

# Lattice QCD on Non-Orientable Manifolds

## Part II

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# Outline

- 1 P-boundaries  $\longrightarrow$  complex determinant
- 2 Dirac operator in Majorana basis
- 3 Construction of  $D$  with CP-boundaries
- 4 Pion propagator
- 5 P-boundaries for 2 degenerate flavors
- 6 Conclusions & Outlook

# Outline

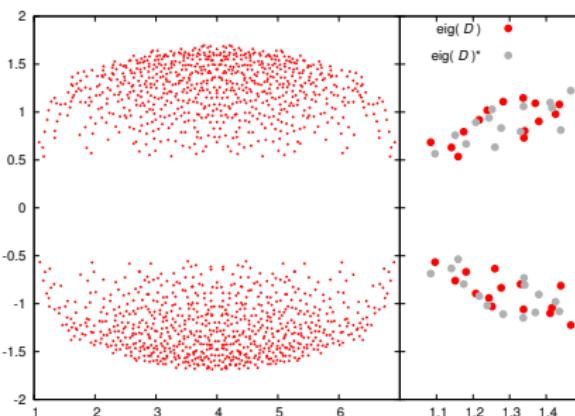
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# P-boundaries: complex determinant problem

- Reflection of fermion fields

$$\psi(x, y, z) \longrightarrow i\gamma_5 \gamma_x \psi(L - x, y, z)$$

- $\gamma_5$ -hermiticity is lost  $\longrightarrow$  complex determinant



- Avoid complex determinant problem: add charge conjugation

$$\psi \rightarrow C \bar{\psi}^T \quad \bar{\psi} \rightarrow -\psi^T C \quad U_\mu \rightarrow U_\mu^*$$

$$\text{with } C = i\gamma_y \gamma_t = C^\dagger = C^{-1} = -C^T$$

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# Fermionic action in antisymmetric form

- Charge conjugation swaps  $\psi$  with  $\bar{\psi}$
- Usual form  $S = \bar{\psi} \mathbf{D} \psi$  not applicable
- Rewrite using vectors containing both  $\psi$  and  $\bar{\psi}$

$$(\psi^T \quad \bar{\psi}) \begin{pmatrix} & -\frac{1}{2} \mathbf{D}^T \\ \frac{1}{2} \mathbf{D} & \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} = \frac{1}{2} \bar{\psi} \mathbf{D} \psi - \frac{1}{2} \psi^T \mathbf{D}^T \bar{\psi}^T = \bar{\psi} \mathbf{D} \psi$$

- Introduce

$$\xi = \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}, \quad \tilde{\mathbf{D}} = \begin{pmatrix} & \mathbf{D}^T \\ -\mathbf{D} & \end{pmatrix} = -\mathbf{D}^T$$

- Then

$$S = \bar{\psi} \mathbf{D} \psi = -\frac{1}{2} \xi^T \tilde{\mathbf{D}} \xi$$

# Grassmann integral $\rightarrow$ Pfaffian

- $S = \bar{\psi} \mathbf{D} \psi = -\frac{1}{2} \xi^T \tilde{\mathbf{D}} \xi$
- $\int d\xi \exp\left(-\frac{1}{2} \xi^T \tilde{\mathbf{D}} \xi\right) = \text{pf}(\tilde{\mathbf{D}})$
- $\text{pf}(M)$ : Pfaffian of  $2n \times 2n$  matrix  $M$

- $\text{pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n M_{\sigma(2i-1), \sigma(2i)}$
- $\text{pf}(M) = \text{pf}\left(\frac{M - M^T}{2}\right)$
- $\text{pf}(A^T M A) = \det(A) \text{pf}(M)$
- if  $M = -M^T$  then  $\text{pf}(M)^2 = \det(M)$

- In our case

$$\text{pf}(\tilde{\mathbf{D}}) = \text{pf}\begin{pmatrix} \mathbf{D}^T \\ -\mathbf{D} \end{pmatrix} = \det(\mathbf{D}) = \int d\psi d\bar{\psi} \exp(\bar{\psi} \mathbf{D} \psi)$$

# Majorana basis

- Change basis:  $\psi, \bar{\psi} \rightarrow$  eigenbasis of charge conjugation

Lucini *et.al.* JHEP 1602 (2016) 076

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi + C\bar{\psi}^T \\ -i\psi + iC\bar{\psi}^T \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & C \\ -i & iC \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} = V\xi$$

- Charge conjugation:  $\eta \rightarrow \begin{pmatrix} \eta_1 \\ -\eta_2 \end{pmatrix} = \rho_3 \eta$ ,  $\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $S = -\frac{1}{2}\xi^T \tilde{\mathbf{D}}\xi = -\frac{1}{2}\eta^T (V^{-1})^T \tilde{\mathbf{D}}(V^{-1})\eta = -\frac{1}{2}\eta^T C\hat{\mathbf{D}}\eta$
- Using  $C\mathbf{D}[U]^T C = \mathbf{D}[U^*]$

$$\begin{aligned} \hat{\mathbf{D}} &= \frac{1}{2} \begin{pmatrix} \mathbf{D}[U] + C\mathbf{D}[U]^T C & i\mathbf{D}[U] - iC\mathbf{D}[U]^T C \\ -i\mathbf{D}[U] + iC\mathbf{D}[U]^T C & \mathbf{D}[U] + C\mathbf{D}[U]^T C \end{pmatrix} = \\ &= \mathbf{D} \begin{bmatrix} \text{Re}(U) & -\text{Im}(U) \\ \text{Im}(U) & \text{Re}(U) \end{bmatrix} = \mathbf{D} \left[ \text{Re}(U) \cdot \mathbf{1}_{2 \times 2} - i \text{Im}(U) \cdot \rho_2 \right] \end{aligned}$$

- $\hat{\mathbf{D}}$  is a usual Dirac operator, but with  $6 \times 6$  real component links

# $\mathcal{O}(a)$ -improvement and $\mathbf{D}_{\text{ov}}$ in C-eigenbasis

$$\bar{\psi} \mathbf{D}[U] \psi = -\frac{1}{2} \eta^T C \mathbf{D} \begin{bmatrix} \text{Re}(U) & -\text{Im}(U) \\ \text{Im}(U) & \text{Re}(U) \end{bmatrix} \eta \quad (1)$$

- Eqn. (1) is valid for  $\mathbf{D}[U]$  linear in links

$U \mapsto \begin{bmatrix} \text{Re}(U) & -\text{Im}(U) \\ \text{Im}(U) & \text{Re}(U) \end{bmatrix}$  is a **representation** equivalent to  $\mathbf{3} \oplus \mathbf{3}^*$

- Eqn. (1) is valid for  $\mathbf{D}[U]$  linear in **products** of links  
→ valid for clover improved Wilson operator
- Eqn. (1) is valid in some more general cases, e.g. for

$$\mathbf{D}_{\text{ov}}[U] = 1 + \gamma_5 \text{sgn}(\gamma_5 (\mathbf{D}_W[U] - m_0))$$

overlap operator:

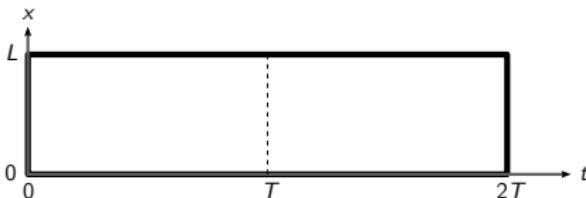
$$\bar{\psi} \mathbf{D}_{\text{ov}}[U] \psi = -\frac{1}{2} \eta^T C \left( 1 + \gamma_5 \text{sgn} \left\{ \gamma_5 \left( \mathbf{D}_W \begin{bmatrix} \text{Re}(U) & -\text{Im}(U) \\ \text{Im}(U) & \text{Re}(U) \end{bmatrix} - m_0 \right) \right\} \right) \eta$$

# Outline

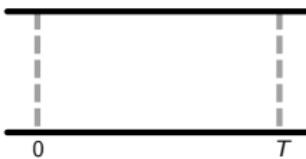
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# Construction using $2\mathcal{M}$

- Start with doubled manifold  $2\mathcal{M}$



- Periodic b.c. for both gauge fields and fermions
- D**: Dirac operator on  $2\mathcal{M}$ , satisfying  $\gamma_5 \mathbf{D} \gamma_5 = \mathbf{D}^\dagger$
- P**: projection onto fermion fields with desired b.c.
- On manifold  $\mathcal{M}$



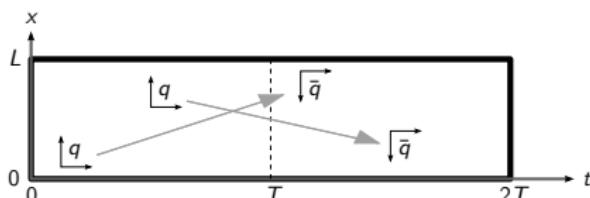
Dirac operator is given as

$$D_{x,y} = 2(\mathbf{PDP})_{x,y} \quad x, y \in \mathcal{M}$$

# Construction of CP-boundaries

- $\tau$ : transformation on  $2\mathcal{M}$

shift in direction  $t$ , reflection in direction  $x$



$$\tau \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} L - x \pmod{L} \\ y \\ z \\ t + T \pmod{2T} \end{pmatrix}$$

- Gauge fields:  $t \rightarrow t + T, x \rightarrow -x$ , charge conjugation

$$U_x(x, y, z, t + T) = U_x^T(L - x - 1, y, z, t),$$

$$U_i(x, y, z, t + T) = U_i^*(L - x, y, z, t), \quad i = y, z, t$$

- $\mathbf{T}$ : transformation of fermion fields:

$t \rightarrow t + T, x \rightarrow -x$ , charge conjugation

$$(\mathbf{T}\eta)(x) = -\gamma_5 \gamma_x \rho_2 \rho_3 \eta(\tau x), \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

# Dirac-operator with CP-boundary condition

- $\mathbf{T}$ : transformation of fermion fields

$$t \rightarrow t + T, \quad x \rightarrow -x, \quad \text{charge conjugation}$$

$$(\mathbf{T}\eta)(x) = -\gamma_5 \gamma_x \rho_2 \rho_3 \eta(\tau x), \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

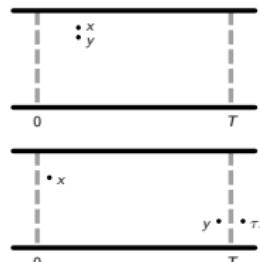
$$\mathbf{T}^\dagger = \mathbf{T}, \quad \mathbf{T}^2 = \mathbf{1}, \quad [\hat{\mathbf{D}}, \mathbf{T}] = 0$$

→ Define  $\mathbf{P}_\pm = \frac{\mathbf{1} \pm \mathbf{T}}{2}$ ,  $\hat{\mathbf{D}}_\pm = \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm$

- $\hat{D}_\pm$ : Dirac-operator on  $\mathcal{M}$   $\hat{D}_\pm \sim \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm|_{\text{Ran}(\mathbf{P}_\pm)}$

$$(\hat{D}_\pm)_{x,y} = (\hat{\mathbf{D}} \pm \mathbf{T} \hat{\mathbf{D}})_{x,y} = \hat{\mathbf{D}}_{x,y} \mp \gamma_5 \gamma_x \rho_2 \rho_3 \hat{\mathbf{D}}_{\tau x, y} \quad x, y \in \mathcal{M}$$

Bulk term:  $\hat{\mathbf{D}}_{x,y}$



Boundary term:  $\mp \gamma_5 \gamma_x \rho_2 \rho_3 \hat{\mathbf{D}}_{\tau x, y}$

# Positivity of fermionic action

- $\gamma_5 \rho_2$ -Hermiticity

$$[\hat{\mathbf{D}}, \rho_2] = 0, \quad \gamma_5 \hat{\mathbf{D}} \gamma_5 = \hat{\mathbf{D}}^\dagger \quad \rightarrow \quad \gamma_5 \rho_2 \hat{\mathbf{D}} \gamma_5 \rho_2 = \hat{\mathbf{D}}^\dagger$$

M. A. Metlitski arXiv:1510.05663

- $\mathbf{T}$ :  $t \rightarrow t + T, \quad x \rightarrow -x, \quad$  charge conjugation

$$(\mathbf{T}\eta)(x) = -\gamma_5 \gamma_x \rho_2 \rho_3 \eta(\tau x), \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- $[\mathbf{T}, \gamma_5 \rho_2] = 0 \quad \rightarrow \quad \mathbf{P}_\pm = \frac{1 \pm \mathbf{T}}{2}$  preserves  $\gamma_5 \rho_2$ -Hermiticity  
 $\rightarrow \det(\hat{\mathbf{D}}_\pm|_{\text{Ran}(\mathbf{P}_\pm)}) \in \mathbb{R}$ , where  $\hat{\mathbf{D}}_\pm = \mathbf{P}_\pm \hat{\mathbf{D}} \mathbf{P}_\pm$

- $\rho_2 \mathbf{T} = -\mathbf{T} \rho_2 \quad \rightarrow \quad \rho_2 \mathbf{P}_\pm = \mathbf{P}_\mp \rho_2 \quad \rightarrow \quad \rho_2 \hat{\mathbf{D}}_\pm \rho_2 = \hat{\mathbf{D}}_\mp$   
 $\rightarrow \hat{\mathbf{D}}_+ \sim \hat{\mathbf{D}}_- \quad \rightarrow \left\{ \text{eig}(\hat{\mathbf{D}}_\pm) \right\} = \left\{ \text{eig}(\hat{\mathbf{D}}) \right\}$  with multiplicity 2

- $\hat{\mathbf{D}}$  is an ordinary Dirac-operator, with  $\text{Re} \left\{ \text{eig}(\hat{\mathbf{D}}) \right\} > 0$  close to the continuum

# Positivity of fermionic action

- $\hat{D}_\pm$ : Dirac-operator on  $\mathcal{M}$        $\hat{D}_\pm \sim \hat{\mathbf{D}}_\pm|_{\text{Ran}(\mathbf{P}_\pm)}$
- $\int d\eta \exp\left(-\frac{1}{2}\eta^T C \hat{D}_\pm \eta\right) = \text{pf}(C \hat{D}_\pm)$
- $(C \hat{D}_\pm)^T = -C \hat{D}_\pm \quad \rightarrow \quad \text{pf}(C \hat{D}_\pm)^2 = \det(C \hat{D}_\pm) = \det(\hat{D}_\pm)$
- $\text{pf}(C(\hat{D}_\pm - s)) = \prod_\alpha (s - \lambda_\alpha)^{m_\alpha}$
- $\det(\hat{D}_\pm - s) = \det(C(\hat{D}_\pm - s)) = \text{pf}(C(\hat{D}_\pm - s))^2 = \prod_\alpha (s - \lambda_\alpha)^{2m_\alpha}$   
 $\rightarrow \lambda_\alpha$  are the eigenvalues of  $\hat{D}_\pm$
- $\gamma_5 \rho_2 \hat{D}_\pm \gamma_5 \rho_2 = \hat{D}_\pm^\dagger, \quad \text{Re} \left\{ \text{eig}(\hat{D}_\pm) \right\} > 0 \quad \rightarrow \quad \lambda_\alpha > 0 \text{ or } \lambda_\alpha^* = \lambda_{\alpha'}$   
 $\rightarrow \text{pf}(C \hat{D}_\pm) = \prod_\alpha \lambda_\alpha > 0 \quad \rightarrow \quad \text{pf}(C \hat{D}_\pm) = \sqrt{\det(\hat{D}_\pm)}$

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P-boundaries



C-eigenbasis



CP-boundaries

 $\pi$  propagator

2 flavors



Conclusions



# Pion propagator

- Interpolating operators in  $\eta$  basis

$$\begin{aligned}\mathcal{O}_{\pi^-} &= \bar{\psi}_u \gamma_5 \psi_d & = -\frac{1}{2} \eta_u^T \gamma_5 C (1 - \rho_2) \eta_d \\ \overline{\mathcal{O}}_{\pi^-} &= -\bar{\psi}_d \gamma_5 \psi_u & = \frac{1}{2} \eta_d^T \gamma_5 C (1 - \rho_2) \eta_u\end{aligned}$$

- Correlator between  $x, y \in \mathcal{M}$

$$\langle \mathcal{O}_{\pi^-}(x) \overline{\mathcal{O}}_{\pi^-}(y) \rangle = -\frac{1}{4} \left\langle (\eta_u^T)_x \gamma_5 C (1 - \rho_2) (\eta_d)_x (\eta_d^T)_y \gamma_5 C (1 - \rho_2) (\eta_u)_y \right\rangle$$

- Grassmann integral for  $\eta$

$$\frac{\int d\eta \, \eta_{i_1} \eta_{i_2} \eta_{i_3} \eta_{i_4} \exp\left(-\frac{1}{2} \eta^T M \eta\right)}{\int d\eta \, \exp\left(-\frac{1}{2} \eta^T M \eta\right)} = \frac{1}{8} \sum_{\sigma \in S_4} \text{sgn}(\sigma) (M^{-1})_{\sigma(i_1), \sigma(i_2)} (M^{-1})_{\sigma(i_3), \sigma(i_4)}$$

- We choose  $m_u = m_d$  (N.B.:  $M = C \hat{D}$ ,  $[\rho_2, \hat{D}] = 0$  but  $[\rho_2, \hat{D}] \neq 0$ )

$$\langle \mathcal{O}_{\pi^-}(x) \overline{\mathcal{O}}_{\pi^-}(y) \rangle = \frac{1}{4} \text{Tr} \left[ (\hat{D}^{-1})_{y,x} \rho_2 (\hat{D}^{-1})_{x,y}^\dagger \rho_2 + (\hat{D}^{-1})_{y,x} (\hat{D}^{-1})_{x,y}^\dagger \right]$$

## 2 remarks

① Strategy to increase statistics

- Usual: put sources at several different  $t \geq 0$
- Here: shift gauge configuration with  $-t$ , and put source at 0

② Correlator can be expressed using  $\hat{\mathbf{D}}^{-1}$

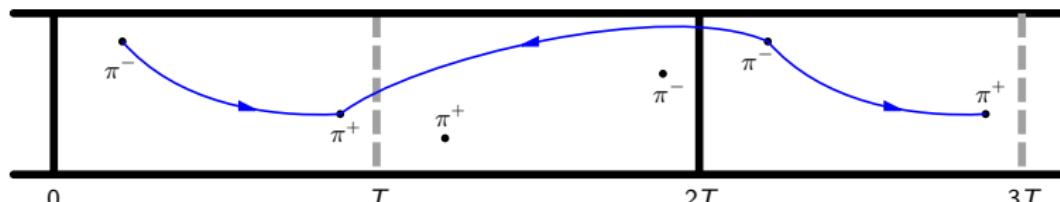
$$(\hat{\mathbf{D}}^{-1})_{x,y} = 2(\mathbf{P}_\pm \hat{\mathbf{D}}^{-1} \mathbf{P}_\pm)_{x,y} \quad x, y \in \mathcal{M}$$

$$\langle \mathcal{O}_{\pi^-}(x) \overline{\mathcal{O}}_{\pi^-}(y) \rangle =$$

$$= \frac{1}{4} \text{Tr} \left[ (\hat{\mathbf{D}}^{-1})_{y,x} \rho_2 (\hat{\mathbf{D}}^{-1})_{x,y}^\dagger \rho_2 + (\hat{\mathbf{D}}^{-1})_{y,x} (\hat{\mathbf{D}}^{-1})_{x,y}^\dagger \right] =$$

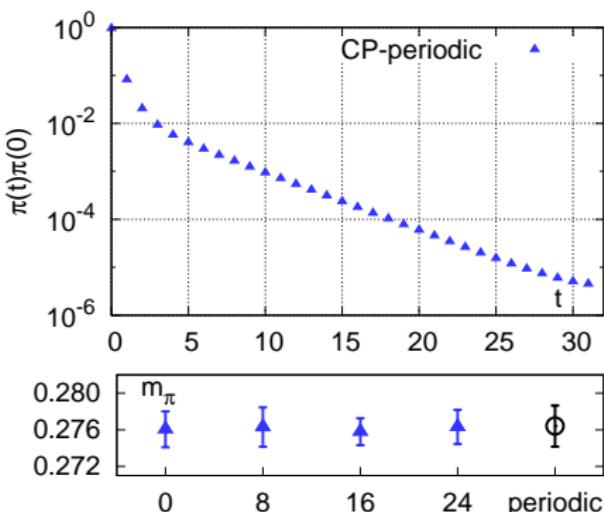
$$= \frac{1}{2} \text{Tr} \left[ (\hat{\mathbf{D}}^{-1})_{y,x} (\hat{\mathbf{D}}^{-1})_{x,y}^\dagger \right]$$

Terms containing  $\mathbf{T}$  cancel



# Numerical test

- $16^3 \times 32$  quenched with CP-boundaries,  $\beta = 4.35$ ,  $w_0 = 1.57$
- Wilson-Dirac operator, 4 steps of stout smearing with  $\varrho = 0.125$ , bare quark mass  $m_0 = -0.16$



- Backward propagation suppressed
- Translational invariance

P-boundaries



C-eigenbasis



CP-boundaries

 $\pi$  propagator

2 flavors



Conclusions



# Suppressed backward propagation

$$\langle \mathcal{O}_{\pi^-}(t) \overline{\mathcal{O}}_{\pi^-}(\bar{t}) \rangle =$$

$$= \text{Tr} [\textcolor{blue}{CP} e^{-(T-t)H} \mathcal{O}_{\pi^-} e^{-(t-\bar{t})H} \overline{\mathcal{O}}_{\pi^-} e^{-\bar{t}H}] =$$

$$= \sum_{n,k} \langle n | \textcolor{blue}{CP} e^{-(T-t)H} \mathcal{O}_{\pi^-} | k \rangle \langle k | e^{-(t-\bar{t})H} \overline{\mathcal{O}}_{\pi^-} e^{-\bar{t}H} | n \rangle =$$

$$= \sum_{n,k} \langle \textcolor{blue}{CP(n)} | \mathcal{O}_{\pi^-} | k \rangle \langle k | \overline{\mathcal{O}}_{\pi^-} | n \rangle \exp [-TE_n - (t-\bar{t})(E_k - E_n)]$$

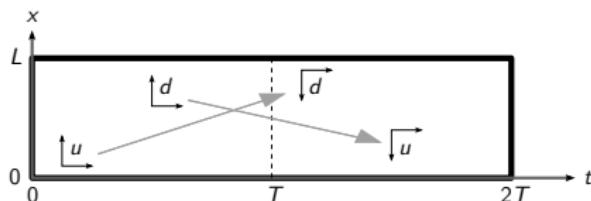
- Lowest term:  $n = \text{vacuum}$ ,  $k = \pi^- \longrightarrow \exp(- (t - \bar{t}) M_\pi)$
- Missing:  $n = \pi^+$ ,  $k = \text{vacuum}$
- 2nd lowest:  $n = \pi^- + \pi^+$ ,  $k = \pi^-$   
 $\longrightarrow \exp [-TE_{\pi^- + \pi^+} + (t - \bar{t})(E_{\pi^- + \pi^+} - M_\pi)]$

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# P-boundaries for 2 degenerate flavors: $u \leftrightarrow d$

- $\tau$ : same as CP-boundaries:  $t \rightarrow t + T, x \rightarrow -x$



$$\tau \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} L - x \pmod{L} \\ y \\ z \\ t + T \pmod{2T} \end{pmatrix}$$

- Gauge fields:  $t \rightarrow t + T, x \rightarrow -x$

$$U_x(x, y, z, t + T) = U_x^\dagger(L - x - 1, y, z, t),$$

$$U_i(x, y, z, t + T) = U_i(L - x, y, z, t), \quad i = y, z, t$$

- $T$ :  $t \rightarrow t + T, x \rightarrow -x, u \leftrightarrow d$

$$(\mathbf{T}\psi)(x) = i\gamma_5\gamma_x \tau_1 \psi(\tau x), \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in flavor space}$$

- If  $m_u = m_d$  then  $[\mathbf{D}, \mathbf{T}] = 0$

$$[\gamma_5\tau_3, \mathbf{T}] = 0 \longrightarrow \gamma_5\tau_3 \mathbf{D}_\pm \gamma_5\tau_3 = \mathbf{D}_\pm^\dagger \longrightarrow \det(\mathbf{D}_\pm) \in \mathbb{R}$$

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# Conclusions

- Conclusions

- P-boundaries for fermions: **complex determinant**
- Fermions on non-orientable manifolds are **possible**
  - 1 flavor: **include C**
  - 2 flavors:  **$u \leftrightarrow d$**
- Pion propagator: suppressed backward propagation

- Outlook

- Implement dynamical HMC
- Investigate effects of 2 sectors for QCD

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# Outline

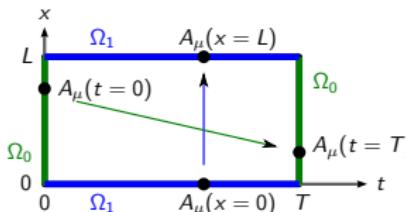
## 7 Two topological sectors

- Sectors of  $SU(3)$  in 4 dimensions
- Sectors of  $U(1)$  in 2 dimensions

# Topology of $SU(3)$ -bundle in 4 dimensions

- Space reflection at  $t = T$

$$\rho_{\mu\nu} = \text{diag}(1, -1, -1, -1), \quad \tau \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} L-x \\ L-y \\ L-z \end{pmatrix}$$



$$\begin{aligned} A_\mu(x = L, y, z, t) &= [\Omega_1(y, z, t)] A_\mu(x = 0, y, z, t) \\ A_\mu(x, y = L, z, t) &= [\Omega_2(x, z, t)] A_\mu(x, y = 0, z, t) \\ A_\mu(x, y, z = L, t) &= [\Omega_3(x, y, t)] A_\mu(x, y, z = 0, t) \\ A_\mu(x, y, z, t = T) &= [\Omega_0(x)] \rho_{\mu\nu} A_\nu(\tau x, t = 0) \end{aligned}$$

- $\Omega_\mu$ : transition functions – gauge transformations we need to apply when crossing the boundary in direction  $\mu$ .
- Cocycle conditions

$$\begin{aligned} \Omega_1(y, z, t = T) \Omega_0(x = 0, y, z) &= \Omega_0(x = L, y, z) \Omega_1(\tau y, \tau z, t = 0) \\ \Omega_2(x, z, t = T) \Omega_0(x, y = 0, z) &= \Omega_0(x, y = L, z) \Omega_2(\tau x, \tau z, t = 0) \\ \Omega_3(x, y, t = T) \Omega_0(x, y, z = 0) &= \Omega_0(x, y, z = L) \Omega_3(\tau x, \tau y, t = 0) \\ \Omega_1(y = L, z, t) \Omega_2(x = 0, z, t) &= \Omega_2(x = L, z, t) \Omega_1(y = 0, z, t) \\ \Omega_1(y, z = L, t) \Omega_3(x = 0, y, t) &= \Omega_3(x = L, y, t) \Omega_1(y, z = 0, t) \\ \Omega_2(x, z = L, t) \Omega_3(x, y = 0, t) &= \Omega_3(x, y = L, t) \Omega_2(x, z = 0, t) \end{aligned}$$

# Sectors

- Gauge transformation

$$\begin{aligned}
 A'_\mu(x) &= g(x) A_\mu(x) g^\dagger(x) + i g(x) \partial_\mu g^\dagger(x) \\
 \Omega'_1(y, z, t) &= g(x = L, y, z, t) \Omega_1(y, z, t) g^\dagger(x = 0, y, z, t) \\
 \Omega'_2(x, z, t) &= g(x, y = L, z, t) \Omega_2(x, z, t) g^\dagger(x, y = 0, z, t) \\
 \Omega'_3(x, y, t) &= g(x, y, z = L, t) \Omega_3(x, y, t) g^\dagger(x, y, z = 0, t) \\
 \Omega'_0(\underline{x}) &= g(\underline{x}, t = T) \Omega_0(\underline{x}) g^\dagger(\tau \underline{x}, t = 0)
 \end{aligned}$$

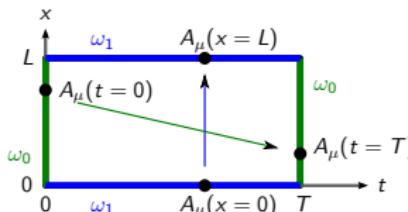
- Choosing suitable  $g$ :  $\Omega_{1,2,3} = 1$  achievable
- In this choice of gauge

$$Q = -\frac{i}{24\pi^2} \epsilon_{ijk} \int d^3\underline{x} \text{Tr} \left[ \left( \Omega_0^\dagger \partial_i \Omega_0 \right) \left( \Omega_0^\dagger \partial_j \Omega_0 \right) \left( \Omega_0^\dagger \partial_k \Omega_0 \right) \right]$$

- $Q \in \mathbb{Z}$
- non-orientability**  $\rightarrow$  via  $g$   $Q \rightarrow Q \pm 2$  possible.
- 2 distinct topological sectors, characterized by  $Q \pmod{2}$ .

# $U(1)$ in 2d: Continuum

- $\Omega_\mu(x) = \exp(i\omega_\mu(x))$



$$\begin{aligned} A_0(x, t = T) &= A_0(L-x, t = 0) \\ A_1(x, t = T) &= -A_1(L-x, t = 0) + \partial_x \omega_0(x) \\ A_0(x = L, t) &= A_0(x = 0, t) + \partial_t \omega_1(t) \\ A_1(x = L, t) &= A_1(x = 0, t) \end{aligned}$$

$$Q = \frac{1}{2\pi} \left[ \int_0^L \partial_x \omega_0(x) dx - \int_0^T \partial_t \omega_1(t) dt - 2\omega_1(t = 0) \right] \pmod{2}$$

- Cocycle condition

$$\omega_1(t = T) + \omega_0(x = 0) = \omega_0(x = L) + \omega_1(t = 0) + 2k\pi, \quad k \in \mathbb{Z}$$

- Gauge transformation

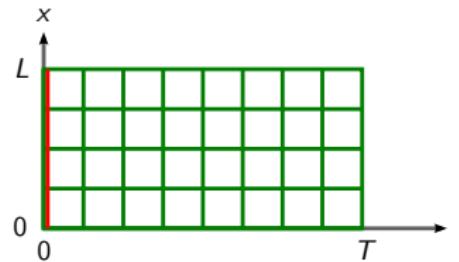
$$\begin{aligned} A'_\mu(x, t) &= A_\mu(x, t) + \partial_\mu \alpha(x, t) \\ \omega'_1(t) &= \omega_1(t) + \alpha(x = L, t) - \alpha(x = 0, t) \\ \omega'_0(x) &= \omega_0(x) + \alpha(x, t = T) - \alpha(L-x, t = 0) \end{aligned}$$

# $U(1)$ in 2d: Lattice

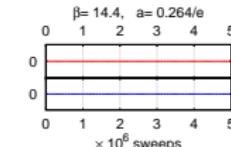
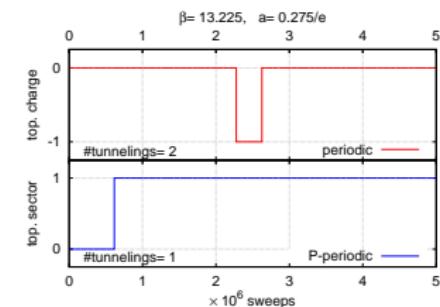
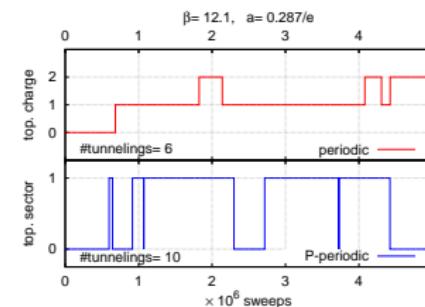
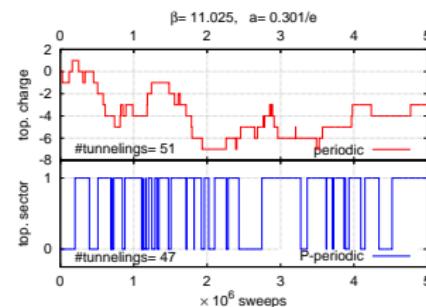
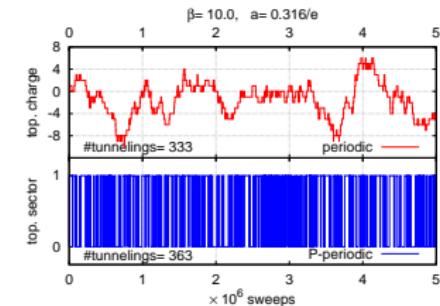
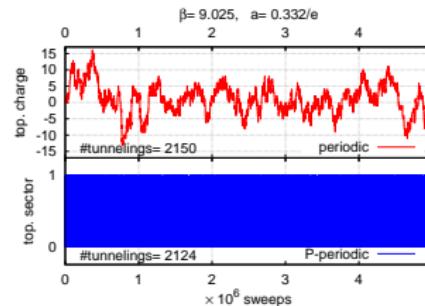
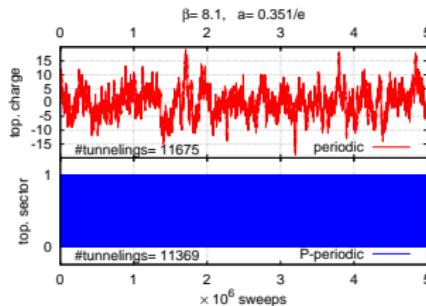
- Links  $U_\mu(x, t) = \exp(iA_\mu(x, t))$
- Plaquette  
 $U_P(x, t) = \exp [iA_1(x, t) + iA_0(x + 1, t) - iA_1(x, t + 1) - iA_0(x, t)]$
- Topological charge has a **bulk term** and a **boundary term**

$$Q = \frac{1}{2\pi} \left[ \sum_{x,t} \arg [U_P(x, t)] - 2 \sum_x A_1(x, t=0) \right] \pmod{2}$$

with  $-\pi \leq \arg [U_P(x, t)] < \pi$ .

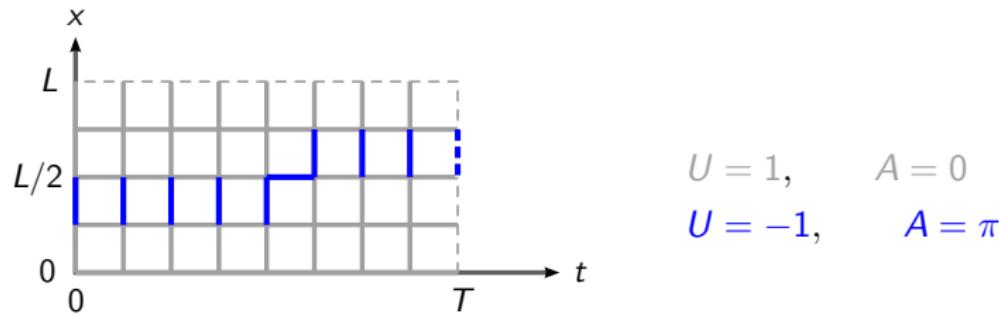


# Tunneling between sectors



# Equivalence of sectors

- Non-orientability  $\longrightarrow S = 0, Q = 1$  configuration:  $U_Z(x, t)$



- The mapping  $U \mapsto U \cdot U_Z$ 
  - bijection between the sectors  $Q = 0$  and  $Q = 1$ ,
  - leaves the action invariant,
  - leaves the integration measure invariant.