The Anatomy of a Calculation of ε'

The Ninth International Workshop on Numerical Analysis and Lattice QFT (QCDNA)

August 2, 2016

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Outline

- Overview of CP violation and $K \rightarrow \pi\pi$ decay
- Lattice calculation of $K \rightarrow \pi\pi$:
  - Lellouch-Luscher method
  - Exploit boundary conditions
  - Choice of $\pi\pi$ operator
  - Non-perturbative renormalization
- Result for ε'
- Outlook

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CP violation and

$K \rightarrow \pi\,\pi$ decay

$K \rightarrow \pi\pi$ and CP violation

- Final $\pi\pi$ states can have $I = 0$ or 2.

$$\langle \pi\pi(I = 2)|H_w|K^0\rangle = A_2 e^{i\delta_2} \qquad \Delta I = 3/2$$
$$\langle \pi\pi(I = 0)|H_w|K^0\rangle = A_0 e^{i\delta_0} \qquad \Delta I = 1/2$$

- CP symmetry requires A_0 and A_2 be real.
- Direct CP violation in this decay is characterized by:

$$\epsilon' = \frac{i e^{\delta_2 - \delta_0}}{\sqrt{2}} \left| \frac{A_2}{A_0} \right| \left(\frac{\mathrm{Im} A_2}{\mathrm{Re} A_2} - \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} \right)$$

Direct CP violation

$K^0 - \overline{K}^0$ mixing

- ΔS=1 weak decays allow K^0 and $\overline{K}^0$ to decay to the same $\pi\pi$ state.
- Resulting mixing described by Wigner-Weisskopf:

$$i\frac{d}{dt}\begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix} = \left\{ \begin{pmatrix} M_{00} & M_{0\bar{0}} \\ M_{\bar{0}0} & M_{\bar{0}\bar{0}} \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{00} & \Gamma_{0\bar{0}} \\ \Gamma_{\bar{0}0} & \Gamma_{\bar{0}\bar{0}} \end{pmatrix} \right\} \begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix}$$

- Decaying states are mixtures of K^0 and $\overline{K}^0$

$$|K_S\rangle = \frac{K_+ + \overline{\epsilon} K_-}{\sqrt{1 + |\overline{\epsilon}|^2}} \qquad \overline{\epsilon} = \frac{i}{2} \left\{ \frac{\mathrm{Im} M_{0\bar{0}} - \frac{i}{2} \mathrm{Im} \Gamma_{0\bar{0}}}{\mathrm{Re} M_{0\bar{0}} - \frac{i}{2} \mathrm{Re} \Gamma_{0\bar{0}}} \right\}$$

$$|K_L\rangle = \frac{K_- + \overline{\epsilon} K_+}{\sqrt{1 + |\overline{\epsilon}|^2}}$$

Indirect CP violation

CP violation

- CP violating, experimental amplitudes:

$$\eta_{+-} \equiv \frac{\langle \pi^+\pi^-|H_w|K_L\rangle}{\langle \pi^+\pi^-|H_w|K_S\rangle} = \epsilon + \epsilon'$$

$$\eta_{00} \equiv \frac{\langle \pi^0\pi^0|H_w|K_L\rangle}{\langle \pi^0\pi^0|H_w|K_S\rangle} = \epsilon - 2\epsilon'$$

- Where: $\epsilon = \overline{\epsilon} + i\,\frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}$

Indirect: $|\varepsilon| = (2.228 \pm 0.011) \times 10^{-3}$

Direct: $\mathrm{Re}(\varepsilon'/\varepsilon) = (1.66 \pm 0.23) \times 10^{-3}$

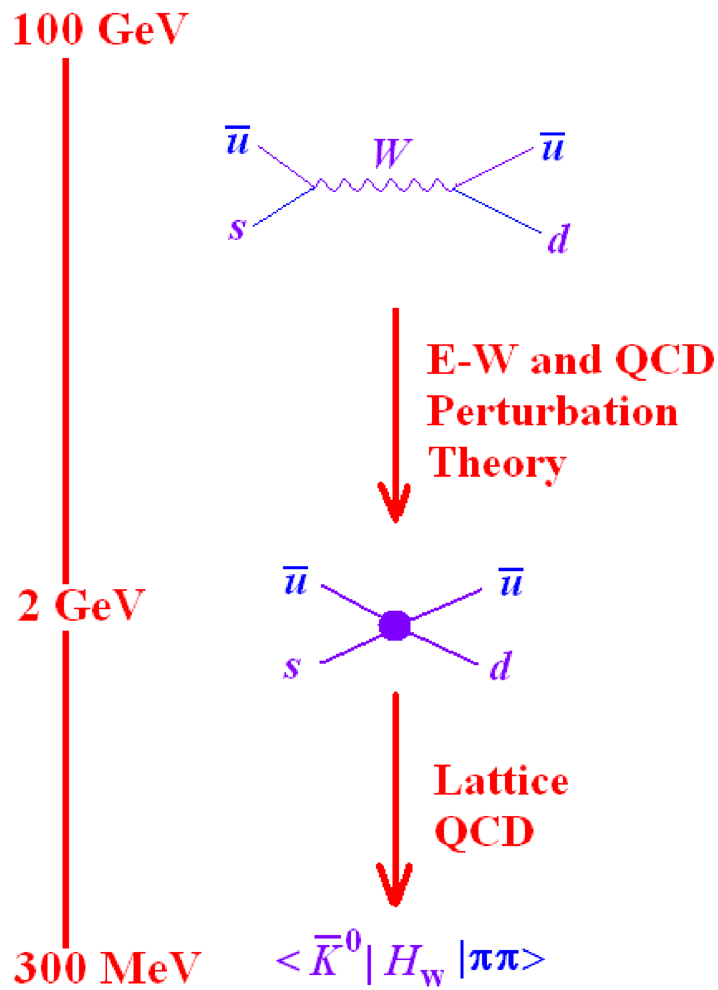
$K \to \pi\pi$ decay from lattice QCD

Low Energy Effective Theory

- Represent weak interactions by local four-quark Lagrangian

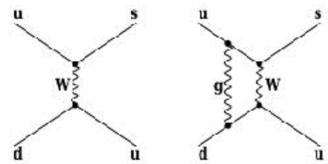
$$\mathcal{H}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ \sum_{i=1}^{10} \left[z_i(\mu) + \tau y_i(\mu) \right] Q_i \right\}$$

- $\tau = -\dfrac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*} = (1.543 + 0.635i) \times 10^{-3}$
- $V_{qq'}$ – CKM matrix elements
- z_i and y_i – Wilson Coefficients
- Q_i – four-quark operators



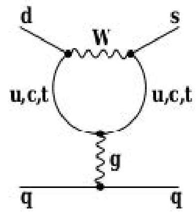
Four quark operators

- **Current-current**



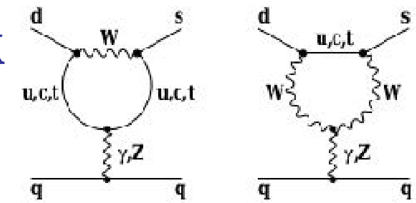
$$Q_1 \equiv (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}$$
$$Q_2 \equiv (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A}$$

- **QCD Penguins**



$$Q_3 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_4 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V-A}$$
$$Q_5 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_6 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V+A}$$

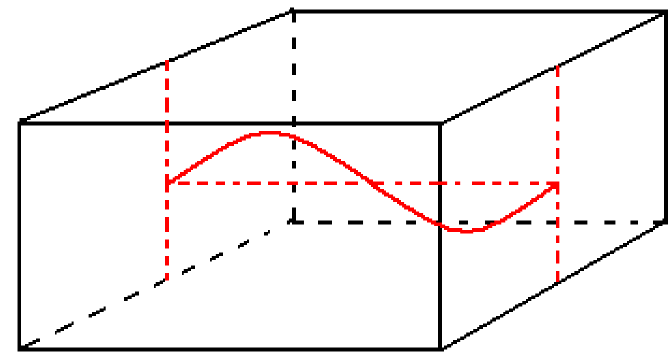
- **Electro-Weak Penguins**



$$Q_7 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_8 \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V+A}$$
$$Q_9 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_{10} \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V-A}$$

Physical $\pi\pi$ states – Lellouch-Luscher

- Euclidean e^{-Ht} projects onto $|\pi\pi(\vec{p}=0)\rangle$
- Use finite-volume quantization.
- Adjust volume so 1st or 2nd excited state has correct *p.*



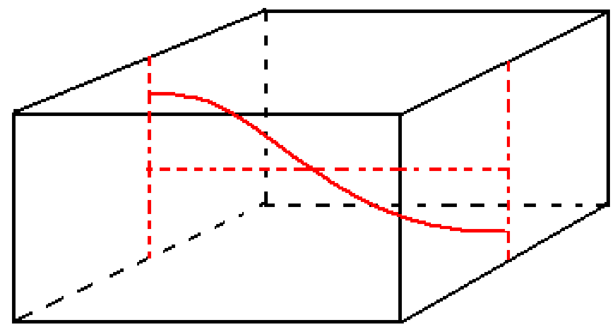
- Correctly include $\pi-\pi$ interactions, including leading $1/L^3$ effects of finite volume.
- Requires extracting signal from non-leading large-***t*** behavior:

$$G(t) \sim c_0 e^{-E_0 t} + c_1 e^{-E_1 t}$$

Exploit boundary conditions

- Remove $\pi\pi$ states with $E_{\pi\pi} < M_K$ by imposing anti-periodic boundary conditions:

  $$2\sqrt{3\left(\frac{\pi}{L}\right)^2 + M_\pi^2} = M_K \quad \rightarrow L = 5.2 \text{ fm}$$



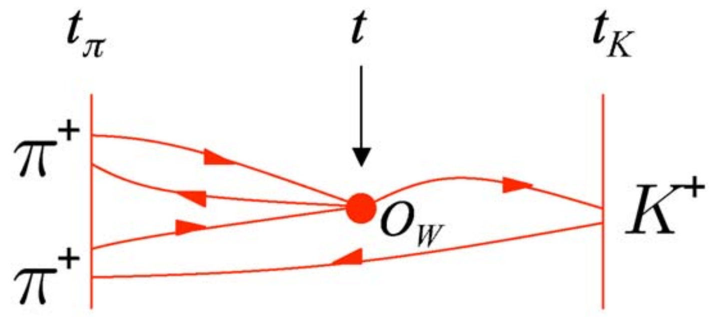
- $I = 2$, Repulsive, $L \rightarrow 5.7$ fm
  - Work with $\pi^+\pi^+$ state, impose anti-periodic BC on d quark
  - $|\pi^+\pi^+>$ unique, charge 2 state, does not mix

- $I = 0$, Attractive, $L \rightarrow 4.5$ fm
  - Must distinguish $I = 0$ state: $|\pi^+\pi^-> -2\,|\pi^0\pi^0> + |\pi^-\pi^+>$
  - Impose G-parity BC, $G = C\,e^{i\pi I_y}$; $[G, \vec{I}\,] = 0$

$$\Delta I = 3/2$$

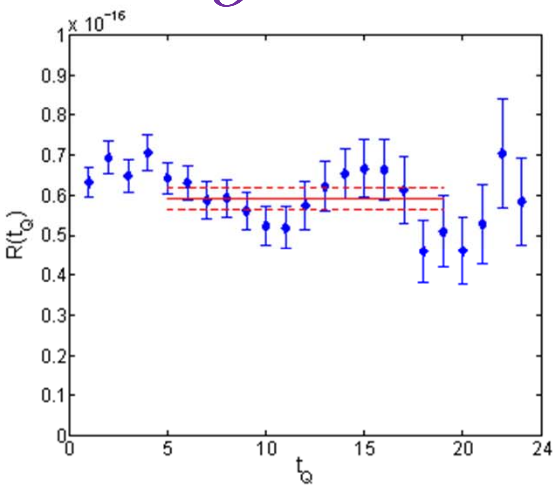
Δ I = 3/2 $K \rightarrow \pi\pi$

- Three operators contribute $O^{(27,1)}$, $O^{(8,8)}$ and $O^{(8,8)m}$.
- Calculated three times:
  - 32^3 x 64, DSDR 1/a=1.38 GeV
  - 48^3 x 96, Iwasaki, 1/a=1.73 GeV
  - 64^3 x 128 Iwasaki, 1/a=2.28 GeV

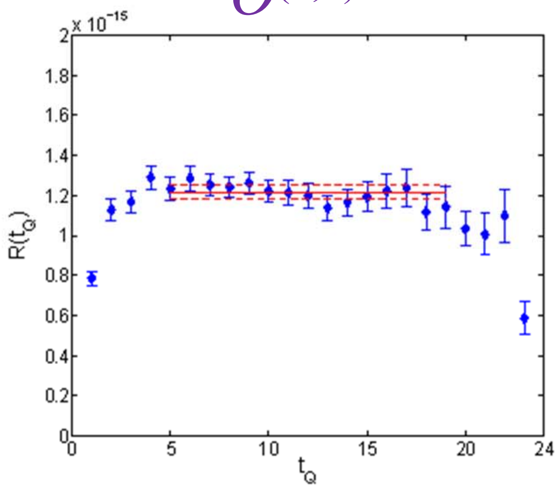


$$\frac{C^i_{K\pi\pi}(t)}{C_K(t_K - t)C_{\pi\pi}(t)} = \frac{\mathcal{M}_i}{Z_K Z_{\pi\pi}}$$

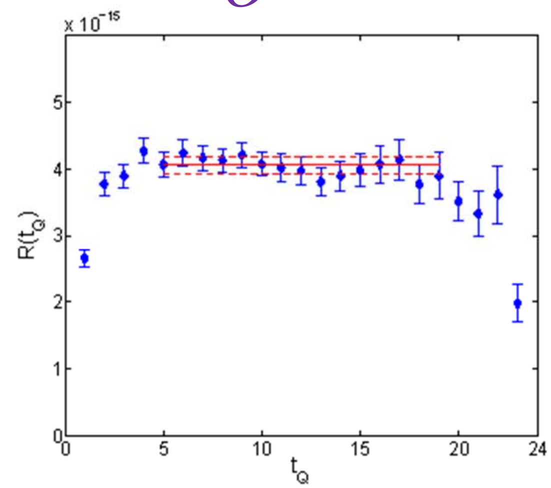
$O^{(27,1)}$



$O^{(8,8)}$

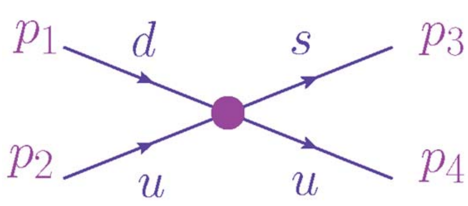


$O^{(8,8)m}$



Operator Normalization
(Rome-Southampton)

- Effective weak Hamiltonian H_W contains four-quark operators normalized in the $\overline{\mathrm{MS}}$ scheme.
- Impose non-perturbative RI scheme on lattice operators:
  - Evaluate Landau-gauge, off-shell Green's functions:



$$\Big(\Gamma(p_1, p_2, p_3, p_4)_j\Big)^{\alpha\beta\gamma\delta}_{abcd} = \prod_{i=1}^{4}\Big(\int d^4x_i e^{ip_i\cdot x_i}\Big)\Big\langle \overline{q}^{\alpha}_a(x_1)\overline{q}^{\beta}_b(x_2) O_j q^{\delta}_c(x_3) q^{\gamma}_d(x_4)\Big\rangle$$

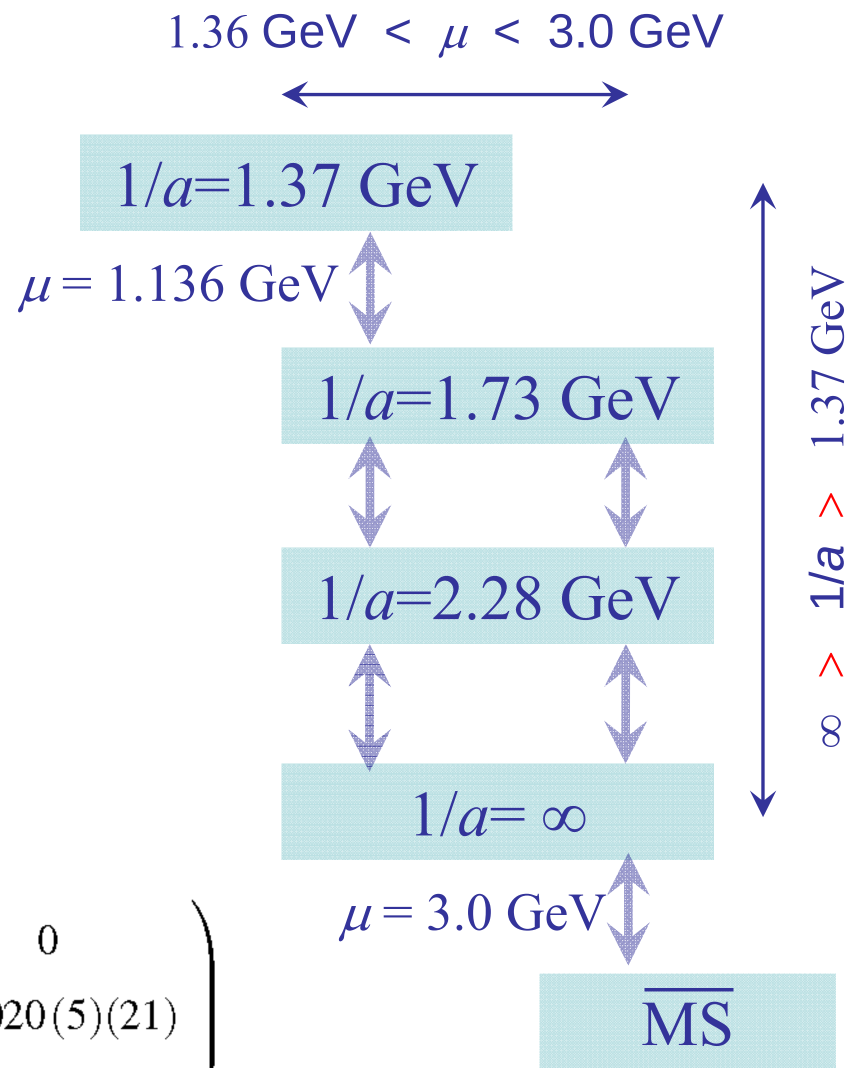
  - Impose normalization conditions: $\mathrm{tr}\{P_i\Gamma_j\} = F_{ij}$
- Use continuum perturbation theory to convert RI to $\overline{\mathrm{MS}}$

Operator Normalization
(Refinements)

- Use chiral fermions (DWF): good short-distance chiral symmetry controls operator mixing (Mobius with L_s=12)
- Impose normalization conditions $\mathrm{tr}\{P_i \Gamma_j\} = F_{ij}$ at infrared-safe, non-exceptional momenta, at a large, Euclidean energy scale μ.
- Use a series of finer lattice ensembles to non-perturbatively run μ up to 3 GeV (or higher) before converting RI to $\overline{\mathrm{MS}}$.
- Use twisted boundary conditions to allow matching between ensembles at equal physical momenta without varying momentum direction – freeze ~~O(4)~~ a^2 artifacts.

Relate lattice and continuum operators

- Normalize off-shell, gauge-fixed 4-quark Greens functions.
- Calculation is performed on $1/a$=1.37 GeV lattice.
- Converting to perturbative $\overline{\mathrm{MS}}$ scheme is unreliable at scale $\mu \sim 1/a$!
- Carry out sequence of NP RI matching steps:

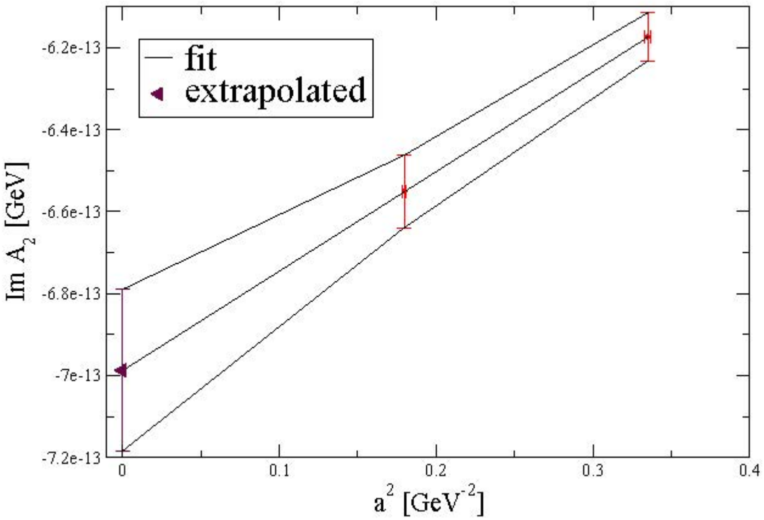


$$Z_{(\not{q},\not{q})}^{\overline{\mathrm{MS}},(\mathrm{latt})}(\mu) = \begin{pmatrix} 0.424(4)(4) & 0 & 0 \\ 0 & 0.472(6)(8) & -0.020(5)(21) \\ 0 & -0.067(23)(30) & 0.572(28)(20) \end{pmatrix}$$

Δ I = 3/2 – Continuum Results

(Tadeusz Janowski)

- Use two new large ensembles to remove a^2 error (m_π=135 MeV, L=5.4 fm)
  - 48^3 x 96, $1/a$=1.73 GeV
  - 64^3 x 128, $1/a$=2.28 GeV

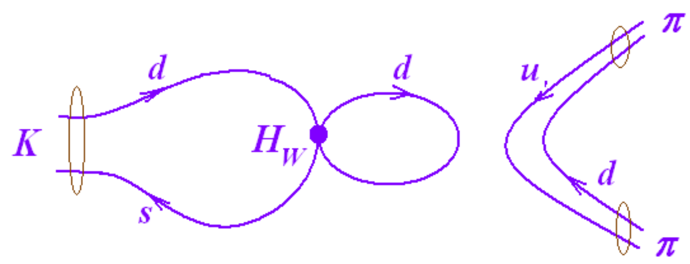


- Continuum results:
  - $\mathrm{Re}(A_2) = 1.50(0.04_{\mathrm{stat}})\,(0.14_{\mathrm{syst}})\times10^{-8}$ GeV
  - $\mathrm{Im}(A_2) = -\,6.99(0.20)_{\mathrm{stat}}\,(0.84)_{\mathrm{syst}}\times10^{-13}$ GeV
- Experiment: $\mathrm{Re}(A_2) = 1.479(4)\ 10^{-8}$ GeV
- $E_{\pi\pi} \rightarrow \delta_2 = -11.6(2.5)(1.2)^{\circ}$
- Phys.Rev. **D91**, 074502 (2015)

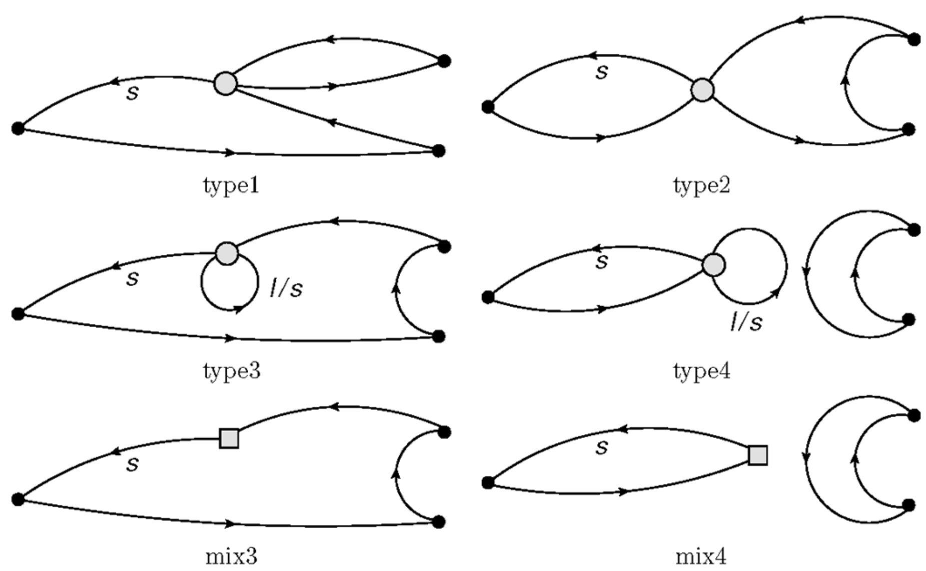
$$\Delta I = 1/2$$

$\Delta I = 1/2 \;\; K \rightarrow \pi\pi$

- Made much more difficult by disconnected diagrams:

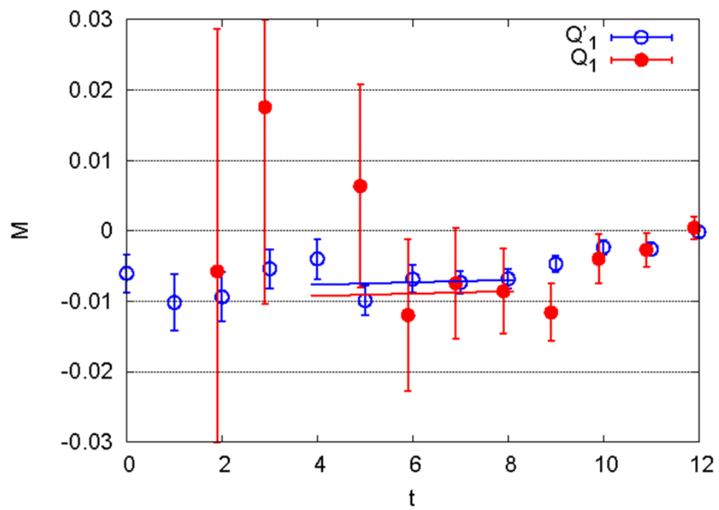


- Many more diagrams (48) than $\Delta I = 3/2$:

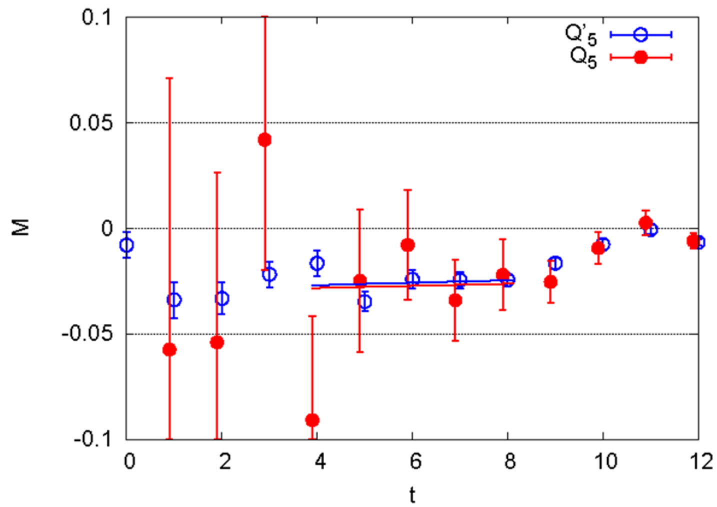


$\Delta I = 1/2$ $K \to \pi\pi$ at threshold
(Qi Liu)

- Initial threshold decay calculation successful
  - Re (A_0): 25% statistical errors
  - Im (A_0): 50% statistical errors



Q2 - largest part of Re(A_0)

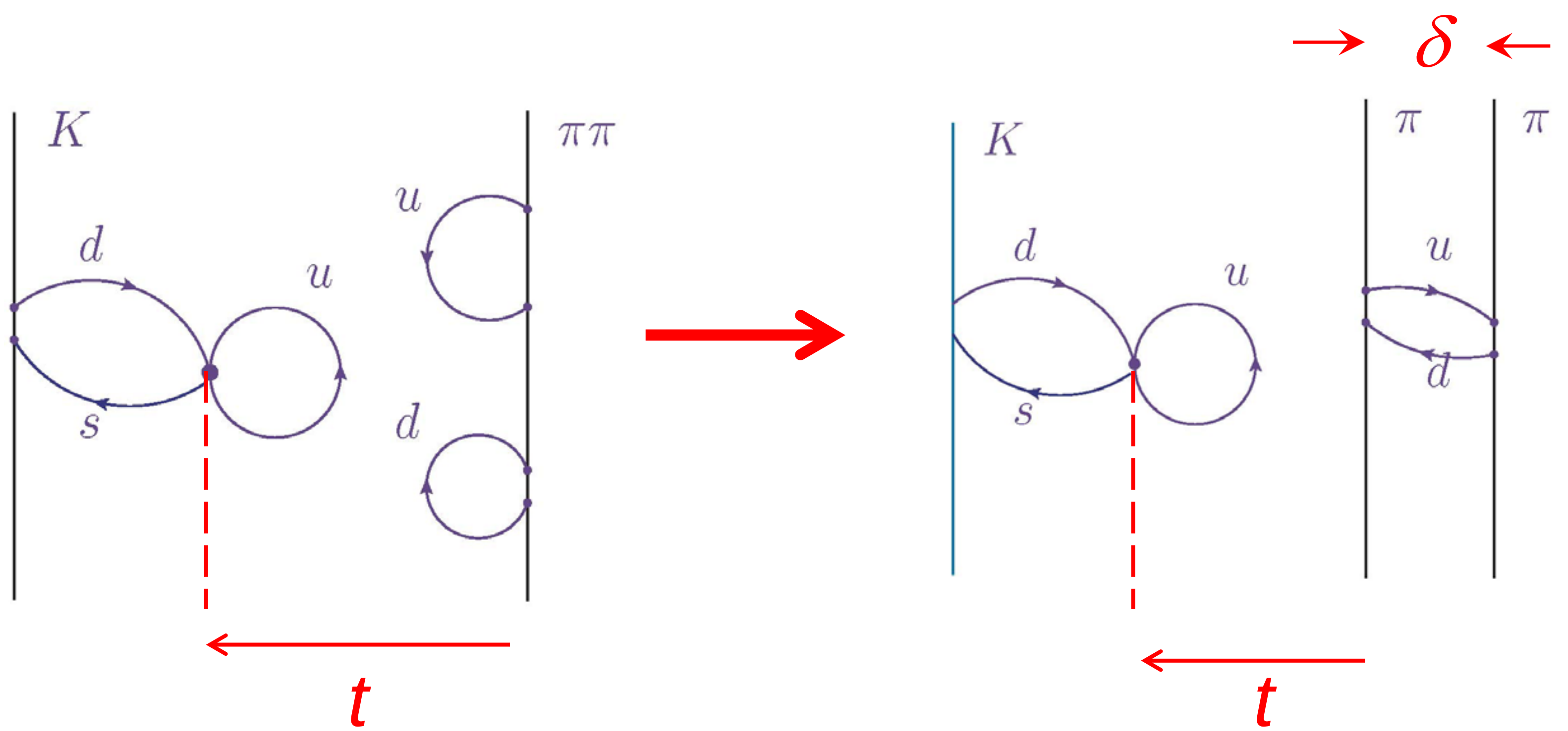


Q6 - largest part of Im(A_0)

$\Delta I = ½$ $K \rightarrow \pi\pi$ – suppress vacuum

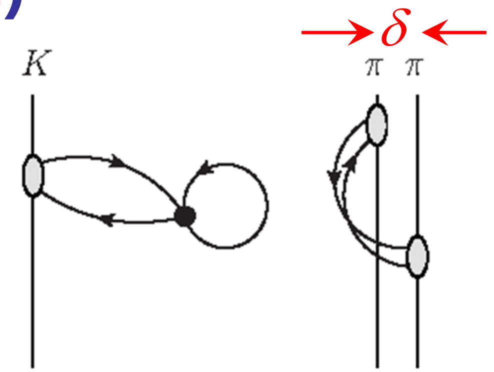
(Qi Liu & Daiqian Zhang)

- Separate two pion operators in time.

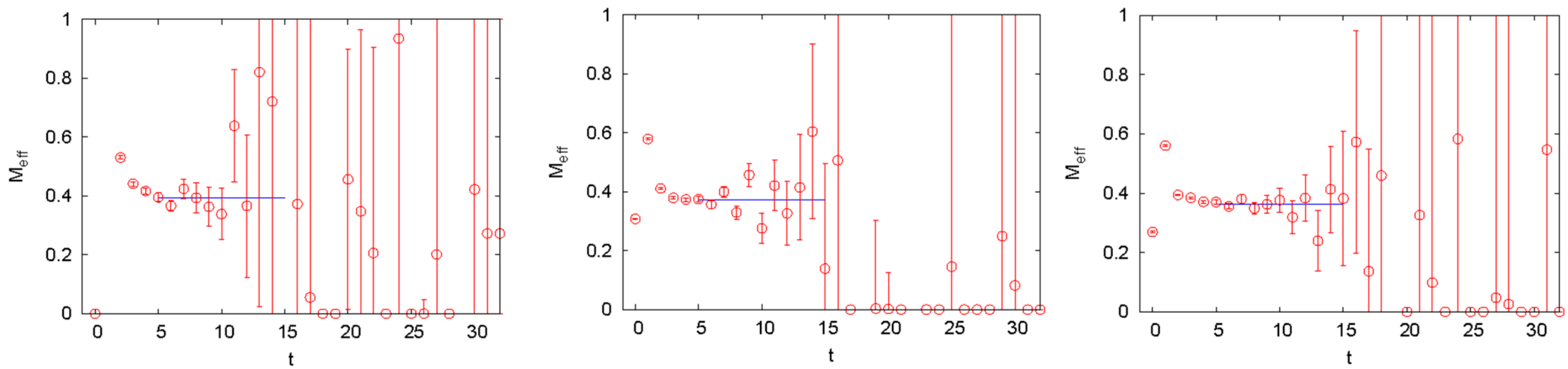


$\Delta I = ½$ $K \to \pi\pi$ – suppress vacuum

(Qi Liu & Daiqian Zhang)



- Obtain 2x decrease in errors



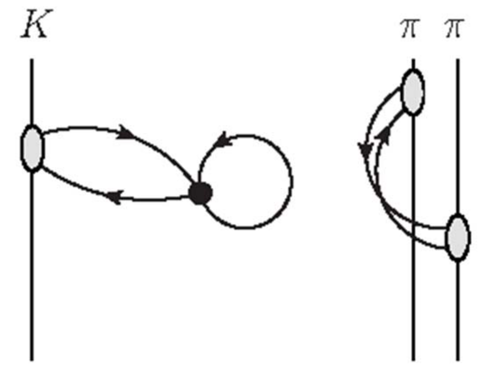
$$\delta = 0 \qquad \delta = 2 \qquad \delta = 4$$

$$m_{\pi\pi} = 0.3922(126) \qquad m_{\pi\pi} = 0.3720(62) \qquad m_{\pi\pi} = 0.3639(55)$$

$\Delta I = ½$ $K \rightarrow \pi\pi$ – suppress vacuum

(Qi Liu & Daiqian Zhang)

- Use **all-2-all** propagators (Trinity/KEK)
  - Use localized sources – further suppress vacuum coupling
  - Sum over source location to fix momentum



$$\langle q(x)\bar{q}(y)\rangle = \langle x|\frac{1}{D_{\mathrm{DWF}}}|y\rangle = \sum_{n=1}^{N_{\mathrm{modes}}} \phi_n(x)\frac{1}{\lambda_n}\phi_n(y)^\dagger + \sum_{k=1}^{N_{\mathrm{noise}}} \langle x|\frac{1}{D}\left(I - P_{n\le N_{\mathrm{modes}}}\right)|\eta_k\rangle\eta_k(y)^\dagger$$

$$= \sum_{l=1}^{N_{\mathrm{modes}}+N_{\mathrm{noise}}} v_l(x)w_l(y)^\dagger$$

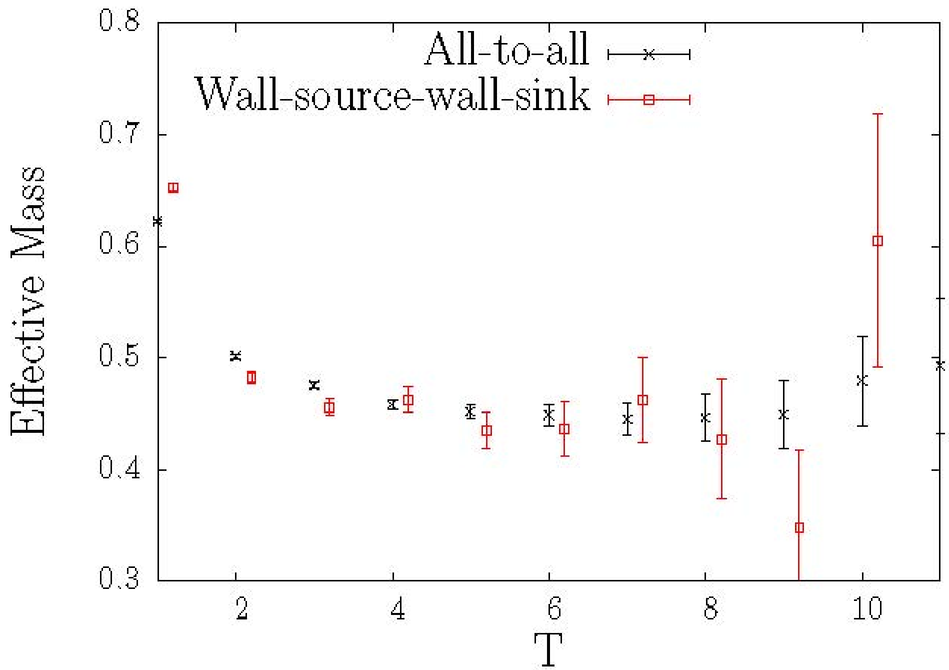
$$\pi_{\mathrm{op}}^{(lj)} = \int d^3x d^3y \; v_l(\vec{y},t)^\dagger \psi_\pi(\vec{x},\vec{y}) w_j(\vec{x},t)$$

$$\psi_\pi(\vec{x},\vec{y}) = e^{i\vec{p}_{q_1}\cdot\vec{x}} e^{-|\vec{x}-\vec{y}|/r} e^{i\vec{p}_{q_2}\cdot\vec{y}}$$

$\Delta I = ½$ $K \rightarrow \pi\pi$ – suppress vacuum
(Qi Liu & Daiqian Zhang)

- See more than 2x reduction in statistical error:

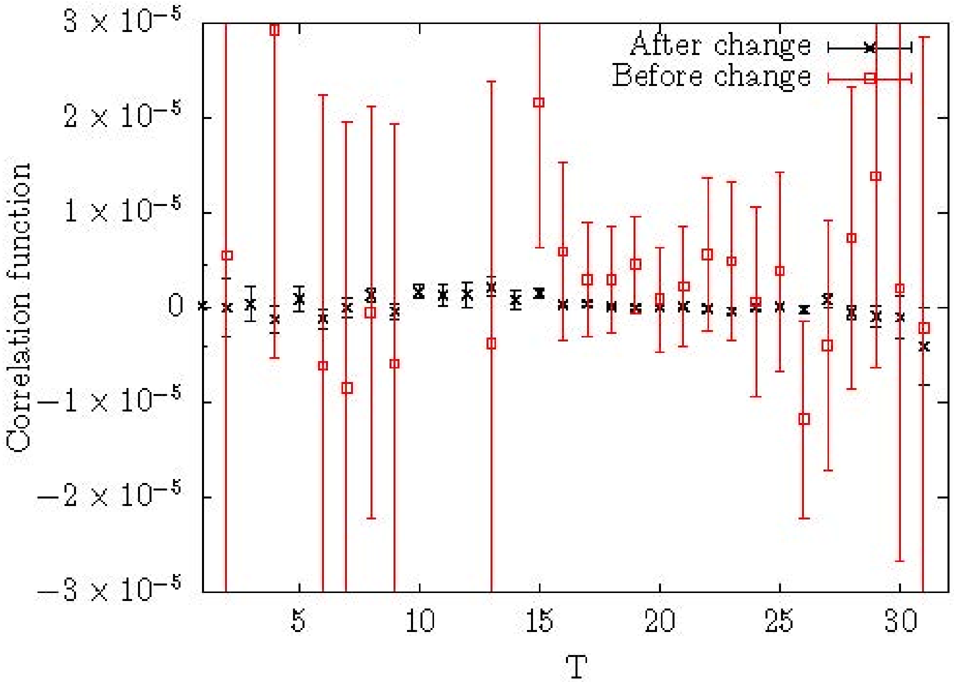
	$E_{\pi\pi}^{I=0}$
All-to-all	0.449(10)
Wall-source-wall-sink	0.438(25)

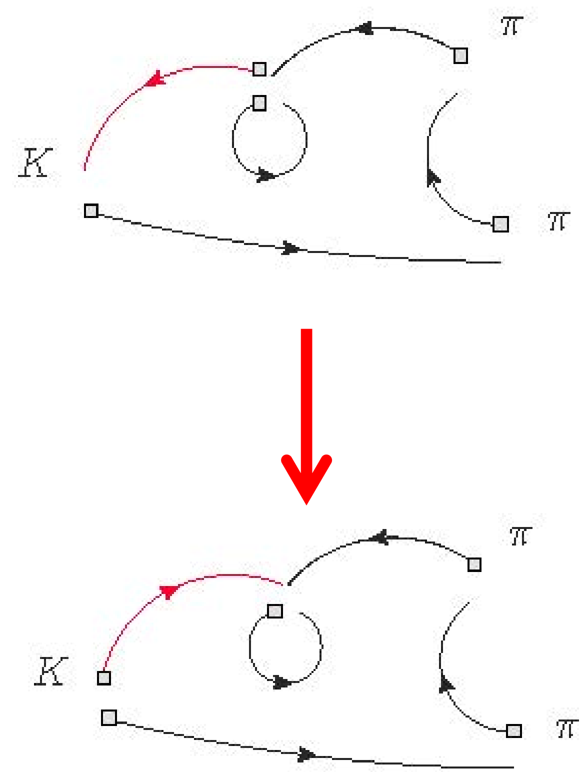


$16^3 \times 32, 1/a=1.73$ GeV

$\Delta I = ½\ K \rightarrow \pi\pi$ – A2A propagators

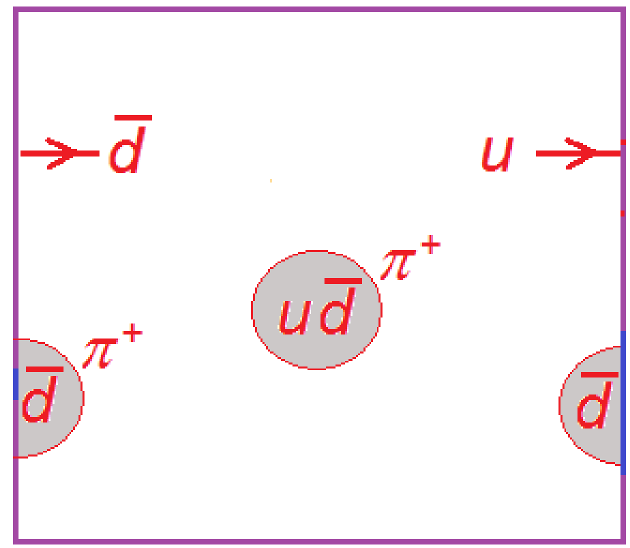
- Use γ^5 hermiticity to locate few A2A sources at H_W.
- Avoid (noise)^2 ~V^2





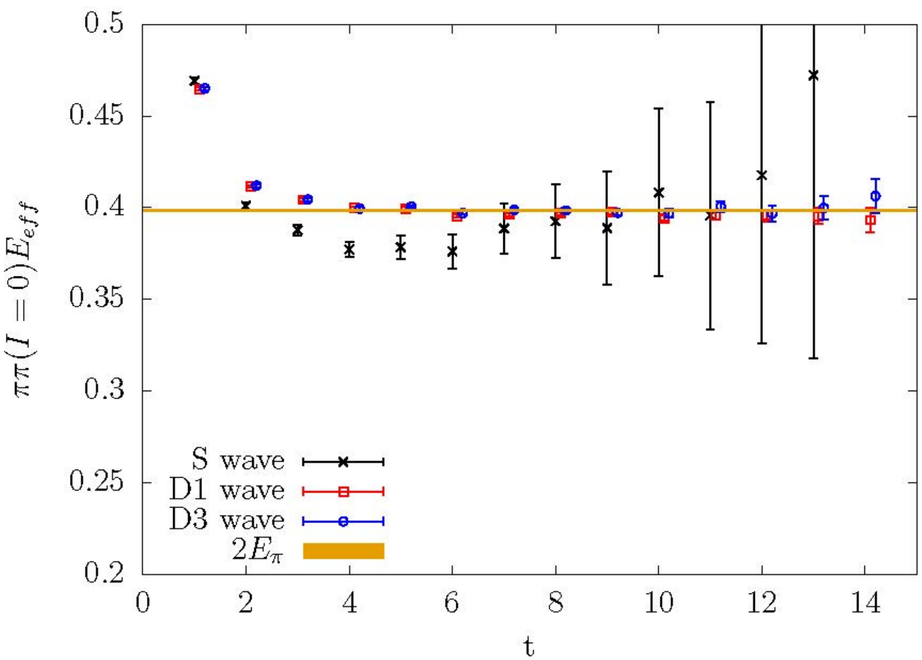
$\Delta I = \frac{1}{2}$ $K \rightarrow \pi\pi$ – above threshold
(Chris Kelly & Daiqian Zhang)

- Use **G-parity** BC to obtain p_π = 205 MeV (Changhoan Kim, hep-lat/0210003)
  - $G = C\, e^{i\pi I_y}$
  - Non-trivial: $\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$
  - Gauge fields obey C BC
  - Extra I = 1/2, s' quark adds $e^{-m_K L}$ error.
  - Must take non-local square root of s-s' determinant
  - Tests: f_K and B_K agree with no G-parity results.



Lattice symmetries – rotation

- Our 2π state has 8 possible non-zero relative spatial momenta: π/L ($\pm1, \pm1, \pm1$).
- Project onto the s-wave, A_1 rep., remove T_2.
  - Splitting of A_1 and T_2 states is caused by finite-volume
  - Results from difference between $\delta_{l=2}(I=0)$ and $\delta_{l=0}(I=0)$
  - Too small to distinguished numerically, ΔE=51 MeV



s-wave (A_1)

$E_{\pi\pi} - 2E_\pi = -0.0371 \pm 0.0075$

d-wave (T_2)

$E_{\pi\pi} - 2E_\pi = -0.0019 \pm 0.0006$

Lattice symmetries – translation

- Essential to exploit momentum conservation to avoid nearby states: $\Delta E = \left((2\pi/L)^2+m_\pi^{\,2}\right)^{1/2} - m_\pi^{\,2}$ = 165 MeV
- Must work with momentum eigenstates:

$$\chi(l-1) = e^{-ipa}\chi(l)$$

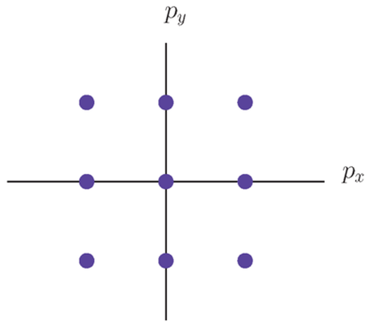
where $\chi(L) = i\sigma_y\chi(0) \qquad \chi = \begin{pmatrix} u \\ d \end{pmatrix} \qquad 0 \le l < L$

- Solved by

$$\chi(l) = e^{\frac{2\pi}{L}i(n\pm\frac{1}{4})l}\,(1 \pm \sigma_y)\begin{pmatrix} a \\ b \end{pmatrix} \qquad p = \frac{2\pi}{L}(n\pm\frac{1}{4})$$

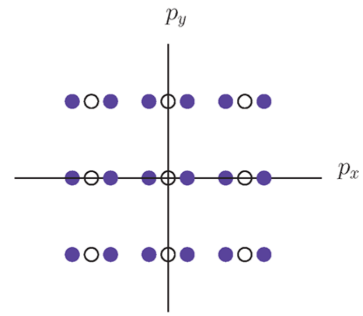
Cubic rotation symmetry broken!

$(2\pi n_x/L\,,\ 2\pi n_x/L)$



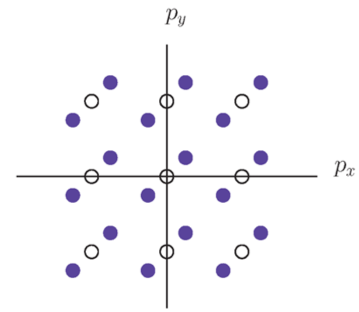
quark: 0 twists

$(2\pi(n_x \pm \tfrac{1}{4})/L\,,\ 2\pi n_x/L)$



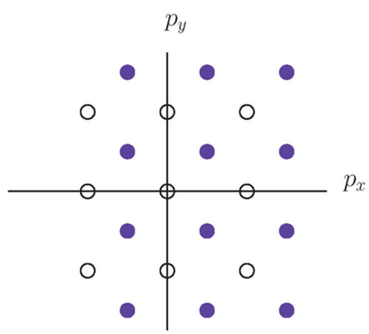
quark: 1 twists

$(2\pi(n_x \pm \tfrac{1}{4})/L\,,\ 2\pi(n_y \pm \tfrac{1}{4})/L)$



quark: 2 twists

$(2\pi(n_x \pm \tfrac{1}{2})/L\,,\ 2\pi(n_y \pm \tfrac{1}{2})/L)$



meson: 2 twists

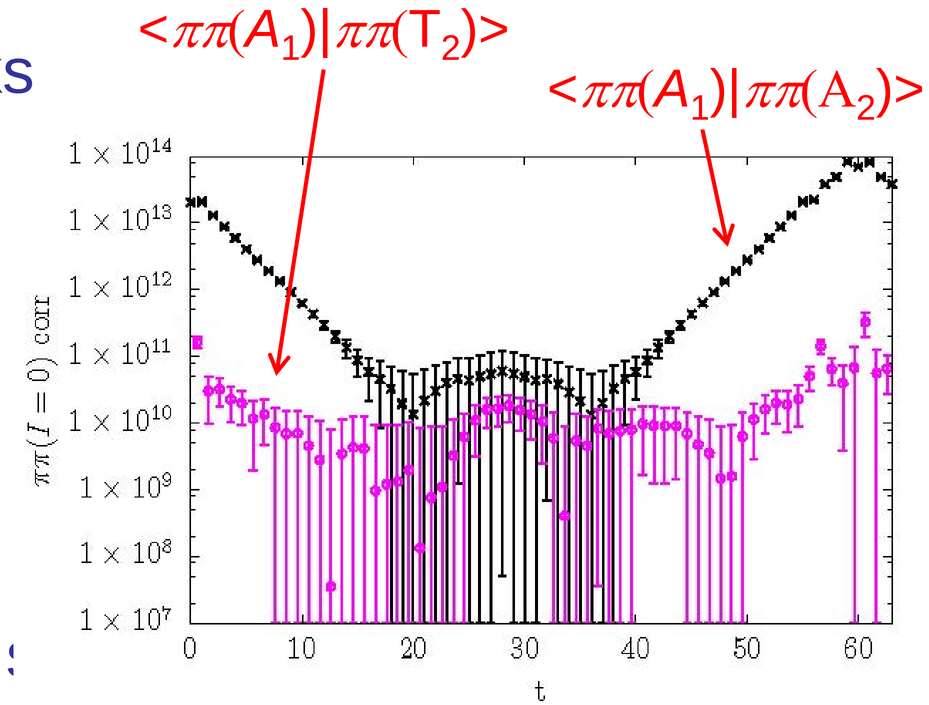
- Allowed momenta with G-parity links *x* and *y*
- Diagonal structure results
- Breaks cubic symmetry

Cubic rotation symmetry broken

- Will appear only when quarks are separated.
- Use symmetrical π wave function, add two $\vec{p}_q$ choices

$$
\begin{aligned}
\left(-\frac{\pi}{L},\frac{\pi}{L},\frac{\pi}{L}\right) &= \left(\frac{\pi}{2L},\frac{\pi}{2L},\frac{\pi}{2L}\right)+\left(\frac{-3\pi}{2L},\frac{\pi}{2L},\frac{\pi}{2L}\right)\\
&= \left(\frac{-\pi}{2L},\frac{-\pi}{2L},\frac{-\pi}{2L}\right)+\left(\frac{-\pi}{2L},\frac{3\pi}{2L},\frac{3\pi}{2L}\right)
\end{aligned}
$$

- Symmetry violation is highly suppressed for our r = 2.



	p=(2,2,2)	p=(-2,2,2)	p=(2,-2,2)	p=(2,2,-2)
E_π	0.19852(85)	0.19823(82)	0.19839(72)	0.19866(88)
Z_π	6.167(69)e+06	6.081(63)e+06	6.183(50)e+06	6.170(61)e+06

Calculation of A_0 and ε'

Overview of calculation

- Use 32^3 x 64 ensemble
  - *1/a* = 1.3784(68) GeV, *L* = 4.53 fm.
  - G-parity boundary condition in 3 directions
    - Usual *u* – *d* iso-doublet
    - Unusual *s* – *s*' with rooted determinant.
  - 216 configurations separated by 4 time units
  - 300 time units discarded for equilibration
  - 900 low modes for all-to-all propagators
  - Solve for $\pi\pi$ and kaon sources on each of 64 time slices
- Computer resources
  - 6 hours/trajectory – BG/Q ½ rack
  - 20 hours/trajectory – BG/Q ½ rack
  - One year to generate configurations, one year for measurements.

Overview of calculation

- Evolution runs 4x slower than without G-parity
  - Now two distinct flavors (2x)
  - Must use RHMC for both light and strange quarks:

$$\det\{M_l M_l^\dagger\} \rightarrow \left[\det\{M_u M_u^\dagger\}\right]^{\frac{1}{2}} \left[\det\{M_d M_d^\dagger\}\right]^{\frac{1}{2}}$$

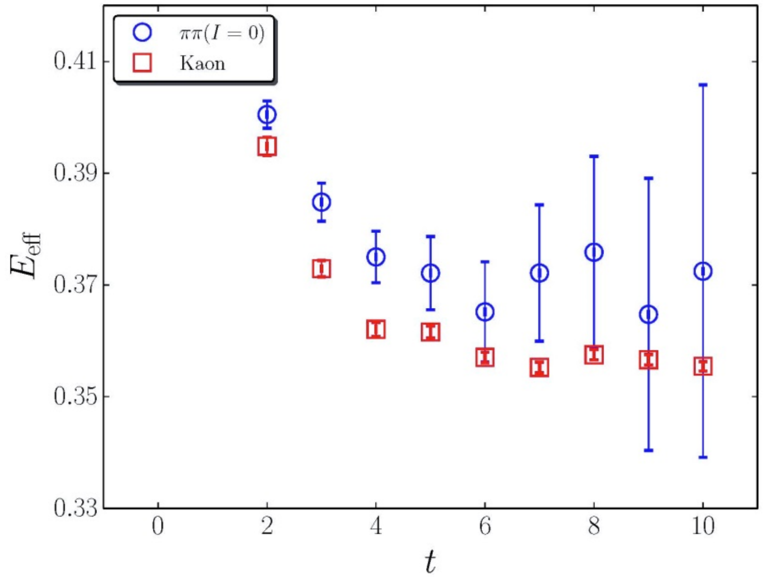
- Breakdown of BG/Q measurement

Lanczos (900 eigen vectors)	3.6h
Light quark CG (900 modes deflation)	4.6h
Strange quark CG	2.9h
Gauge fixing	0.33h
Computing meson field(900 low modes)	3.0h
Pion(s),Kaon spectrum	1.1h
Type1 contraction	0.79h
Type2 contraction	0.54h
Type3 contraction	1.97h
Type4 contraction	0.50h
Total	$\sim$19.5h

Overview of calculation

- Achieve essentially physical kinematics:
  - M_π = 143.1(2.0)
  - M_K = 490.6(2.2) MeV
  - $E_{\pi\pi}$ = 498(11) MeV
  - m_{res} = 0.001842(7) (90% of physical light quark mass)

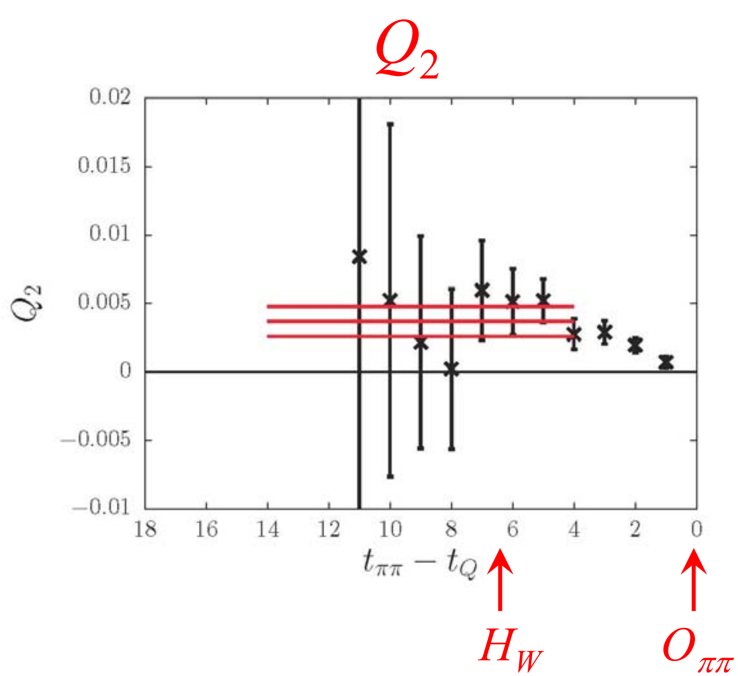
I = 0, $\pi\pi - \pi\pi$ correlator

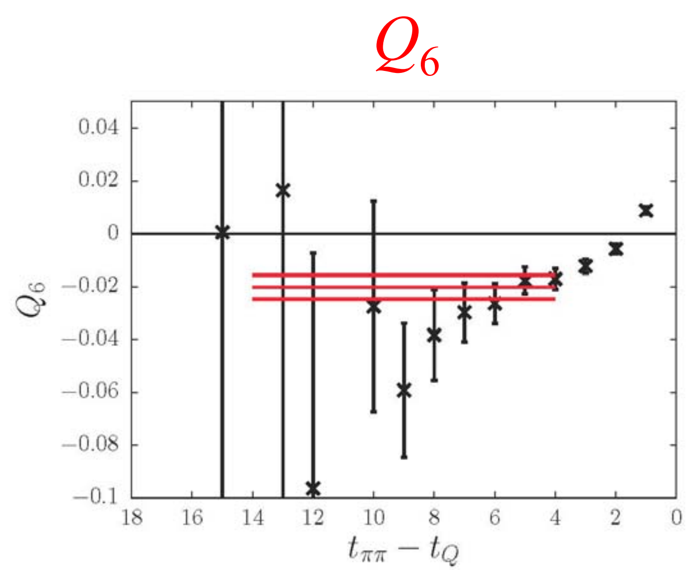


- Determine normalization of $\pi\pi$ interpolating operator.
- Determine energy of finite volume, I=0, $\pi\pi$ state: $E_{\pi\pi}$ = 498(11) MeV.
- Determine I = 0 $\pi\pi$ phase shift: δ_0 = 23.8(4.9)(2.2)°.
- $E_{\pi\pi}$ from a correlated 1-state fit, $6 \leq t \leq 25$, χ^2/dof=1.56(68)
- Consistent result obtained from 2-state fit, $3 \leq t \leq 25$.
- Leading-term amplitude changes by 5% between these two fits.

$\Delta I = ½$ $K \rightarrow \pi\,\pi$ matrix elements

- Vary time separation between H_W and $\pi\pi$ operator.
- Show data for all $K - H_W$ separations t_Q - $t_K \geq 6$ and $t_{\pi\pi}$ - t_K = 10, 12, 14, 16 and 18.
- Fit correlators with $t_{\pi\pi}$ - $t_Q \geq 4$
- Obtain consistent results for $t_{\pi\pi}$ - $t_Q \geq 3$ or 5





Lattice matrix elements

Chiral basis

$Q'_1 = 3Q_1 + 2Q_2 - Q_3$

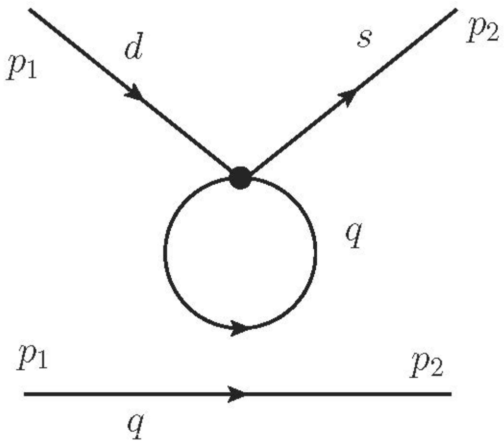
$Q'_2 = (2Q_1 - 2Q_2 + Q_3)/5$

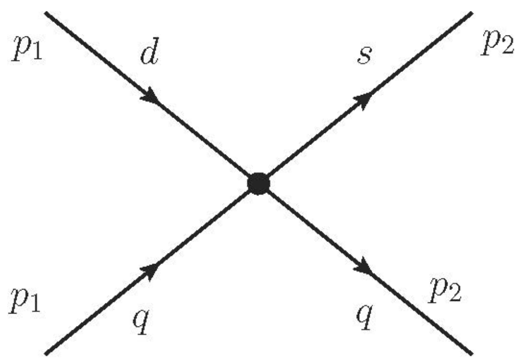
$Q'_3 = (3Q_1 - 3Q_2 + Q_3)/5$

	Conventional 10 operators	Chiral basis	
i	$\mathcal{M}^{(i)}_{\text{lat}}$ (GeV)^3	$\mathcal{M}'^{(i)}_{\text{lat}}$ (GeV)^3	
1	-0.247(62)	-0.147(242)	(27,1)
2	0.266(72)	-0.218(54)	(8,1)
3	-0.064(183)	0.295(59)	(8,1)
4	0.444(189)	—	(8,1)
5	-0.601(146)	-0.601(146)	(8,1)
6	-1.188(287)	-1.188(287)	(8,1)
7	1.33(8)	1.33(8)	(8,8)
8	4.65(14)	4.65(15)	(8,8)
9	-0.345(97)	—	
10	0.176(100)	—	

RI/SMOM normalization of chiral operators

- For (8,1) operators must include disconnected diagrams.
- Use $p_1 = 2\pi(4,4,0,0)/L$ and $p_2 = 2\pi(0,4,4,0)/L$
- $p_1^2 = p_2^2 = (p_1 - p_2)^2 = 1.531$ GeV^2
- Use 100 configurations





RI/SMOM normalization of chiral operators

- For (8,1) operators must include disconnected diagrams.

_7 x 7_

$$
Z_{\mathrm{lat}\to\mathrm{RI}} = \begin{pmatrix}
\mathbf{0.45828(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{0.293(30)} & -0.313(35) & -0.012(11) & 0.006(7) & 0 & 0 \\
0 & 0.137(48) & \mathbf{0.835(57)} & -0.005(21) & 0.006(14) & 0 & 0 \\
0 & -0.160(134) & -0.160(143) & \mathbf{0.439(49)} & -0.071(296) & 0 & 0 \\
0 & 0.034(79) & 0.106(100) & -0.036(37) & \mathbf{0.346(24)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{0.49235(3)} & -0.09216(3) \\
0 & 0 & 0 & 0 & 0 & -0.05931(6) & \mathbf{0.3724(1)}
\end{pmatrix}
$$

_10 x 7_

$$
Z_{\mathrm{RI}\to\mathrm{MS}} = \begin{pmatrix}
0.19744 & 1.07389 & 0.13830 & 0 & 0 & 0 & 0 \\
0.19744 & -0.12667 & 0.75559 & 0.00208 & -0.00625 & 0 & 0 \\
0 & 2.96832 & 1.92607 & 0.00417 & -0.01250 & 0 & 0 \\
0 & 1.74719 & 2.49297 & 0.01554 & -0.04636 & 0 & 0 \\
0 & 0 & 0 & 1.00121 & -0.00364 & 0 & 0 \\
0 & -0.04687 & -0.10936 & -0.00164 & 0.99519 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.00121 & -0.00364 \\
0 & 0 & 0 & 0 & 0 & -0.01726 & 1.04206 \\
0.29616 & 0.12667 & -0.75559 & -0.00208 & 0.00625 & 0 & 0 \\
0.29616 & -1.07389 & -0.13830 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Physical matrix elements

i	$\mathcal{M}'^{(i)}_{\mathrm{SMOM}}$ $(\mathrm{GeV})^3$	$\mathcal{M}^{(i)}_{\overline{\mathrm{MS}}}$ $(\mathrm{GeV})^3$
1	-0.0675(1109)(128)	-0.151(29)(36)
2	-0.156(27)(30)	0.169(42)(41)
3	0.212(52)(40)	-0.0492(652)(118)
4	—	0.271(93)(65)
5	-0.193(62)(37)	-0.191(48)(46)
6	-0.366(103)(70)	-0.379(97)(91)
7	0.225(37)(43)	0.219(37)(53)
8	1.65(5)(31)	1.72(6)(41)
9	—	-0.202(54)(49)
10	—	0.118(42)(28)

Contributions to A_0

i	$\mathrm{Re}(A_0)(\mathrm{GeV})$	$\mathrm{Im}(A_0)(\mathrm{GeV})$
1	$1.02(0.20)(0.07) \times 10^{-7}$	0
2	$3.63(0.91)(0.28) \times 10^{-7}$	0
3	$-1.19(1.58)(1.12) \times 10^{-10}$	$1.54(2.04)(1.45) \times 10^{-12}$
4	$-1.86(0.63)(0.33) \times 10^{-9}$	$1.82(0.62)(0.32) \times 10^{-11}$
5	$-8.72(2.17)(1.80) \times 10^{-10}$	$1.57(0.39)(0.32) \times 10^{-12}$
6	$3.33(0.85)(0.22) \times 10^{-9}$	$-3.57(0.91)(0.24) \times 10^{-11}$
7	$2.40(0.41)(0.00) \times 10^{-11}$	$8.55(1.45)(0.00) \times 10^{-14}$
8	$-1.33(0.04)(0.00) \times 10^{-10}$	$-1.71(0.05)(0.00) \times 10^{-12}$
9	$-7.12(1.90)(0.46) \times 10^{-12}$	$-2.43(0.65)(0.16) \times 10^{-12}$
10	$7.57(2.72)(0.71) \times 10^{-12}$	$-4.74(1.70)(0.44) \times 10^{-13}$
Tot	$4.66(0.96)(0.27) \times 10^{-7}$	$-1.90(1.19)(0.32) \times 10^{-11}$

Stat. error NPR

Stat. error ME

$\mathrm{Re}(A_0) = 4.66(1.00)_{\mathrm{stat}}(1.26)_{\mathrm{sys}} \times 10^{-7}$ GeV

Expt: $3.3201(0.0018) \times 10^{-7}$ GeV

$\mathrm{Im}(A_0) = -1.90(1.23)_{\mathrm{stat}}(1.08)_{\mathrm{sys}} \times 10^{-11}$ GeV

Systematic errors

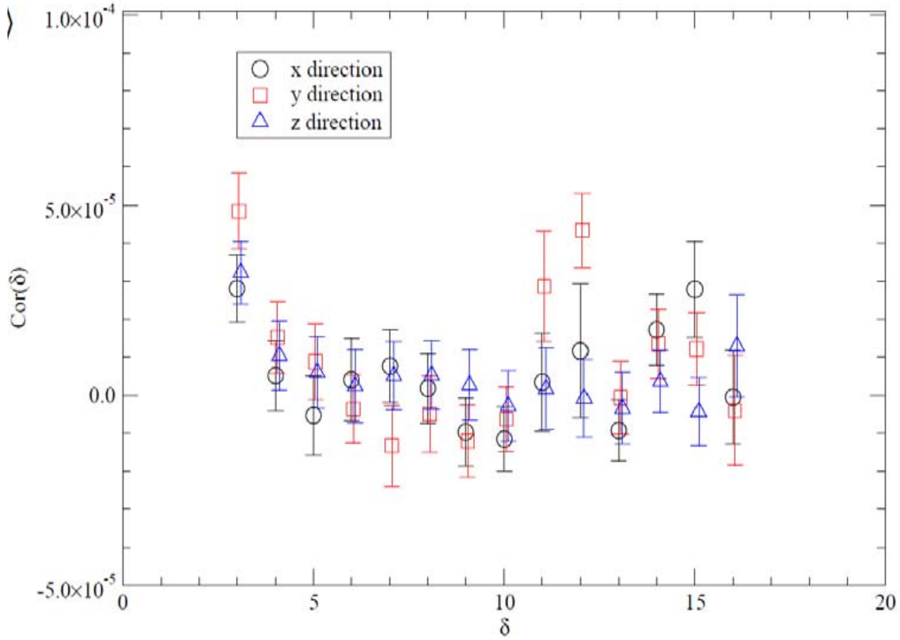
Description	Error
Operator renormalization	15%
Wilson coefficients	12%
Finite lattice spacing	12%
Lellouch-Luscher factor	11%
Finite volume	7%
Parametric errors	5%
Excited states	5%
Unphysical kinematics	3%
Total	27%

Testing Correctness

- RHMC: G-parity and “doubled lattice” evolutions agree
- Results for f_K and B_K agree with earlier DSDR values
- Calculation of matrix elements done by two people with largely independent code
- G-parity code applied to $\Delta I = 3/2$ amplitudes and results agreed with earlier method
- G-parity and standard RBC/UKQCD code agreed for a free field calculation with large mass and large volume to remove effects of boundary (with anti-periodic time boundary to ensure that loop graphs are non-zero)

Error in ensemble generation

- Up and down quark forces computed from the same random numbers after shift by 12 in y-direction.



Average plaquette:

Correct ensemble 0.512239(3)(7)

Incorrect ensemble 0.512239(6)

Calculate Re(ε'/ε)

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right]\right\}$$

$= (1.38 \pm 5.15_{stat} \pm 4.59_{sys}) \times 10^{-4}$

Expt: $= (16.6 \pm 2.3) \times 10^{-4}$ [2.1 σ difference]

- Im(A_0), Im(A_2), δ_0 and δ_2 from lattice QCD
- Re(A_2) and Re(A_0) from measured decay rates
- $|\varepsilon| = 2.228(0.011) \times 10^{-3}$ from experiment
- arg(ε) = arctan($2\Delta M_K/\Gamma_S$) = 42.52^o (Bell-Steinberger relation)

Improve Operator Renormalization

- Add 5th, dimension-6 (8,1) operator that had been dropped [still missing dimension-6 (8,1) operators which enter at two loops]: [McGlynn arXiv:1605.08807]

$$G_1(x) = \bar{s}(x)\left[D^{\mu} F^{\mu\nu}(x)\gamma^{\nu}(1-\gamma^{5})\right] d(x)$$

- G_1 vanishes by equations of motion; enters at one loop
- Effect of inclusion ~1%

$$\frac{1}{Z_q^2}\Delta Z^{\mathrm{lat}\to\mathrm{RI}} = \begin{pmatrix} 0 & & & & & & \\ & -0.00090(41) & -0.0025(12) & 0.00044(20) & -0.00090(42) & & \\ & 0.00510(24) & 0.01423(65) & -0.00248(11) & 0.00512(23) & & \\ & -0.0005(13) & -0.0013(36) & 0.00023(63) & -0.0005(13) & & \\ & 0.01470(76) & 0.0410(20) & -0.00717(35) & 0.01476(73) & & \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}$$

Improve Operator Renormalization

- **Old:**
  - Find $Z_{RI}^{lat\rightarrow RI}$ on original lattice at 1.53 GeV
  - Use PT to find $Z_{RI}^{RI\rightarrow\overline{MS}}$ at 1.53 GeV
- **New:**
  - Find $Z^{lat\rightarrow RI}$ on original lattice at 1.32 GeV
  - Find $Z^{1.33\text{ GeV}\rightarrow 2.29\text{ GeV}}$ on 24^3x64, 1/a=1.73 GeV lattice
  - Use PT to find $Z^{RI\rightarrow MS}$ at 2.29 GeV
- Test by comparing two RI schemes: gamma-gamma and $\not{q}$-$\not{q}$

Improve Operator Renormalization

- Examine $1 - Z^{\phi\phi}/Z^{\gamma\gamma}$
- Discard (33) element [large PT coefficient]
- Use ½ the largest difference for error estimate:
  - Old: 18%
  - New: 9%

Op idx.	$\mu = 1.53$ GeV 32ID	$\mu = 2.29$ GeV stepscaled no G_1	$\mu = 2.29$ GeV stepscaled with G_1
(1,1)	0.05978(13)	0.03948(16)	0.03948(16)
(2,2)	0.300(68)	0.080(33)	0.092(35)
(2,3)	0.363(76)	0.153(36)	0.161(42)
(2,5)	0.030(22)	0.0083(89)	0.0086(93)
(2,6)	0.015(20)	0.0074(53)	0.0081(77)
(3,2)	0.264(87)	0.143(23)	0.174(26)
(3,3)	0.44(11)	0.238(25)	0.297(32)
(3,5)	0.019(27)	0.0057(59)	0.0017(66)
(3,6)	0.008(27)	0.0247(46)	0.0090(68)
(5,2)	0.24(25)	0.07(10)	0.06(11)
(5,3)	0.26(30)	0.08(11)	0.09(13)
(5,5)	0.076(88)	0.024(30)	0.023(31)
(5,6)	0.046(79)	0.033(19)	0.027(26)
(6,2)	0.19(17)	0.048(63)	0.033(74)
(6,3)	0.25(20)	0.101(77)	0.075(99)
(6,5)	0.039(62)	0.044(20)	0.031(23)
(6,6)	0.016(56)	0.012(15)	0.062(21)
(7,7)	0.006154(96)	0.00185(24)	0.00185(24)
(7,8)	0.002073(59)	0.002002(43)	0.002002(43)
(8,7)	0.024131(91)	0.02260(20)	0.02260(20)
(8,8)	0.12284(25)	0.08705(11)	0.08705(11)

Outlook

- Present calculation of $\mathrm{Im}(A_0)$ and ε' can be improved with added statistics:
  - Reduce statistical error 2 x; gauge evolution running:
    - DIRAC Edinburgh
    - BNL/RBRC
    - KEK
    - Blue Waters (soon)
  - Step scale to higher energies and thru charm threshold
- Accurate NPR and theoretical control of rescattering effects allow many critical kaon quantities to be computed:
  - $K \to \pi\pi$, ΔI = 3/2 and 1/2, ε'
  - $m_{K_L} - m_{K_S}$
  - Long-distance parts of ε and $K^0 \to \pi^0 l \bar{l}$, $K^+ \to \pi^+ \nu\bar{\nu}$