\boxed{\tilde{l} = a\sqrt{N}}$$

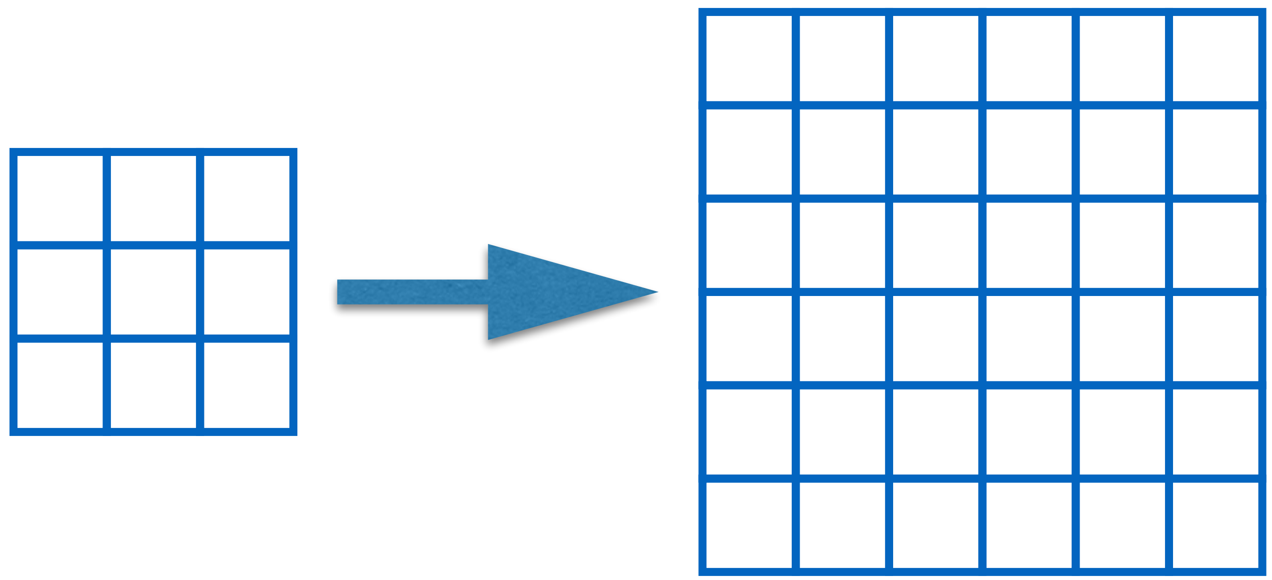
Implement step-scaling by changing the rank of the group

Implement step-scaling by changing the rank of the group $\quad \tilde{l} = a\sqrt{N}$

Fix a value of the renormalized coupling u

$u = \lambda(b, L)$

$L = \sqrt{N}$

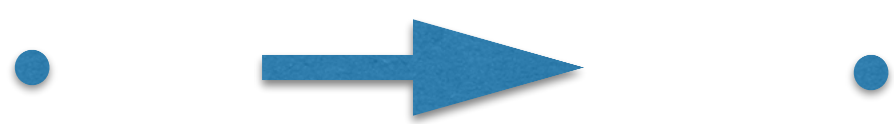


$\Sigma(u, s, L) = \lambda(b, sL)$

$sL = s\sqrt{N}$

$SU(N) \qquad\qquad SU(s^2 N)$

For TEK

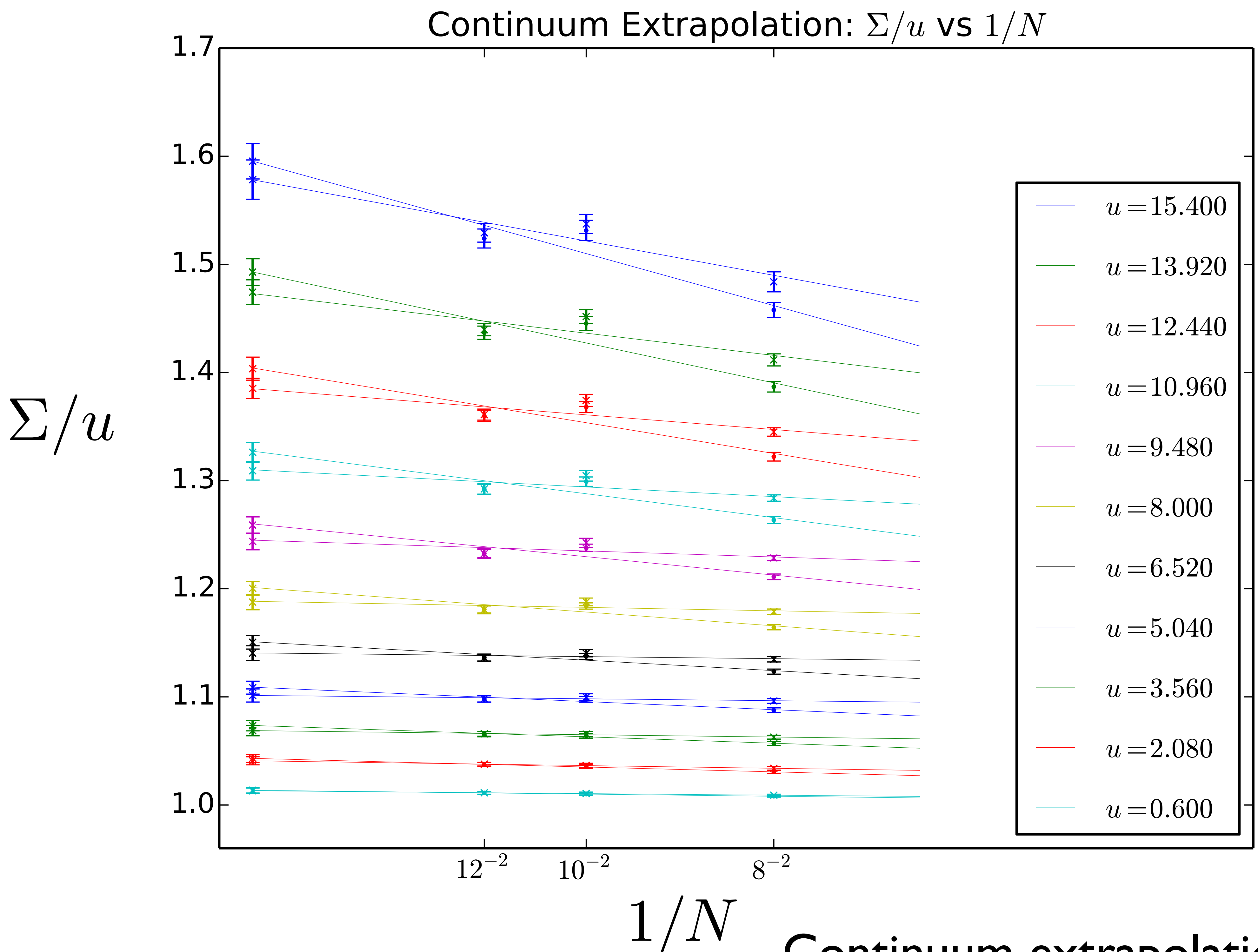


$$\sigma(u, s) = \lim_{1/N \to 0} \Sigma(u, s, \sqrt{N})$$ at fixed u (at fixed $\tilde{l}$)

Repeat for a sequence $\quad u_{k+1} = \sigma(u_k, s) \longrightarrow s^n \tilde{l}$

$\sqrt{N} = 8 \to 12,\ 10 \to 15,\ 12 \to 18 \qquad s = 3/2$

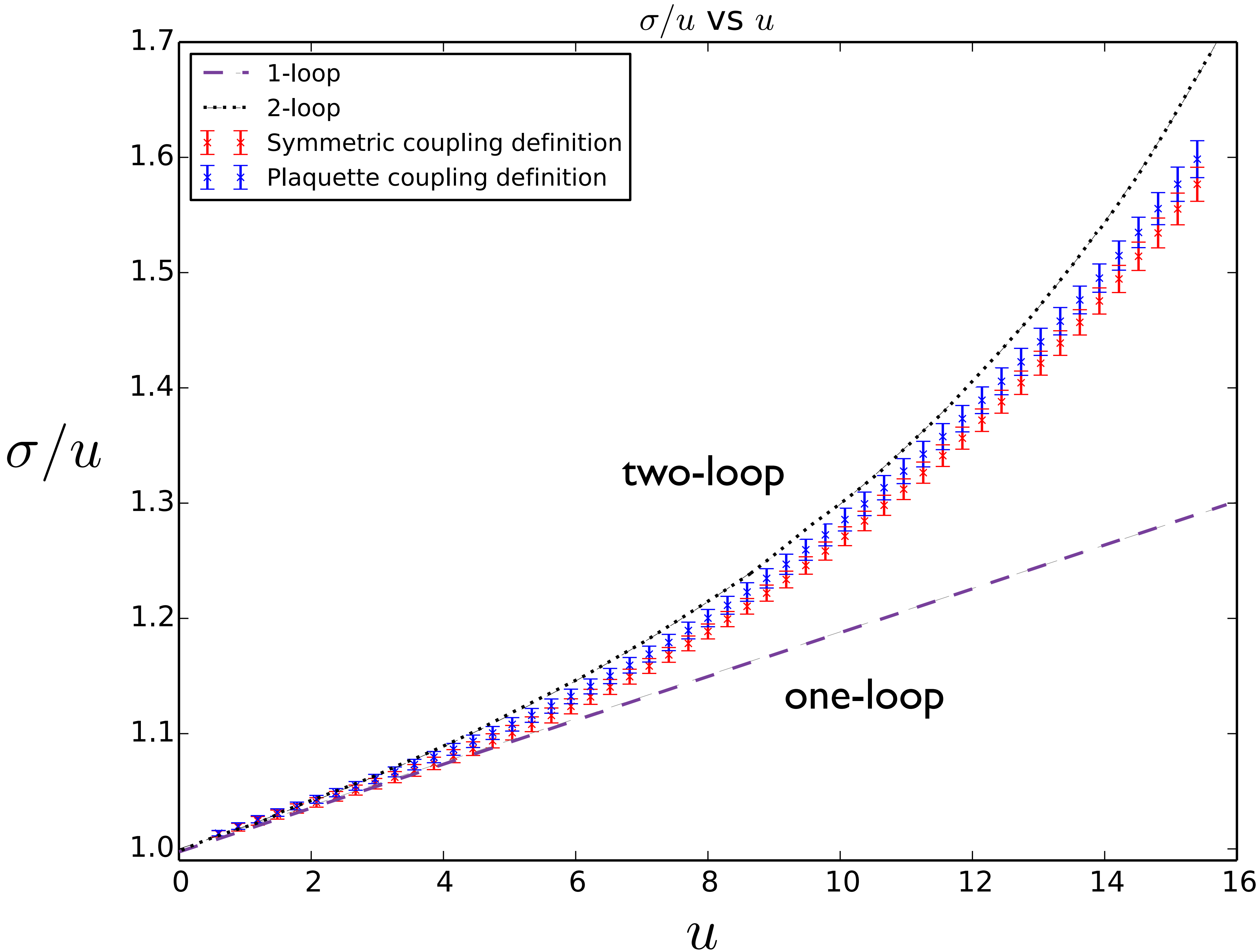
Continuum limit at fixed $\tilde{l}$ by sending $N \to \infty$ c=0.3



Continuum extrapolation implies a large N limit at fixed renormalized 't Hooft coupling

Step scaling function SU(∞)

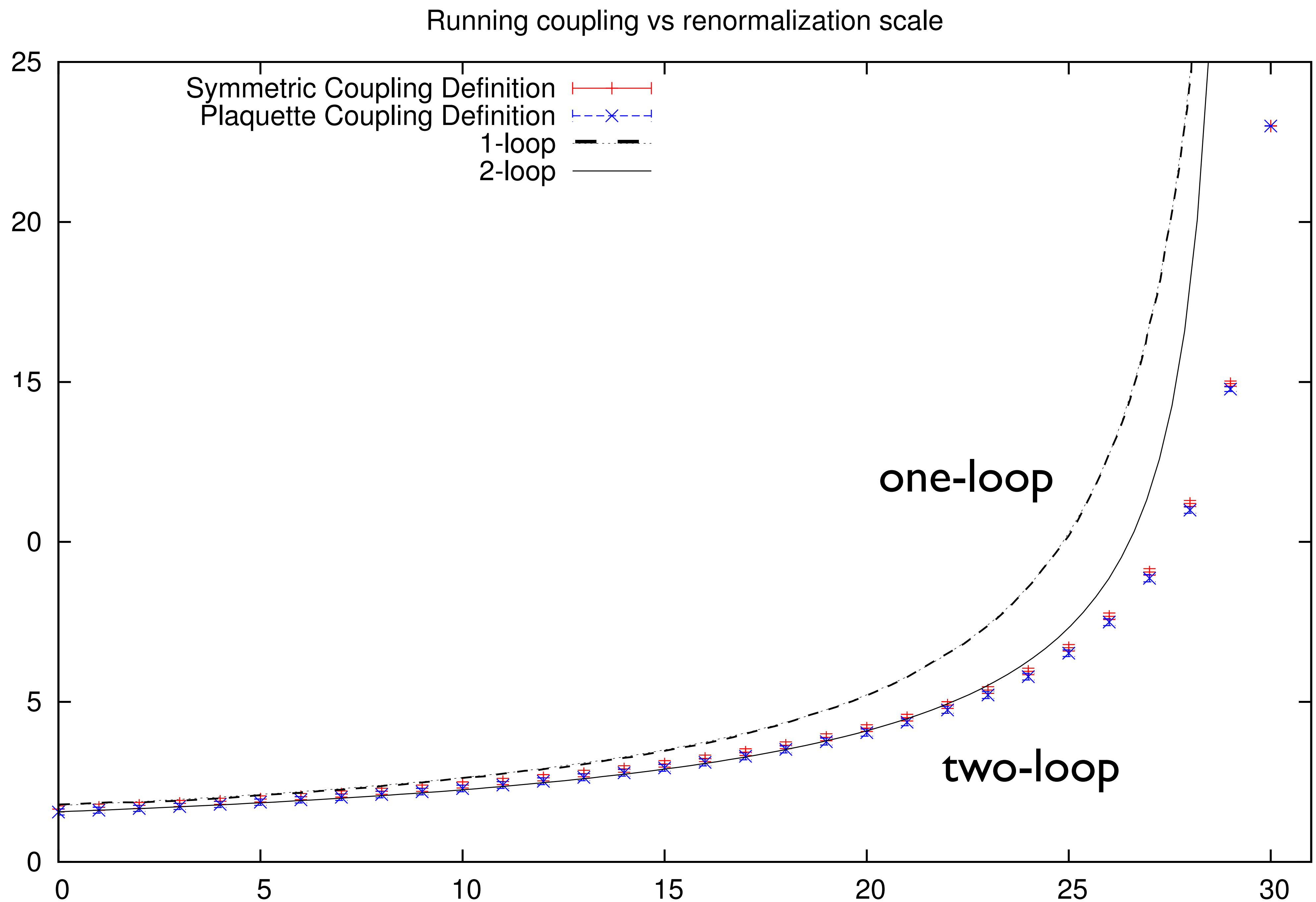
c=0.3



Renormalized coupling SU(∞)

c=0.3

$\lambda(\tilde{l})$



$s = 3/2$ $\log_{3/2}(\tilde{l}/\tilde{l}_{\min})$

Results in 2+1 d

MGP, González-Arroyo, Koren, Okawa

in 2-d

$$\tilde{\theta} = \frac{2\pi \bar{k}}{N}$$

$$n_{ij} = \epsilon_{ij} k$$

$$k\bar{k} = 1 \,(\mathrm{mod}\, N)$$

$$l_{\mathrm{eff}} = lN$$

Dimensionful 't Hooft coupling

$$x = \lambda N l/(4\pi)$$

Relevant scaling parameter for fixed $\tilde{\theta} = \frac{2\pi \bar{k}}{N}$

$\boxed{\text{Mass Gap in PT}}$ $\quad \frac{2\pi|\vec{n}|}{Nl} \quad \vec{n} \neq \vec{0} \pmod N$

one-gluon states $\longrightarrow$ electric flux $\boxed{e_i = -\bar{k}\epsilon_{ij}n_j}$

one to one relation between flux and momentum

Lowest state has flux $\bar{k}$ $\quad \boxed{\frac{\mathcal{E}_1}{\lambda} = \frac{1}{2x}}$

$\boxed{\text{Glueball mass in PT}}$ $\quad \frac{4\pi}{Nl} \quad \boxed{\frac{\mathcal{E}_G}{\lambda} = \frac{1}{x}}$

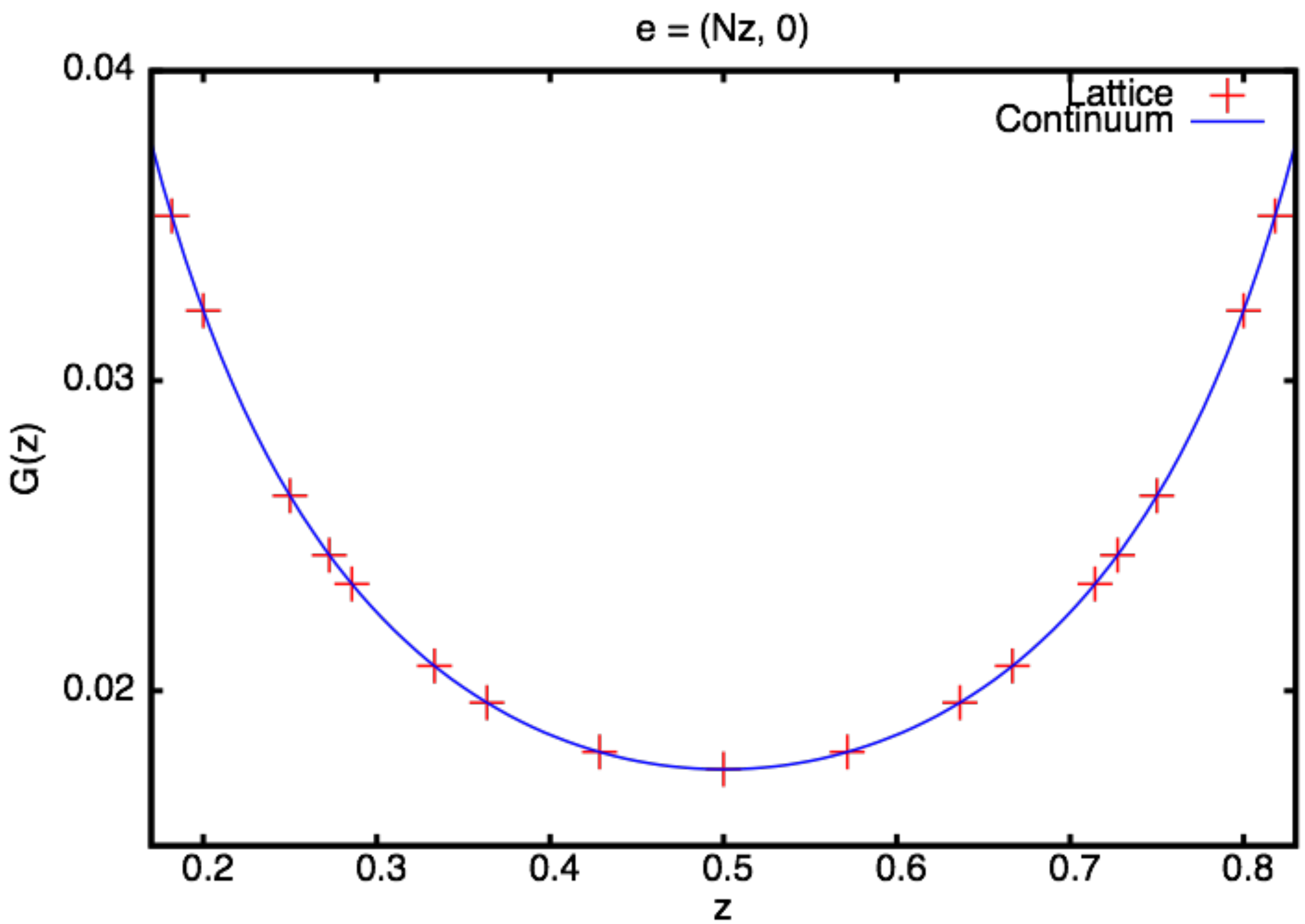
Gluon self-energy - Perturbation theory

$$\frac{\mathcal{E}^2}{\lambda^2} = \frac{|\vec{n}|^2}{4x^2} - \frac{1}{x} G\Big(\frac{\vec{e}}{N}\Big)$$

$$\boxed{G(\vec{z}) = -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{\sqrt{t}} \Big(\theta_3^2(0,it) - \prod_{i=1}^{2}\theta_3(z_i,it) - \frac{1}{t}\Big)}$$

$$\theta_3(z,it) = \sum_{k\in\mathbf{Z}} \exp\{-t\pi k^2 + 2\pi i k z\}$$

Tachyonic instability



$$\frac{\vec{e}}{N} = (z, 0)$$

$$G\left(\frac{\vec{e}}{N}\right) \propto \frac{N}{|\vec{e}|}$$

Mass squared negative at

$$x_t(\vec{e}) = \frac{4\pi^2|\vec{n}|^2|\vec{e}|}{N}$$

$$x_T = \frac{4\pi^2 k^2}{N} \qquad |\vec{e}| = 1$$

$$x_T = \frac{4\pi^2 \bar{k}}{N} \qquad |\vec{n}| = 1$$

$$\boxed{k \text{ and } \bar{k} \propto N \quad \text{as} \quad N \to \infty}$$

Gonzalez-Arroyo & Chamizo

Electric-flux energies grow linearly with l in the confined region

$$\boxed{\frac{\mathcal{E}}{\lambda} = \frac{\sigma_{\vec{e}}\, l}{\lambda}}$$

$$\sigma_{\vec{e}} = N \sigma' \phi\Big(\frac{\vec{e}}{N}\Big) \qquad \phi(z) = \phi(1-z)$$

$$\phi(z) = z(1-z) \qquad \phi(z) = \sin(\pi z)/\pi$$

- Compatible with reduction
- The linear growth can overcome the tachyonic behaviour

$$\boxed{\frac{\mathcal{E}(z)}{\lambda} = 4\pi x \,\frac{\sigma'}{\lambda^2}\,\phi(z)}$$

Ansatz for electric flux energies

$$\frac{\mathcal{E}_n^2}{\lambda^2} = \frac{|\vec{n}|^2}{4x^2} + \frac{\alpha}{x} + \beta + \gamma^2 x^2$$

Nambu-Goto with winding e on Kalb-Ramond B-field background

Guralnik

$$\frac{\mathcal{E}^2(\vec{e})}{\lambda^2} = \sum_i \left(\frac{\epsilon_{ij} e_j B}{\lambda l}\right)^2 - \frac{\pi\sigma}{3\lambda^2} + \left(\frac{4\pi\sigma\vec{e}}{\lambda^2 N}\right)^2 x^2$$

$$B = -\frac{2\pi k}{N}$$

Gives the tree level term

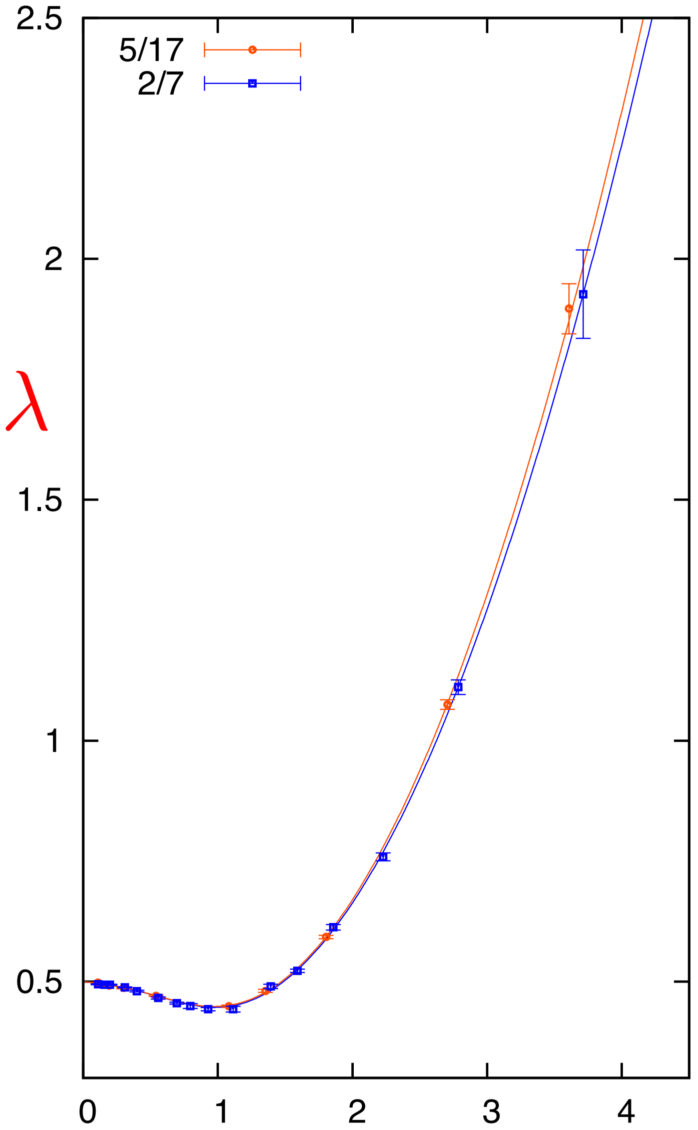
Low energy - non-commutative field theory

$$\theta_{ij} = -\epsilon_{ij}\, l^2 \,\frac{1}{B} \quad \longrightarrow \quad \theta_{\mu\nu} = \left(\frac{l_{\mathrm{eff}}}{2\pi}\right)^2 \times\ \tilde{\epsilon}_{\mu\nu}\, \tilde{\theta}$$

Energy of electric flux - lightest perturbative state

$$\frac{\tilde{\theta}}{2\pi} = \frac{5}{17}$$

$$\frac{\tilde{\theta}}{2\pi} = \frac{2}{7}$$



$x\mathcal{E}_1/\lambda$

$x = \lambda N l/(4\pi)$

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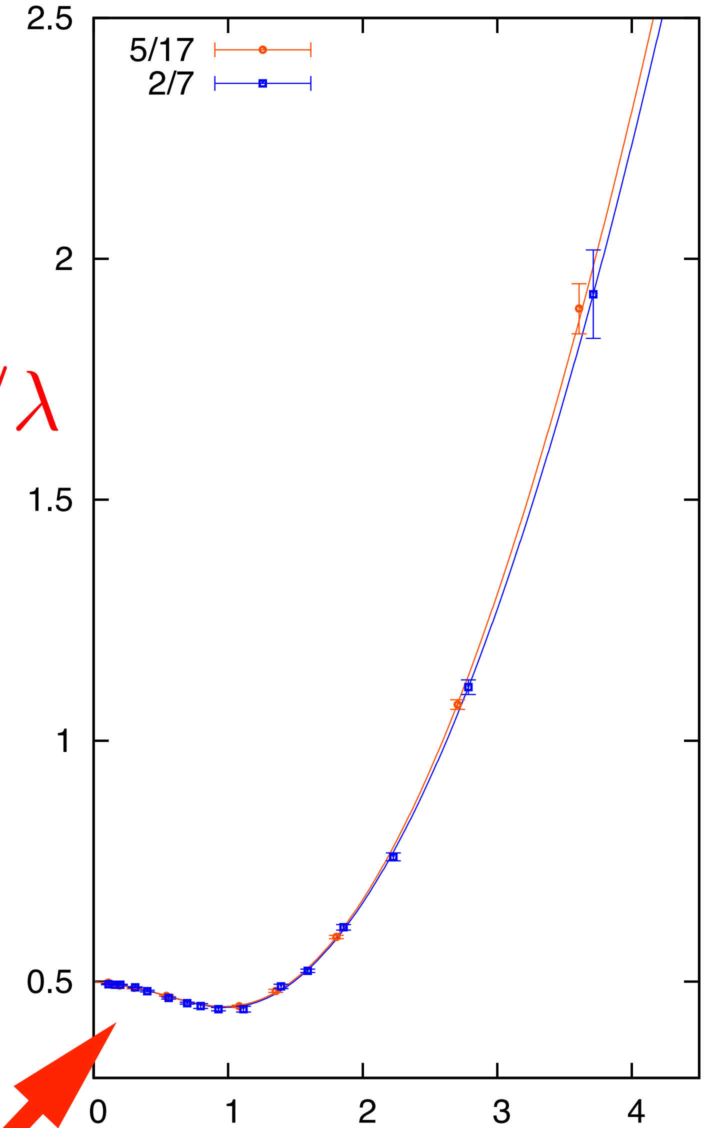

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Perturbation theory

$$x\frac{\mathcal{E}_1}{\lambda} = \frac{1}{2}\sqrt{1 - 4xG\left(\frac{\vec{e}}{N}\right)}$$

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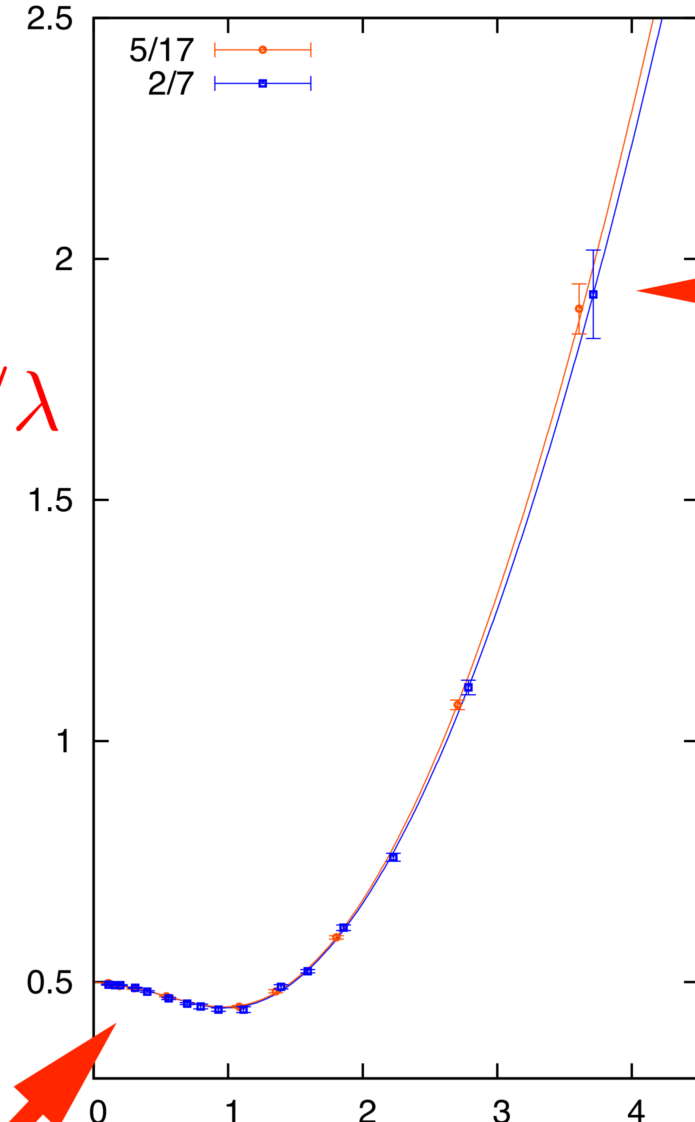

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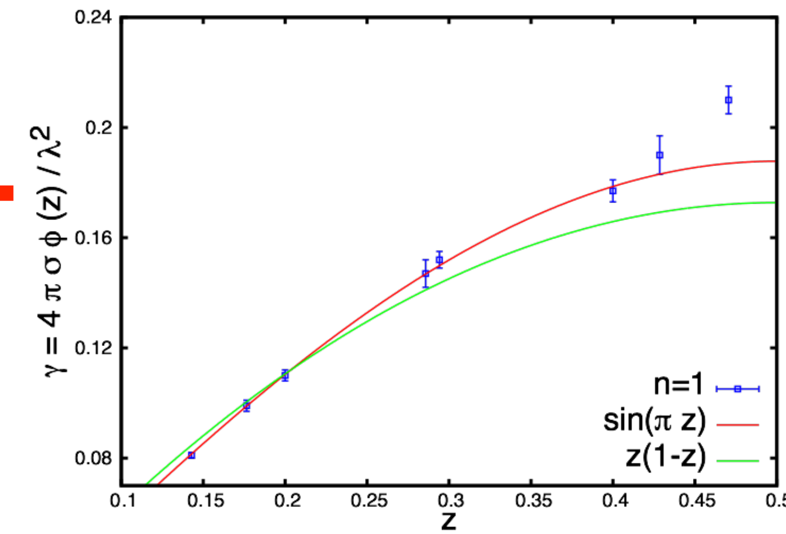
Confinement



$$\frac{\mathcal{E}(z)}{\lambda x} = 4\pi \frac{\sigma}{\lambda^2} \phi(z)$$

Perturbation theory

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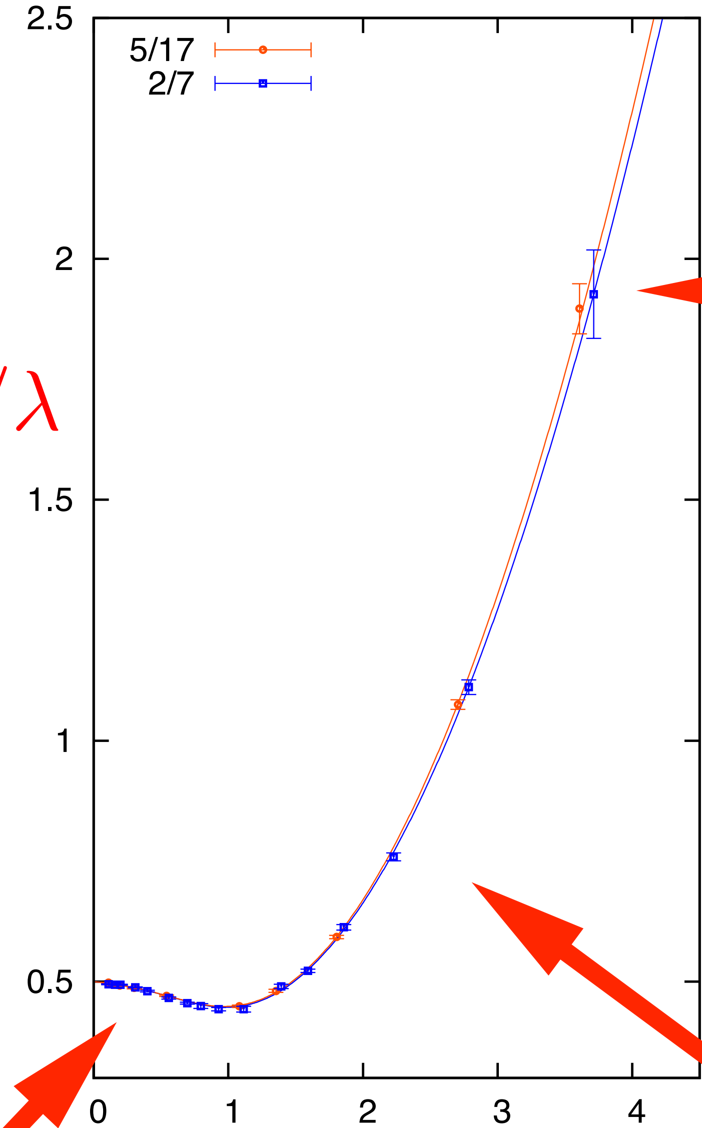

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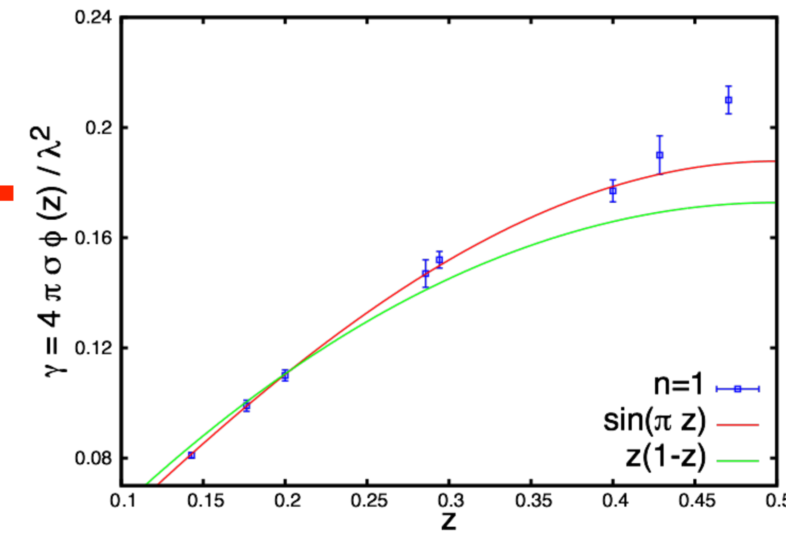

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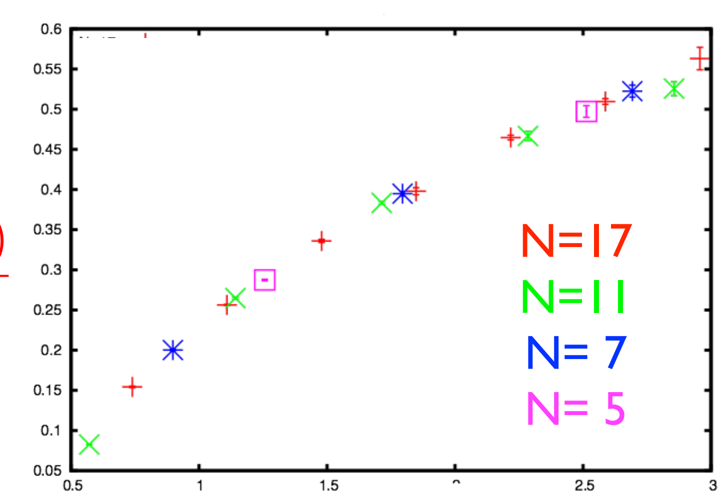
Perturbation theory

$$x\frac{\mathcal{E}_1}{\lambda} = \frac{1}{2}\sqrt{1 - 4xG\left(\frac{\vec{e}}{N}\right)}$$

$\tilde{\theta} = 2\pi \frac{\bar{k}}{N}$ dependence (x=2.8)

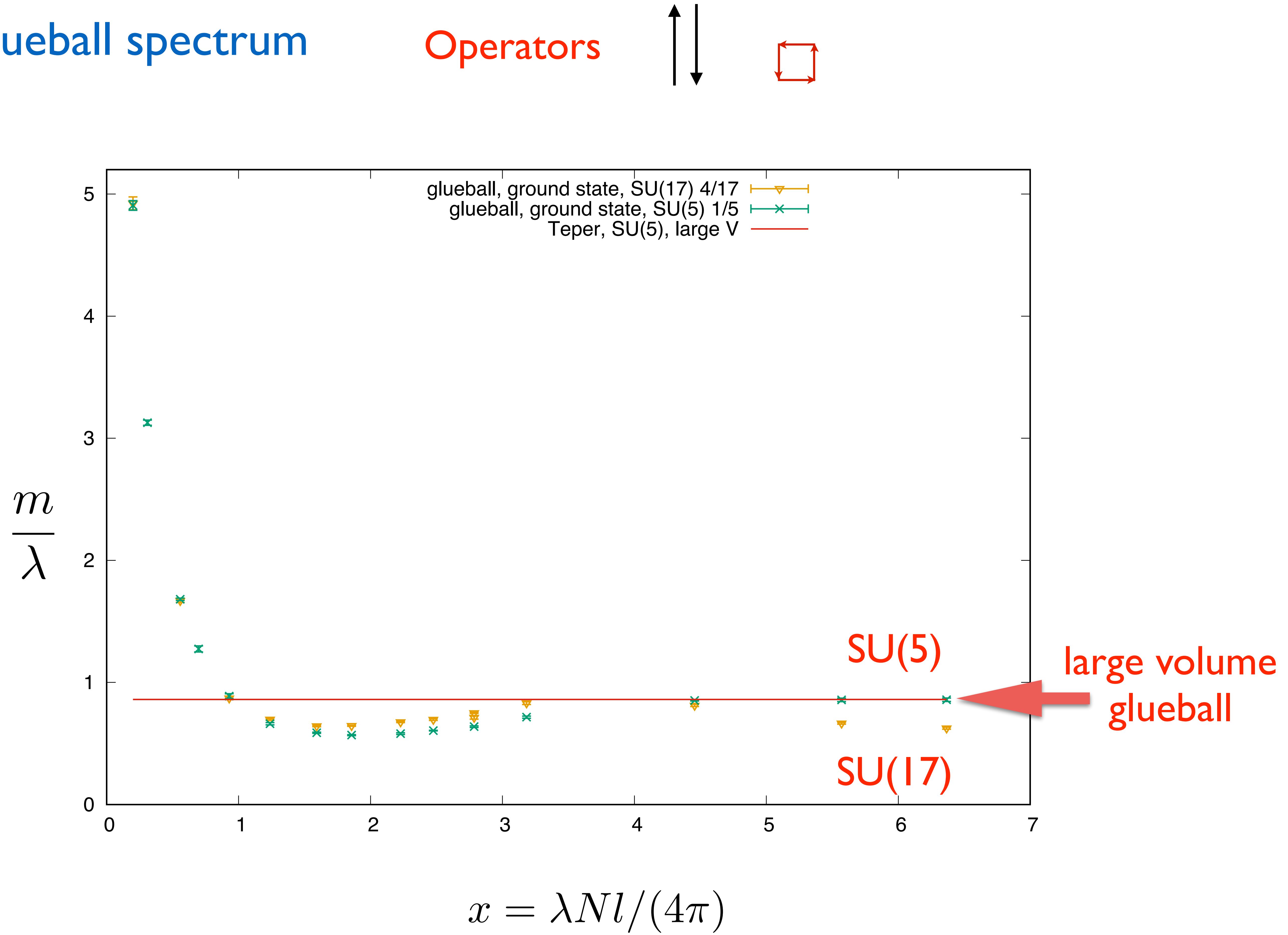
$\frac{\mathcal{E}_1(z)}{\lambda}$



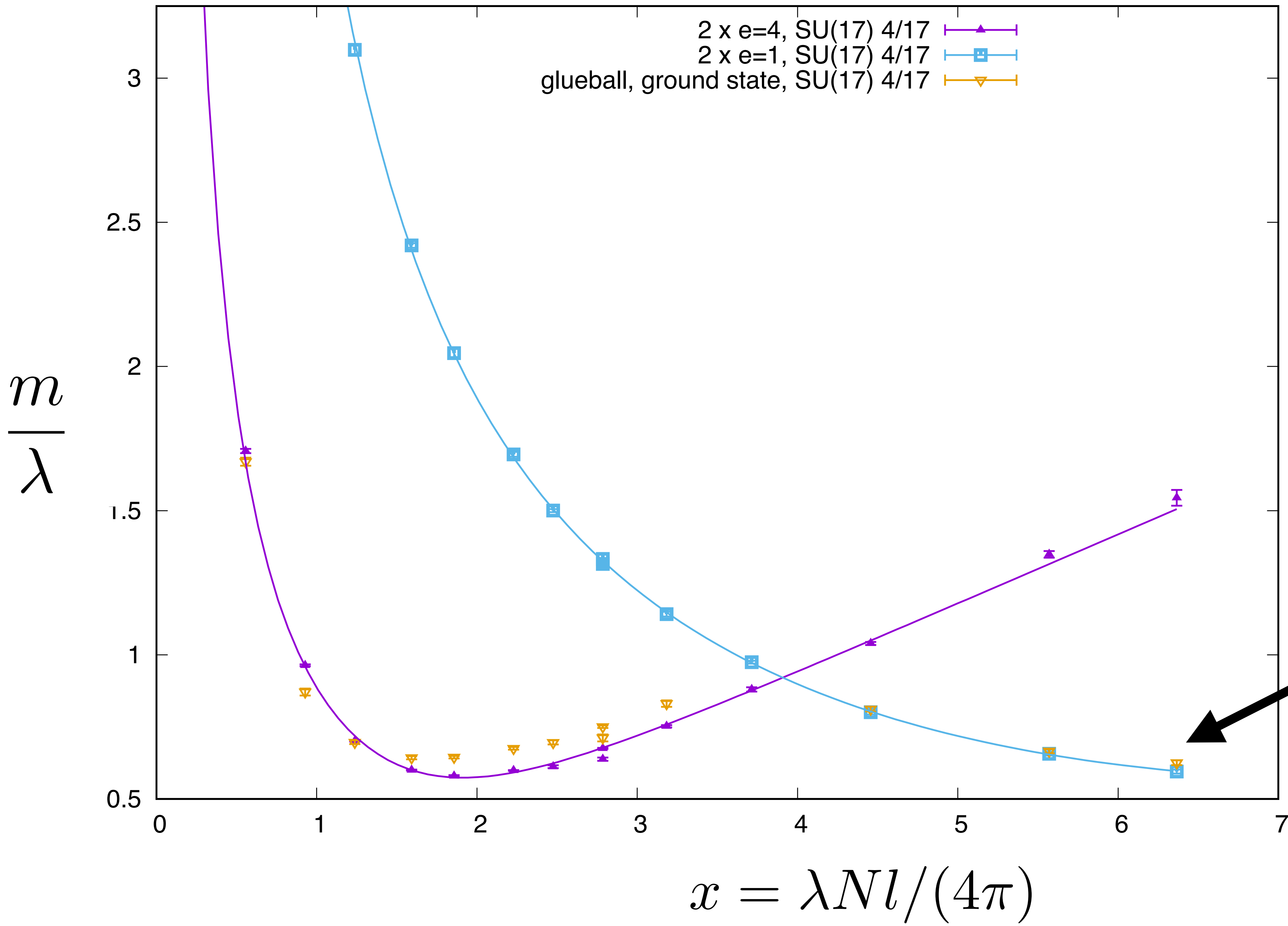
$\tilde{\theta}$

Glueball spectrum

Operators



Glueball spectrum 0^{++} $SU(17), \bar{k} = 4$ vs $SU(5), \bar{k} = 1$

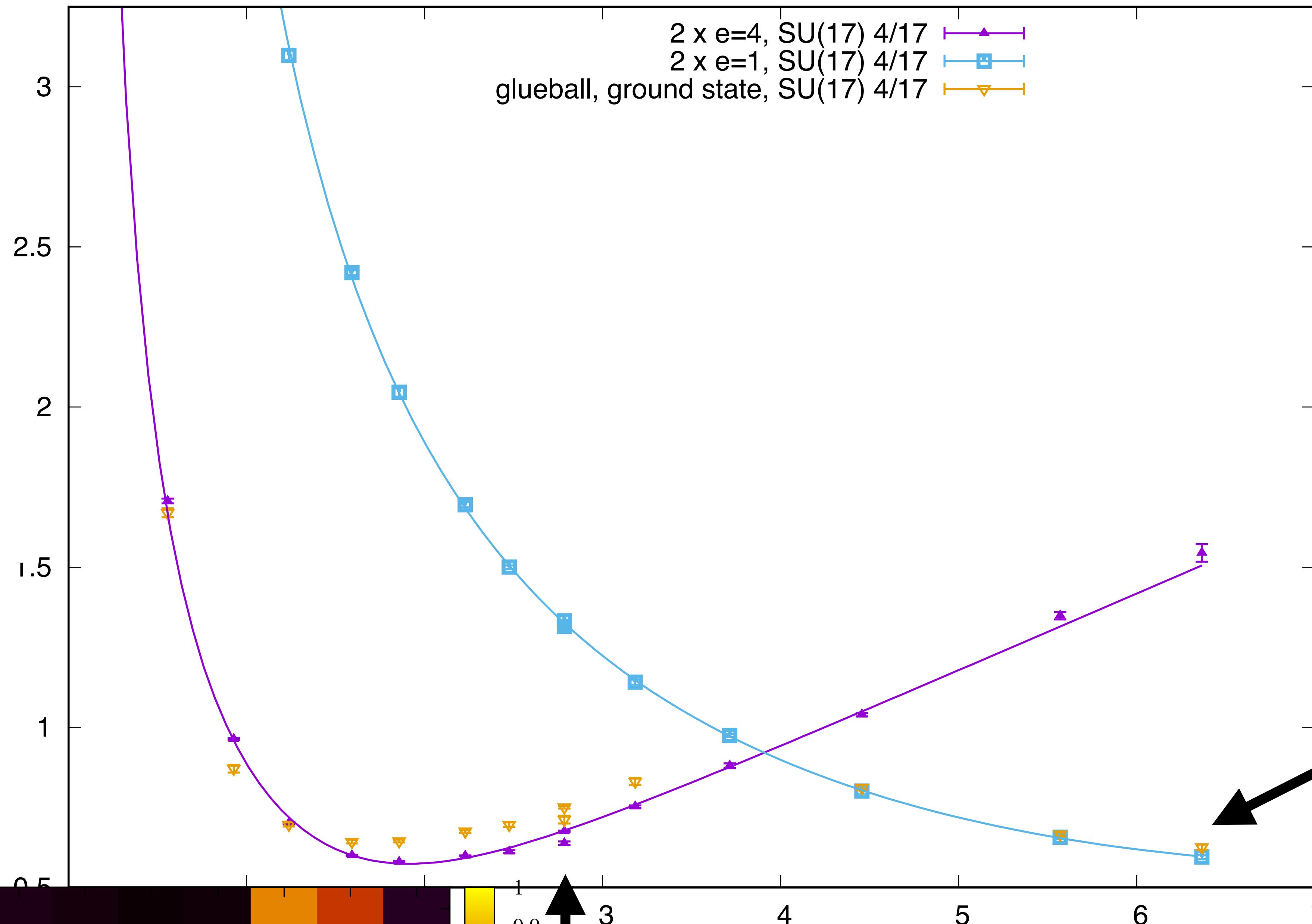


$1/17$ cannot be matched for N<17

For larger N more flux states, not all of them can be matched with the small N spectra

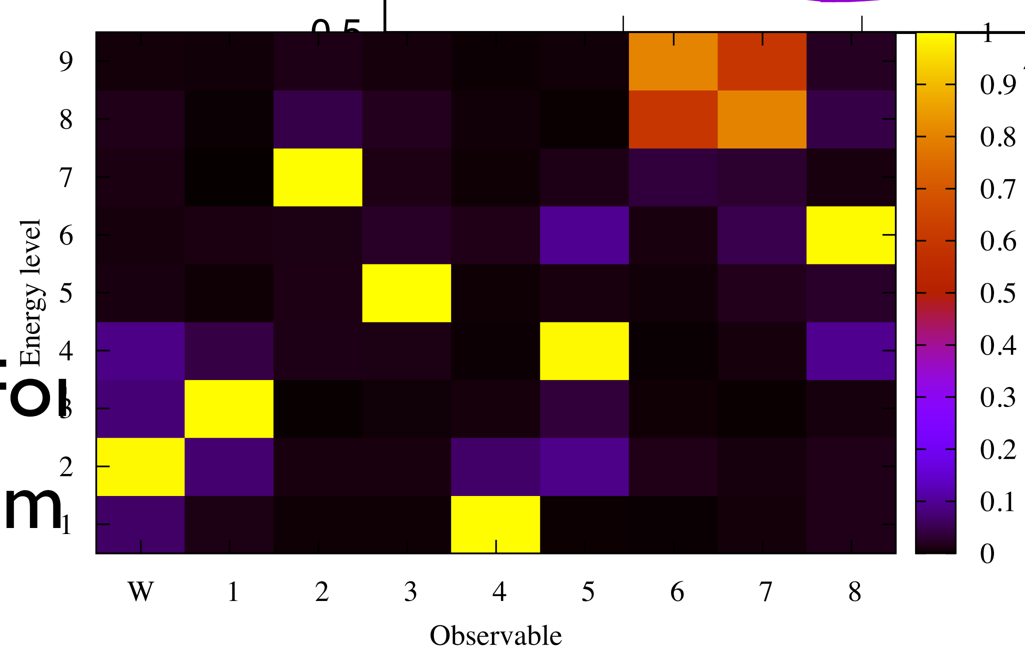
Glueball spectrum 0^{++} $SU(17), \bar{k} = 4$ vs $SU(5), \bar{k} = 1$

$\frac{m}{\lambda}$



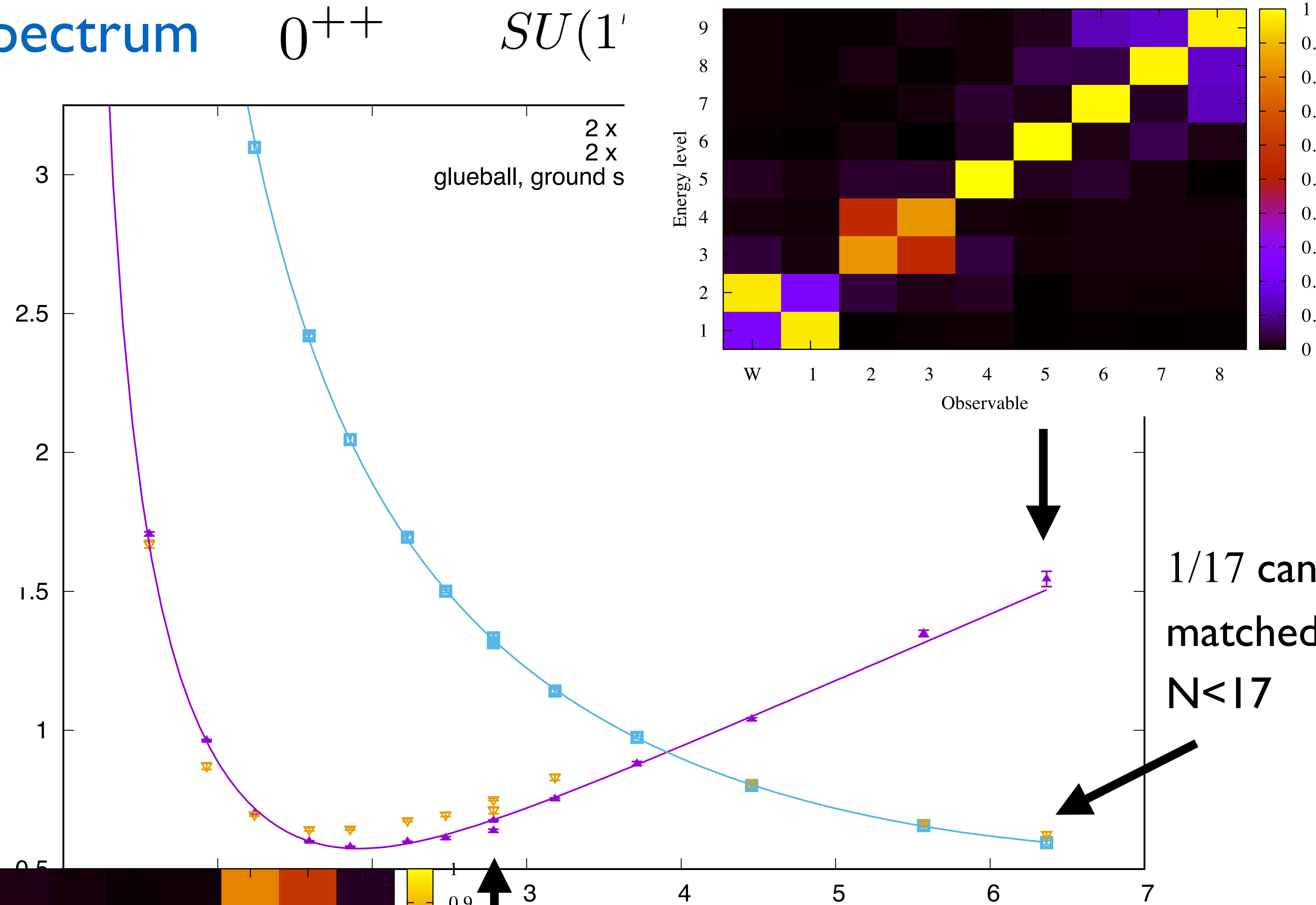
$x = \lambda N l/(4\pi)$

1/17 cannot be matched for N<17



Fo… ot all of them can be matched with the sm…

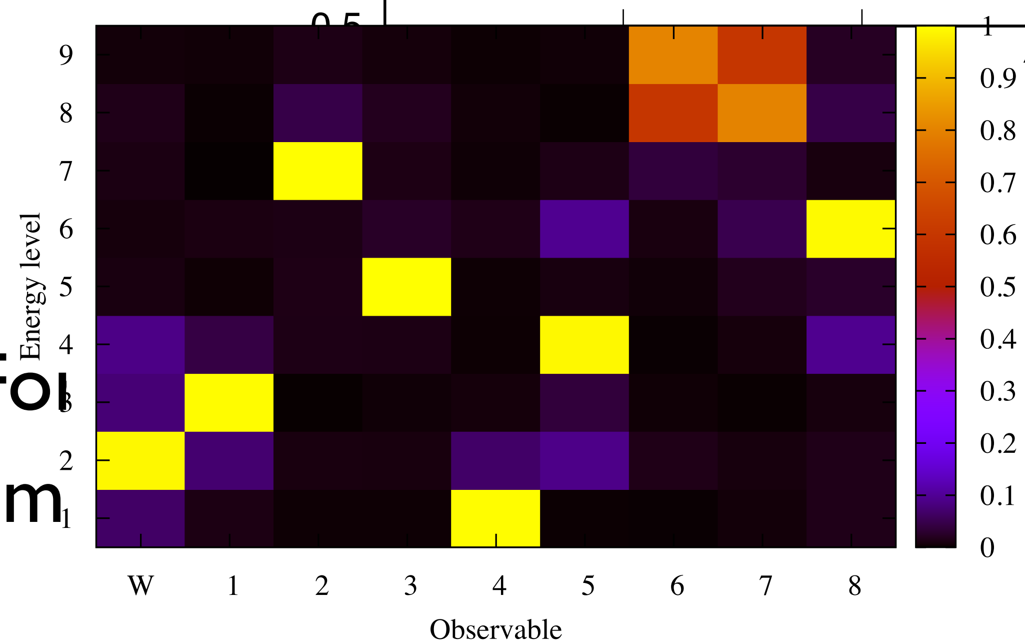
Glueball spectrum 0^{++} $SU(1$



$$\frac{m}{\lambda}$$

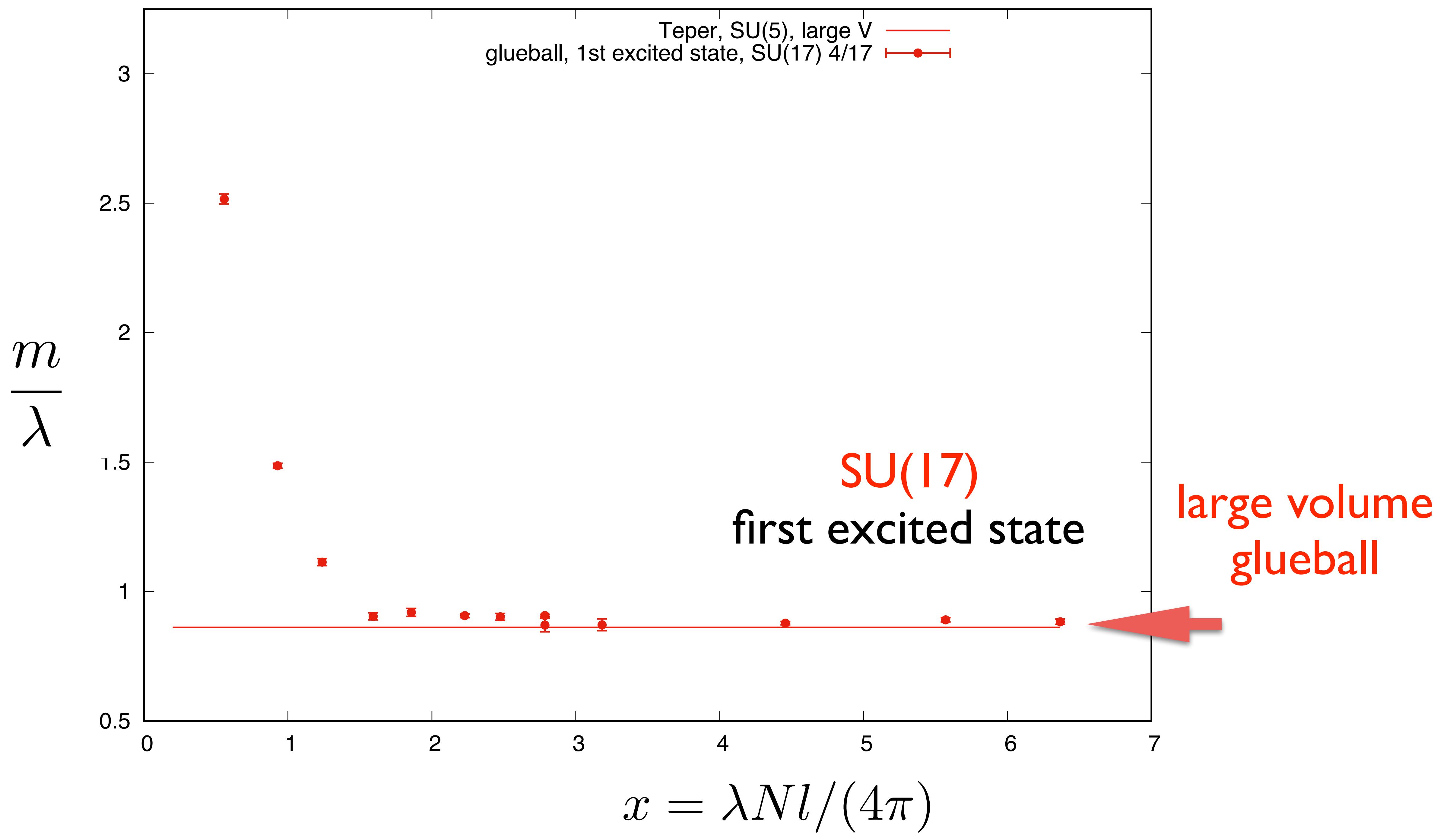
$$x = \lambda N l/(4\pi)$$

1/17 cannot be matched for N<17

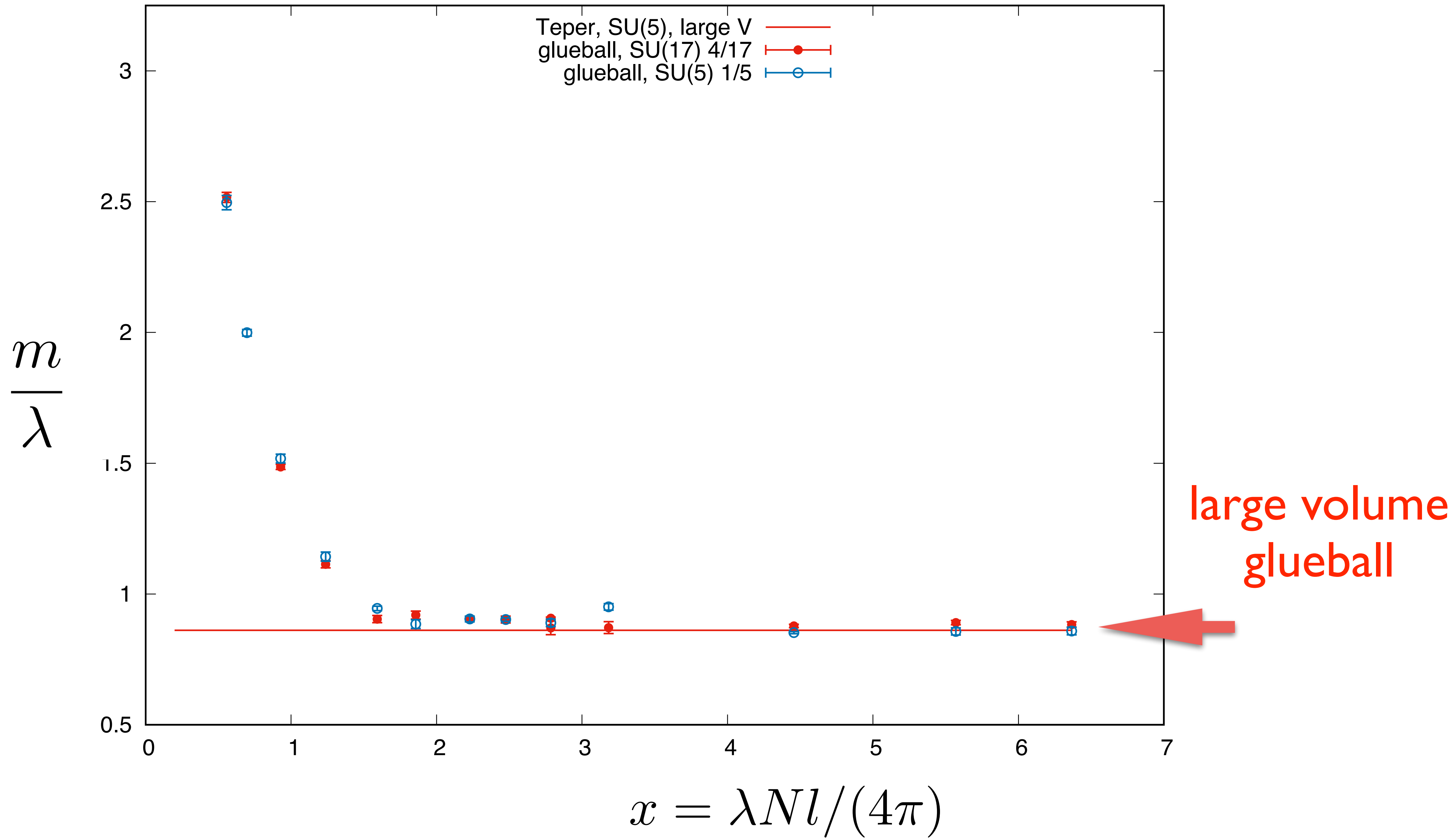


For ... ot all of them can be matched with the sm...

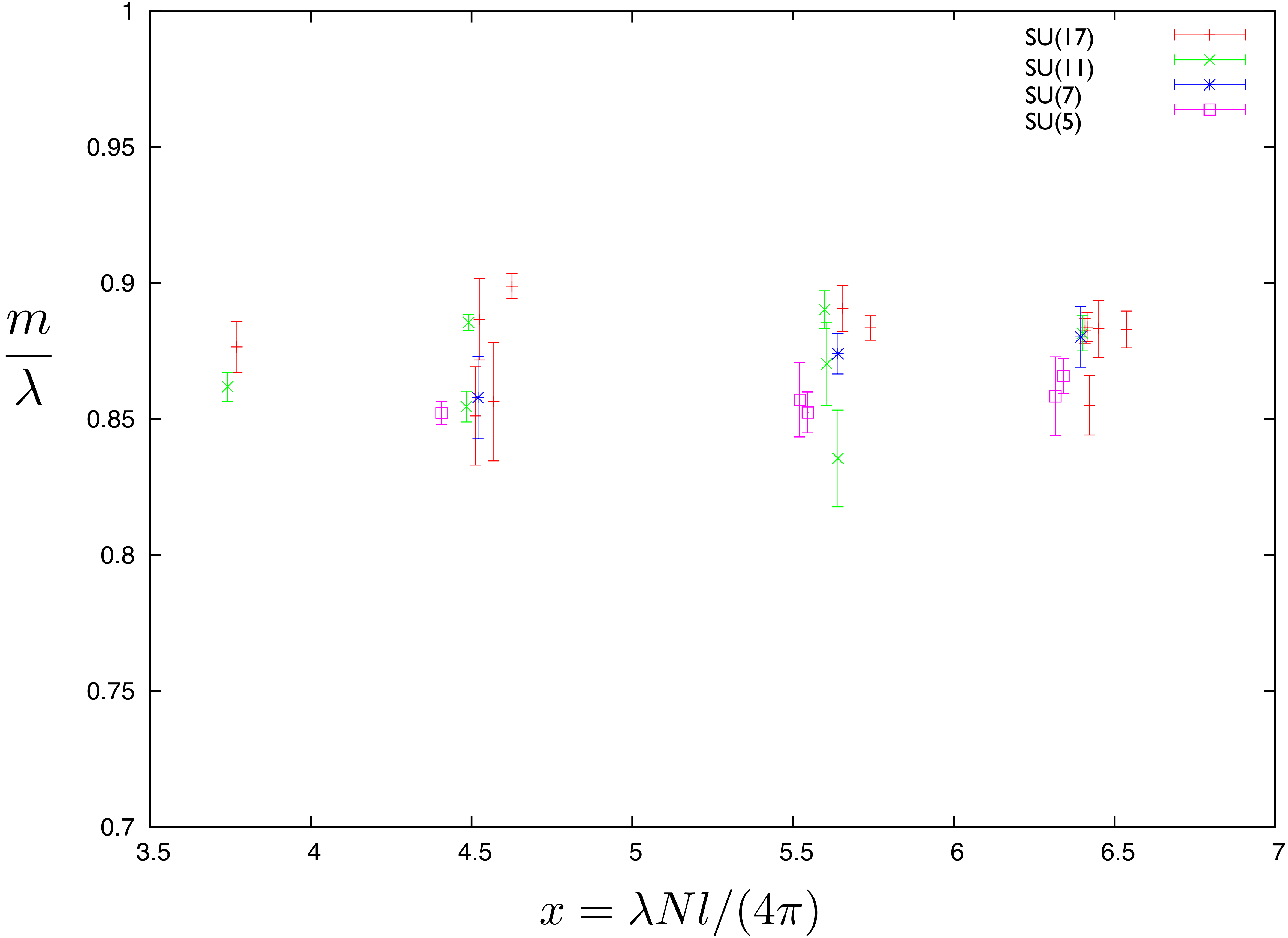
Glueball spectrum $\quad 0^{++} \qquad SU(17), \bar{k} = 4 \ \text{ vs } \ SU(5), \bar{k} = 1$



Glueball spectrum 0^{++} $SU(17), \bar{k}=4$ vs $SU(5), \bar{k}=1$



Glueball spectrum 0^{++}



Summary

- For finite N - Perturbation theory indicates physical quantities depend on

$$\tilde{\theta}\,,\ \lambda\,,\ l_{\rm eff}$$

(up to possible $1/N^2$ corrections)

- $l_{\rm eff}$ combines N and l dependence
- Non-perturbative tests