AMG for Staggered Fermions

Progress Report

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Staggered Dirac Operator:

$$D\psi(x) = m\psi(x) + \sum_{\mu=1}^{4} \alpha_\mu(x) \left(U_\mu(x)\psi(x+\hat{\mu}) - U_\mu(x-\hat{\mu})^\dagger \psi(x-\hat{\mu}) \right)$$

with mass shift m and *staggered phase*

$$\alpha_\mu(x) = (-1)^{\sum_{\nu=1}^{\mu-1} x_\nu}$$

and $x \in \mathbb{Z}^4$

Properties of the Staggered Operator D

(1) D can be written as

$$D = mI + \hat{D} \quad \text{with} \quad \hat{D} = \begin{pmatrix} & D_{eo} \\ -D_{eo}^{\dagger} & \end{pmatrix}, \; \hat{D} = -\hat{D}^{\dagger}$$

and each EV $v = \begin{pmatrix} v_e \\ v_o \end{pmatrix}$ with $\hat{D}v = \lambda v$ yields

an EV $\hat{v} = \begin{pmatrix} v_e \\ -v_o \end{pmatrix}$ with $\hat{D}\hat{v} = -\lambda\hat{v} = \bar{\lambda}\hat{v}$

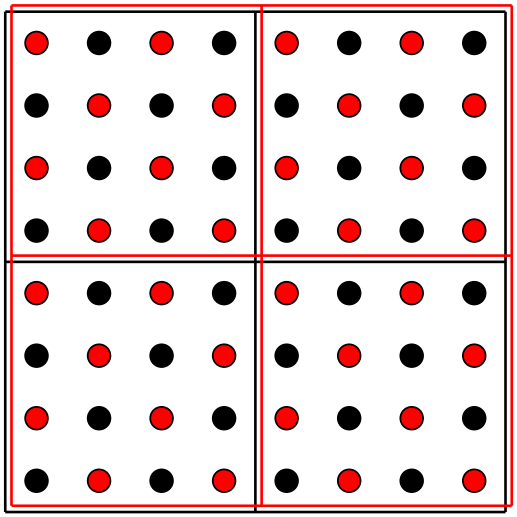
(2) Solving the normal equation

$$D^{\dagger}D = m^2 I + \hat{D}^{\dagger}\hat{D} = \begin{pmatrix} m^2 I + D_{eo}D_{eo}^{\dagger} & \\ & m^2 I + D_{eo}^{\dagger}D_{eo} \end{pmatrix}$$

is equivalent to applying odd-even preconditioning to D

1st Choice of Aggregation

Property (1) $\Longrightarrow$ natural choice of aggregation type:

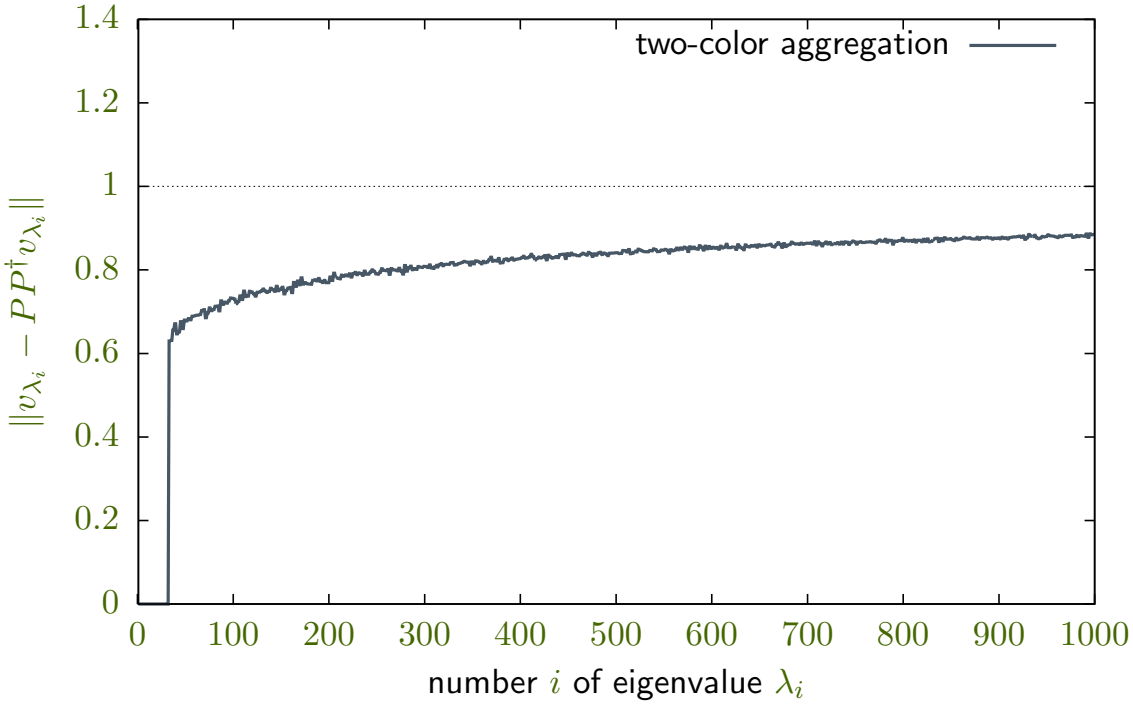


aggregate even and odd points separately

- aggregate size 4^4
- 16 test vectors $=$ 16 smallest EVs v_λ, $\lambda > 0$
- GMRES as smoother

Numerical Tests

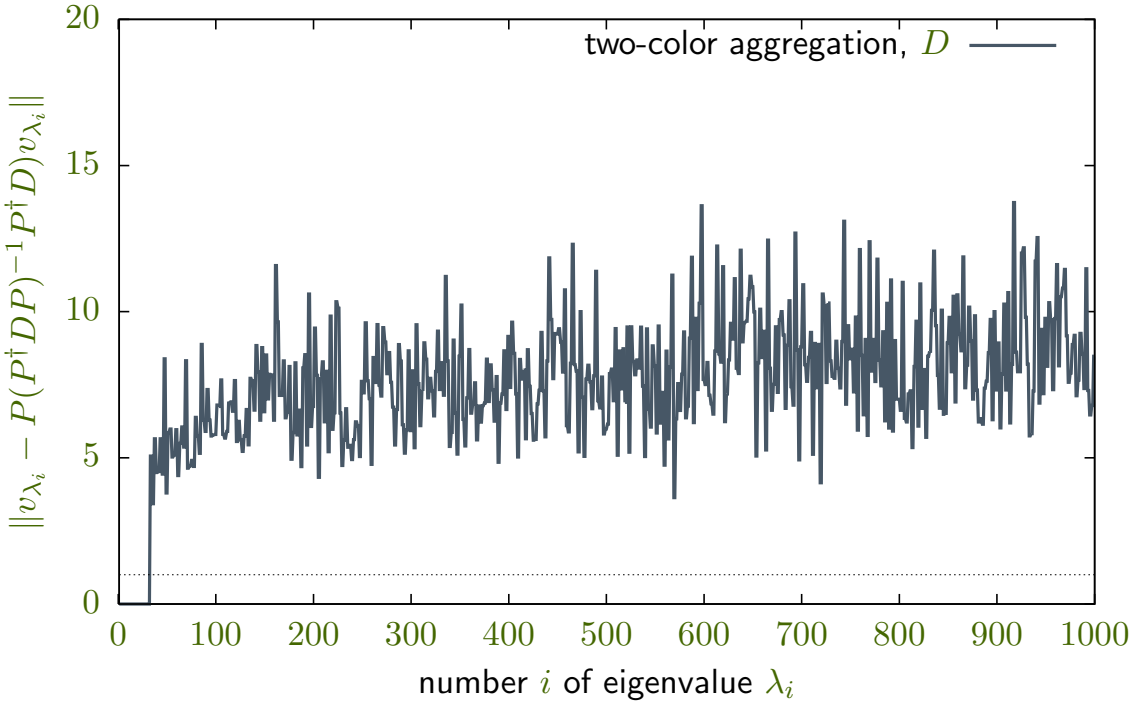
$$\|(1 - PP^\dagger)v_{\lambda_i}\|$$



- lattice size 12^4, unsmeared
- only slight local coherence of small EVs
- smearing causes only slight improvement

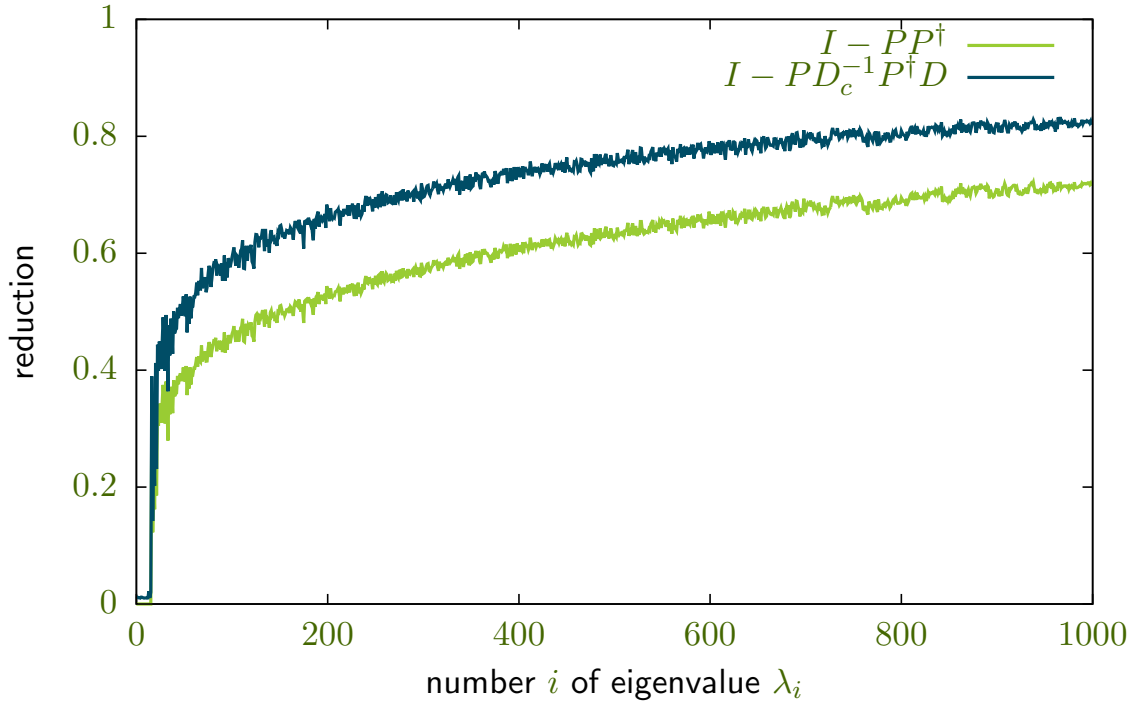
Numerical Tests

$$\|(1 - P(P^\dagger DP)^{-1}P^\dagger D)v_{\lambda_i}\|$$



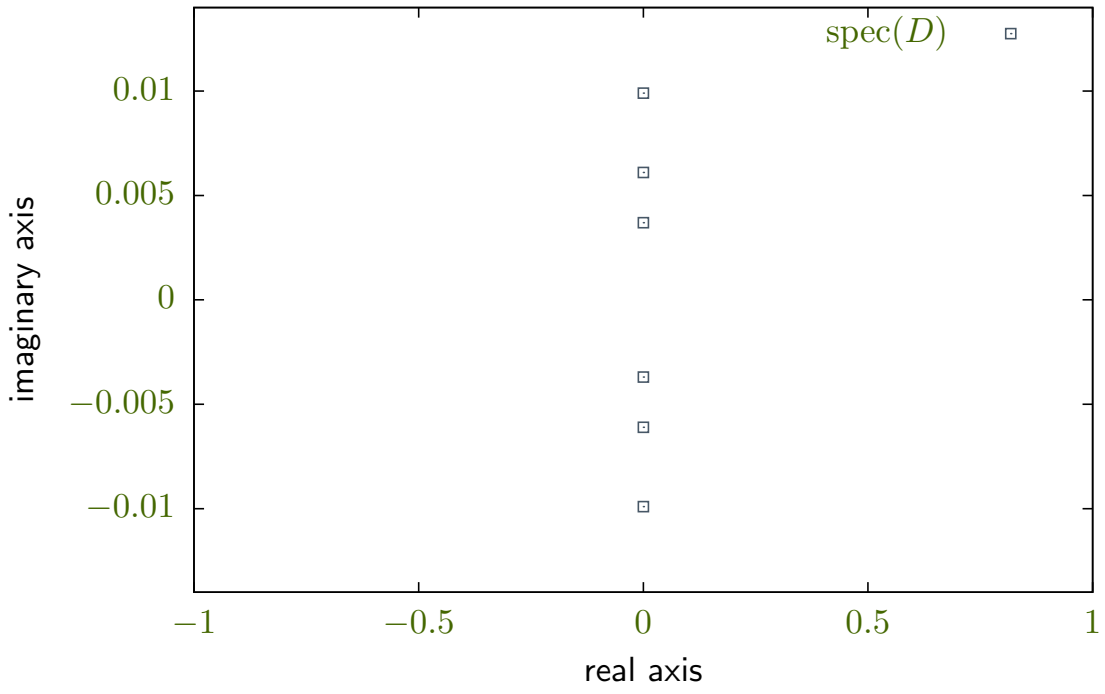
- coarse grid correction not able to make use of local coherence
- strongly amplifies all EVs

The Wilson case:



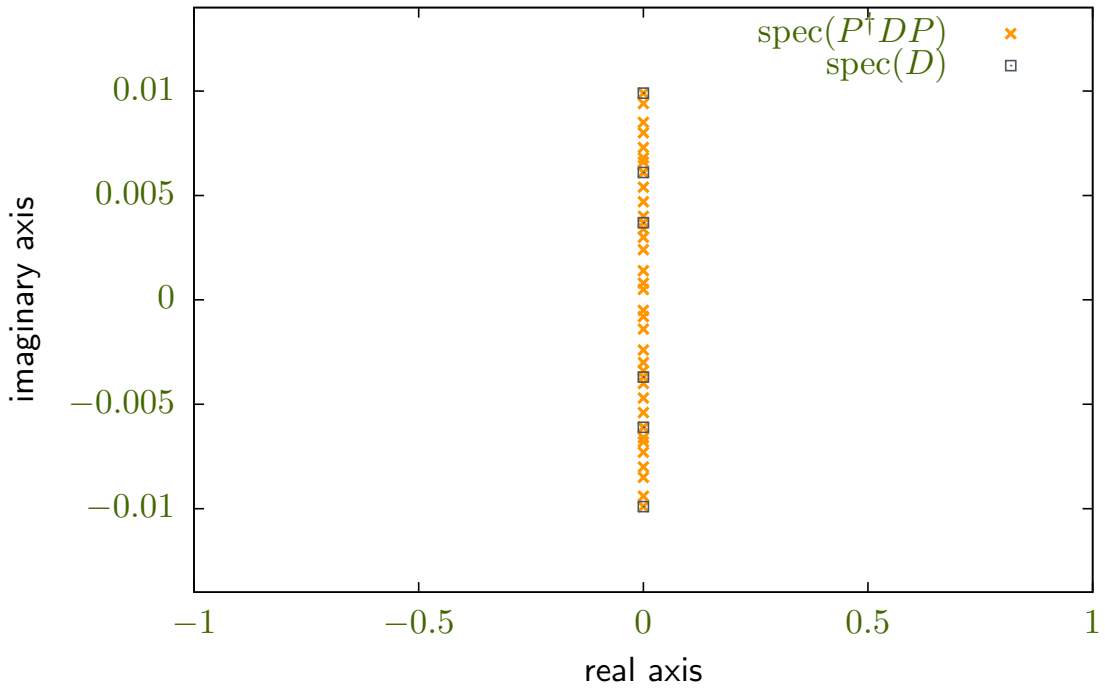
- more local coherence
- coarse grid correction is able to make use of it

Numerical Tests



- 6 smallest EVs of D

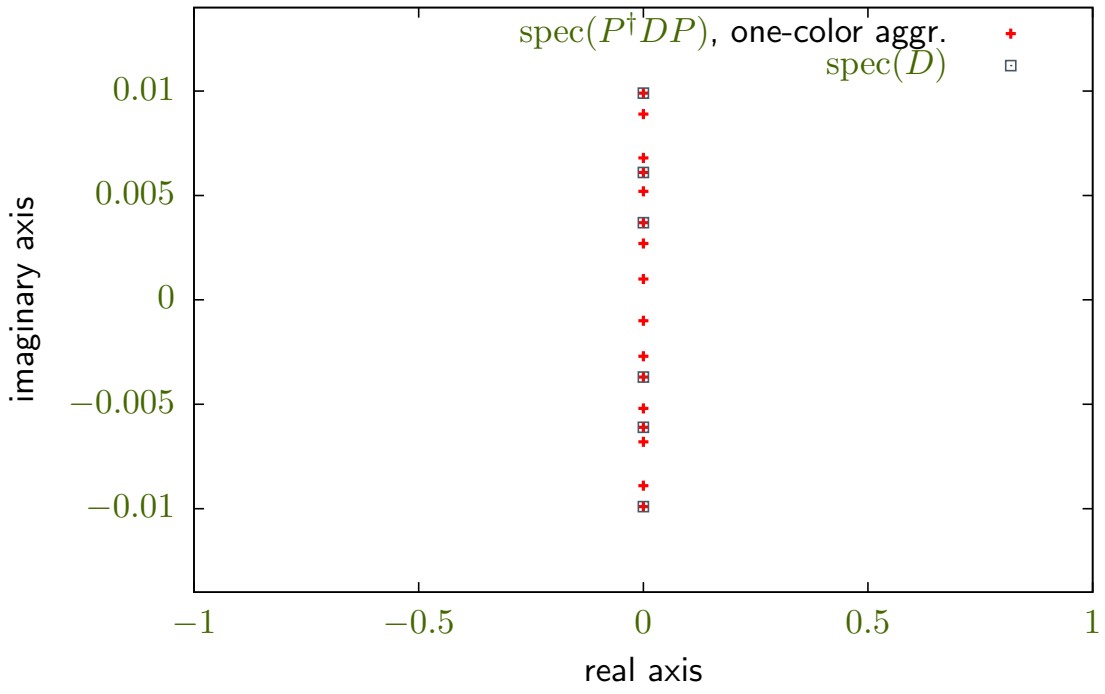
Numerical Tests



- smallest EVs of $P^\dagger DP$
- massive fill-in at small EVs
- positive and negative EVs of D seem to mix to artificial small EVs on coarse grid

  $\Longrightarrow$ maybe aggregate even and odd points together?

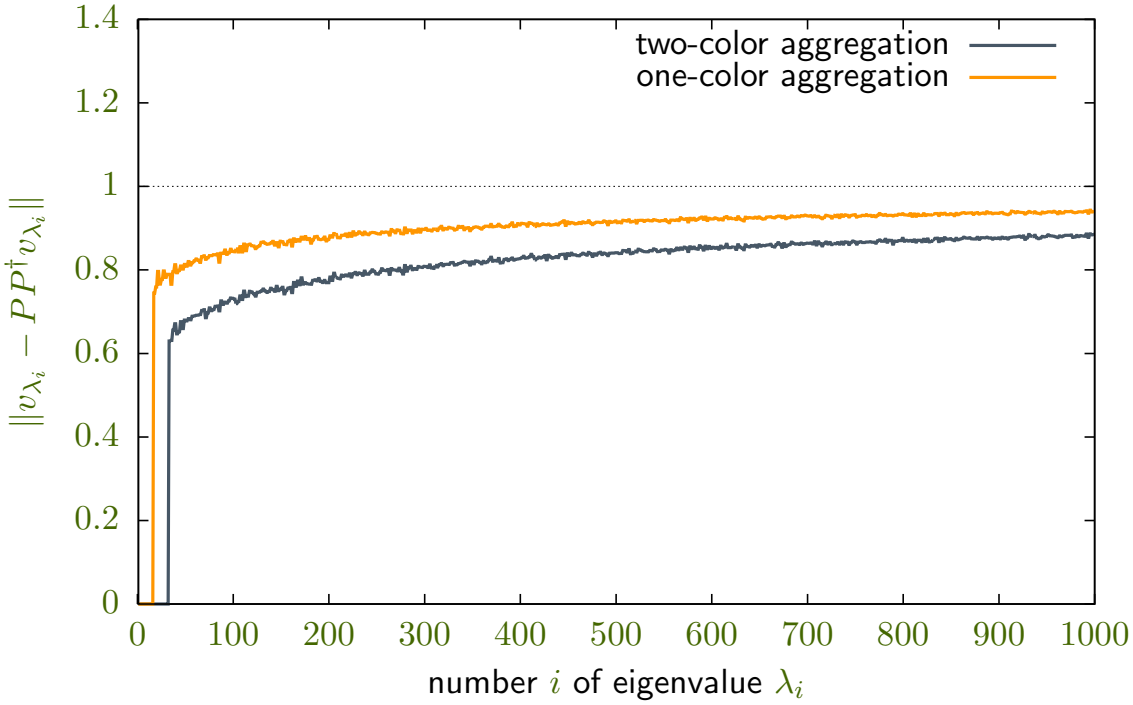
Numerical Tests



- less fill-in at small EVs
- but still fill-in

Numerical Tests

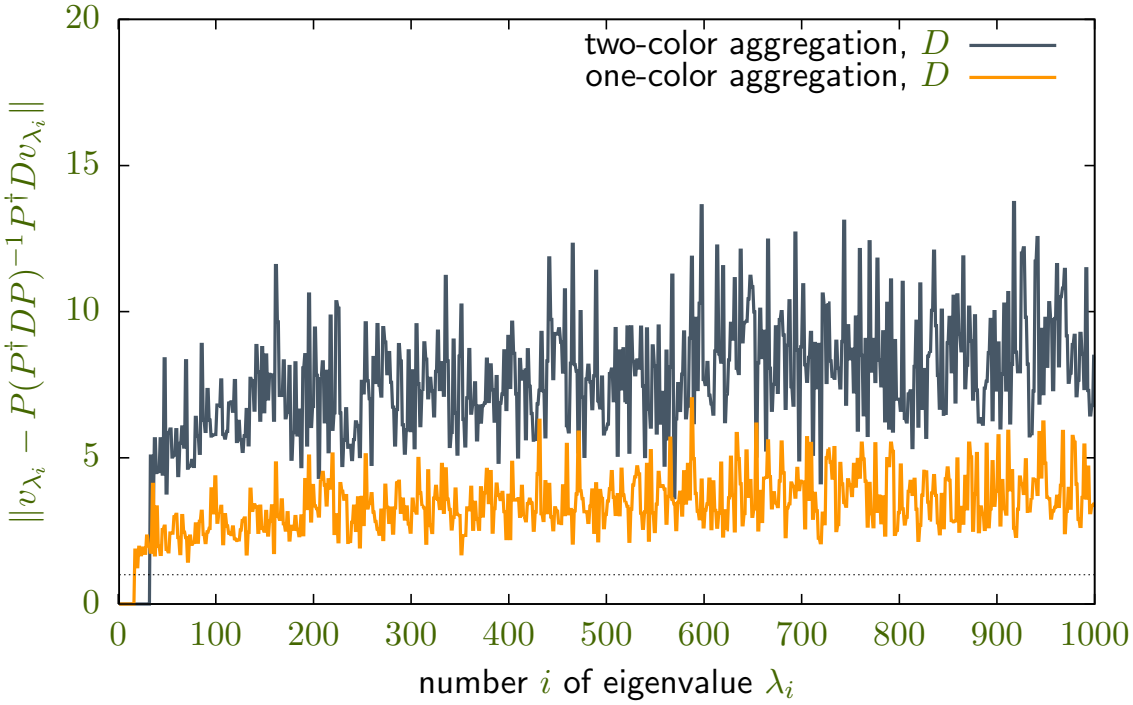
$$\|(1 - PP^\dagger)v_{\lambda_i}\|$$



- even less local coherence of small EVs

Numerical Tests

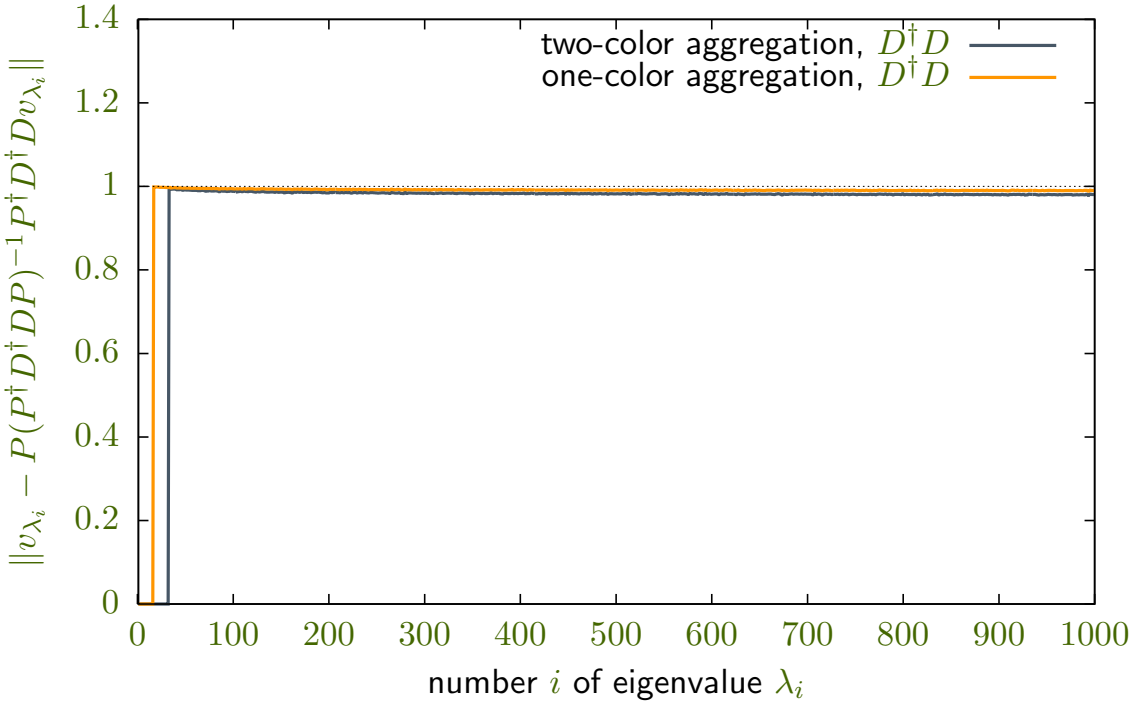
$$\|(1 - P(P^\dagger DP)^{-1}P^\dagger D)v_{\lambda_i}\|$$



- less amplification but still bad
  $\Longrightarrow$ let's try the normal equation $D^\dagger D\psi = D^\dagger \eta$

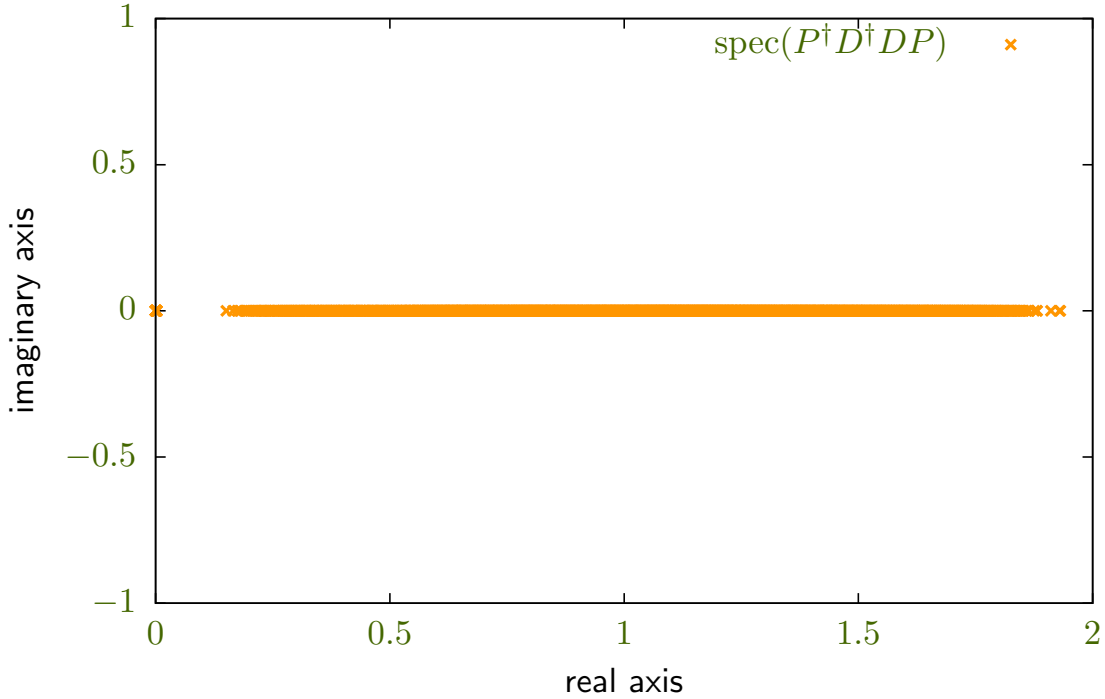
Numerical Tests

$$\|(1 - P(P^\dagger D^\dagger DP)^{-1} P^\dagger D^\dagger D) v_{\lambda_i}\|$$



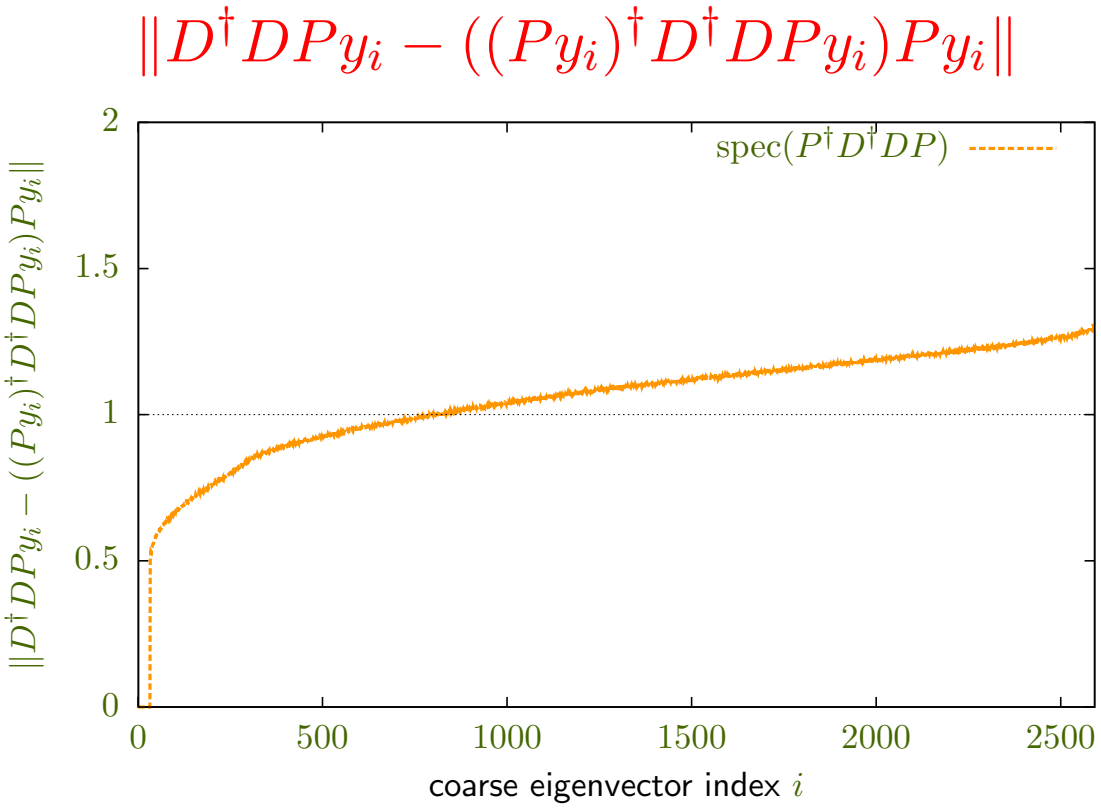
- does not make use of local coherence
- should perform just as good as explicit deflation

Numerical Tests



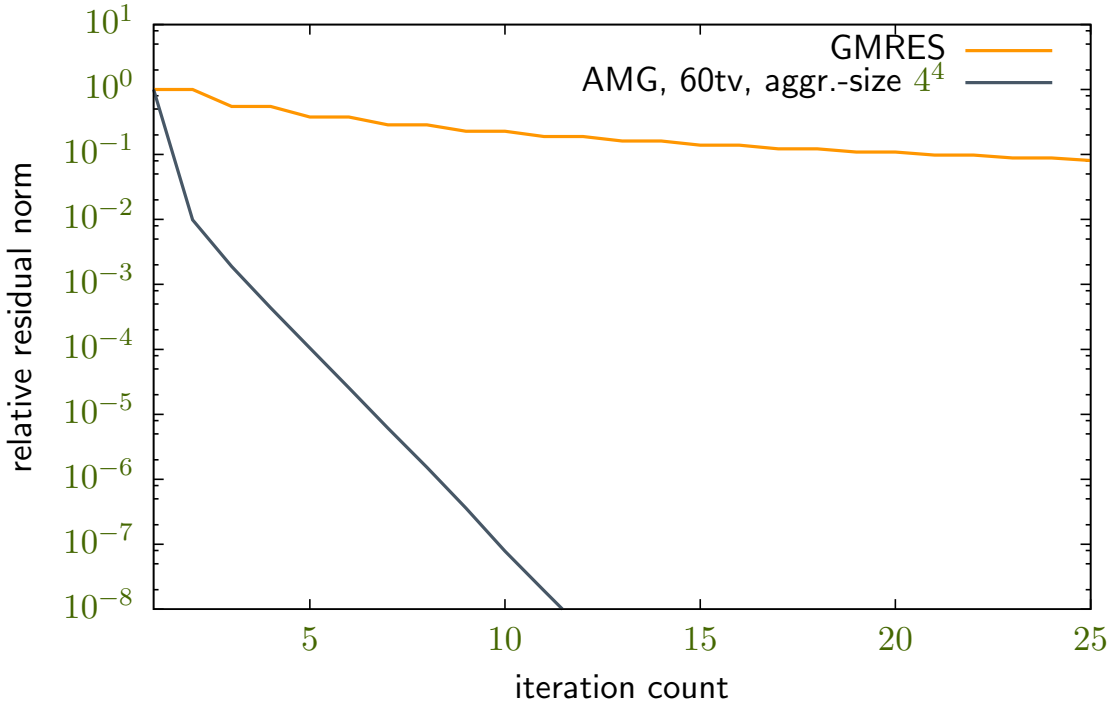
- 32 exact EVs, then a gap, then the rest

Numerical Tests



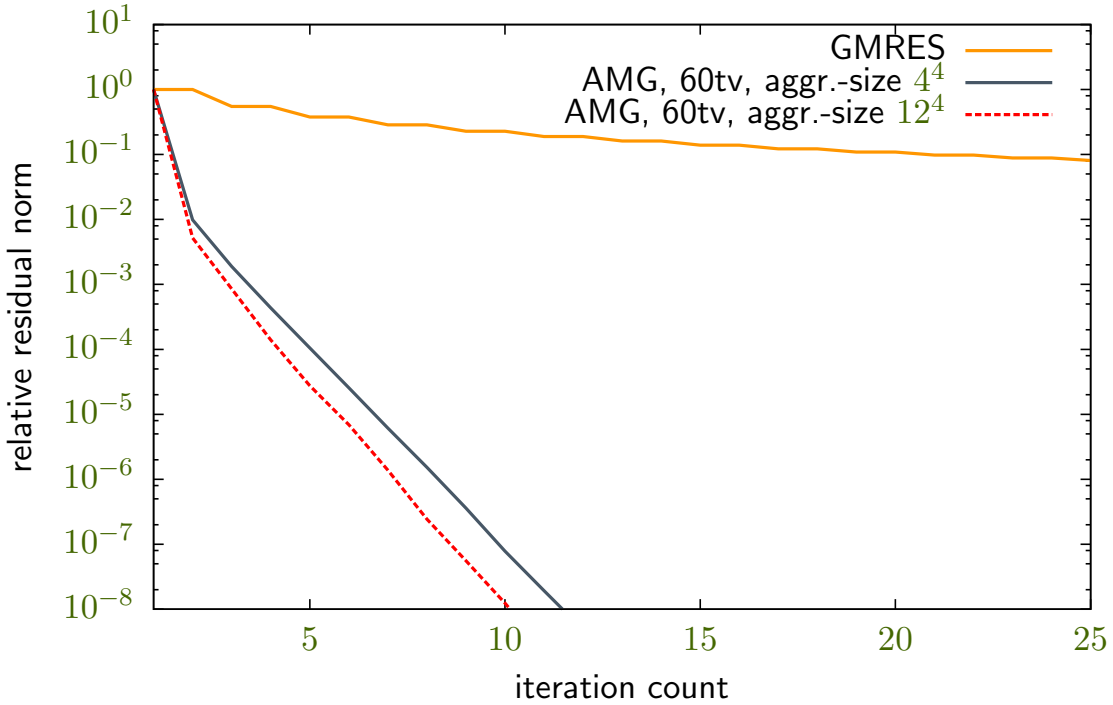
- EVs beyond the gap do not approximate any EVs of D

Numerical Tests



- AMG, P built with 60 smallest (positive) EVs, 4 iterations GMRES as smoother
- lattice size 12^4

Numerical Tests



- lattice size 12^4
- aggregate size $12^4 =$ 1 red & 1 black aggregate
  $\Longrightarrow$ performs about the same like aggregate size 4^4
- maybe we should preserve more structure of D on the coarse grid?

Phase-Respecting Aggregation

Idea:

build aggregation which preserves staggered phases

$$\alpha_\mu(x) = (-1)^{\sum_{\nu=1}^{\mu-1} x_\nu} = (-1)^{\sum_{\nu=1}^{\mu-1} x_\nu + 2y_\nu}$$

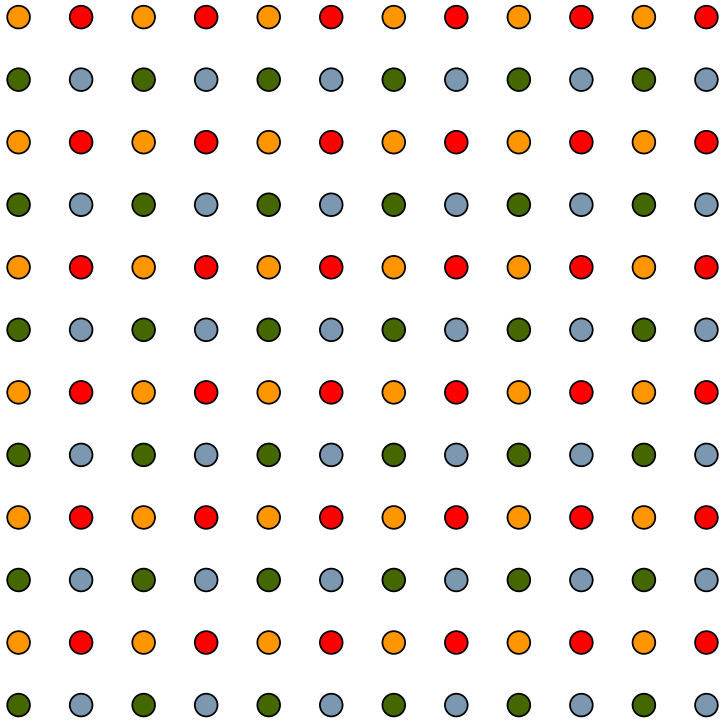
where $y_\nu \in \mathbb{Z}^4$, i.e., adding multiples of two to x_ν does not change staggered phase $\alpha_\mu(x)$

$\Longrightarrow$ 16 color aggregation

proposed by T. Kalkreuter, DESY preprint (1990)

16 Color Phase-Respecting Aggregation

2D illustration of 16 color aggregation which respects staggered phases and coupling structure



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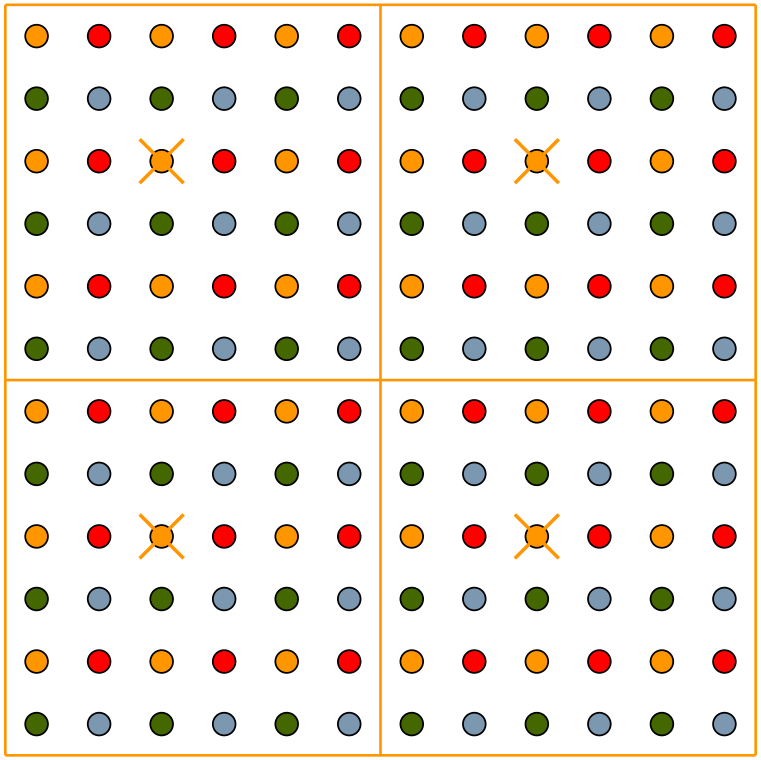

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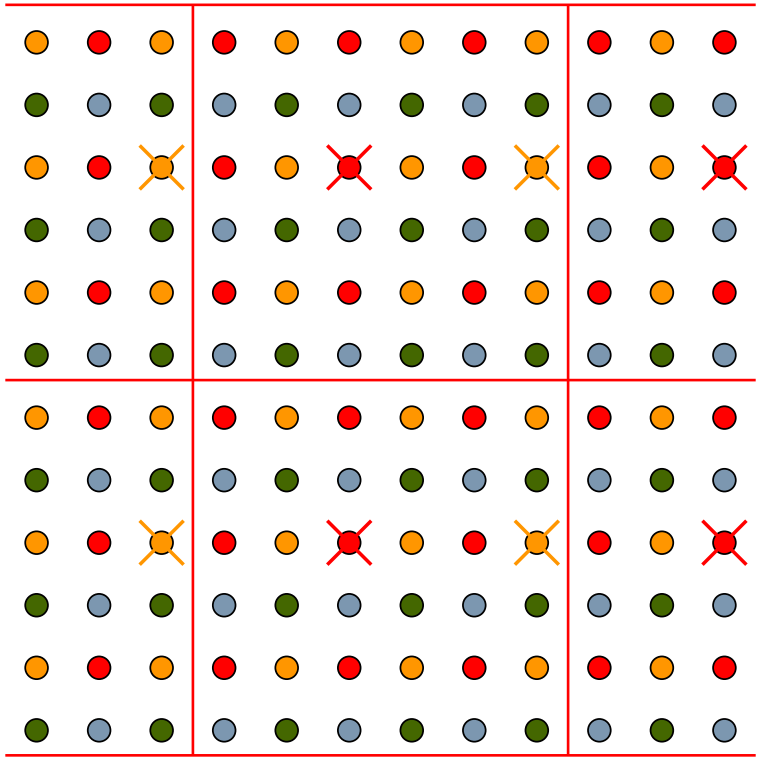

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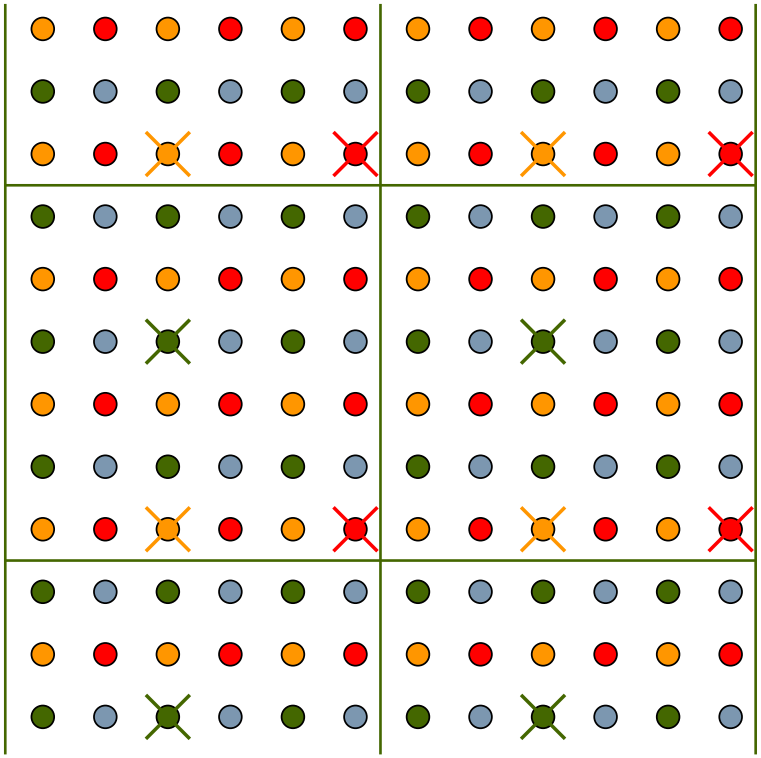

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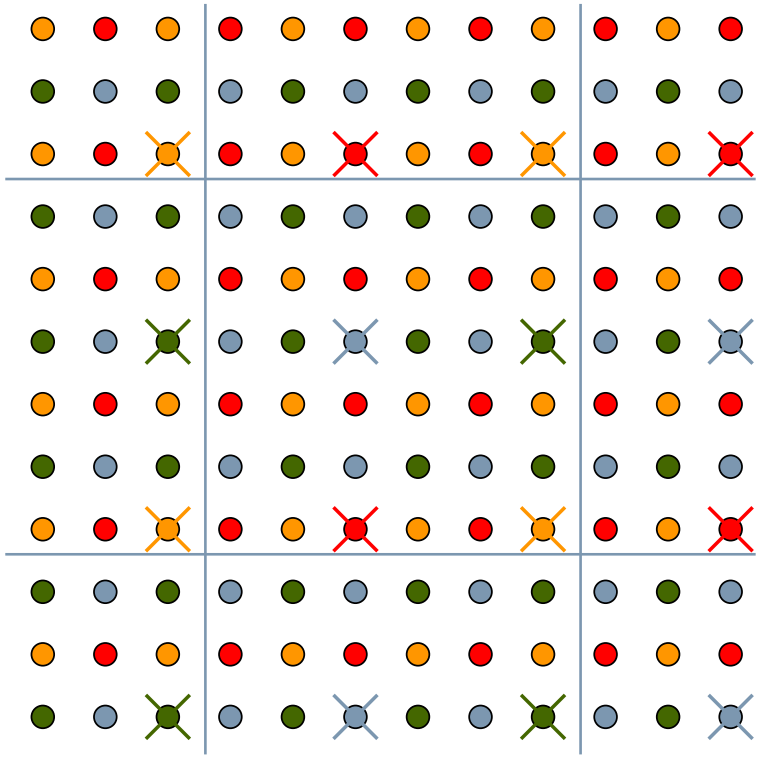

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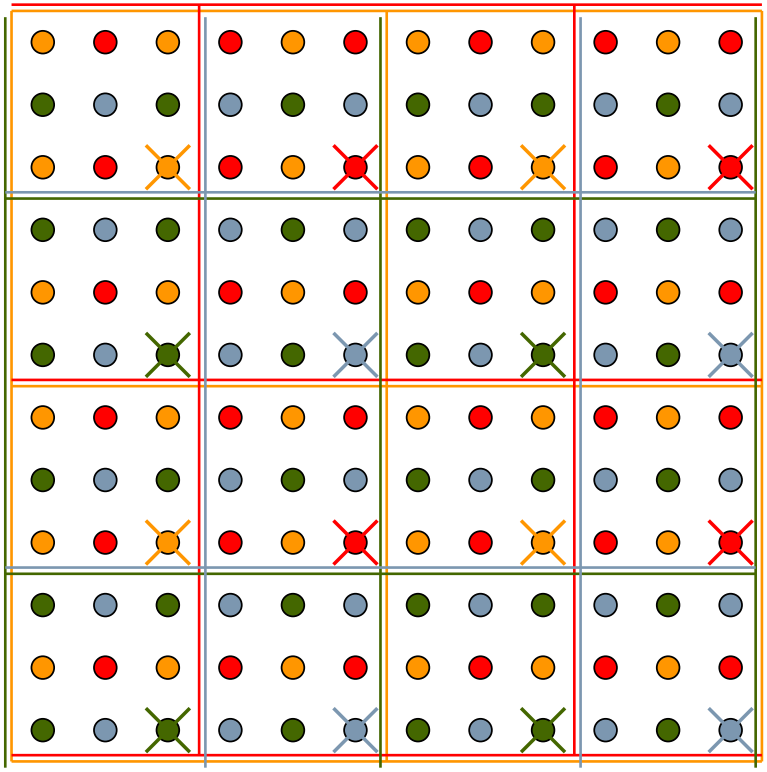

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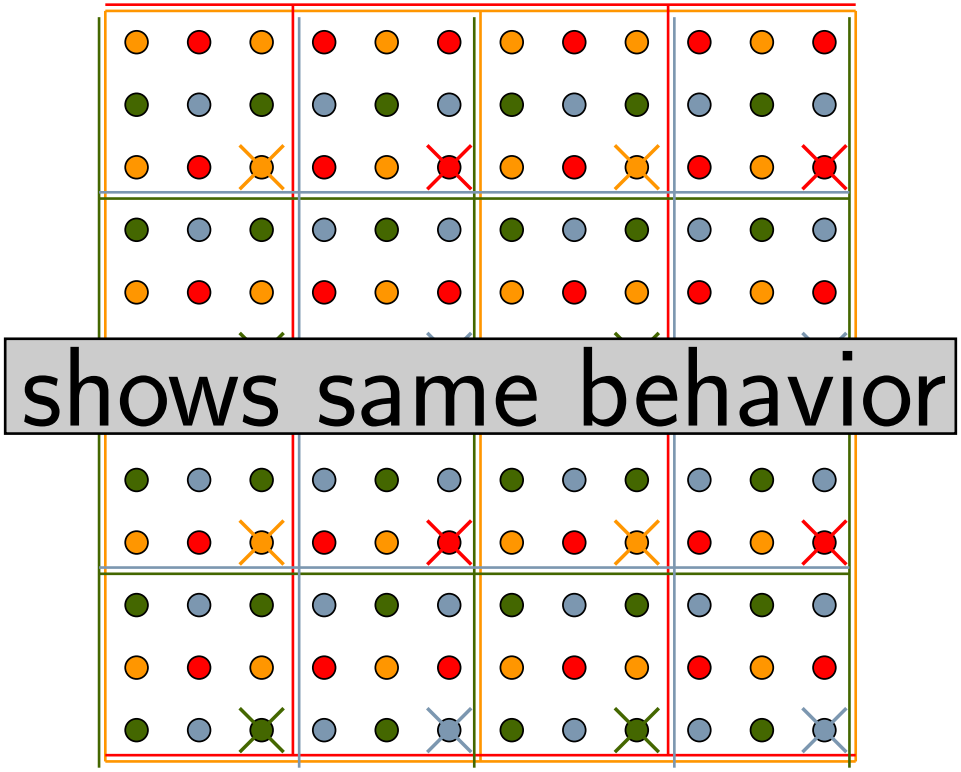

Other Approaches that did not work

- introduce second derivative stabilization term (Wilson like)
  - $A\psi(x) = \frac{1}{4}\sum_{\mu=1}^{4}(2\psi(x) - U_\mu(x)U_\mu(x+\hat{\mu})\psi(x+2\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})U_\mu^\dagger(x-2\hat{\mu})\psi(x-2\hat{\mu}))$
  - $A+D$ shows improved local coherence of small eigenvectors $\implies$ use $A+D$ as preconditioner for D (auxiliary space type preconditioner, cf. overlap 2014)
  - fails; small eigenvectors of $A+D$ and D not similar enough
- different parameter sets
  - smaller aggregates and/or more test vectors
  - $I - PP^\dagger$ looks better but coarse grid corrections do not work
- SAP as smoother/preconditioner does not work

Questions

- does anyone observe same/different behavior?
- any ideas how to make it better?