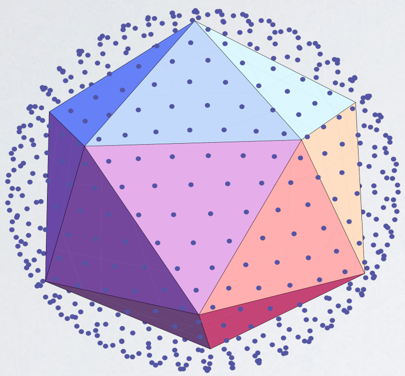
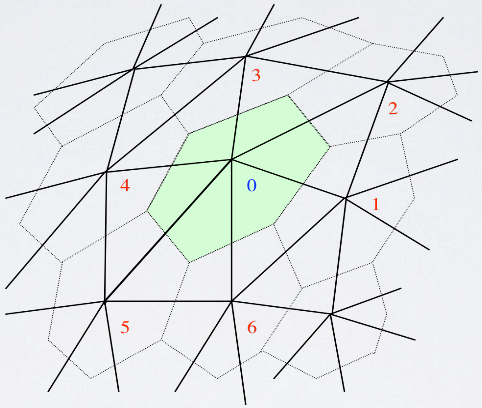
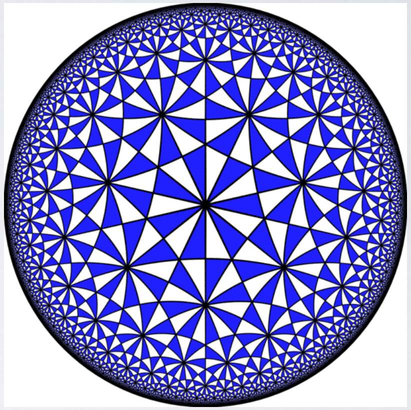
THE QUANTUM SIMPLICIAL LATTICE FOR CONFORMAL FIELD THEORY*







Rich Brower, Boston University

Holography, conformal field theories, and lattice

Monday, 27 June 2016

*collaboration with G. Fleming, A. Gabarro, T. Ruben, C-I Tan and E. Weinberg

BACK TO THE BOOTSTRAP

Stanley Mandelstam (born 12 December 1928) is a South African-born American theoretical physicist. He introduced the relativistically invariant Mandelstam variables into particle physics in 1958 as a convenient coordinate system for formulating his double dispersion relations. **The double dispersion relations were a central tool in the bootstrap program which sought to formulate a consistent theory of infinitely many particle types of increasing spin.**

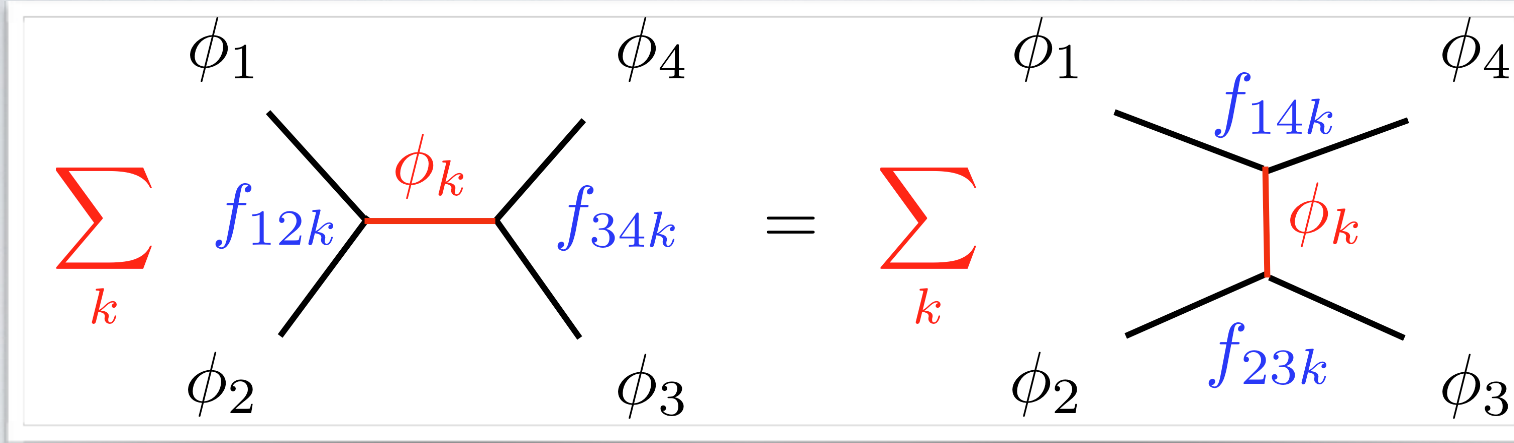
Mandelstam show that the **two dimensional quantum Sine-Gordon model is equivalently described by a Thirring model whose fermions are the kinks**. He also demonstrated that the 4d N=4 supersymmetric gauge theory is power counting finite,

Dec 12 1928 - June 24, 2016

Pioneer of S-matrix Theory. the Boostrap, String Theory etc.

Dynamics Based on Rising Regge Trajectories,1968

CFT OPE expansion and Conformal Bootstap



$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Exact 2 and 3 correlators

(i.e. Data: spectra + couplings to conformal blocks)

CFT Bootstrap: OPE & factorization completely fixed the theory

One Motivation:Radial Quantization for CFT

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

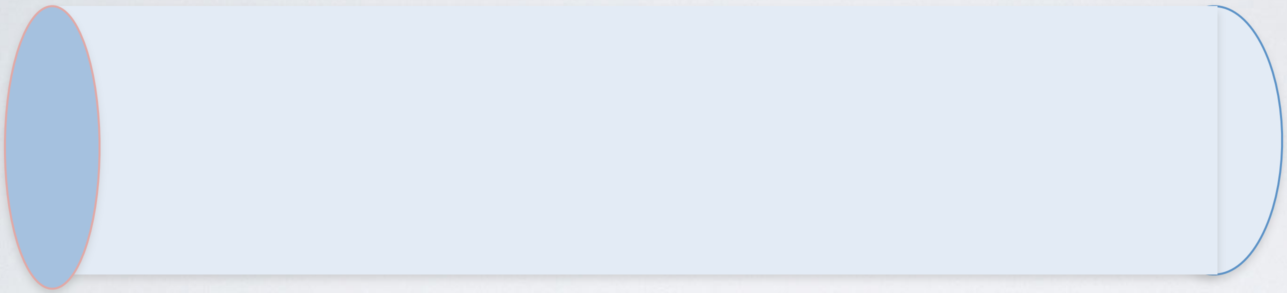
Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$\text{"time"} \quad \tau = log(r)\ , \quad \text{"mass"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

RADIAL QUANTIZATION OF CONFORMAL FIELD THEORY

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$ ← Boundary of Global AdS5



← Isometries are the conformal group O(5,1)

On lattice scales exponentially

$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

Applications:
(1) Near IR conformality — composite Higgs
(2) AdS/CFT weak-strong duality,
(3) CFT c-theorems, anomalies on sphere,
(4) Quantum Phase Transitions Critical Phenomena etc.

WHAT ABOUT LATTICE FIELDS ON RIEMANNIAN MANIFOLD?

Lattice Field theory on $\mathbb{R}^D$ Euclian Manifold provides a rigorous solutions to UV complete (renormalizable) quantum field theory.

Can this be generalized to a smooth Riemann manifold $(\mathcal{M}, g)$?

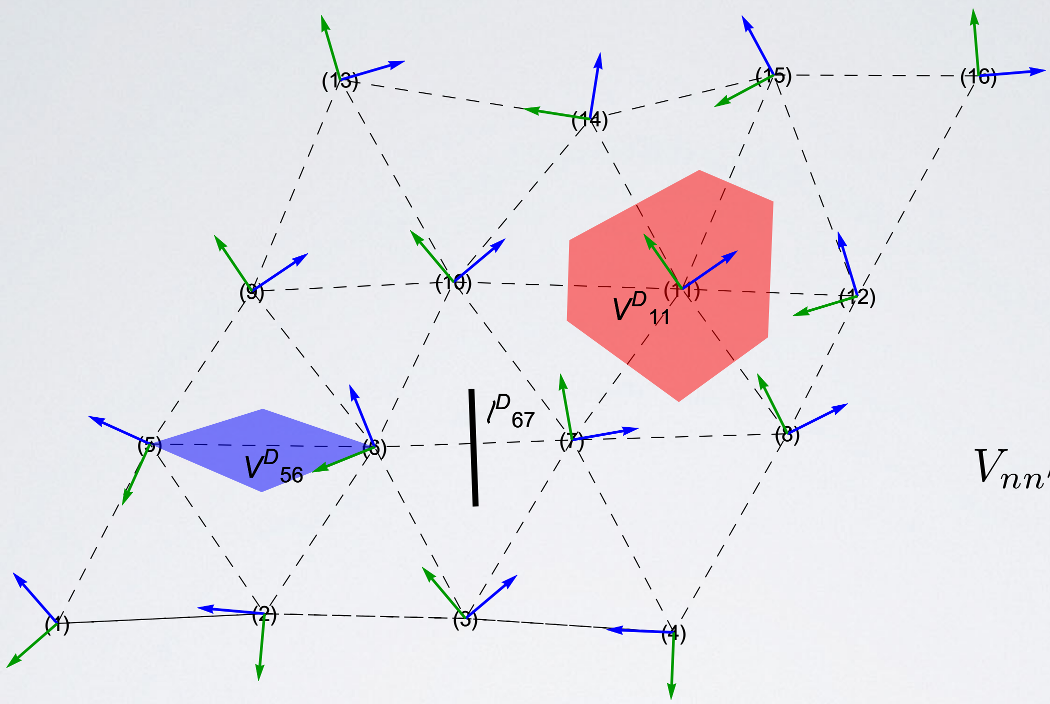
Renormalized perturbation theory does exist on $(\mathcal{M}, g)$

e.g. See Jack and Osborn "Background Field Calculations on Curved Space Time" NP 1984

OUTLINE

- ***FUNAMENTAL QUESTION:***
  - Is it possible to construct a Simplicial Lattice Path Integral that convergences exactly to the Renormalized Quantum Theory Field in the continuum on any Smooth Euclidian Riemann Manifold?*
- ***PROGRESS TO DATE:***
  - Scalars and Fermion on Simplicial lattice .
  - Quantum Correction for UV divergence in phi 4th theory.
  - Reproduce exact solution to 2D Ising (CFT) on Riemann Sphere
- ***FUTURE AMBITIONS:***
  - 3d Ising CFT & 3D QED & IR Fixed Pt for BSM Theories

Lattice Simplex vs dual Lattice Complex



$$V_{nn'} = \int \sigma_n \wedge \sigma^*_{n'}$$
$$= \frac{n!(D-n)!}{D!} |\sigma_n||\sigma^*_{n'}|$$

1. First replace the smooth Riemann manifold $(\mathcal{M}, g)$ by an approximating piecewise flat manifold $(\mathcal{M}_\sigma, g_\sigma)$ composed of elementary simplices.
2. Second expand the field, $\phi(x)$, in a finite element basis on each simplex: $\phi(x) \to \phi_\sigma(x) = W^i(x)\phi_i$.
3. Linear Elements $\phi(x) = E^i(x)\phi_i$ where $E^i(x) = \xi^i$ inside each simplex

Possible Solution: **Quantum Finite Element (QFE)**
which draws on two traditions

I. CLASSICAL FEM for PDEs on smooth Riemann Manifolds*

FEM: Alexander Hrennikoff (1941) Richard Courant (1943)*

Discrete Exterior Calculus* (de Rahm Complex, Whitney, etc, etc.),

II. QUANTUM FEILDS on "random" Lattices.

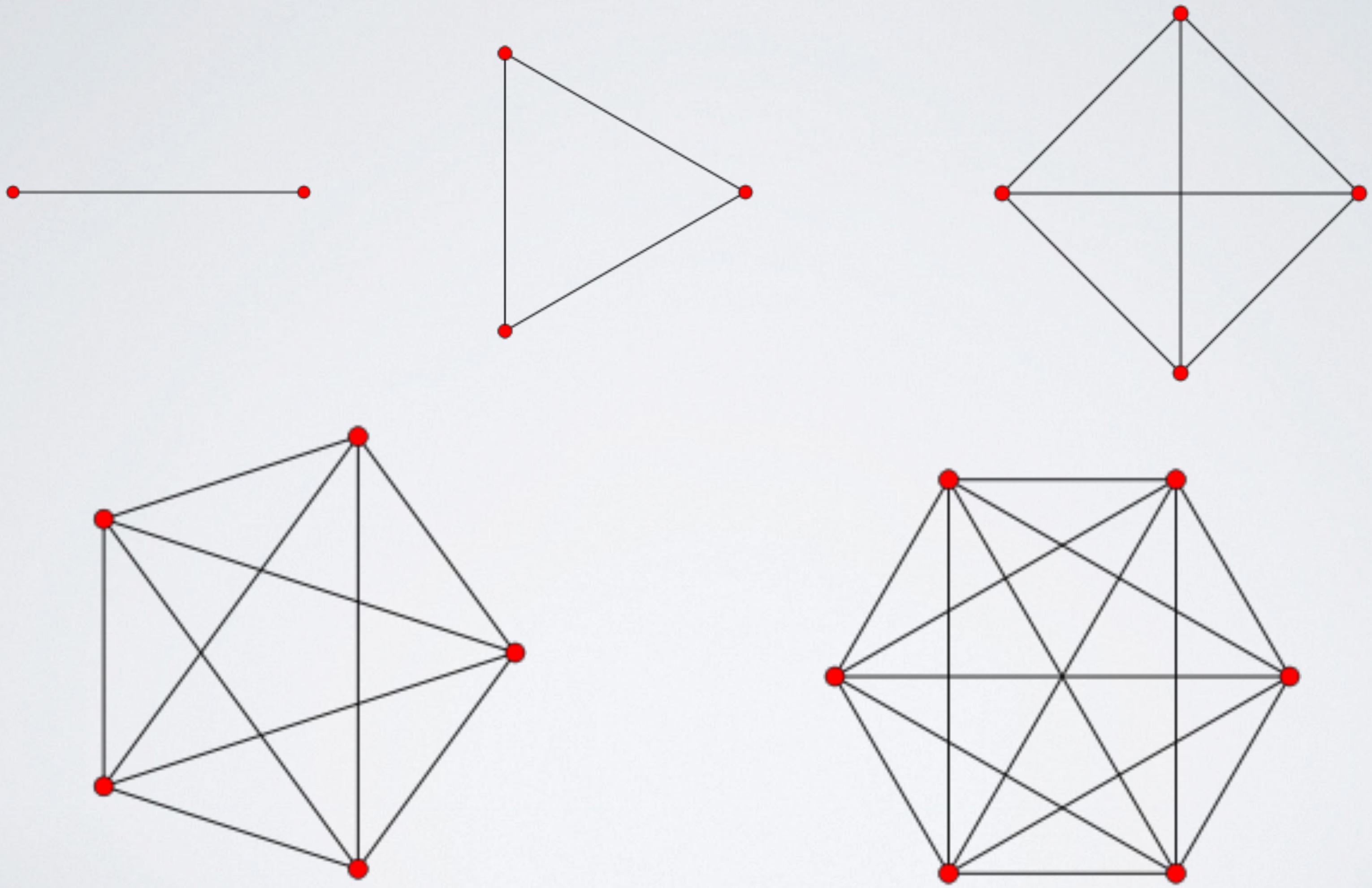
Regge Calculus T. Regge, Nuovo Cimento 19 (1961) 558. *

Random Lattices: N. H. Christ, R. Friedberg, and T. D. Lee, Nucl. Phys. B 202, 89 (1982).* Fermion Fields on a Random Lattice: R. Friedberg, T.D.Lee and Hai-Cang Ren Prog. of Th. Physics 86 (1986).

Topology/Chirality 'tHooft, Leuscher et al for QCD

***Google: "Finite Element Method" ==> 25,500,000 results (0.46 seconds)**

K-SIMPLICIES



IS THERE A SYSTEMATIC APPROACH ?

- **Topology:** The manifold $\mathcal{M}$ is replaced by a simplicial complex $\mathcal{M}_\sigma$ composed of elementary D-simplices, which is homeomorphic to the target manifold.

- **Geometry:** The metric on the target manifold $(\mathcal{M}, g)$ is approximated on the simplicial complex to form $(\mathcal{M}_\sigma, g_\sigma)$ by assigning lengths l_{ij} between neighboring sites and extending the metric with piecewise flat volumes into the interior of each simplex. This is the Regge calculus construction of the metric.

- **Hilbert Space:** The Hilbert space of continuum fields, $\phi(x)$, is truncated by expanding the fields in a finite element basis on each simplex, $\phi_\sigma(x) = E^i(x)\phi_i$, where the summation over i is implied and i runs over the vertices of each simplex.

$$(\mathcal{M}, g) \to (\mathcal{M}_\sigma, g_\sigma)$$

CLASSICAL SCALAR ACTION

$$S = \frac{1}{2}\int_{\mathcal{M}} d^D x\sqrt{g}\;[g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

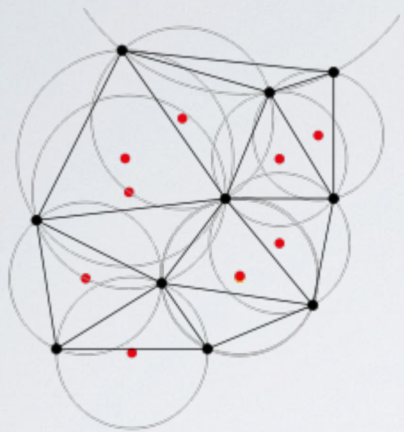
$$\downarrow$$

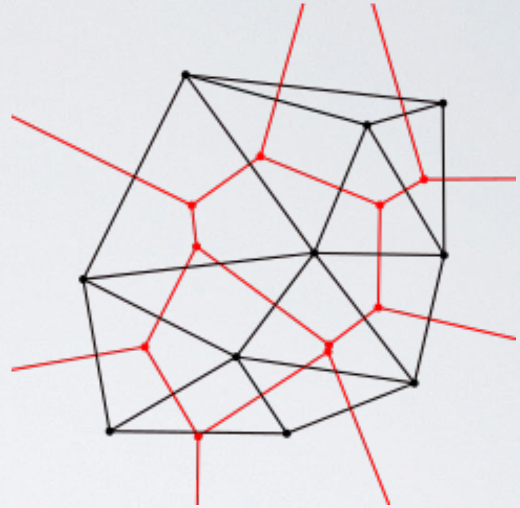
$$\begin{aligned} I_\sigma &= \frac{1}{2}\int_\sigma d^D y[\vec{\nabla}\phi(y)\cdot\vec{\nabla}\phi(y) + m^2\phi^2(y) + \lambda\phi^4(y)] \\ &= \frac{1}{2}\int_\sigma d^D\xi\sqrt{g}\;[g^{ij}\partial_i\phi(\xi)\partial_j\phi^2(\xi) + m^2\phi^2(\xi) + \lambda\phi^4(\xi)] \end{aligned}$$

$$\downarrow$$

$$I_\sigma \simeq \sqrt{g_0}\left[g_0^{ij}\,\frac{(\phi_i - \phi_0)(\phi_j - \phi_0)}{l_{i0}\,l_{j0}} + m^2\phi_0^2 + \lambda\phi_0^4\right]$$

TOPOLOGY : SIMPLICIAL COMPLEX





Set of points, lines, triangles, etc.

$$\mathcal{M}_\sigma = (\sigma_0, \sigma_1, \sigma_2, \cdots \sigma_D)$$

Boundary Matrix : $\partial \sigma_n \in \sigma_{n-1}$

∂^T Co- Boundary Matrix :

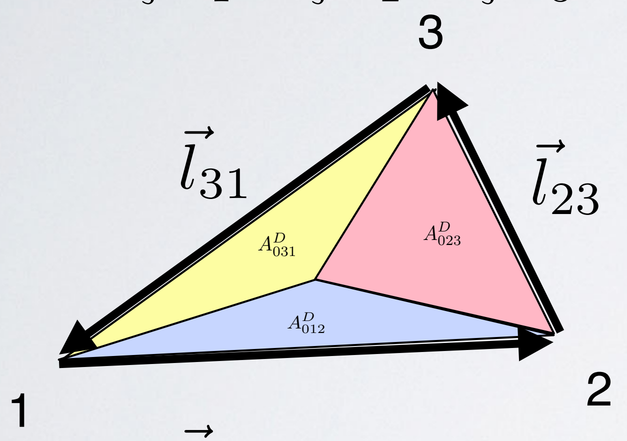
Dual Simplex: $\mathcal{M}^*_\sigma = (\sigma^*_D, \cdots , \sigma^*_2, \sigma^*_1, \sigma^*_0,)$

Beautiful Discrete Topology with chains, duality, homology sequences etc. NO metric necessary

GEOMETRY: REGGE CALCULUS

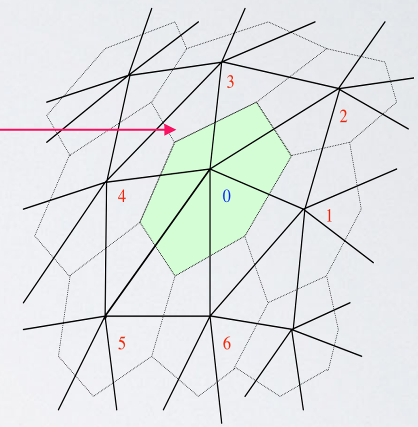
Add lengths l_{ij} of all edges constant interpolations to interior of each simplex. This provides both an intrinsic geometry

$$\vec{x} = \xi^1 \vec{r}_1 + \xi^2 \vec{r}_2 + \xi^3 \vec{r}_3$$



$$\vec{l}_{12} = \vec{r}_2 - \vec{r}_1$$

SingularCurvature at Vertex!



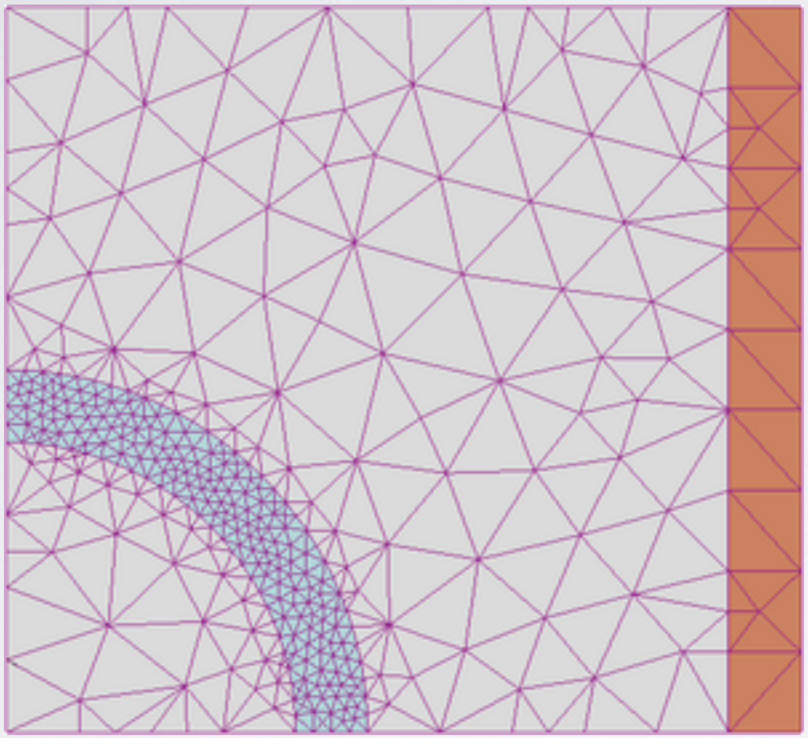
Baricentric co-ordinates:
Interior of each simplex is flat
Obvious generalization to D-simplex!

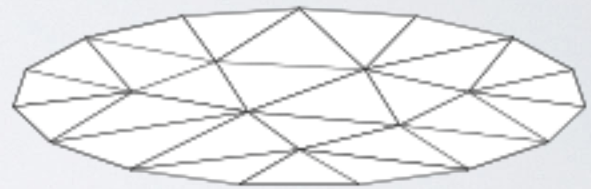
$$\xi^1 + \xi^2 + \xi^3 = 1$$

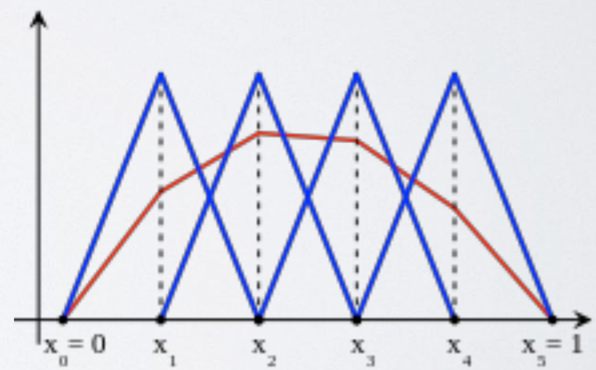
$$(\mathcal{M}, g) \to (\mathcal{M}_\sigma, g_\sigma)$$

Hilbert Space: Finite Element Interpolates

$$\phi(x) \to \phi_\sigma(x) = \sum_i W^i(x)\phi_i$$



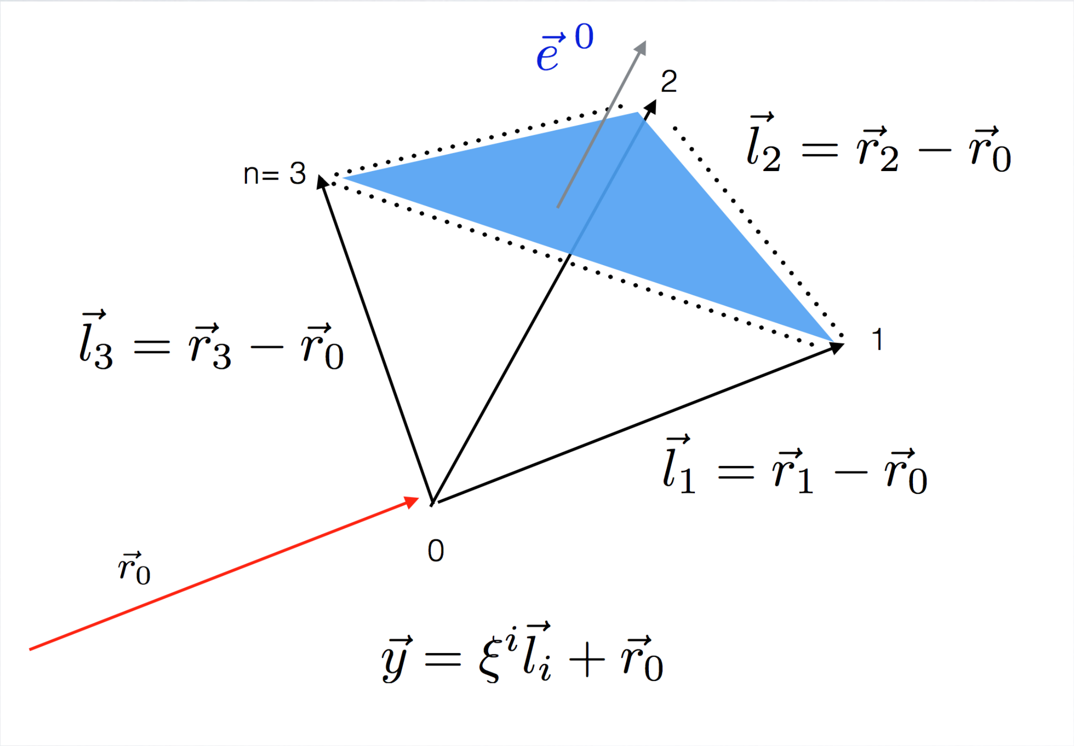




RCB, M. Cheng and G.T. Fleming,
"Improved Lattice Radial Quantization" PoS LATTICE2013 (2013) 335

BARICENTRIC CO-ORD AND SIMPLICIAL METRIC

$$d\vec{y} = \vec{l}_i d\xi^i \implies ds = g_{ij} d\xi^i d\xi^j = \vec{l}_i \cdot \vec{l}_j$$



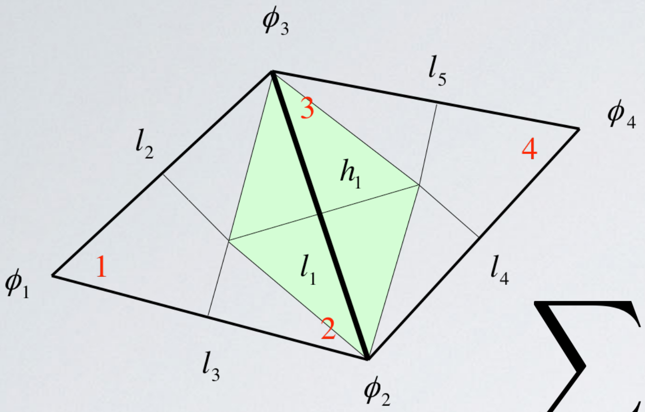
$$\vec{\nabla} = \vec{n}^{\,i} \partial_i$$

$$g^{ij} = \vec{n}^{\,i} \cdot \vec{n}^{\,j}$$

$$\vec{n}^{\,i} \cdot \vec{l}_j = \delta^i_j$$

$$\vec{y} = \xi^0 \vec{r}_0 + \xi^1 \vec{r}_1 + \cdots + \xi^D \vec{r}_D$$

REGGE CALCULUS FORMULATION



LINEAR FEM/ REGGE CALCULUS *

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

Only for D = 2

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

DISCRETE EXTERIOR CALCULUS

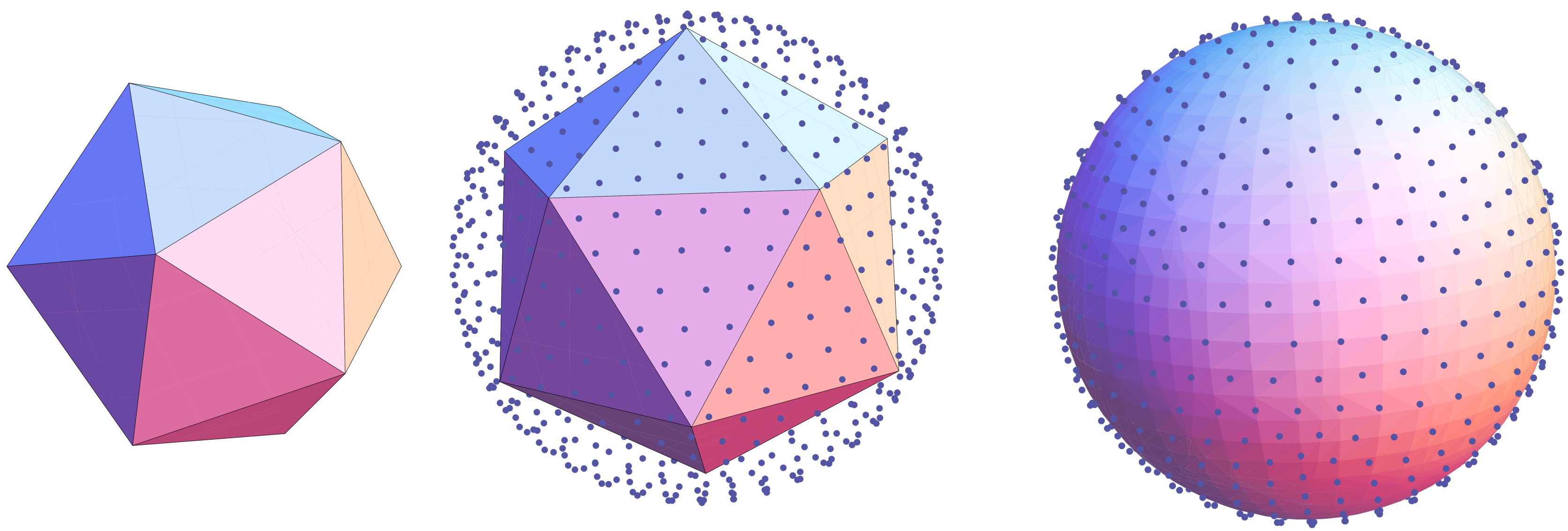
$$\nabla^2 \phi \to (\delta d + d\delta)\phi$$

$$\delta \equiv *d*$$

* H. Hamber, S. Liu, **Feynman rules for simplicial gravity,** NP B475 (1996)

2D SphereTest Case

$s = 1$ $\qquad$ $s = 8$



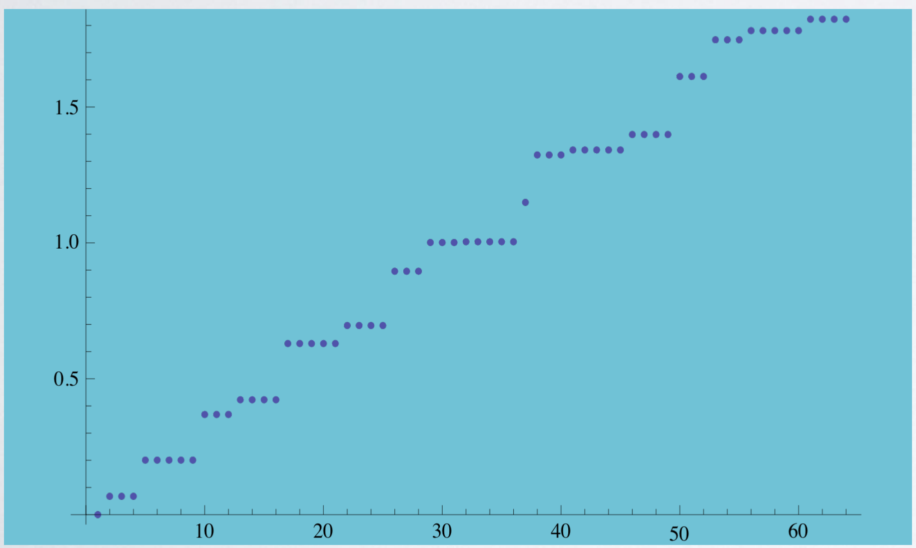
I = 0 (A),1 (T1) , 2 (H) are irreducible 120
Iscosahedral subgroup of O(3)

FEM FIXES THE HUGE SPECTRAL DEFECTS OF THE LAPLACIAN ON THE SPHERE

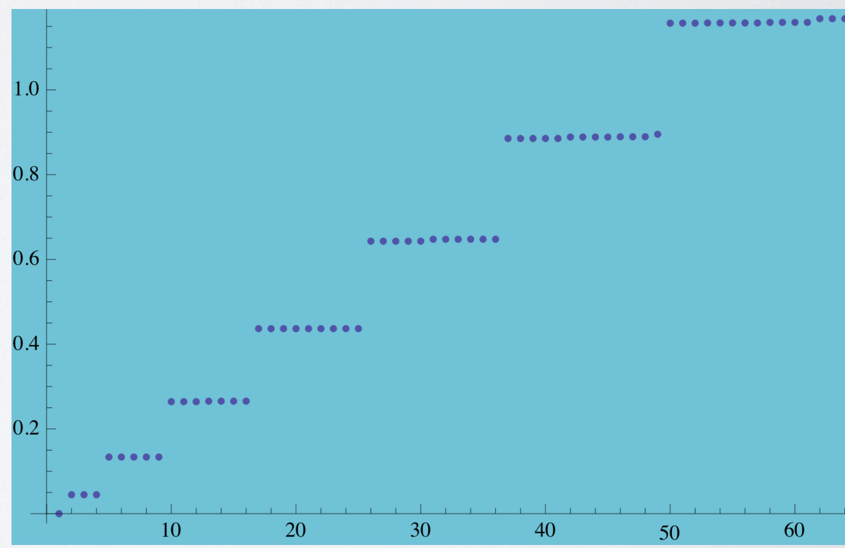
For s = 8 first (l+1)*(l+1) = 64 eigenvalues

BEFORE (K = 1)

AFTER (FEM K's)

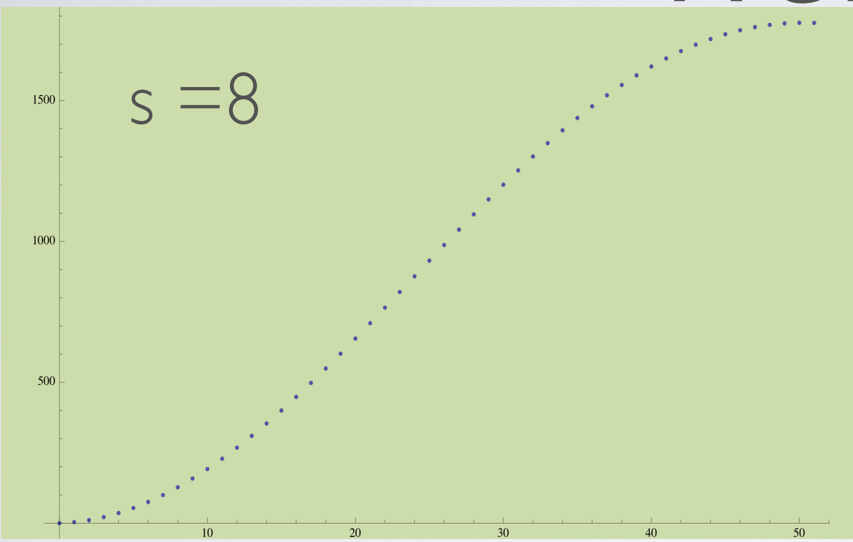


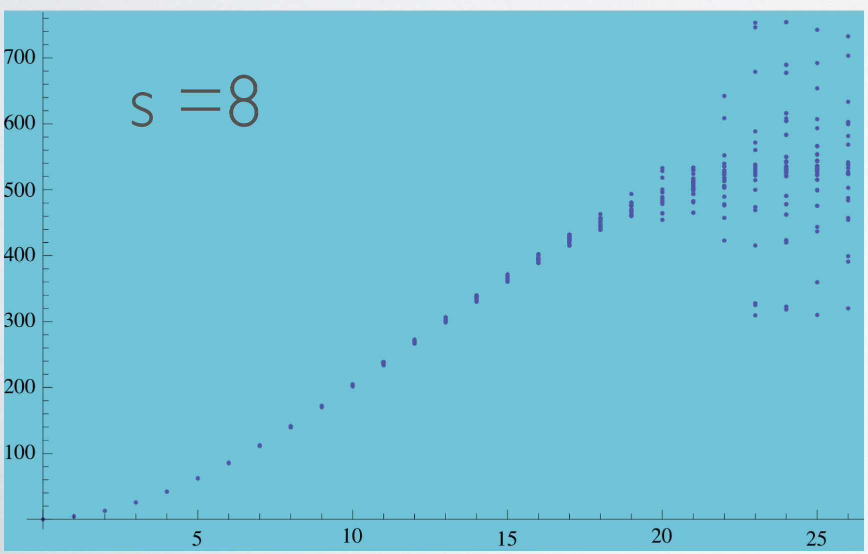
l, m

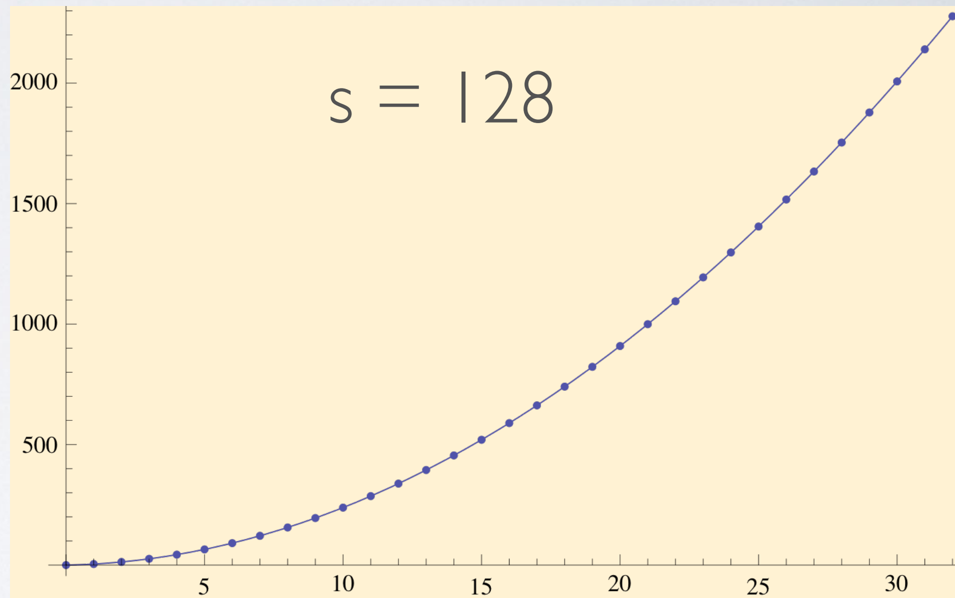


l, m

SPECTRUM OF FE LAPLACIAN ON A SPHERE







Fit

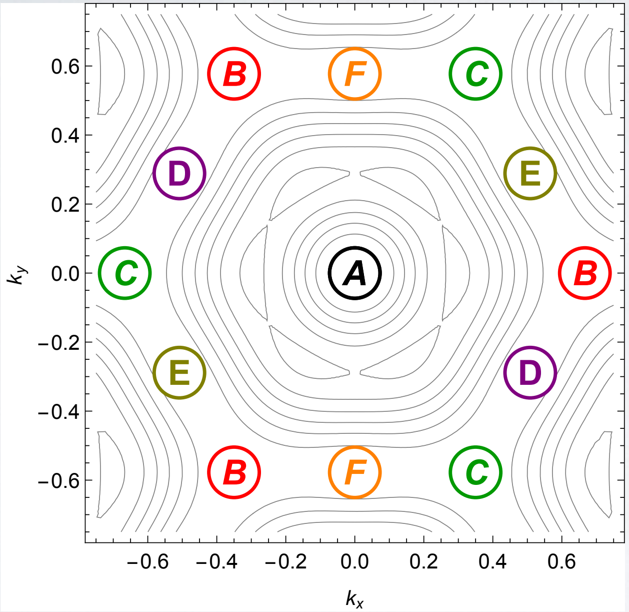
$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

REGGE + FEM: NOT ENOUGH

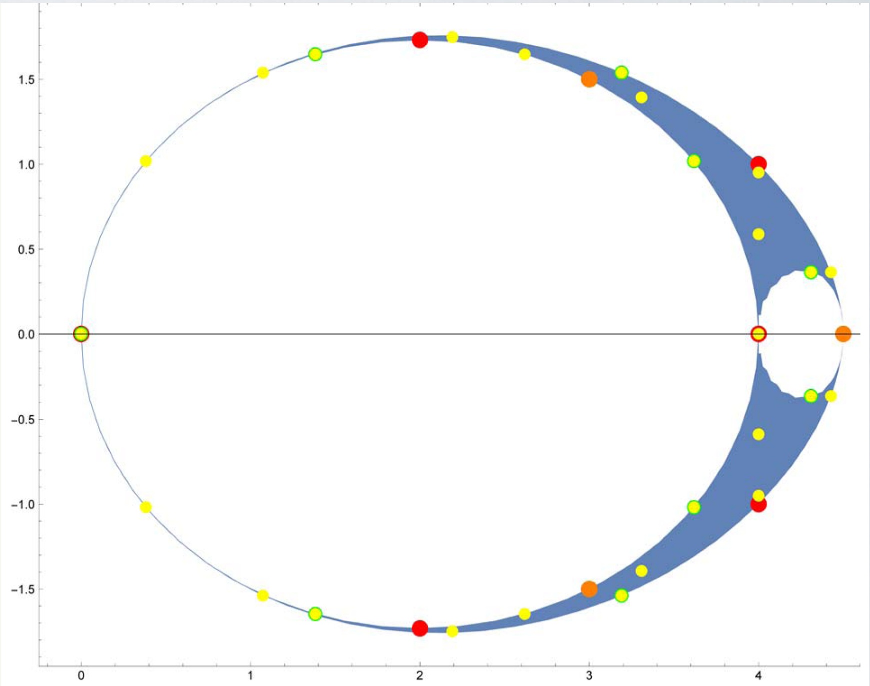
- FEM Convergence theorems: With a "proper" refinement of the Regge complex convergence of all classical so to EOM is guaranteed.
- But D> 2 linear elements may not be suitable. Better DEC or higher order elements maybe needed.
- Very little FEM consideration of Dirac and Non-abelian Gauge fields
- No treatment UV diverges for a renormalized quantum theory.
- Still renormalized perturbation theory on Riemann manifold is well understood. (e.g. see Jack and Osborn "Background Field Calculations on Curved Space Time" NP 1984)

DIRAC FIELD ON REIMANN MANIFOLD

2D DIRAC TRIAGULAR LATTICE



Torus: NAIVE



Torus: With Wilson Term

9 pts (orange) 16 pts (red) 25 pts(green) 100 pts (yellow)

QFEM DIRAC EQUATION : MUCH HARDER

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Vierbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab} \quad , \quad \sigma_{ab} = i[\gamma_a, \gamma_a]/2$

(1) New spin structure “knows” about intrinsic geometry

(2) Need to avoid simplex curvature singularities at sites.

(3) Spinors rotations: Spin(D) is double of Lorentz O(D).

e.g. D = 2 as $\theta \to 2\pi$ $e^{i(\theta/2)\sigma_3/2} \to -1$

Continuum Acton

$$S = \int d^D x \sqrt{g}\, \bar{\psi}[\mathbf{e}^{\mu}(\partial_\mu - i\boldsymbol{\omega}_\mu(x)) + m]\psi(x)\ ,$$

$$\boldsymbol{D}_\mu$$

Tetrad Postulate

$$\partial_\mu \mathbf{e}^\nu + \Gamma^\nu_{\mu,\lambda}\mathbf{e}^\lambda = i[\boldsymbol{\omega}_\mu, \mathbf{e}^\nu]\ .$$

Simplicial Lattice Action

$$S_\sigma = \frac{1}{2}\sum_{\langle ij\rangle} \frac{V^D_{ij}}{l_{ij}} (\bar{\psi}_i e^{(i)j}_a \gamma^a \Omega_{ij}\psi_j + \bar{\psi}_j e^{(j)i}_a \gamma^a \Omega_{ji}\psi_i) + \cdots$$

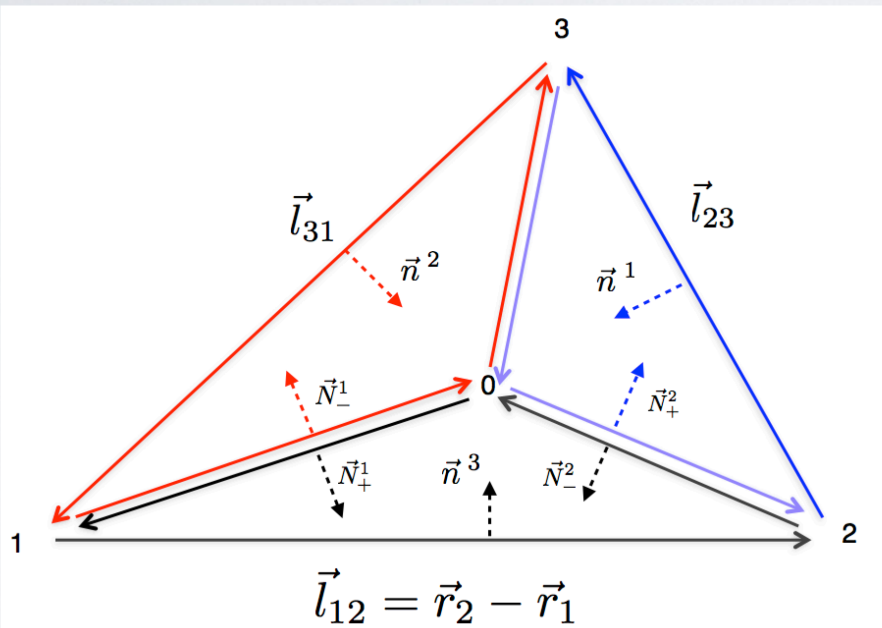
$$\psi_i \to \Lambda_i \psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j \Lambda^\dagger_j \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i \mathbf{e}^{(i)j}\Lambda^\dagger_i \quad , \quad \Omega_{ij} \to \Lambda_i \Omega_{ij} \Lambda^\dagger_j$$

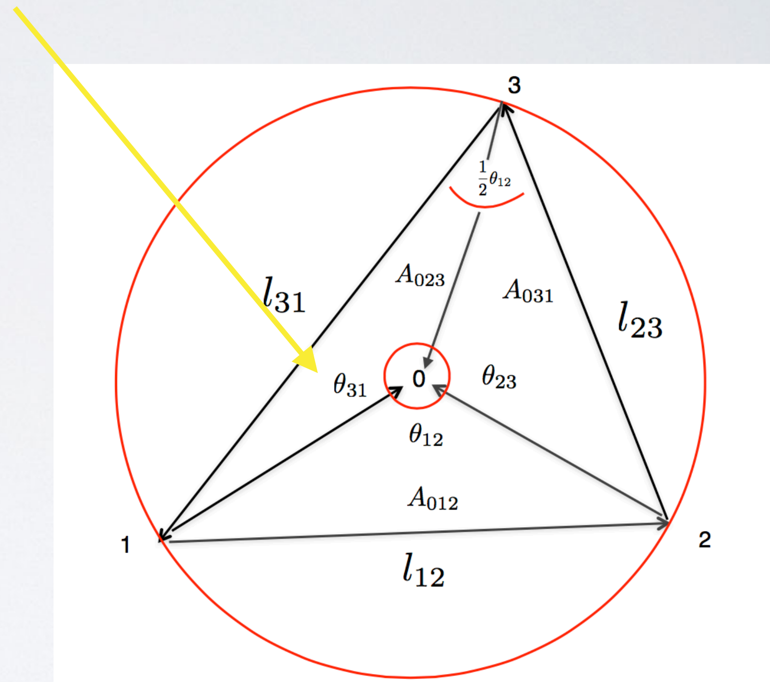
COMMENT: NOT USING LINEAR FEM

$$S_{linear} = \frac{A_{123}}{6} \sum_{\langle i,,j \rangle} \bar{\psi}_i(\vec{n}^{\,j} - \vec{n}^{\,i}) \cdot \vec{\sigma}\psi_j$$

New Dirac Element is 3 linear elements meeting ghost sites at Circumcenter

$$\phi_0 = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$$





$$c_k = \frac{4A_{0ij}}{l_{ij}^2}\frac{4A_{0ik}}{l_{ik}^2} = \cot(\theta_{ik}/2)\cot(\theta_{jk}/2)$$

Sort of ? $\sqrt{\delta d + d\delta} = \delta + d$

WILSON/CLOVER TERM

$$[\gamma_\mu(\partial_\mu - iA_\mu)]^2 = (\partial_\mu - iA_\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\ ,$$

$$\Downarrow$$

$$[\mathbf{e}_a^\mu(\partial_\mu - i\boldsymbol{\omega}_\mu)]^2 = \frac{1}{\sqrt{g}}\boldsymbol{D}_\mu\sqrt{g}g^{\mu\nu}\boldsymbol{D}_\nu - \frac{1}{2}\sigma^{ab}e_a^\mu e_b^\nu\boldsymbol{R}_{\mu\nu}$$

$$\Downarrow$$

$$S_{Wilson} = \frac{r}{2}\sum_{\langle i,j\rangle}\frac{aV_{ij}}{l_{ij}^2}(\bar{\psi}_i - \bar{\psi}_j\Omega_{ji})(\psi_i - \Omega_{ij}\psi_j)$$

Construction Procedure for Discrete Spin connection

(1) Assume Elements with Spherical Triangles (i,j,k) or boundaries give by geodesics on an 2D manifold

(Angles at each vertex add to 2 pi exactly)

(2) Calculate discrete "curl" around the triangle

$$\Omega_{ij}\Omega_{jk}\Omega_{ki} = e^{i(2\pi - \delta_\triangle)\sigma_3/2}$$

(3) Fix $\Omega_{ij} \to \pm\Omega_{ij}$ so $\delta_\triangle \sim A_{ijk}/4\pi R$

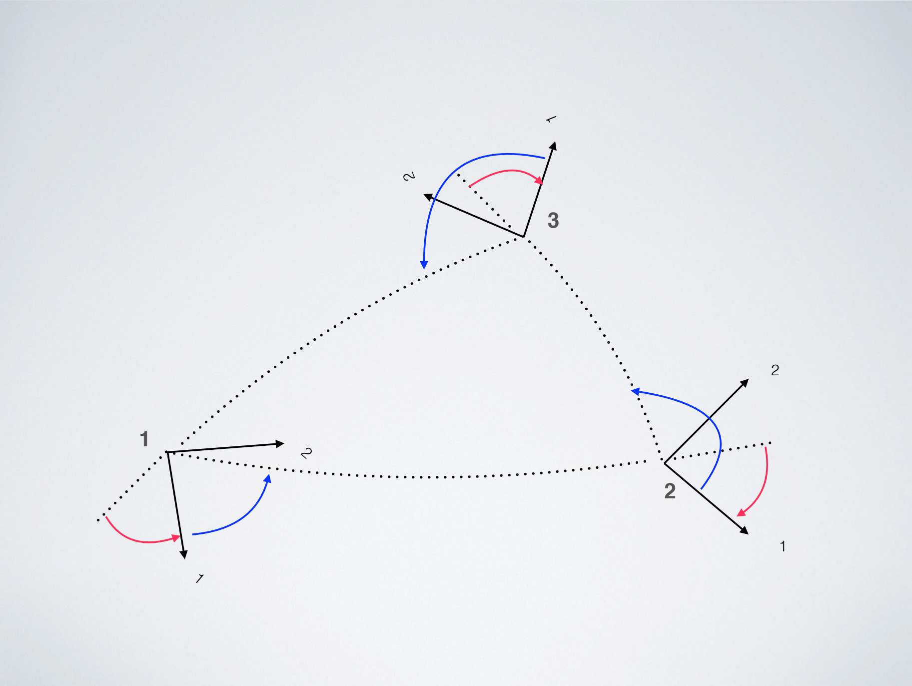
Sphere: or any manifold with this topology has a unique lattice spin connection up to gauge Lorentz transformation on spinors

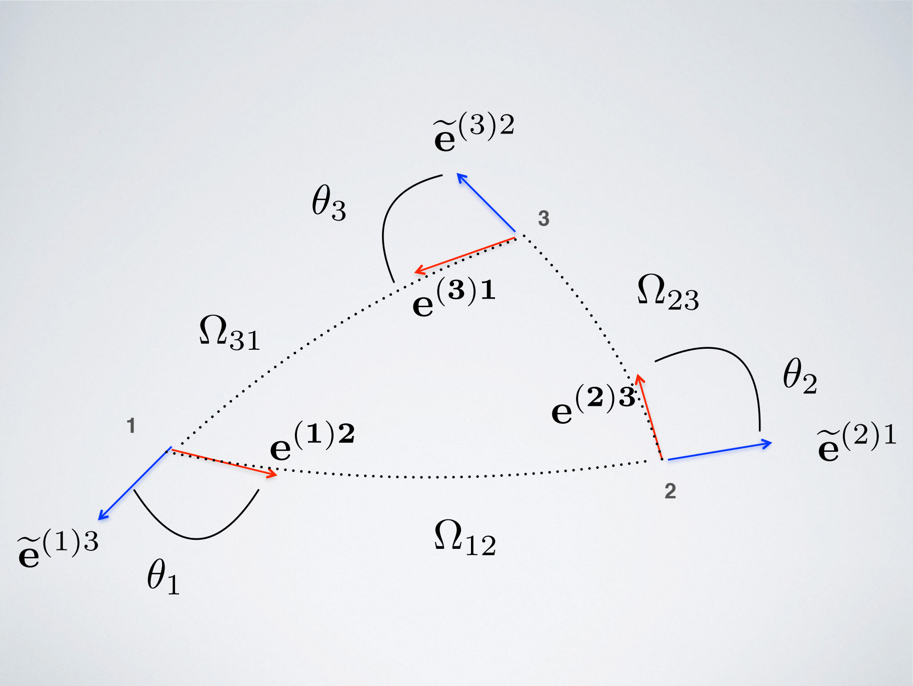
Cylindar: There are 2 solutions (periodic or anti periodic)

Torus: There are 4 solutions: (periodic/anti-periodic): Non-contractible loops.

Category Theory: A spin structure is a property shared between any simplicial complex and

Lattice Spin Connection on simplicial lattice



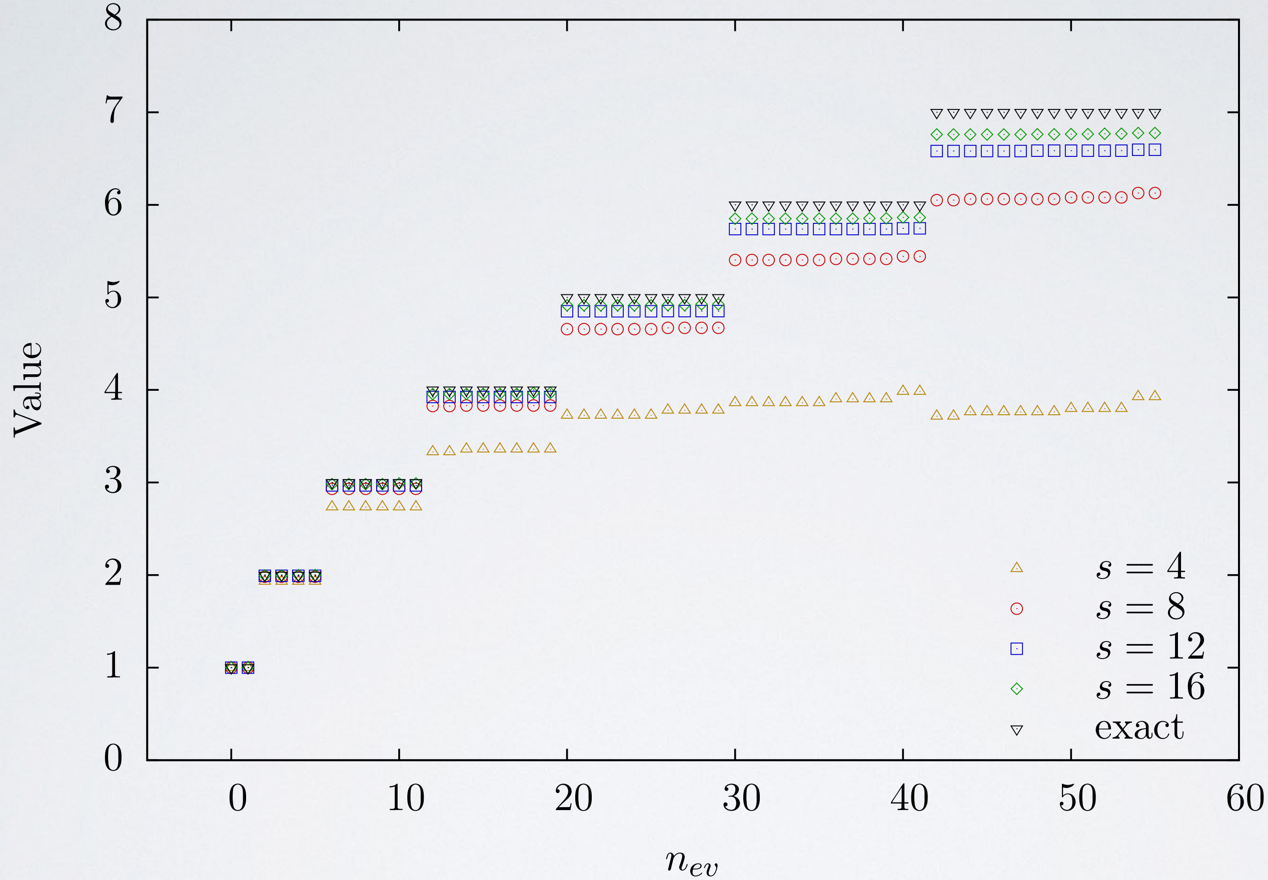


The spin connection is gauge field whose curl gives the local curvature or deficit angle

Geodesics and Parallel Transport is easy on a Sphere: In general use a Relaxation to fix Gauge Field

$$S^{(i)}_{\Delta} \equiv e^{i A^{\mu\nu}_{\Delta} \boldsymbol{R}_{\mu\nu}(i)} \quad \leftrightarrow \quad \Omega^{(i)}_{\Delta_{ijk}} \equiv \Omega_{ij}\Omega_{jk}\Omega_{ki}$$

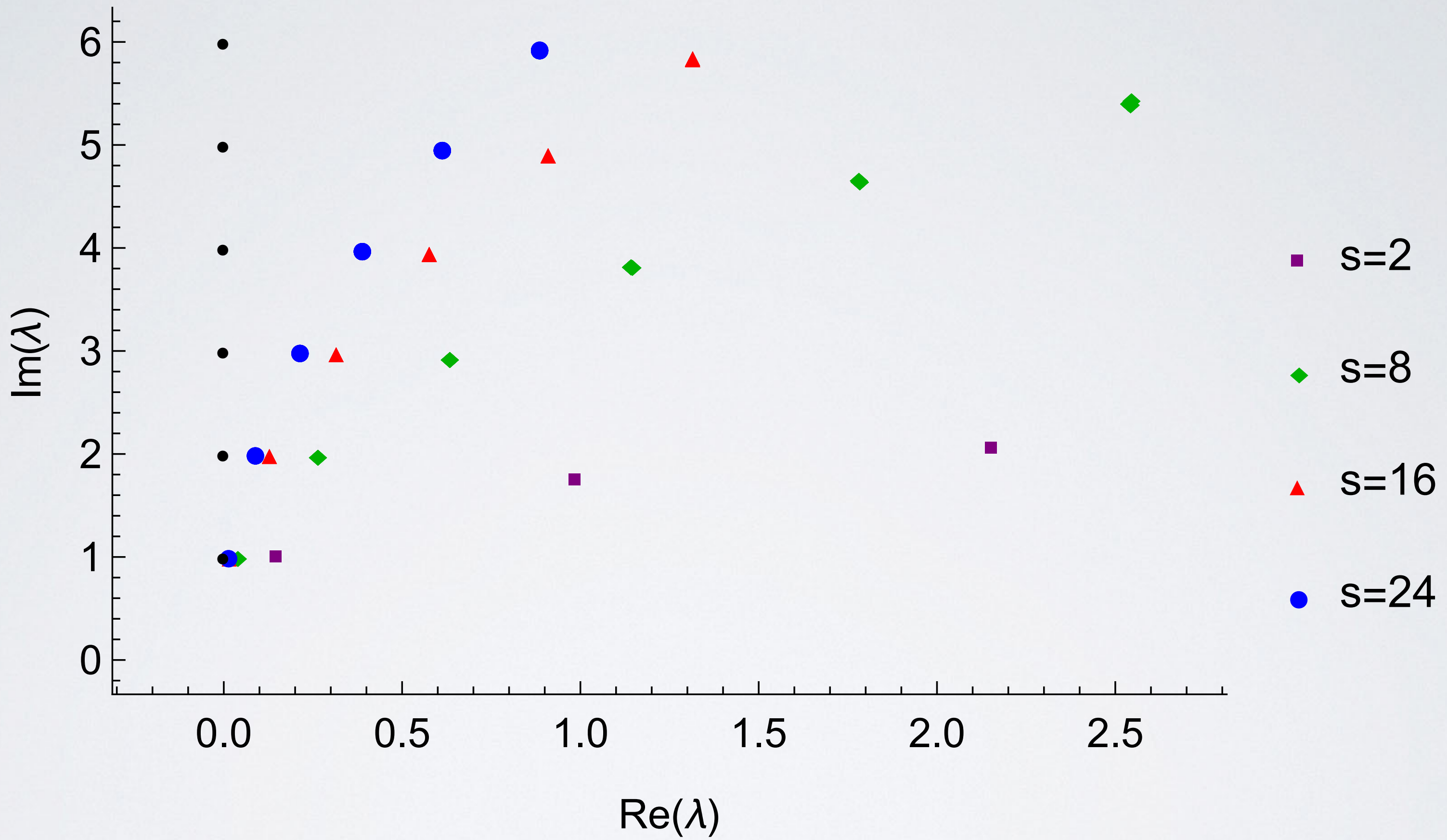
SPECTRUM OF DIRAC ON SPHERE



Exact is integer spacing for j = 1/2, 3/2, 5/2 ...

Exact degeneracy 2j + 1: No zero mode in chiral limit!.

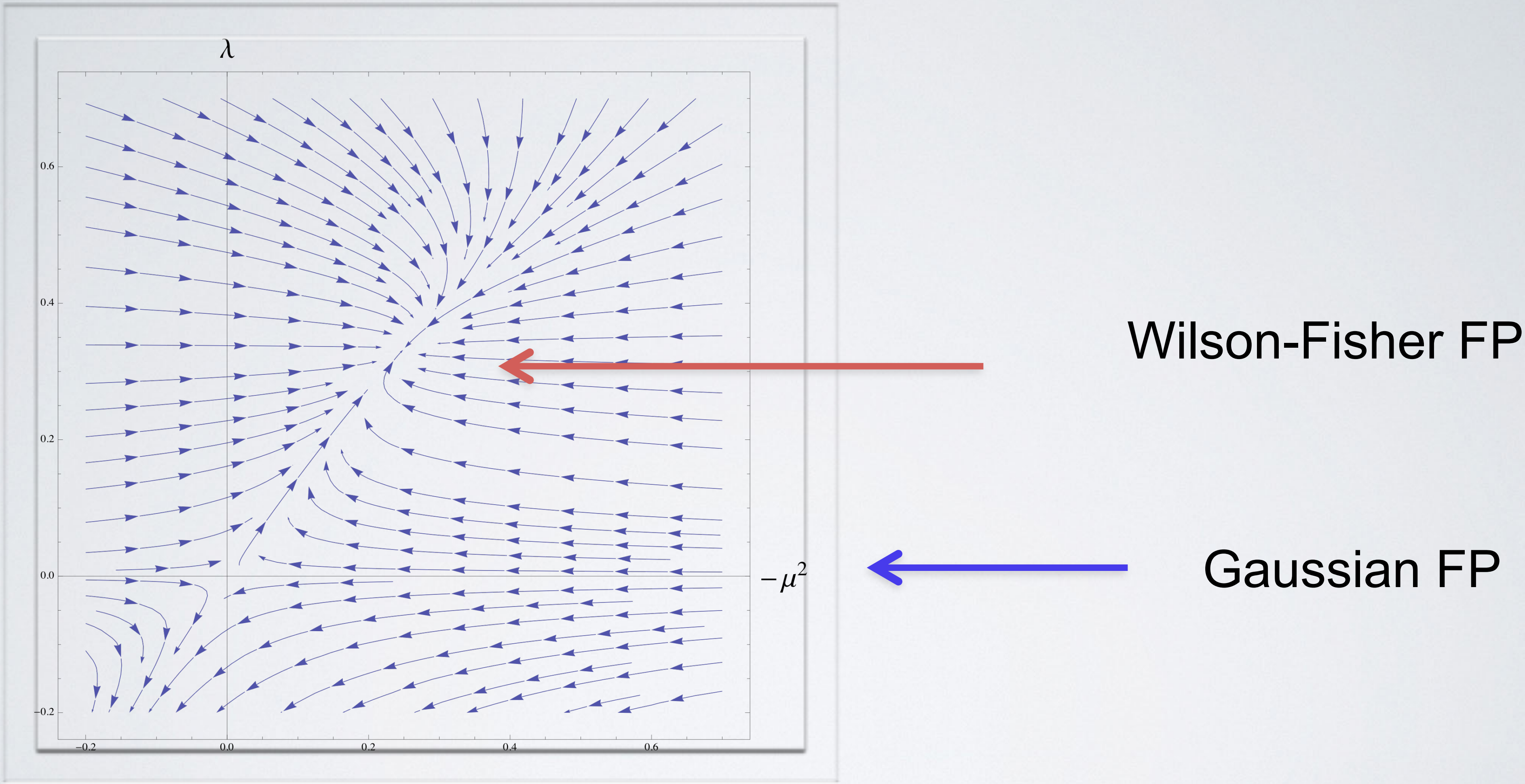
CONVERGENCE TO CONTINUUM



$$Min \sum_x |\Psi_x - e^{i\omega_x \sigma_3/2} U \psi_x|^2$$

UV DIVERGENCE AND QUANTUM FINITE ELEMENTS

Replace Ising Model by phi 4th

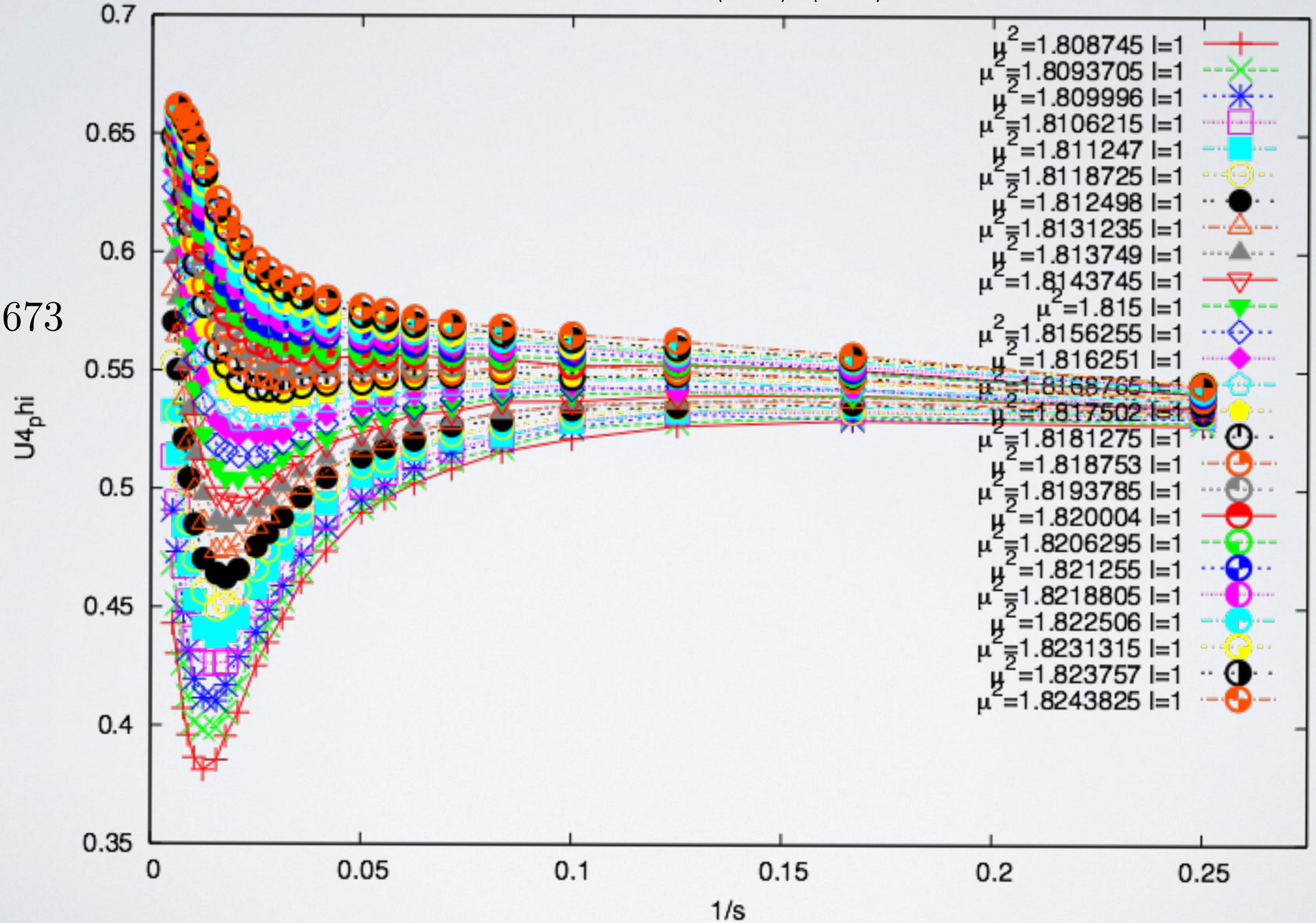


$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

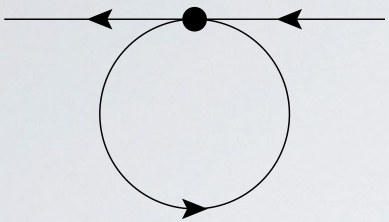
BINDER CUMULANT NEVER CONVERGES

$$U_B = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \, \langle M^2 \rangle}$$

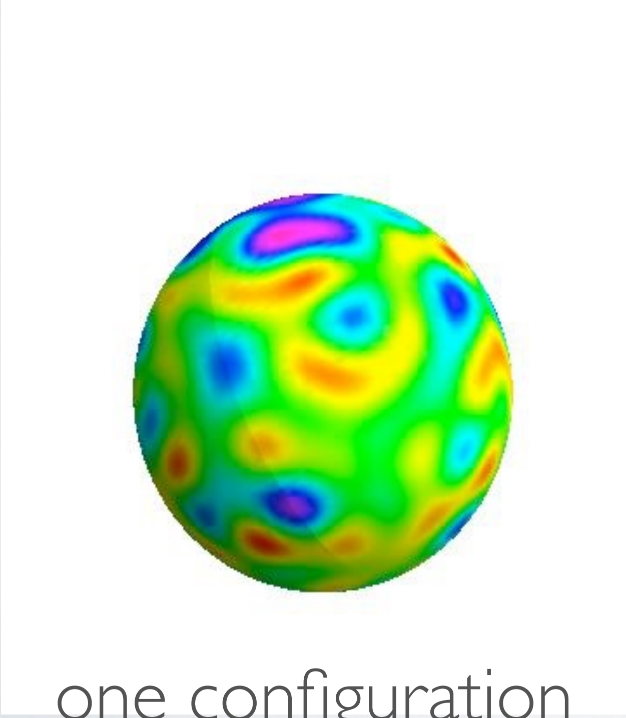
0.5673



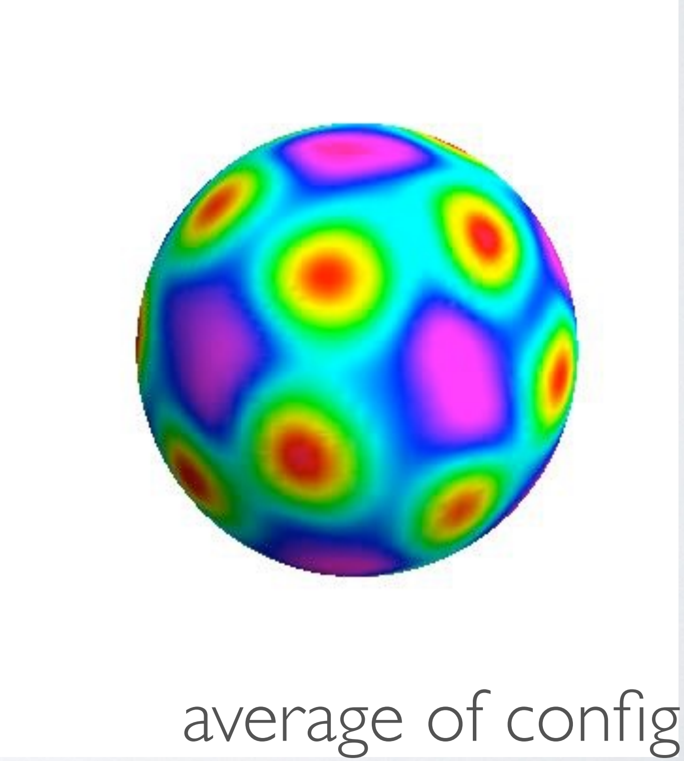
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \rightarrow \frac{1}{K_{xx}}$$



one configuration

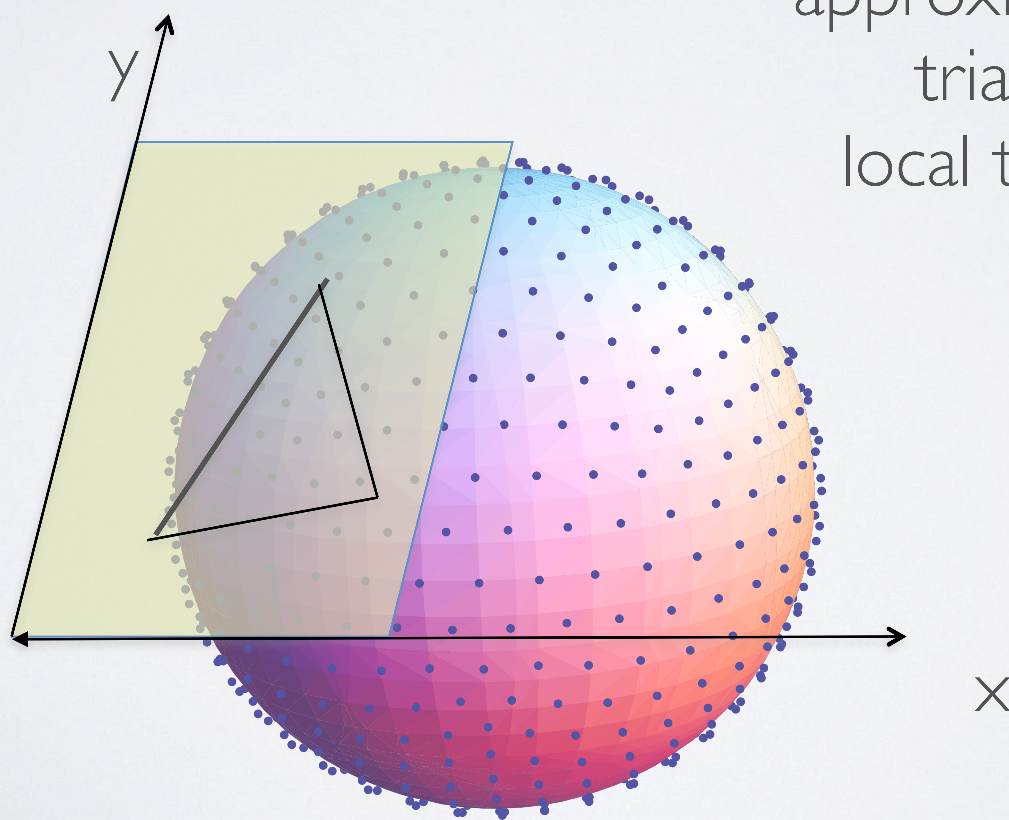


average of config.

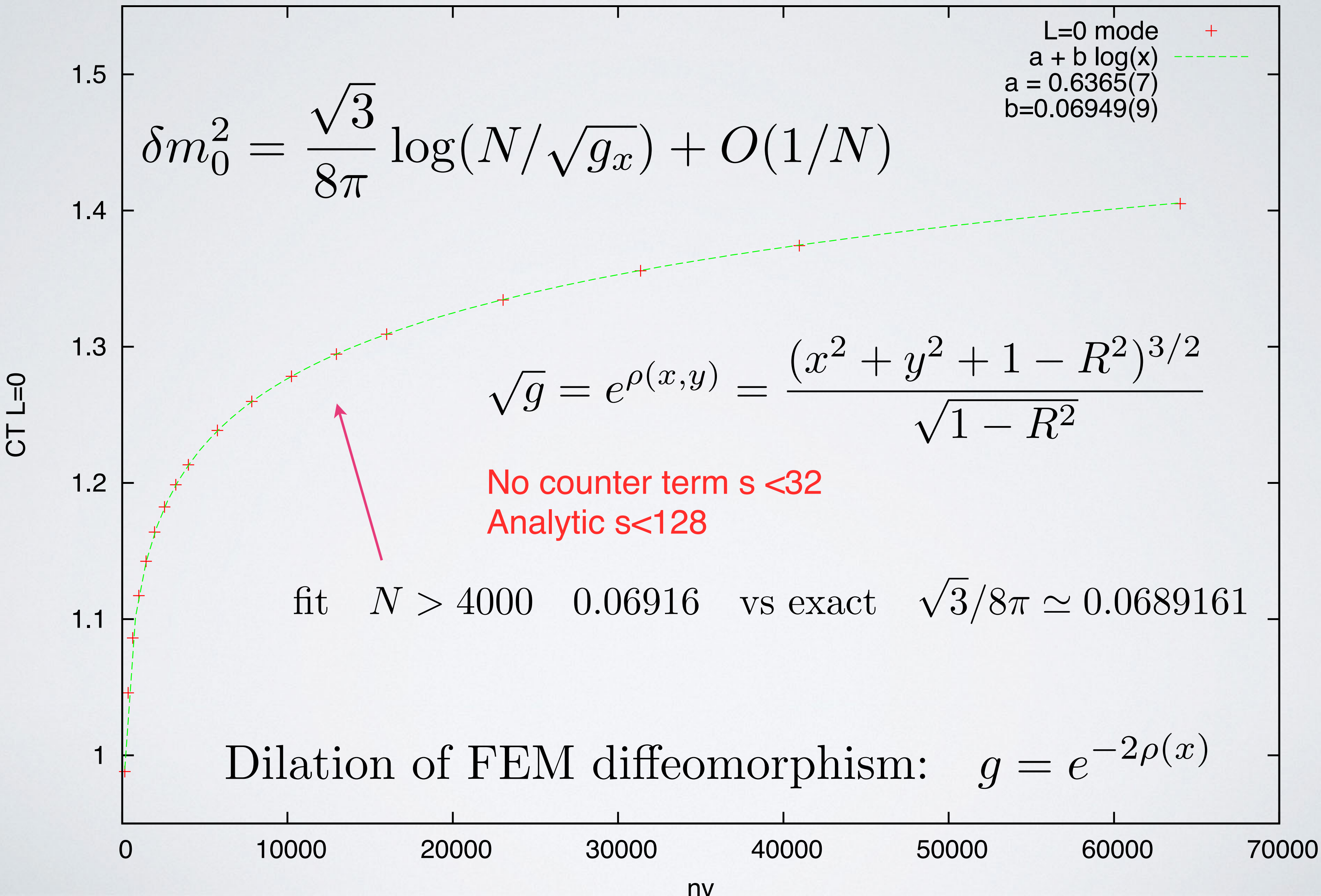
TEST CFT: PHI 4TH WILSON-FISHER FIXED POINT IN 3D.

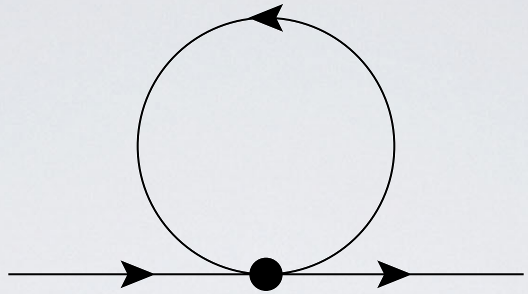
$$L = \int d^3x[\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\sqrt{g}(\phi^2 - \mu^2/2\lambda)^2]$$

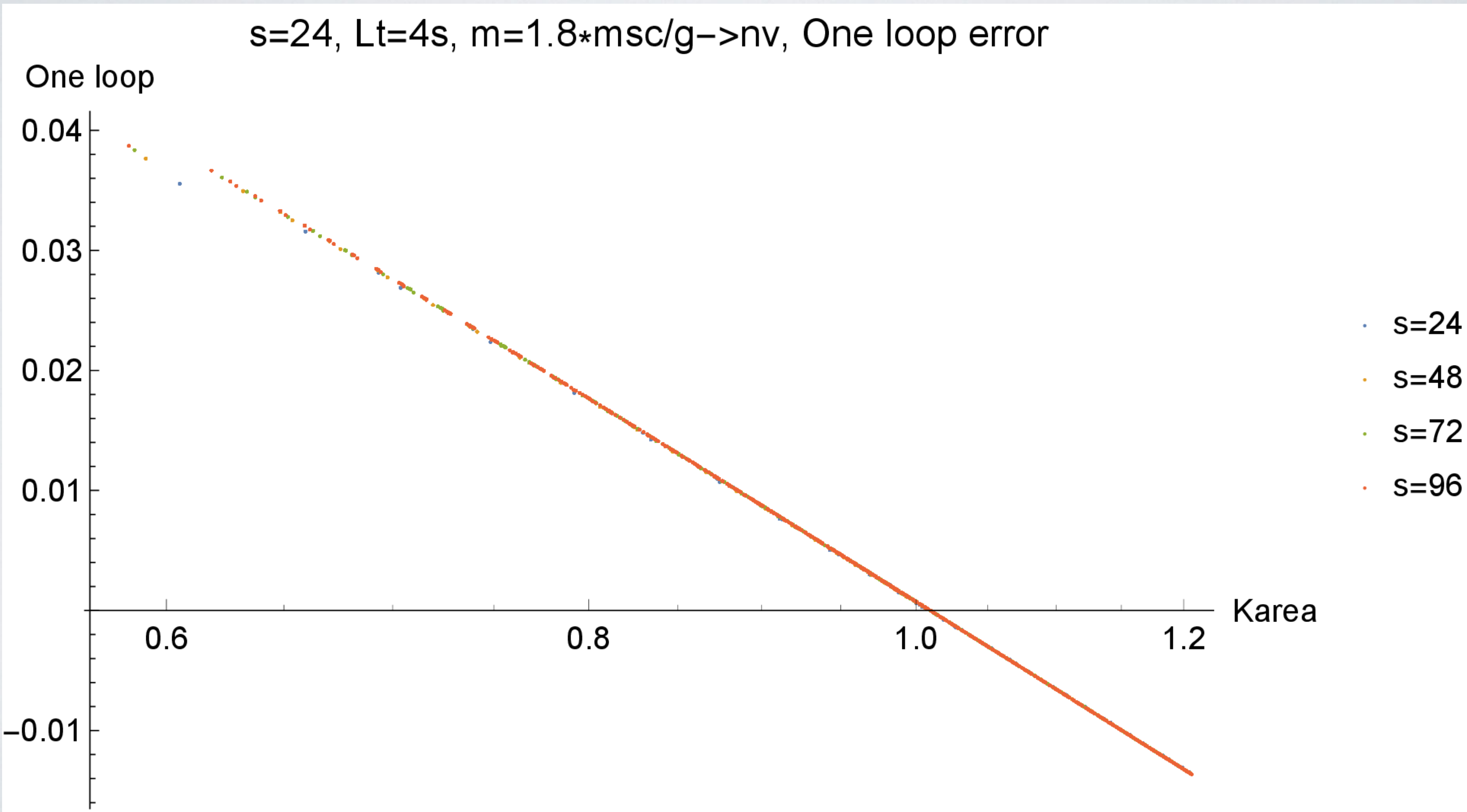
approximate spherical triangles onto local tangent plane



MODEL OF COUNTER TERM







s=24, Lt=4s, m=1.8*msc/g->nv, One loop error
One loop
0.04
0.03
0.02
0.01
-0.01
0.6
0.8
1.0
1.2
Karea
s=24
s=48
s=72
s=96

ONE LOOP COUNTER TERM

$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$

NOW BINDER CUMULANT CONVERGES

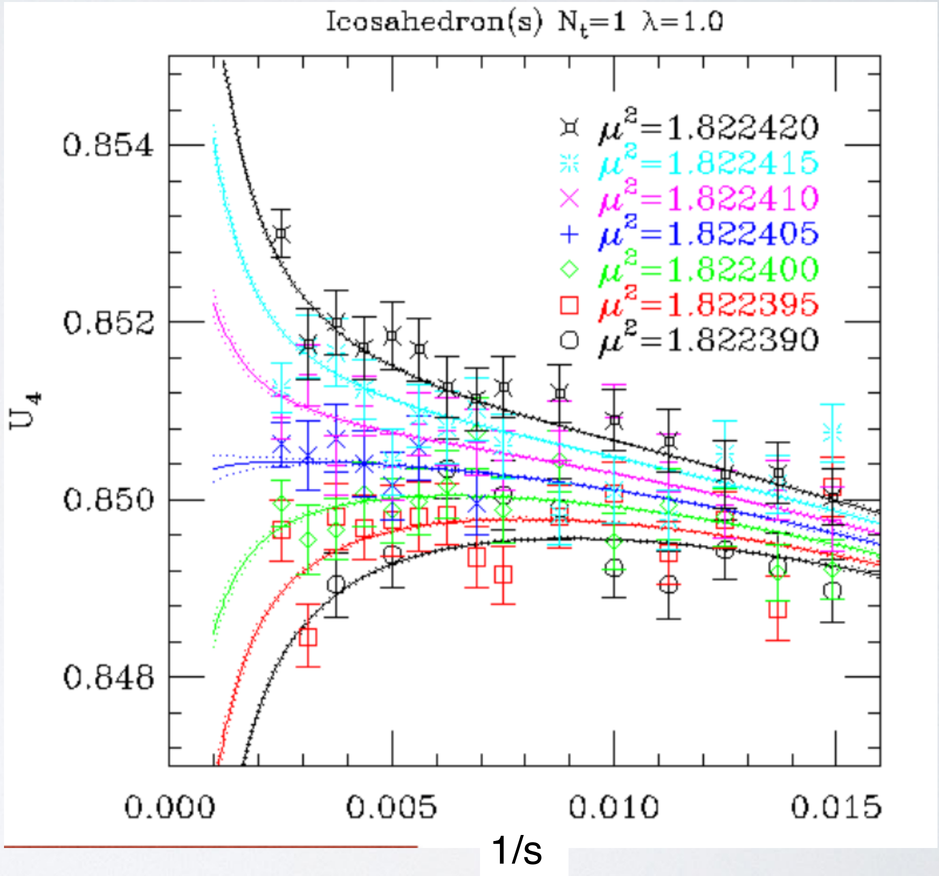
$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu^2_{cr}]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT: $U_4 = 0.85081(10)$

EXACT: $U_4^{exact} = 0.851021(5)$

HIGHER MOMENT 2n = 4,6,8,10,12

$U_6 = 0.77280(13)$



$$U_4 = \frac{3}{2}[1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \langle M^2 \rangle}]$$

$$\mu^2_{cr} = 1.82240070(34)$$

Simultaneous fit for s up 800: E.G. 6,400,002 Sites on Sphere

$$dof = 1701 \quad , \quad \chi^2/dof = 1.026$$

Using Binder Cumulants

$$U_4 = \frac{3}{2}\left(1 - \frac{m_4}{3\ m_2^2}\right) \qquad m_n = \langle \phi^n \rangle$$

$$U_6 = \frac{15}{8}\left(1 + \frac{m_6}{30\ m_2^3} - \frac{m_4}{2\ m_2^2}\right)$$

$$U_8 = \frac{315}{136}\left(1 - \frac{m_8}{630\ m_2^4} + \frac{2\ m_6}{45\ m_2^3} + \frac{m_4^2}{18\ m_2^4} - \frac{2\ m_4}{3\ m_2^2}\right)$$

$$U_{10} = \frac{2835}{992}\left(1 + \frac{m_{10}}{22680\ m_2^5} - \frac{m_8}{504\ m_2^4} - \frac{m_6\ m_4}{108 m_2^5} + \frac{m_6}{18\ m_2^3} + \frac{5\ m_4^2}{36\ m_2^4} - \frac{5\ m_4}{6\ m_2^2}\right)$$

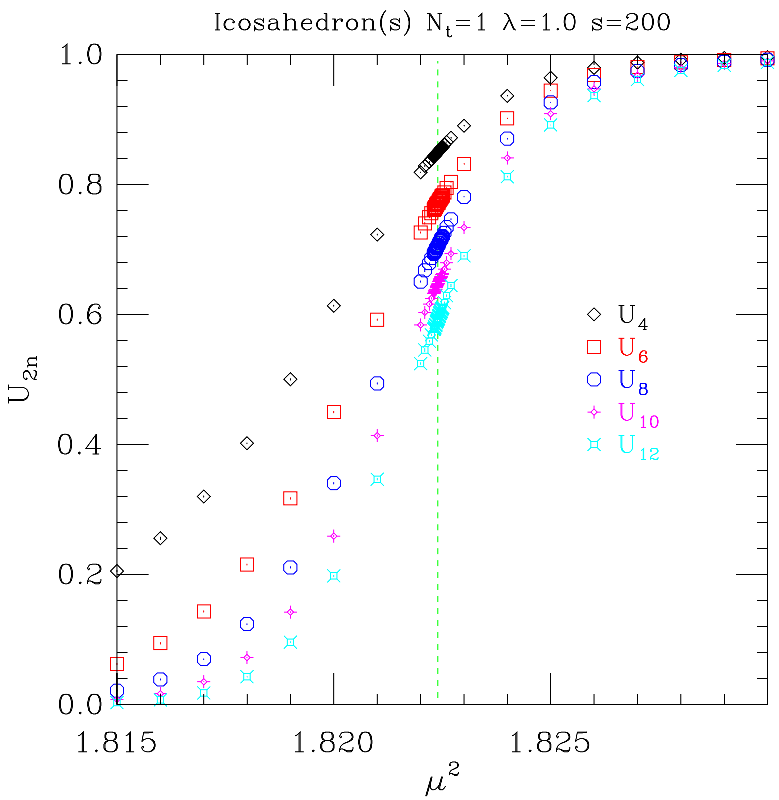
$$U_{12} = \frac{155925}{44224}\left(1 - \frac{m_{12}}{1247400\ m_2^6} + \frac{m_{10}}{18900\ m_2^5} + \frac{m_8\ m_4}{2520\ m_2^6} - \frac{m_8}{420\ m_2^4} + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45\ m_2^5} + \frac{m_6}{15\ m_2^3} - \frac{m_4^3}{108\ m_2^6} + \frac{m_4^2}{4\ m_2^4} - \frac{m_4}{m_2^2}\right)$$

In infinite volume

$U_{2n}=0$ in disordered phase

$U_{2n}=1$ in ordered phase

$0<U_{2n}<1$ on critical surface



- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}$=0.851001
- Higher critical cumulants computable using conformal 2n-point functions: Luther and Peschel (1975) Dotsenko and Fateev (1984)

LESSON: **QFE** NEEDS COUNTER TERMS

APPROACHES

(i) Explicitly Subtract Finite Terms for Super Renormalized Theories

(ii) Pauli-Villars* 1949 (or Feynman and Stuekelberg)

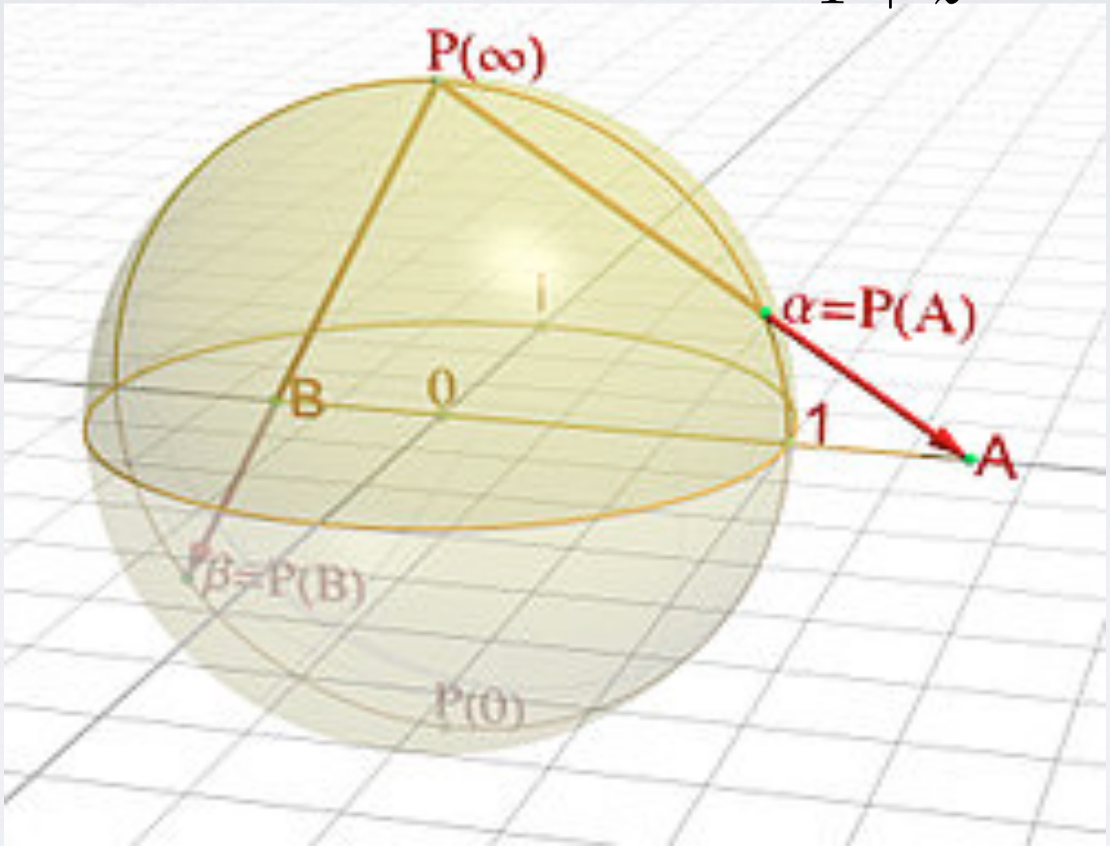
$$\frac{1}{p^2} - \frac{1}{p^2 + M_{PV}^2} \equiv \frac{1}{p^2 + p^4/M_{PV}^2} \qquad 1/\xi \ll M_{PV} \ll \pi/a$$

(iii) Perhaps Wilson Flow for D = 4 & Non-Abelian Gauge Theory

(iv) Classify should reflect the perturbative CT

TEST 2D ISING/PHI 4TH ON THE RIEMANN SPHERE

projection $$\xi = \tan(\theta/2) e^{-i\phi} = \frac{x + iy}{1 + z}$$



Conformal Projection + Weyl Rescaling to the Sphere

EXACT SOLUTION TO CFT

Exact Two point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \rightarrow \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

$$x^2 + y^2 + z^2 = 1$$

4 pt function

$$(x_1, x_2, x_3, x_4) = (0, z, 1, \infty)$$

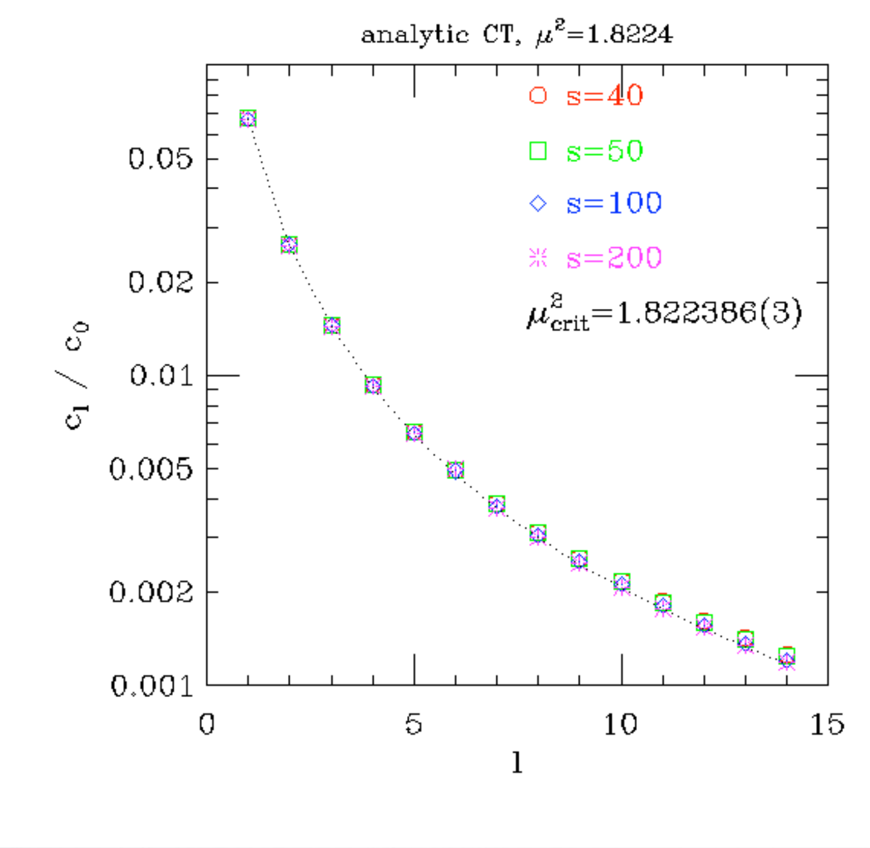
$$g(0, z, 1, \infty) = \frac{1}{2|z|^{1/4}|1-z|^{1/4}}[|1+\sqrt{1-z}| + |1-\sqrt{1-z}|]$$

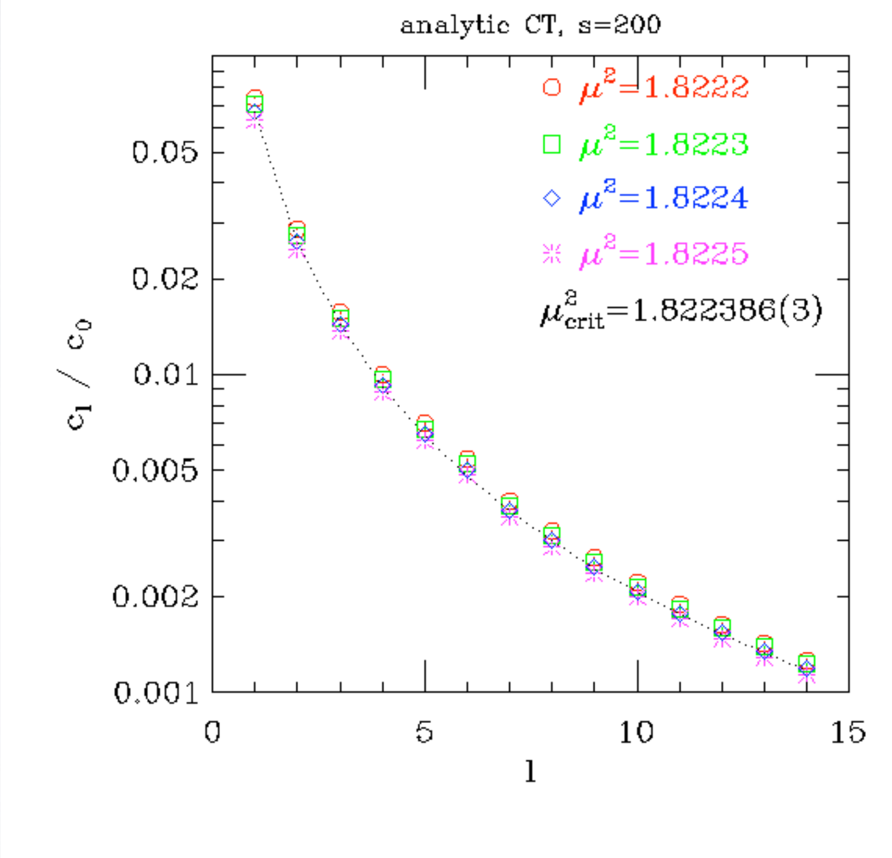
Critical Binder Cumulant

$$U_B^* = 1 - \frac{\langle M^4 \rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$$

Dual to Free Fermion

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 x_{24}^2} \quad , \quad v = \frac{r_{14}^2 r_{32}^2}{r_{13}^2 r_{24}^2} \qquad \text{where} \quad r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2 = 2(1 - \cos\theta_{ij})$$





$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z)$$

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

Very fast cluster algorithm:

Brower,Tamayo 'Embedded Dynamics for phi 4th Theory" PRL 1989. Wolff single cluster + plus Improved Estimators etc

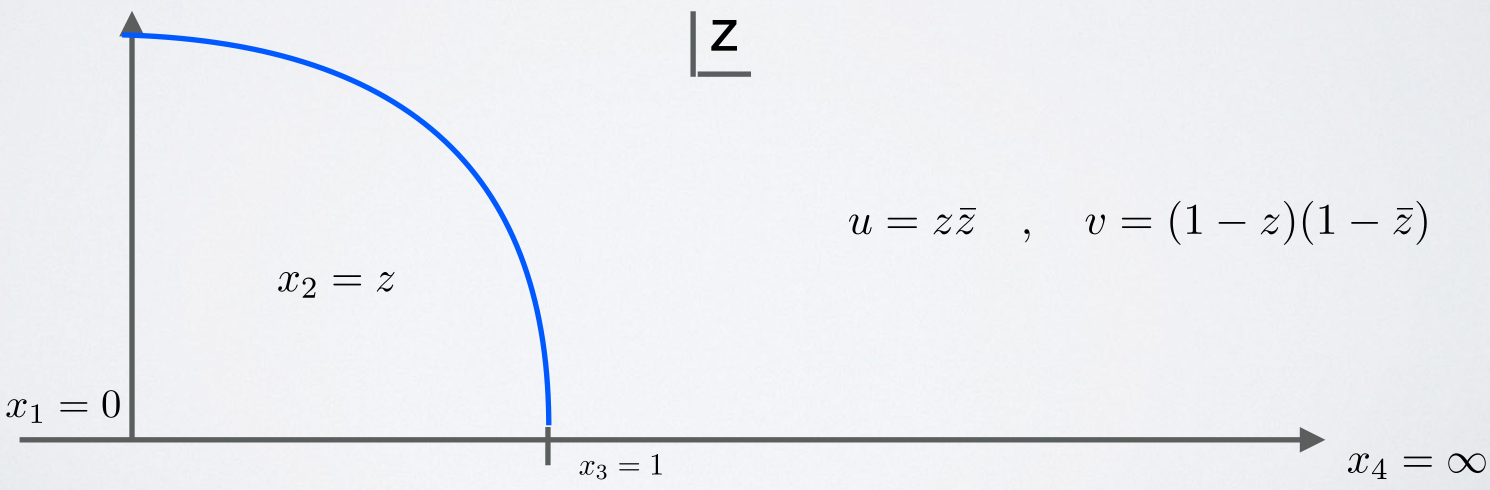
EXACT FOUR POINT FUNCTION

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} + \epsilon$

$$g(u,v) = \frac{\langle \phi_1\phi_2\phi_3\phi_4 \rangle}{\langle \phi_1\phi_3 \rangle \, \langle \phi_2\phi_4 \rangle}$$

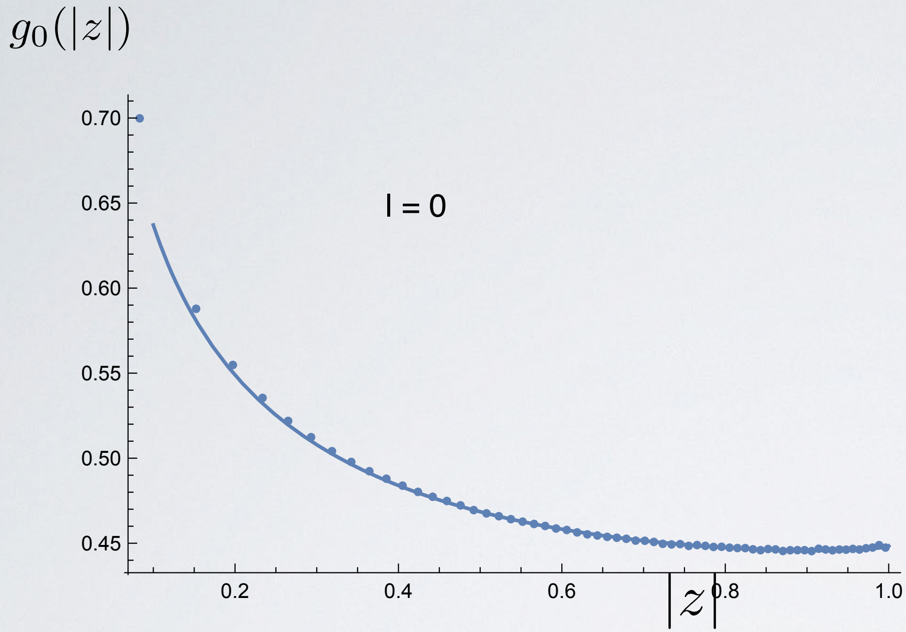
$$= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} \left[|1+\sqrt{1-z}| + |1-\sqrt{1-z}| \right]$$

Crossing Sym: $|1+\sqrt{1-z}| + |1-\sqrt{1-z}| = \sqrt{2+2\sqrt{(1-z)(1-\bar{z})}+2\sqrt{z\bar{z}}}$



$u = z\bar{z}$, $v = (1-z)(1-\bar{z})$

2 TO 2 SCATTERING DATA

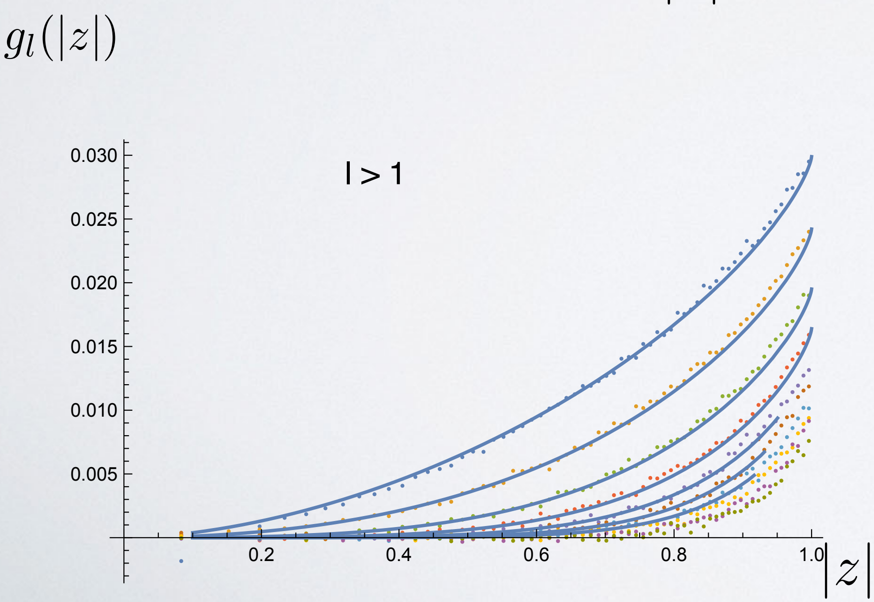


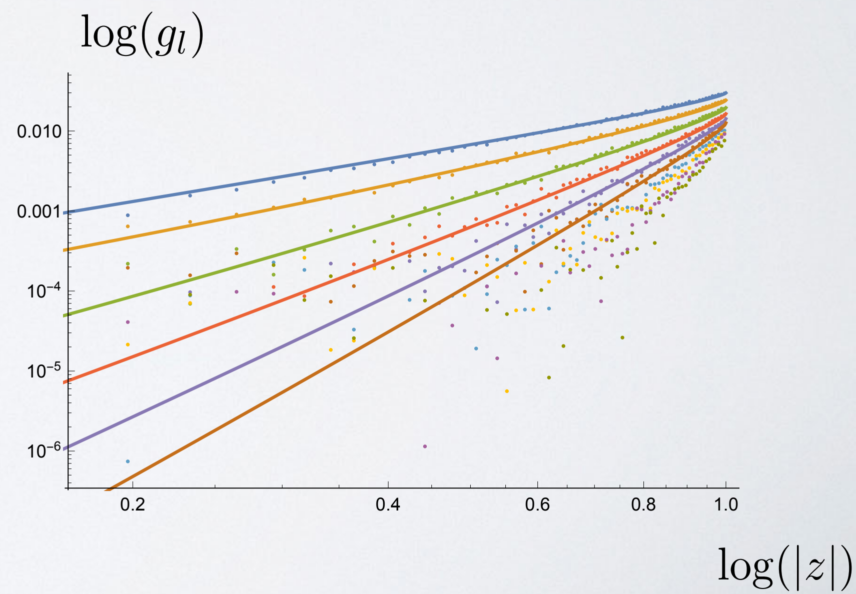
ZERO PARAMETER FIT

s= 10 Run for 1/2 hour

$$g(u,v) = \sum_l g_l(|z|)\cos(l\theta)$$

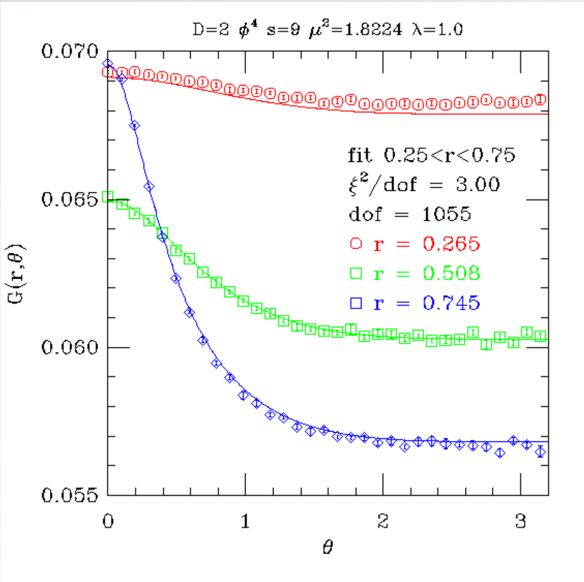
$$z = |z|e^{il\theta}$$



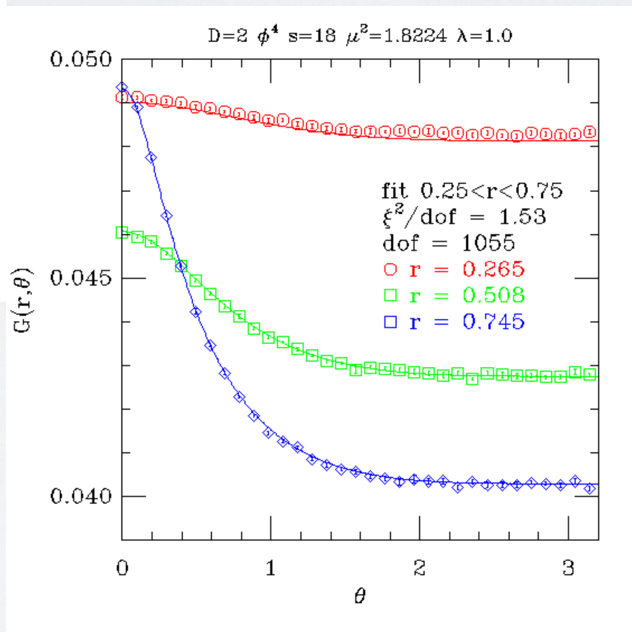


4PT CONVERGENCE

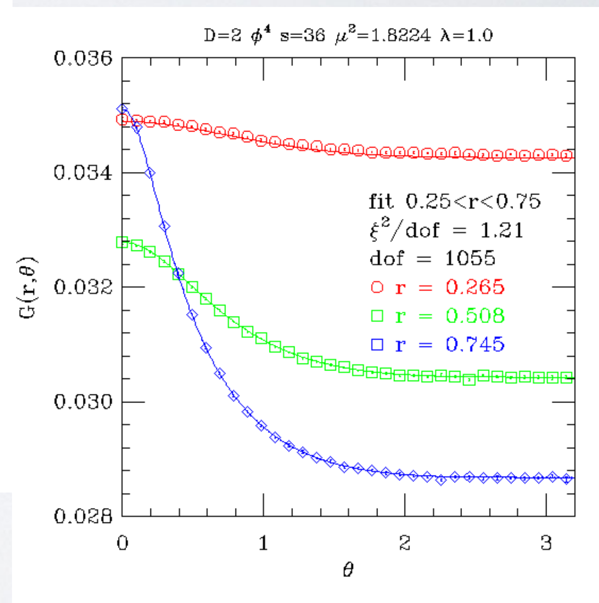
$s = 8 \quad \xi^2/dof = 3.00$



$s = 18 \quad \xi^2/dof = 1.53$



$s = 36 \quad \xi^2/dof = 1.21$



ISING: FREE MAJORANA FERMIONS

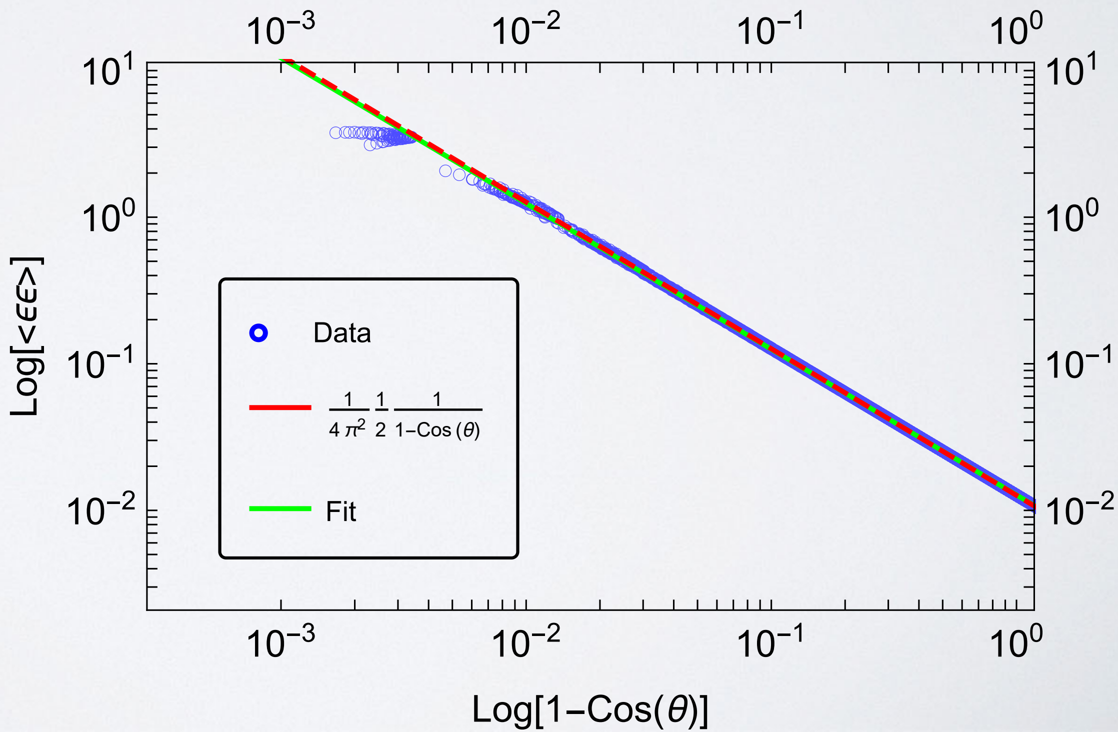
c =1/2 Minimal Model OPE: $\sigma \times \sigma = \mathbf{1} + \epsilon \quad , \quad \epsilon \times \sigma = \epsilon \quad , \quad \epsilon \times \epsilon = \mathbf{1}$

Even Ope $\epsilon(z) = i\bar{\psi}(z)\psi(z)$ Odd operator is twist $\sigma(z)$

$$S_{Dirac} = \int d^2x[\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi}]$$

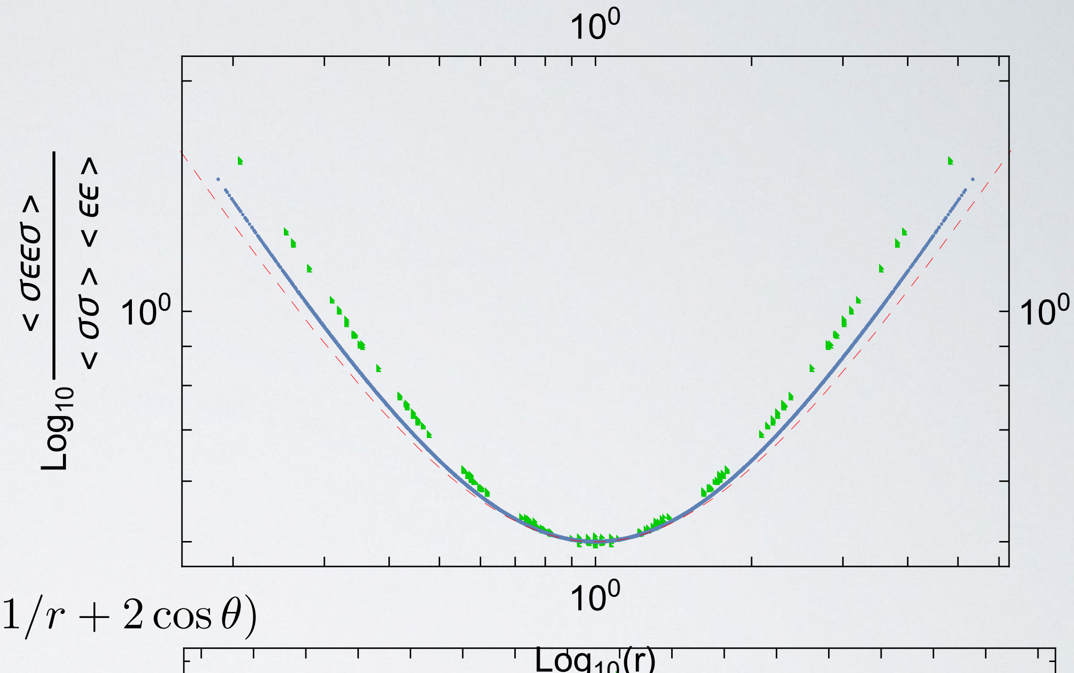
$$\langle\psi(z_1)\bar{\psi}(z_1)\bar{\psi}(z_1)\psi(z_2)\rangle = \left[\frac{1}{\partial}\right]_{z_1,z_2}\left[\frac{1}{\bar{\partial}}\right]_{z_1,z_2} = \frac{1}{4\pi^2}\frac{1}{|z_1 - z_2|^2}$$

$$\langle\epsilon(z_1)\epsilon(z_2)\rangle$$



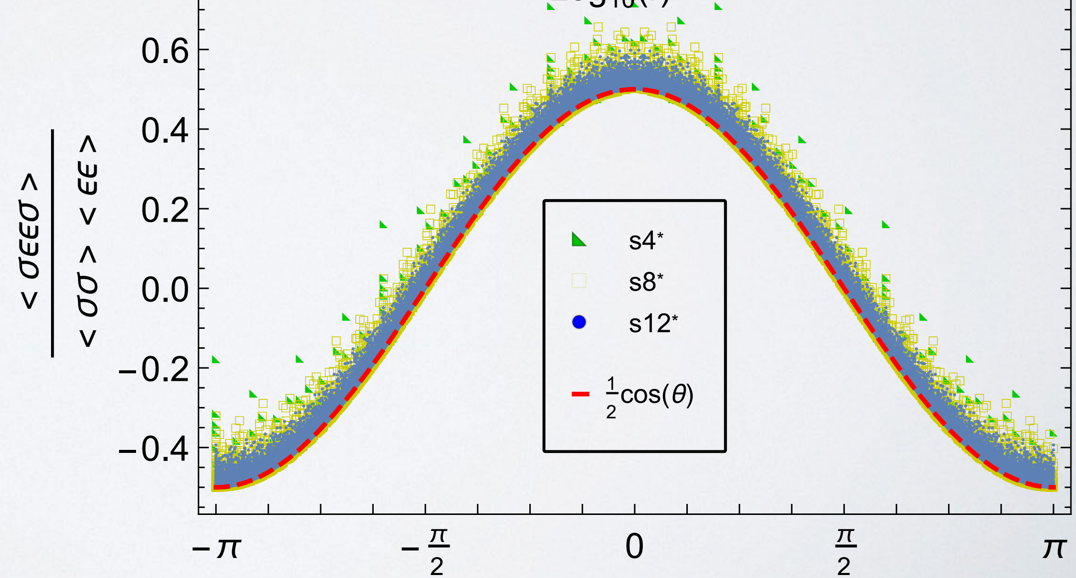
ADD TWISTS AT N/S POLES

Subtract Cos term



$$\frac{\langle\sigma(0)\epsilon(z_2)\epsilon(z_3)\sigma(\infty)\rangle}{\langle\epsilon(z_2)\epsilon(z_3)\rangle}$$

$$= \frac{1}{4}|\sqrt{z_1/z_2} + \sqrt{z_2/z_1}|^2 = \frac{1}{4}(r + 1/r + 2\cos\theta)$$



Substract r term

QFE PLANS

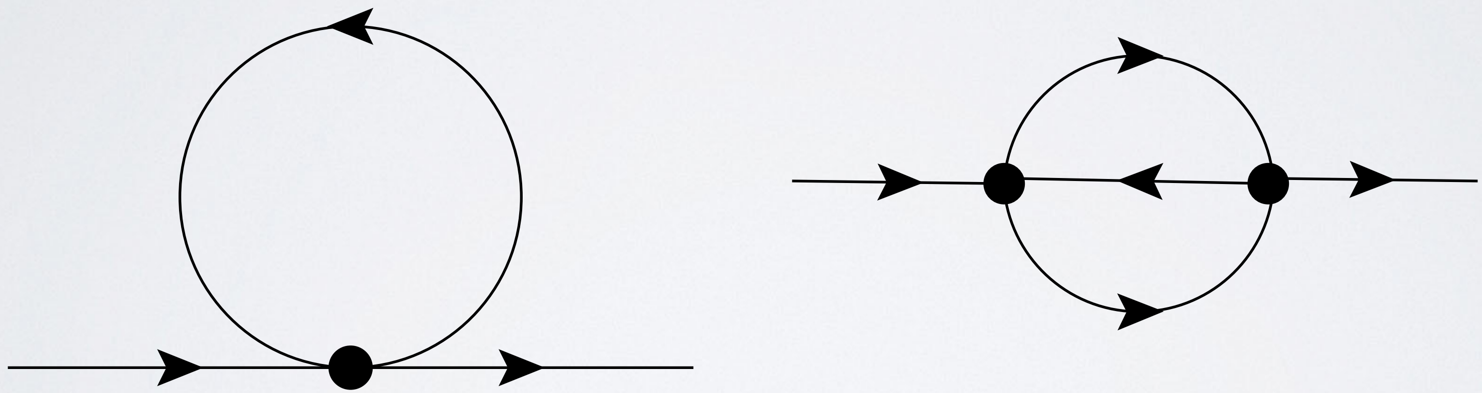
- COMPUTATION:
  - 2+1 Radial Phi 4th/3D Ising CFT (with cluster algorithm)
  - Extend Peter Boyle's GRID to HMC on Simplicial Spheres (Interesting 3D Problem for Dirac/Scalar Theories.)
  - 3 Sphere starting with 600 cell: 4 Sphere ?
- THEORY:

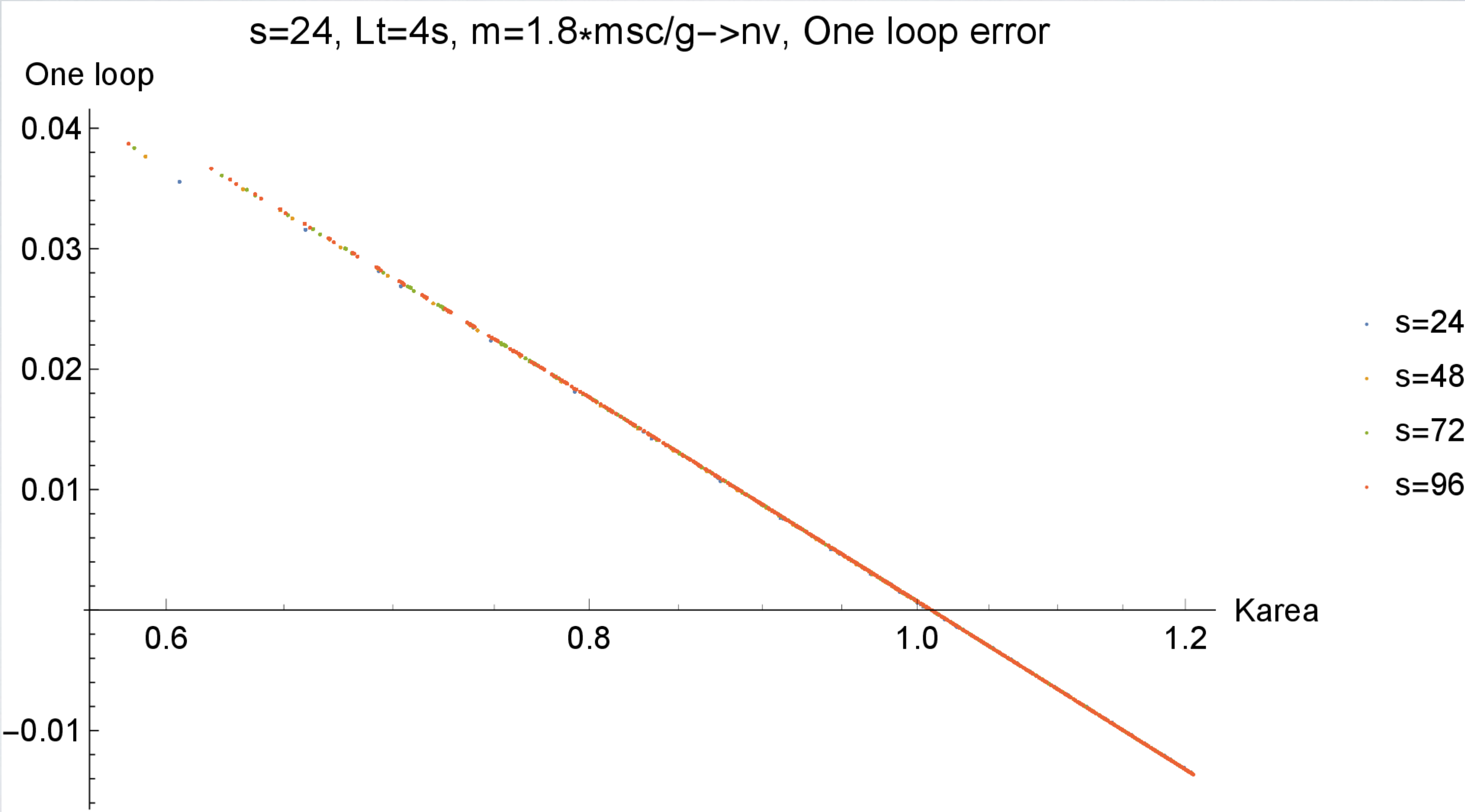
$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

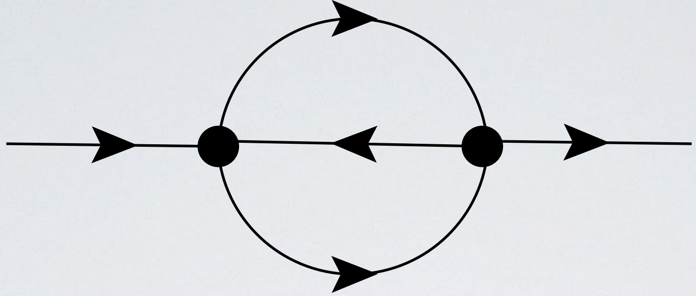
  - Prove QFE for super renormalizable theories
  - Classify all CT that break diffeomorphism invariance.
  - Renormalization of 4d non-Abelian FT
  - Clarity DEC for Quantum FT

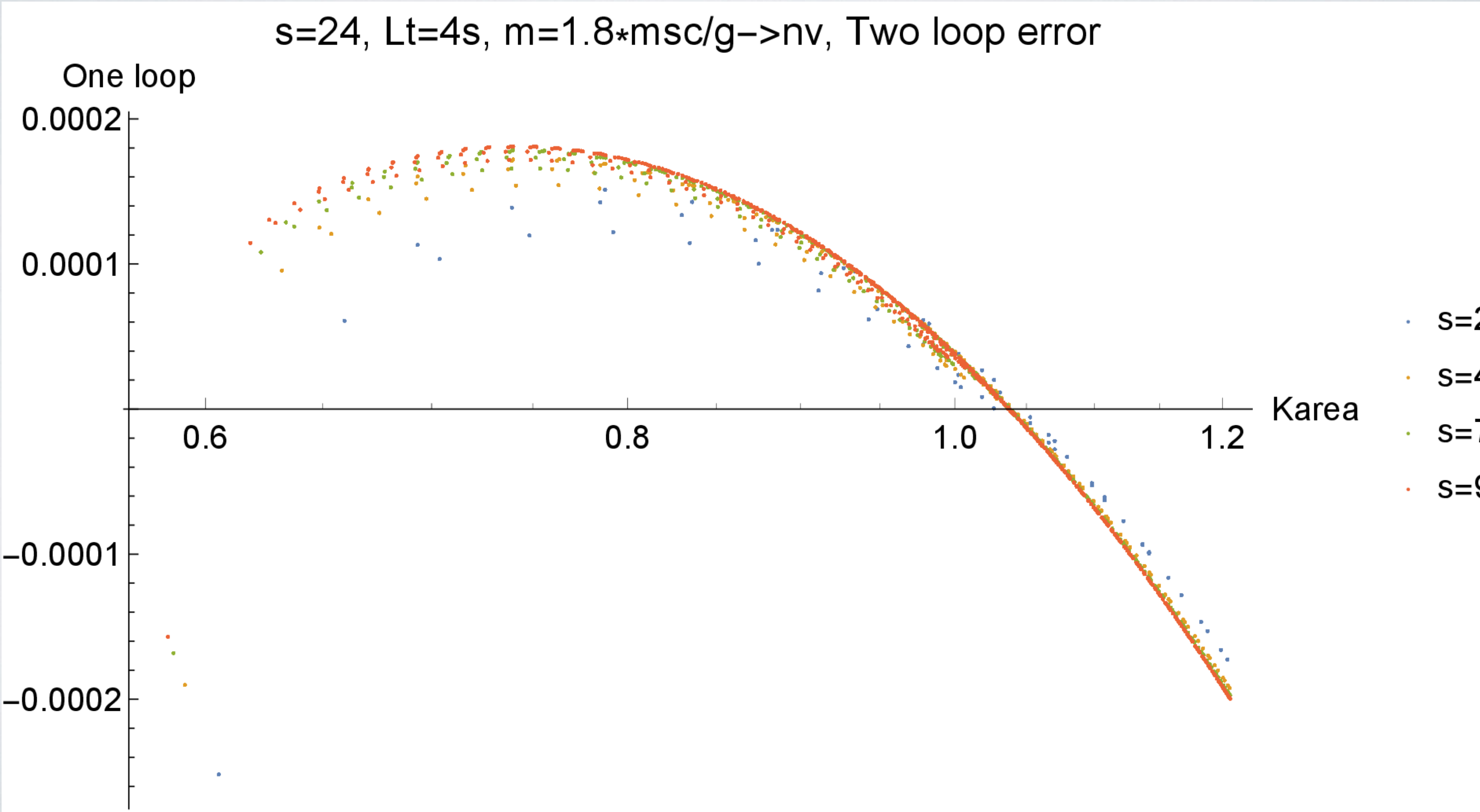
BACKUP SLIDES

COUNTER TERM IN 3D




s=24, Lt=4s, m=1.8*msc/g->nv, One loop error
One loop
0.04
0.03
0.02
0.01
-0.01
0.6
0.8
1.0
1.2
Karea
s=24
s=48
s=72
s=96

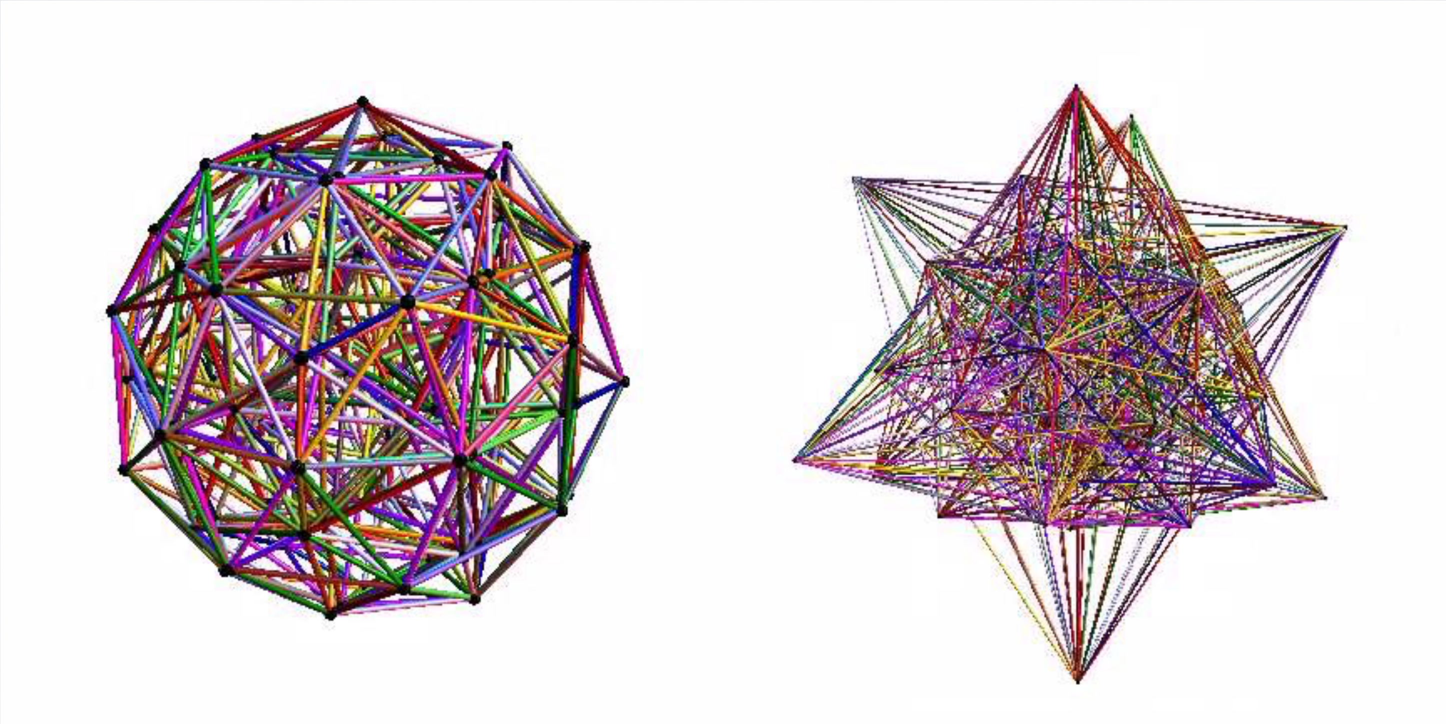




s=24, Lt=4s, m=1.8*msc/g->nv, Two loop error
One loop
0.0002
0.0001
-0.0001
-0.0002
0.6
0.8
1.0
1.2
Karea

600 CELL ON S3

HTTPS://EN.WIKIPEDIA.ORG/WIKI/600-CELL



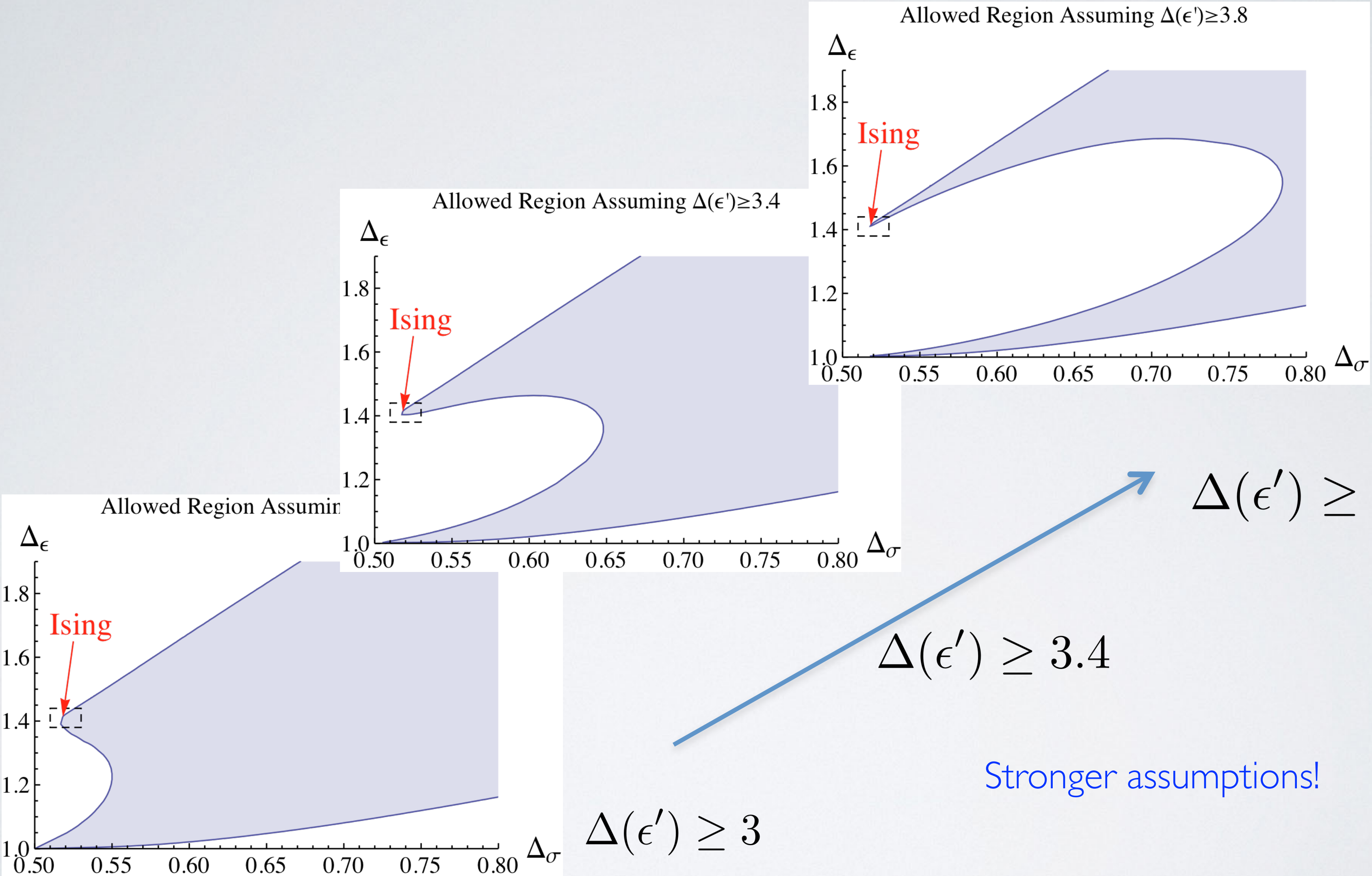
16 vertices of the form:[3] (±½, ±½, ±½, ±½),

8 vertices obtained from (0, 0, 0, ±1) by permuting coordinates.

96 vertices are obtained by taking even permutations of ½ (±ϕ, ±1, ±1/ϕ, 0).

https://en.wikipedia.org/wiki/List_of_regular_polytopes_and_compounds#Five-dimensional_regular_polytopes_and_higher

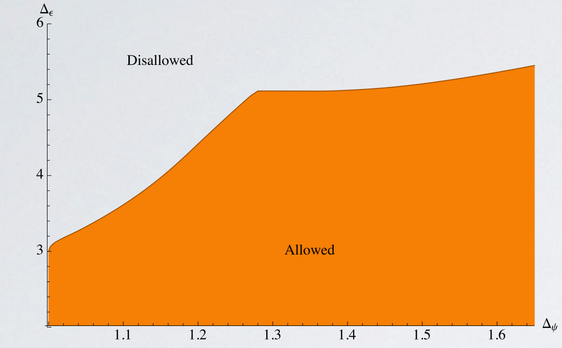
INEQUALITIES FROM BOOTSTRAP*

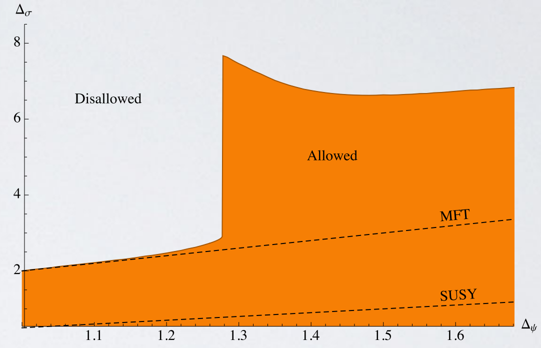


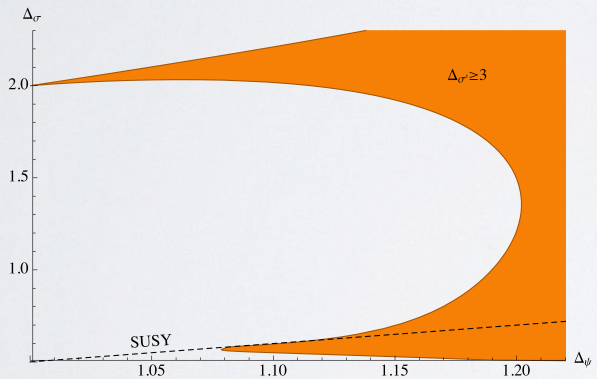
- *“**Solving the 3D Ising Model with the Conformal Bootstrap**” (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

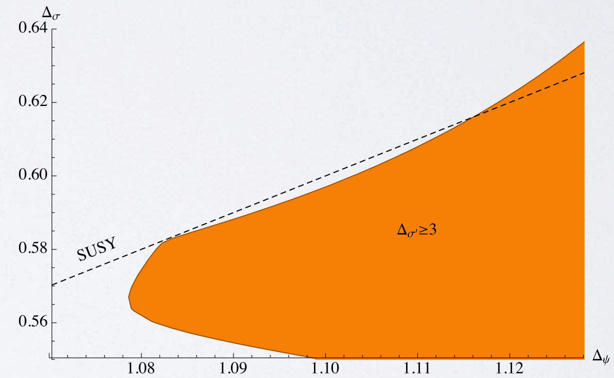
Bootstrapping 3D Fermions

Luca Iliesiu[a], Filip Kos[b], David Poland[b], 1508.00012v1.









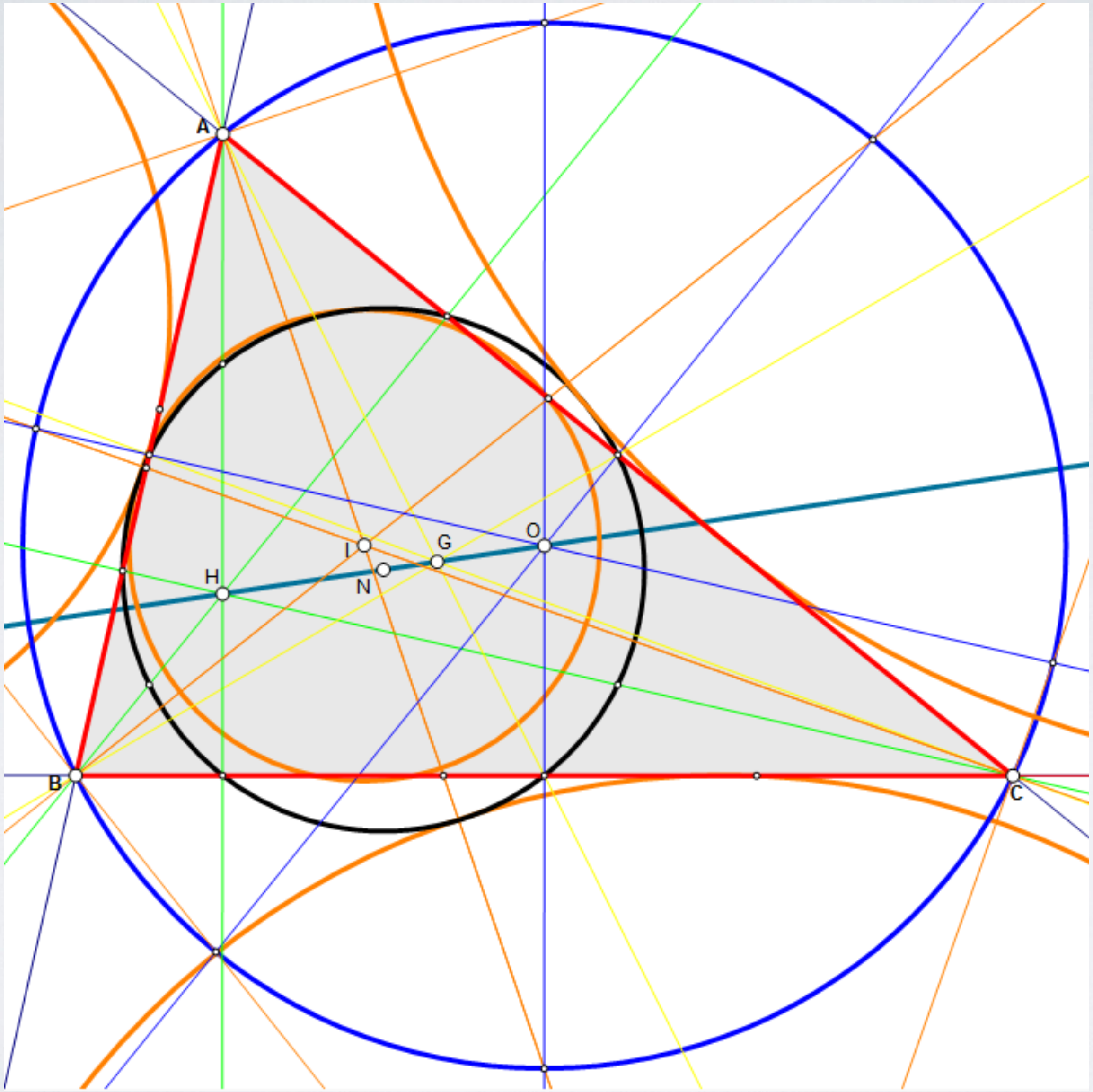
$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{N}\bar{\psi}_i(\gamma^\mu\partial_\mu + g\phi)\psi_i - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4$$

Gross Neuve (-Yukawa)

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{g}{2}\phi\bar{\psi}\psi - \frac{1}{8}\left(g\phi^2 + h\right)^2$$

N =1 SUSY-Ising

CENTERS OF A TRIANGLE



https://en.wikipedia.org/wiki/Triangle_center