A Massive Momentum-Subtraction Scheme

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Motivation for a Mass Dependent Scheme

Lattice	$48^3 \times 96$ Physical	$64^3 \times 128$ Physical	$48^3 \times 96$ Fine
$1/a$ (GeV)	1.73	2.36	2.8
M_{D_l} (GeV)	1.42 – 1.68	1.49 – 2.12	1.51– 2.42

PDG $M_{D_l^\pm} = 1.86957(16)$ GeV, $M_{D_l^0} = 1.86480(14)$ GeV [1]

- massless quarks: $am \ll a\mu \ll \pi$
- Reduction in lattice artefacts when performing continuum extrapolation in a massive scheme, by potentially removing mass dependent $\mathcal{O}(a^2)$ terms

[1] arXiv: hep-lat/1602.04118v1 and hep-lat/1511.09328

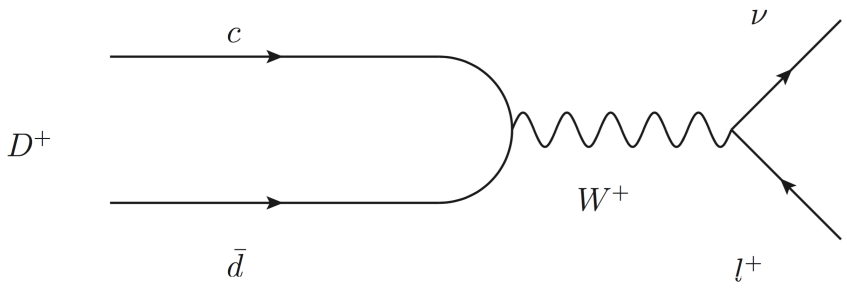
The Charm Project

Determine the decay constants f_D and f_{D_s} using

$$\langle 0|A^{\mu}_{cq}|D_q(p)\rangle = f_{D_q} p^{\mu}_{D_q}$$

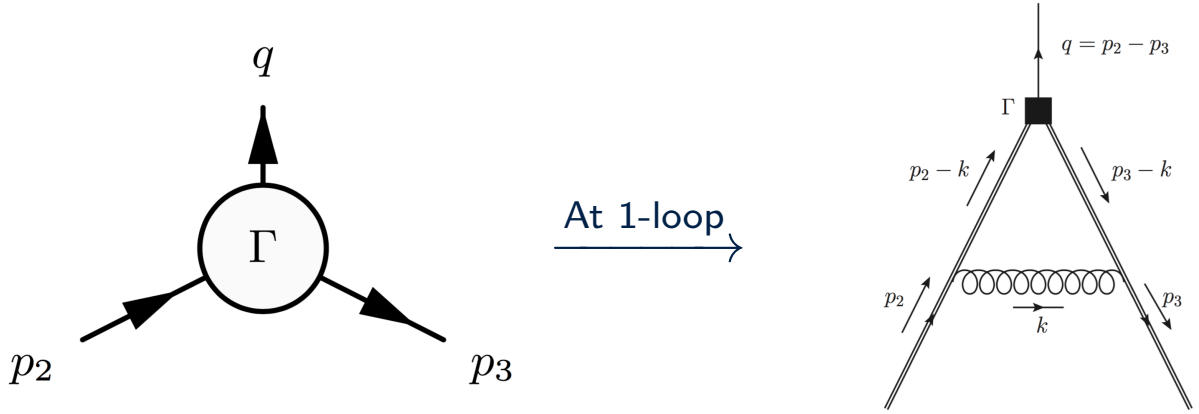
where $q = d, s$ and the axial current $A^{\mu}_{cq} = \bar{c}\gamma_{\mu}\gamma_5 q$.

To obtain the decay constant, we need to renormalize the bare axial current.



Kinematics

- Symmetric Minkowski momentum $p_2^2 = p_3^2 = q^2 = -\mu^2$ with $\mu^2 > 0$
- Vertex $G^a_\Gamma(p_3, p_2) = \langle O^a_\Gamma(q)\bar{\psi}(p_3)\psi(p_2)\rangle$, fermion bilinear $O^a_\Gamma = \bar{\psi}\Gamma\tau^a\psi$
- Γ spans all the element of the basis of the Clifford algebra, $\Gamma = \mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$
- Propagator $S(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon}$
- Amputated vertex function $\Lambda^a_\Gamma(p_2, p_3) = S(p_3)^{-1} G^a_\Gamma(p_3, p_2) S(p_2)^{-1}$



Continuum Ward identities

- Consider chiral transformations with a regulator that does not break the symmetry, e.g. dim-reg
- Vector and axial transformations on $\bar{\psi}_i, \psi_j$ in the path integral imply:

  - Vector WI: $q \cdot \Lambda^a_{\mathrm{V}} = iS(p_2)^{-1} - iS(p_3)^{-1}$
  - Axial WI: $q \cdot \Lambda^a_{\mathrm{A}} = 2mi\Lambda^a_{\mathrm{P}} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$

- Flavor non-singlet $\tau^a = \sigma^+/2$

Renormalization

$$\psi_R = Z_q^{1/2}\psi\,, \quad S_R(p) = Z_q S(p)\,, \quad m_R = Z_m m$$

$$\left[\bar{\psi}\Gamma\psi\right]_R = Z_\Gamma \bar{\psi}\Gamma\psi\,, \quad A_R^\mu = Z_A A^\mu\,, \quad V_R^\mu = Z_V V^\mu$$

Renormalization of Λ_Γ: $\Lambda_{\Gamma,R} = \frac{Z_\Gamma}{Z_q}\Lambda_\Gamma$

- In general, $Z = Z(g, a\mu, am)$
- Regulator a
- Renormalization scale μ
- Renormalization constants are determined by imposing renormalization conditions. e.g. RI/SMOM.

RI/SMOM Conditions

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1}\right]\big|_{p^2=-\mu^2} - \frac{1}{2}\ \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\,\gamma_5\right]\big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R})\ \not{q}\right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\,\gamma_5\ \not{q}\right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R}\gamma_5\right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R}\right]\big|_{\mathrm{sym}} = 1$$

Tree level values

RC are consistent with trivial renormalizations at tree level e.g.

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12p^2} Z_q^{-1} \mathrm{Tr}\left[iS(p)^{-1} \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12p^2} Z_q^{-1} \mathrm{Tr}\left[(\not{p} - m) \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

at tree level $Z_q = 1$, same for all the others.

This is a property we wish to preserve in the massive scheme

$Z_V = 1$ in SMOM

Bare Vector WI: $\quad q \cdot \Lambda_{\mathrm{V}} = iS(p_2)^{-1} - iS(p_3)^{-1}$

Rewriting in terms of renormalized quantities using,

$$S_R(p) = Z_q S(p) \quad \text{and} \quad \Lambda_{\mathrm{V},R} = \frac{Z_{\mathrm{V}}}{Z_q}\Lambda_{\mathrm{V}} \quad \Rightarrow$$

$$\frac{Z_q}{Z_{\mathrm{V}}}\ q \cdot \Lambda_{\mathrm{V},R} = i\ Z_q\ S_R(p_2)^{-1} - i\ Z_q\ S_R(p_3)^{-1}$$

multiplying by $\not{q}$ and taking trace, using

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$
$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R})\ \not{q}\right]\Big|_{\mathrm{sym}} = 1$$

gives $\frac{Z_q}{Z_{\mathrm{V}}} = Z_q \quad \Rightarrow Z_{\mathrm{V}} = 1$

Heavy-Heavy RI/mSMOM Conditions

$$\lim_{m_R \to \bar{m}} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1}\right]\Big|_{p^2=-\mu^2} - \frac{1}{2}\ \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\,\gamma_5\right]\Big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R})\ \not{q}\right]\Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R}\ -2m_R i\Lambda_{\mathrm{P},R})\,\gamma_5\ \not{q}\right]\Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R}\gamma_5\right]\Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R}\right]\ -\frac{1}{6q^2} \mathrm{Tr}\left[2im_R\Lambda_{P,R}\gamma_5\not{q}\right]\Big|_{\mathrm{sym}} = 1$$

Example: $Z_A = 1$ for Heavy-Heavy Vertex

Bare axial WI:

$$q \cdot \Lambda_{\mathrm{A}} = 2mi\Lambda_{\mathrm{P}} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$$

Rewriting in terms of renormalized quantities

$$\frac{1}{Z_{\mathrm{A}}} q \cdot \Lambda_{\mathrm{A},R} - \frac{1}{Z_m Z_{\mathrm{P}}} 2m_R i\Lambda_{\mathrm{P},R} = -\left\{\gamma_5 iS_R(p_2)^{-1} + iS_R(p_3)^{-1}\gamma_5\right\}$$

Example: $Z_A = 1$ for Heavy-Heavy Vertex

1. Trace with $\gamma^5 \not{q}$

$$(Z_{\mathrm{A}} - 1) = \left(1 - \frac{Z_A}{Z_m Z_{\mathrm{P}}}\right) C_{mP} \,,$$

$$C_{m\mathrm{P}} = \lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[2im_R \Lambda_{\mathrm{P},R} \gamma_5 \not{q}\right]\Big|_{\mathrm{sym}}$$

2. Trace with γ^5

$$(Z_{\mathrm{A}} - 1) C_{qA} = -2Z_{\mathrm{A}} \left(1 - \frac{1}{Z_m Z_{\mathrm{P}}}\right) \,,$$

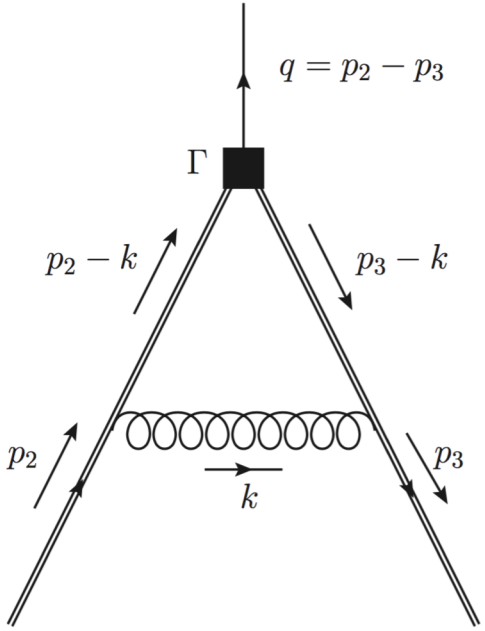
$$C_{q\mathrm{A}} = \lim_{m_R \to \bar{m}} \frac{1}{12m_R} \mathrm{Tr}\left[q \cdot \Lambda_{\mathrm{A},R} \gamma_5\right]\Big|_{\mathrm{sym}}$$

Together give $Z_A = 1$ and $Z_m Z_p = 1$.

Check at 1-loop in perturbation theory using dim reg

Dimensional Regularization, $D = 4 - 2\epsilon$

$$\Lambda_\Gamma^{(1)} = -ig^2 C_2(F) \int_k \frac{\gamma_\mu[\not{p}_2 - \not{k} + m]\Gamma[\not{p}_3 - \not{k} + m]\gamma^\mu}{k^2[(p_2 - k)^2 - m^2][(p_3 - k)^2 - m^2]}$$



Results

$$Z_q = 1 + \frac{\alpha}{4\pi} C_2(F) \left(\frac{1}{\epsilon} - \gamma_E + 1 - \frac{m^2}{\mu^2} - \frac{m^4}{\mu^4} \ln\left(\frac{m^2}{m^2+\mu^2} \right) - \ln\left(\frac{m^2+\mu^2}{\tilde{\mu}^2} \right) \right)$$

$$\Lambda_{\mathrm{V}}^{(1)\sigma}(p_2, p_3) = \frac{\alpha}{4\pi} C_2(F) \Big[A_{\mathrm{V}} \frac{1}{\mu^2} \left(i\epsilon^{\sigma\rho\alpha\beta} \gamma_\rho \gamma^5 p_{3\alpha} p_{2\beta} \right) + B_{\mathrm{V}} \gamma^\sigma + C_{\mathrm{V}} \frac{1}{\mu^2} \left(p_2^\sigma \not{p}_2 + p_3^\sigma \not{p}_3 \right)$$
$$+ D_{\mathrm{V}} \frac{1}{\mu^2} \left(p_2^\sigma \not{p}_3 + p_3^\sigma \not{p}_2 \right) + E_{\mathrm{V}} \frac{1}{\mu} \left(p_2^\sigma + p_3^\sigma \right) \Big]$$

$$A_{\mathrm{V}} = \frac{4}{3} \left[\left(\frac{1}{2} - \frac{m^2}{\mu^2} \right) C_0 + \left(1 + \frac{m^2}{\mu^2} \right) \log\left(\frac{m^2}{m^2+\mu^2} \right) - \sqrt{1 + 4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1 + 4\frac{m^2}{\mu^2}} - 1}{\sqrt{1 + 4\frac{m^2}{\mu^2}} + 1} \right) \right]$$

$$B_{\mathrm{V}} = \frac{1}{\epsilon} - \gamma_E + \frac{1}{3} \Bigg[-C_0 \left(1 - 4\frac{m^2}{\mu^2} - 2\frac{m^4}{\mu^4} \right) + 2\left(3 - \frac{m^2}{\mu^2} \right) \frac{m^2}{\mu^2} \log\left(\frac{m^2}{m^2+\mu^2} \right) + \left(1 - 4\frac{m^2}{\mu^2} \right) \log\left(\frac{m^2}{\tilde{\mu}^2} \right)$$
$$-4\left(1 - \frac{m^2}{\mu^2} \right) \log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2} \right) - \left(1 - 2\frac{m^2}{\mu^2} \right) \sqrt{1 + 4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1 + 4\frac{m^2}{\mu^2}} - 1}{\sqrt{1 + 4\frac{m^2}{\mu^2}} + 1} \right) \Bigg]$$

Results

$$C_{\mathrm{V}} = -\frac{2}{3}\left[\left(1-\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)+\left(1-2\frac{m^2}{\mu^2}\right)\sqrt{1+4\frac{m^2}{\mu^2}}\log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}}-1}{\sqrt{1+4\frac{m^2}{\mu^2}}+1}\right)\right.$$
$$\left.+\left(2-\frac{m^2}{\mu^2}\right)-2C_0\frac{m^2}{\mu^2}\left(1+\frac{m^2}{\mu^2}\right)-\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2}{\tilde{\mu}^2}\right)+\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2}\right)\right]$$

$$D_{\mathrm{V}} = \frac{2}{3}\left[(1+C_0)\left(1-2\frac{m^2}{\mu^2}\right)-2\left(1+\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)\right]$$

satisfies bare WI, **and** ... $Z_V = 1$!
Similarly for Z_A and all other identities.

In particular no μ dependence for the renormalization constant of Noether currents.

Heavy-Light RI/mSMOM Conditions

$$\lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{V},R}-(M_R-m_R)\Lambda_{\mathcal{S},R}\right)\not{q}\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(i\zeta^{-1}S_{H,R}(p_2)^{-1}-i\zeta S_{l,R}(p_3)^{-1}\right)\not{q}\right]$$

$$\lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R}-(M_R+m_R)i\Lambda_{\mathcal{P},R}\right)\gamma_5\not{q}\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(-i\gamma^5\zeta^{-1}S_{H,R}(p_2)^{-1}-i\zeta S_{l,R}(p_3)^{-1}\gamma^5\right)\gamma_5\not{q}\right]$$

$$\lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \frac{1}{12i}\mathrm{Tr}\left[\Lambda_{\mathcal{P},R}\gamma_5\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to \bar{m}}} \left\{\frac{1}{12(M_R+m_R)}\left\{\mathrm{Tr}\left[-i\zeta^{-1}S_{H,R}(p)^{-1}\right]\Big|_{p^2=-\mu^2}-\frac{1}{2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R}\right)\gamma_5\right]\Big|_{\mathrm{sym}}\right\}+\right.$$

$$\left.\frac{1}{12(M_R+m_R)}\left\{\mathrm{Tr}\left[-i\zeta S_{l,R}(p)^{-1}\right]\Big|_{p^2=-\mu^2}-\frac{1}{2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R}\right)\gamma_5\right]\Big|_{\mathrm{sym}}\right\}\right\}.$$

where M and m refer to heavy and light quark masses respectively and

$$\zeta = \frac{\sqrt{Z_l}}{\sqrt{Z_H}}$$

Lattice Regularization

Lattice WI for chiral symmetry

$$\nabla^*_\mu \langle A^a_\mu(x)\psi(y)\bar{\psi}(z)\rangle = 2m\langle P^a(x)\psi(y)\bar{\psi}(z)\rangle + \text{contact terms}$$
$$+\langle X^a(x)\psi(y)\bar{\psi}(z)\rangle$$

- X^a explicit chiral symmetry breaking by lattice regulator
- Reproduces usual continuum result when regulator is removed
  $\Rightarrow\ X^a(x) = aO^a_5(x)$
- Renormalize operators, $O^a_5(x)$ mixed with lower-dim operators
- Testa[2]: power divergencies do not contribute to the anomalous dimensions
  $\Rightarrow\ A_{R,\mu} = Z_A\,(g, am)\,A_\mu$

$$O^a_{5R}(x) = Z_5\left[O^a_5(x) + \underbrace{\frac{\bar{m}}{a}P^a(x) + \frac{Z_A - 1}{a}\nabla^*_\mu A^a_\mu(x)}_{\frac{\tilde{Z}}{a}\tilde{O}(x)}\right]$$

[2] arXiv: hep-th/9803147

Summary

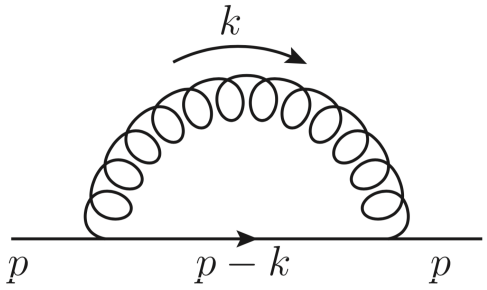
- Generalised SMOM to non-vanishing fermion mass
- Derived non-perturbatively, checked at 1-loop in perturbation theory
- Both for heavy-heavy and heavy-light vertex functions such $Z_{V,A}^{\text{cons}} = 1$
- Obtain $Z_{V,A}^{\text{local}}$ by taking ratios of vertex function with appropriate projectors
- Numerical implementation and tests will be performed on renormalizing matrix elements used to obtain decay constants and form factors in semi-leptonics

Backup Slides

Finiteness of the ζ ratio

$$\zeta = \frac{\sqrt{Z_I}}{\sqrt{Z_H}}$$

- BPHZ theorem: Remove all the divergences of a graph, G, using **local subtractions** only $\Longrightarrow$
  coeffs. multiplying the divergent part are local.
- Possible structure of the coeffs: $1, \underbrace{p^2/m^2}_{\text{IR div}}, \underbrace{m^2/p^2}_{\text{non-local}}, \underbrace{\ln\left(\frac{m^2}{p^2}\right)}_{\text{non-local}}$



Numerical Implementation

$$\frac{\left(\frac{C_{\mathcal{A}(Mm)}-C_{Mm\mathcal{P}}}{\Delta_{H-L}}\Big|_{\text{mixed}}\right)^2}{\left(C_{\mathrm{A}(MM)}-C_{MMP}\right)C_{\mathrm{A}(mm)}}=1$$

$$C_{\mathcal{A}(Mm)}=\lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}}\frac{1}{12q^2}\mathrm{Tr}\left[q\cdot\Lambda_{\mathcal{A},R}\gamma_5\ \not{q}\right]\Big|_{\mathrm{sym}}$$

$$\lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}}\frac{1}{12q^2}\frac{Z_H^{1/2}Z_I^{1/2}}{Z_{\mathcal{A}}}\mathrm{Tr}\left[q\cdot\Lambda_{\mathcal{A}}\gamma_5\ \not{q}\right]\Big|_{\mathrm{sym}}$$

Numerical Implementation

$$\Delta_{H-L} = \lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr}\left[\left(-i\gamma^5 \zeta^{-1} S_{H,R}(p_2)^{-1} - i\zeta S_{l,R}(p_3)^{-1}\gamma^5\right)\gamma_5 \ \not{q}\right]$$

$$C_{Mm\mathcal{P}} = \lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr}\left[i(M_R + m_R)\Lambda_{\mathcal{P},R}\gamma_5 \ \not{q}\right]\Big|_{\mathrm{sym}}$$

$$C_{MM\mathrm{P}} = \lim_{M_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[2iM_R\Lambda_{\mathrm{P},R}\gamma_5 \ \not{q}\right]\Big|_{\mathrm{sym}}$$

$$C_{\mathrm{A}(MM)} = \lim_{M_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[q \cdot \Lambda_{\mathrm{A},R}\gamma_5 \ \not{q}\right]\Big|_{\mathrm{sym}}$$

$$C_{\mathrm{A}(mm)} = \lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[q \cdot \Lambda_{\mathrm{A},R}\gamma_5 \ \not{q}\right]\Big|_{\mathrm{sym}}$$

Numerical Implementation

Wish to compute $Z_{\mathcal{A}}(HL)$, all ingredients are available.

$$\frac{Z_A Z_P \langle A(x) P(0) \rangle}{Z_{A^{\text{cons}}} Z_P \langle A^{\text{cons}} P(0) \rangle} = 1 \implies Z_A = \frac{\langle A^{\text{cons}} P(0) \rangle}{\langle A(x) P(0) \rangle}$$

where $Z_{A^{\text{cons}}} = 1$.

$$Z_P = \lim_{m_R \to \bar{m}} \frac{i}{p^2} \frac{\text{Tr}\left[i S(p)^{-1} \not{p} \right]\big|_{p^2 = -\mu^2}}{\text{Tr}\left[\Lambda_{\text{P}} \gamma_5 \right]\big|_{\text{sym}}}$$

$$\zeta = \left(\frac{\text{Tr}\left[i S_l(p)^{-1} \not{p} \right]\big|_{p^2 = -\mu^2}}{\text{Tr}\left[i S_H(p)^{-1} \not{p} \right]\big|_{p^2 = -\mu^2}} \right)^{1/2}$$

Continuum 1-Loop Calculation: Example

$$I_{111} = g^2 \int_k \frac{1}{(p_2-k)^2-m^2}\frac{1}{(p_3-k)^2-m^2}\frac{1}{k^2} \propto$$

$$\Gamma(3) g^2 \int_k \int_0^1 \left(\prod_{i=1}^{3} dx_i\right) \delta\left(1-\sum_{i=1}^{3} x_i\right) \frac{1}{[x_2((p_2-k)^2-m^2)+x_3((p_3-k)^2-m^2)+x_1 k^2]^3}$$

- Integrate loop momentum
- Integration over the Feynman parameters is the difficult part
- Certain techniques have become standard in the past few years

Mathematica packages: C. Duhr and Fire5 (Smirnov), Feynman Integral Calculus (V.A. Smirnov),
F. Chavez, C. Duhr (2012): Three-mass triangle integrals and single-valued polylogarithms.