A Lattice Formulation for Chiral Gauge Theory?
One of the great surprises of 20th century particle physics was the discovery of parity violation: LH and RH fermions do not carry the same gauge interactions:

**Question of Parity Conservation in Weak Interactions**

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AND

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(Received June 22, 1956)

The question of parity conservation in β decays and in hyperon and meson decays is examined. Possible experiments are suggested which might test parity conservation in these interactions.

**APPARENT EVIDENCE OF POLARIZATION IN A BEAM OF β-RAYS**

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The SM is the only chiral gauge theory we have seen…
… but GUTs are χGTs, as are many other speculative models of BSM physics

Theoretically chiral gauge theories are in bad shape:

- There does not exist a nonperturbative regulator
- There isn’t an all-orders proof for a perturbative regulator

Nonperturbative definition ⇒

- unexpected phenomenology?
- answers to outstanding puzzles (e.g., CP problem)?
The Problem:

A \textit{vector-like gauge theory} like QCD consists of Dirac fermions, = Weyl fermions in a real representation of the gauge group.

\begin{equation}
Z_V = \int [dA] e^{-S_{YM}} \prod_{i=1}^{N_f} \det(\mathcal{D} - m_i)
\end{equation}

A \textit{chiral gauge theory} consists of Weyl fermions in a complex representation of the gauge group.

\begin{equation}
Z_\chi = \int [dA] e^{-S_{YM}} \Delta[A]
\end{equation}

Witten: “We often call the fermion path integral a ‘determinant’ or a ‘Pfaffian’, but this is a term of art.”

We mean a product of eigenvalues...

...but there is no well-defined eigenvalue problem for a chiral theory
Vector-like gauge theory with Dirac fermions:

\[ \mathcal{D} \psi = \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ D_\mu \bar{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \lambda \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \]

Chiral gauge theory with Weyl fermions:

\[ \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix} \]

Independent unitary basis changes in LH and RH spaces leads to a phase ambiguity for the determinant.

So:  \[ \Delta[A] = e^{i\delta[A]} \sqrt{|\det \mathcal{D}|} \]
The fermion integral for a $\chi$GT:
\[
\Delta[A] = e^{i\delta[A]} \sqrt{|\det \mathcal{D}|}
\]

\[
\Delta[A] \equiv \det \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ \partial_\mu \bar{\sigma}_\mu & 0 \end{pmatrix}
\]

Well-defined eigenvalue problem with complex (gauge variant) eigenvalues
Extra RH fermions decouple
But only perturbative...amenable to lattice regularization?
The basic problem with regulating chiral gauge theories:

Need to render phase space for particles finite

This typically requires a mass scale

Fermion masses violate chiral symmetry...which has been gauged

One could break gauge symmetry explicitly...but how to restore it?
One could introduce mirror fermions to allow mass terms...but how to decouple them?

At the perturbative level, even dimensional regularization is problematic: chirality only exists in even spacetime dimensions
The anomaly is key to representing chiral symmetry on the lattice (global or gauged)

One way of looking at anomalies:

massless Dirac fermions in an electric field $E$, 1+1 dim

$$dp = qE dt$$

$$dn_R = +\frac{dp}{2\pi}$$

$$dn_L = -\frac{dp}{2\pi}$$

$$\frac{dn_5}{dt} = \frac{qE}{\pi}$$

$$\partial_\mu j_5^\mu = \frac{qE}{\pi}$$

infinite source & sink for fermions

quantum violation of a classical $U(1)_A$ symmetry
In the continuum, the Dirac sea is filled…but is a Hilbert Hotel which always has room for more.
Not so on the lattice:
Can reproduce continuum physics for long wavelength modes…

…but no anomalies in a system with a finite number of degrees of freedom

\[ \partial_\mu j^\mu_5 = 0 \]

anomalous symmetry in the continuum must be

explicitly broken symmetry on the lattice
The Nielsen-Ninomiya Theorem:

The Euclidian fermion action: \[ S = \int \frac{d^{2k}p}{(2\pi)^4} \bar{\Psi}_{-p} \tilde{D}(p) \Psi(p) \]

cannot have a kinetic operator \( D \) satisfying all four of the following properties simultaneously:

1. \( \tilde{D}(p) \) is a periodic, analytic function of \( p_\mu \);
2. \( D(p) \propto \gamma_\mu p_\mu \) for \( a|p_\mu| \ll 1 \);
3. \( \tilde{D}(p) \) invertible everywhere except \( p_\mu = 0 \);
4. \( \{\Gamma, \tilde{D}(p)\} = 0 \).

Advances in the 1990s showed us how to break global chiral symmetry in just the right way for QCD…but will be problematic when chiral symmetry is gauged!
How **Wilson fermions** reproduce the $U(1)_A$ anomaly in QCD:

Karsten, Smit 1980

\[
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu + m + a D^2 \right) \psi
\]

- Wilson fermions eliminate doublers by giving them a big mass
- Price paid: mass & Wilson terms explicitly break the (global) chiral flavor symmetries
- Fine tune $m \sim 1/a$ to continuum limit... find some of the chiral symmetry breaking does not decouple & correct anomalous Ward identities are found
- **Lost**: the benefits of chiral symmetry - multiplicative mass renormalization, non-mixing of operators...
Domain Wall Fermions

Consider fermions in Euclidian 5d with a 4d gauge field:

Compact extra dimension: \(-L < s < L\), \(\psi(-L) = \psi(L)\)

\[
\text{mass} = \Lambda \epsilon(s) \quad \text{mass flips sign at } s=0, \pm L
\]

\[
S_5 = \int d^4 x \int ds \bar{\Psi} \left[ D_4 + \gamma_5 \partial_s - \Lambda \epsilon(s) \right] \Psi
\]

Expand \(\Psi\) in terms of 4d fields:

Define: \(Q \equiv -\partial_s - \Lambda \epsilon(s)\)

Solve: \(Q f_n(s) = \mu_n b_n(s)\), \(Q^\dagger b_n(s) = \mu_n f_n(s)\)

Expand:\n\[
\Psi(x, s) = \sum_{n=0}^{\infty} \left[ P_- \psi_n(x) b_n(s) + P_+ \psi_n(x) f_n(s) \right]
\]

4d Dirac spinors

\[
P_{\pm} = \frac{1}{2} (1 \pm \gamma_5)
\]
\[ S_5 = \int d^4 x \int ds \bar{\Psi} \left[ \mathcal{D}_4 + \gamma_5 \partial_s - \Lambda \epsilon(s) \right] \Psi \]
\[ = \sum_{n=0}^{\infty} \int d^4 x \bar{\psi}_n \left[ \mathcal{D}_4 - \mu_n \right] \psi_n \]

Special zero mode solutions (\(\mu_0=0\)):
\[ f_0 \propto e^{-\int^s ds' \Lambda \epsilon(s')} \]
Localized at \(s=0\)

\[ b_0 \propto e^{+\int^s ds' \Lambda \epsilon(s')} \]
Localized at \(s=\pm L\)

Looks like a 4d theory with one massless fermion and an infinite tower of heavy states.

The massless fermion has a **chiral symmetry**

So what?

In 5d, no chiral symmetry!

LH and RH parts of the massless fermion are physically separated in \(s\)!
Light fermion mass stable against radiative corrections. E.g. add:

$$\int d^5 x \, \frac{\alpha}{4\pi} \Lambda \bar{\Psi} \Psi$$

To $O(\alpha)$ the mass induced for the surface modes is:

$$\frac{\alpha}{4\pi} \Lambda \int ds \, f_0(s) b_0(s) \approx \frac{\alpha}{4\pi} L \Lambda^2 e^{-L \Lambda}$$

In large $L$ limit, surface modes remain massless, even with radiative corrections.
But how do anomalies work (e.g., \( U(1)_A \))?

For effective 4d theory, must have:

\[
\partial_\mu j^\mu_5 \propto F \tilde{F}, \quad \partial_\mu j^\mu = 0
\]

…but no \( U(1) \) anomalies in 5d theory!

Callan-Harvey (1984):
- integrating out heavy modes induces Chern-Simons term in bulk
- allows fermion current between defects
Callan Harvey argument:

Integrating out heavy fermion modes induces CS term. (For now: 5d background gauge fields)

\[
\frac{m(s)}{|m(s)|} \epsilon_{abcd} A_a \partial_b A_c \partial_d A_e
\]

Differentiate w.r.t. $A_5$ to get $J_5$

\[
J_5 \propto \frac{m(s)}{|m(s)|} F \tilde{F} \implies \partial_5 J_5 \propto [\delta(s) - \delta(s - L)] F \tilde{F}
\]

Bulk current explains anomalous disappearance of charge on one defect and reappearance on the other
Can implement on the lattice without doublers:

$$S_5 = \overline{\Psi} \left[ \partial_s \gamma_5 + \mathcal{D}_4 - \Lambda \epsilon(s) + \frac{1}{\Lambda} (\partial_s \gamma_5 + \mathcal{D}_4)^2 \right] \Psi$$

\(\Lambda = (\text{lattice spacing})^{-1}\)

Nielsen- Ninomiya?

⇒ chiral symmetry is explicitly broken

Only relevant effect of explicit chiral symmetry breaking?

⇒ the \(U(1)_A\) anomaly

In general lattice theory can have various numbers of surface modes

The number of surface modes is determined by the topology of the bulk fermion dispersion relation in momentum space

Golterman, Jansen, DBK 1992
So far: only discussed **global** chiral symmetries

Suppose we arrange charges and chiralities of zeromodes so that the bulk CS current vanishes for the *gauge* current

Still have CS current for **global** symmetry charges

Simplest: $U(1)$ gauge theory with massless Dirac fermion at each defect

Or: $5^* + 10$ for $SU(5)$ gauge theory

Then each defect looks like a healthy (chiral) gauge theory by itself

Can we localize gauge fields on one defect?
Aside: translator for communication between particle & condensed matter physicists:

Material with topologically determined surface modes, but mass gap in the bulk = “topological insulator”

Quantized Chern-Simons current in the bulk = “integer quantum Hall current”

Dirac fermion surface modes with only bulk $U(1)_A$ current = “Quantum spin Hall effect” (Kane-Mele model, 2005)
Two main routes toward trying to put chiral gauge theories on the lattice:

Start with Dirac fermions = LH chiral fermions + RH mirror fermions and:

- Break the gauge symmetry explicitly to give the mirror fermions mass
- Make mirror fermions decouple in a gauge invariant way

Either way must fail if the LH chiral fermions are in an anomalous rep of gauge symmetry, since then the continuum theory does not exist.
Two main routes toward trying to put chiral gauge theories on the lattice:

Start with Dirac fermions = LH chiral fermions + RH mirror fermions and:

Make mirror fermions decouple in a gauge invariant way

This talk:

- Use domain wall fermions to separate LH and RH modes in 5th dimension
- Include 4d gauge fields that depend on extra dimension so that RH mirror modes don’t couple
Motivation: $D_\chi = \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ \partial_\mu \bar{\sigma}_\mu & 0 \end{pmatrix}$

Alvarez-Gaume et al. (1984,1986)

1st attempt: localizing gauge fields near one domain wall

Requires “Higgs” field at boundary to maintain gauge invariance
**Never works.**
One finds a Dirac fermion & vector-like gauge theory.

Golterman, Jansen, Vink 1993
New proposal: “localize” gauge fields using gradient flow

Dorota Grabowska, D.B.K.

Gradient flow smooths out fields by evolving them classically in an extra dimension via a heat equation

\[ \frac{\partial \bar{A}_\mu(x, t)}{\partial t} = -D_\nu \bar{F}_{\mu\nu} \]

**covariant flow eq.**

Boundary condition:

\[ \bar{A}_\mu(x, 0) = A_\mu(x) \]

4d world \( \xrightarrow{t} \) 5d bulk

\( A_\mu(x) \) lives on 4d boundary of 5d world

\( \bar{A}_\mu(x, t) \) lives in 5d bulk
Gradient flow (continuum version):

\[ \bar{A}_\mu(x, t) \text{ lives in 5d bulk} \]

\[ \begin{align*}
\frac{\partial \bar{A}_\mu(x, t)}{\partial t} &= -D_\nu \bar{F}_{\mu\nu} \\
\bar{A}_\mu(x, 0) &= A_\mu(x) \\
A_\mu(x) &\text{ lives on 4d boundary of 5d world}
\end{align*} \]

4d world \[ \rightarrow \]

\[ t \]

2d/3d U(1) example:

\[ A_\mu \equiv \partial_\mu \omega + \epsilon_{\mu\nu} \partial_\nu \lambda \Rightarrow \partial_t \bar{\omega} = 0, \quad \partial_t \bar{\lambda} = \Box \bar{\lambda} \]

Evolution in \( t \) damps out high momentum modes in physical degree of freedom only

\[ \bar{\lambda}(p, t) = \lambda(p) e^{-p^2 t} \]

This will allow \( \lambda(p) \) to be localized near \( t=0 \) while maintaining gauge invariance
Combining gradient flow gauge fields with domain wall fermions:

• quantum gauge field $A_\mu(x)$ lives at defect at $s=0$ where LH fermions live

• gauge field $A_\mu(x,s)$ defined as solution to gradient flow equation with BC: $A_\mu(x,0) = A_\mu(x)$

• flow equation is symmetric on both sides of defect

• RH mirror fermions behave as if with very soft form factor… “Fluff”… and decouple from gauge bosons

• gauge invariance maintained
• Mirror top quark (fluff)
• mass = 170 GeV
• couples only to radio waves?

• lattice gauge theorist
Decoupling mirror fermions as soft fluff in a gauge invariant way:

- Can show that this could only lead to a local 4d quantum field theory if the fermion representation has no gauge anomalies
- ...but exp(-p²t) form factors are a problem in Minkowski spacetime!
- Gradient flow doesn’t damp out instantons, which can induce interactions with fluff

\[
\frac{\partial \bar{A}_\mu(x,t)}{\partial t} = -D_\nu \bar{F}_{\mu\nu}
\]

Suggests taking L → ∞ limit first, before lattice spacing → 0
...gradient flow like a projection operator A → A★
We now have one weird theory in $L \rightarrow \infty$ limit!

Ordinary chiral gauge theory + mirror fermions

- mirror fermions have infinitely soft form factors (very nonlocal interactions) so they decouple from gauge bosons
- ...but still couple to gauge field topology
- ...but not in an extensive way (configuration with 201 instantons + 200 anti-instantons flows to 1 instanton)

Is $L \rightarrow \infty$ limit even practical? Yes, using “the overlap operator”
Remarkable 4D effective theory exists for surface modes for ordinary ordinary domain wall fermions (vector-like gauge theories) in $L \to \infty$ limit:

$$S_4 = \int d^4x \bar{\psi} D_V \psi$$

$$D_V = 1 + \gamma_5 \epsilon \quad \epsilon \equiv \epsilon(H_w) = \frac{H_w}{\sqrt{H_w^2}}$$

$$\gamma_5 H_w = \frac{1}{2} \gamma \mu (\nabla_\mu + \nabla^*_\mu) - \frac{1}{2} \nabla_\mu \nabla^*_\mu - m$$

properties:

$$\{D_V^{-1}, \gamma_5\} = a \gamma_5$$

$$\lim_{a \to 0} D_V = \frac{1}{am} \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ D_\mu \bar{\sigma}_\mu & 0 \end{pmatrix}$$

Neuberger, Narayanan 1993-1998

just right for U(1)_A anomaly!

desired continuum limit!
Follow same procedure for system with gauge field that changes in the extra dimension & take the $L \rightarrow \infty$ limit:

$$S_4 = \int d^4 x \bar{\psi} D_\chi \psi$$

$$D_\chi = 1 + \gamma_5 \left[ 1 - (1 - \epsilon_*) \frac{1}{\epsilon \epsilon_* + 1} (1 - \epsilon) \right]$$

$$\epsilon \equiv \epsilon(H_w[A]) \quad \epsilon_* \equiv \epsilon(H_w[A_*])$$

**properties:**

$$\{D_{V}^{-1}, \gamma_5\} = a \gamma_5$$

$$\lim_{a \rightarrow 0} D_V = \frac{1}{am} \begin{pmatrix} 0 & D_{\mu} \sigma_{\mu} \\ D_{\mu} \bar{\sigma}_{\mu} & 0 \end{pmatrix}$$

Neuberger, Narayanan 1993-1998
Can construct a gauge invariant overlap operator (DG, DBK, to appear):

\[ D_\chi = 1 + \gamma_5 \left[ 1 - (1 - \epsilon_*) \frac{1}{\epsilon \epsilon_* + 1} (1 - \epsilon) \right] \]

\[ \epsilon \equiv \epsilon(H_w[A]) \quad \epsilon_* \equiv \epsilon(H_w[A_*]) \]

- Obeys Ginsparg-Wilson eq. - U(1)_A
- Has continuum limit:

\[ D_\chi \to \begin{pmatrix} 0 & \sigma_\mu D_\mu(A) \\ \bar{\sigma}_\mu D_\mu(A_*) & 0 \end{pmatrix} \]

D. B. Kaplan ~ New Directions / Higgs Centre ~ 1/13/17
\[ D_{\chi} = 1 + \gamma_5 \left[ 1 - (1 - \epsilon_*) \frac{1}{\epsilon \epsilon_* + 1} (1 - \epsilon) \right] \]

\[ D_{\chi} \rightarrow \begin{pmatrix} 0 & \sigma_\mu D_\mu (A) \\ \bar{\sigma}_\mu D_\mu (A_*) & 0 \end{pmatrix} \]

- Looks like a LH Weyl fermion interacting with gauge field A,
- RH Weyl fermion interacting with gauge field A★

(This approximation of a sudden jump from A to A★ is an oversimplification of gradient flow, but tractable analytically)
Could fluff be real in the SM?
- Mirror fermions we cannot transfer momentum to
- Possible similarly nonstandard gravitational interactions
- Fluff does see global gauge field topology...solution to strong CP problem?

- What could go wrong throwing away locality?

Not clear this approach will work for regulating chiral gauge theories on the lattice...but if it does, should we take fluff seriously?