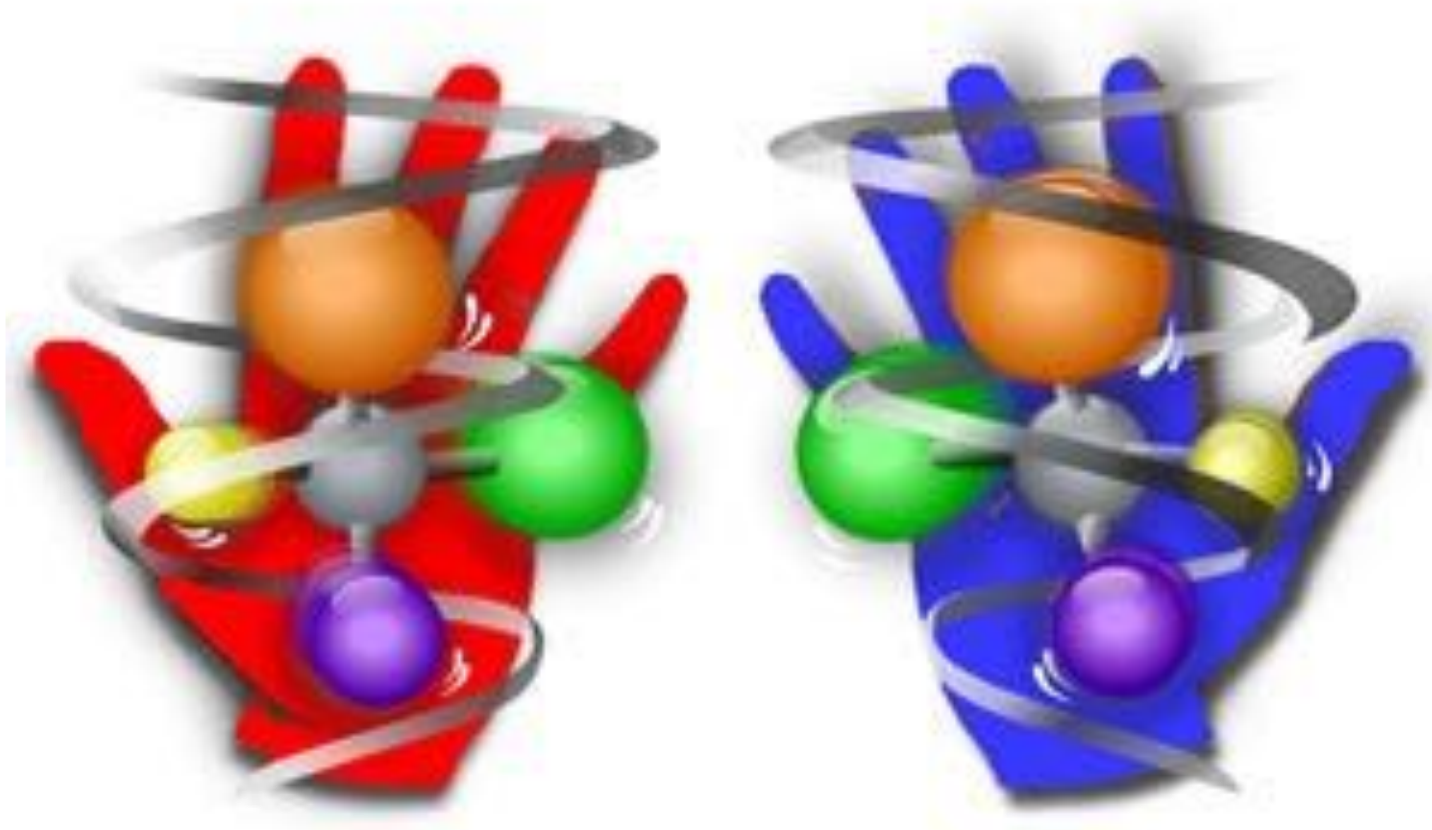
A Lattice Formulation for Chiral Gauge Theory?



One of the great surprises of 20th century particle physics was the discovery of parity violation: LH and RH fermions do not carry the same gauge interactions:

PHYSICAL REVIEW VOLUME 104, NUMBER 1 OCTOBER 1, 1956

Question of Parity Conservation in Weak Interactions*

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AND

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(Received June 22, 1956)

The question of parity conservation in β decays and in hyperon and meson decays is examined. Possible experiments are suggested which might test parity conservation in these interactions.

PNAS 1928 14 (7) 544-549

APPARENT EVIDENCE OF POLARIZATION IN A BEAM OF β-RAYS

BY R. T. COX, C. G. MCILWRAITH AND B. KURRELMEYER*

NEW YORK UNIVERSITY AND COLUMBIA UNIVERSITY†

Communicated June 6, 1928

The SM is the only chiral gauge theory we have seen…
… but GUTs are χGTs, as are many other speculative models of BSM physics

Theoretically chiral gauge theories are in bad shape:

- There does not exist a nonperturbative regulator
- There isn't an all-orders proof for a *perturbative* regulator

Nonperturbative definition ⇒

- unexpected phenomenology?
- answers to outstanding puzzles (e.g., CP problem)?

The Problem:

A vector-like gauge theory like QCD consists of Dirac fermions,
= Weyl fermions in a real representation of the gauge group.

$$Z_V = \int [dA] e^{-S_{YM}} \prod_{i=1}^{N_f} \det(\not{D} - m_i)$$

A chiral gauge theory consists of Weyl fermions in a complex representation of the gauge group.

$$Z_\chi = \int [dA] e^{-S_{YM}} \Delta[A]$$

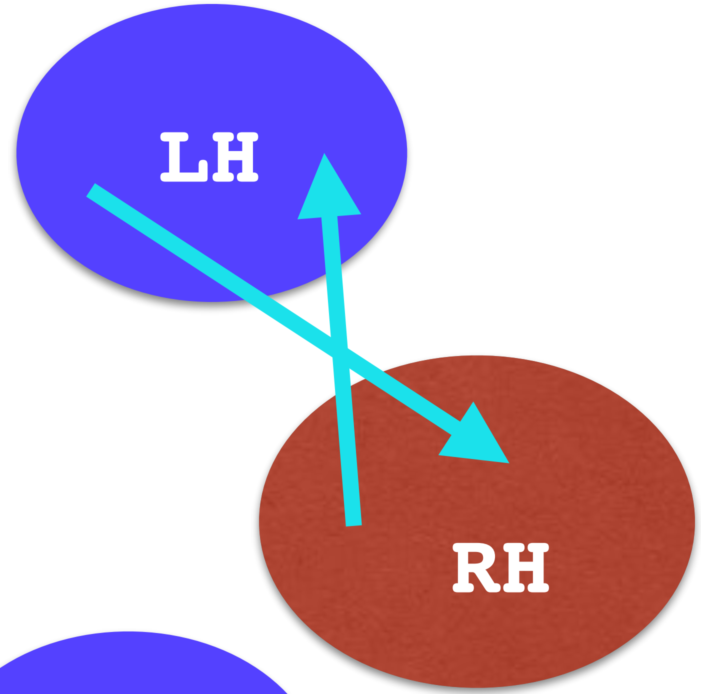
?

Witten: "We often call the fermion path integral a 'determinant' or a 'Pfaffian', but this is a term of art."

We mean a product of eigenvalues...
...but there is no well-defined eigenvalue problem for a chiral theory

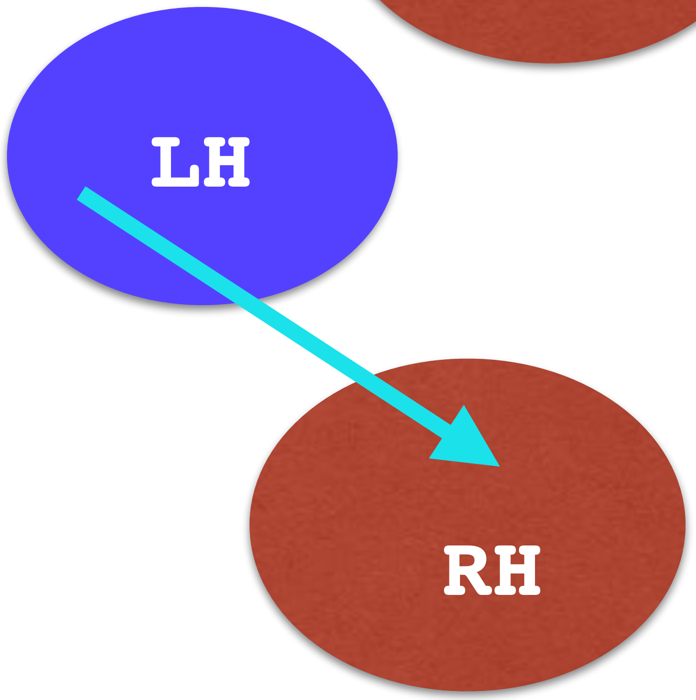
Vector-like gauge theory with Dirac fermions:

$$\not{D}\psi = \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ D_\mu \bar{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \lambda \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$



Chiral gauge theory with Weyl fermions:

$$\begin{pmatrix} 0 & D_\mu \sigma_\mu \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix}$$



Independent unitary basis changes in LH and RH spaces leads to a phase ambiguity for the determinant.

So: $\Delta[A] = e^{i\delta[A]} \sqrt{|\det \not{D}|}$

The fermion integral for a χGT: $$\Delta[A] = e^{i\delta[A]}\sqrt{|\det \not{D}|}$$

Alvarez-Gaume et al. proposal for perturbative definition (1984,1986):

$$\Delta[A] \equiv \det \begin{pmatrix} 0 & D_\mu\sigma_\mu \\ \partial_\mu\bar{\sigma}_\mu & 0 \end{pmatrix}$$

gauged LH Weyl fermion

neutral RH Weyl fermion

Well-defined eigenvalue problem with complex (gauge variant) eigenvalues

Extra RH fermions decouple

But only perturbative...amenable to lattice regularization?

The basic problem with regulating chiral gauge theories:

Need to render phase space for particles finite

This typically requires a mass scale

Fermion masses violate chiral symmetry...which has been gauged

One could break gauge symmetry explicitly...but how to restore it?

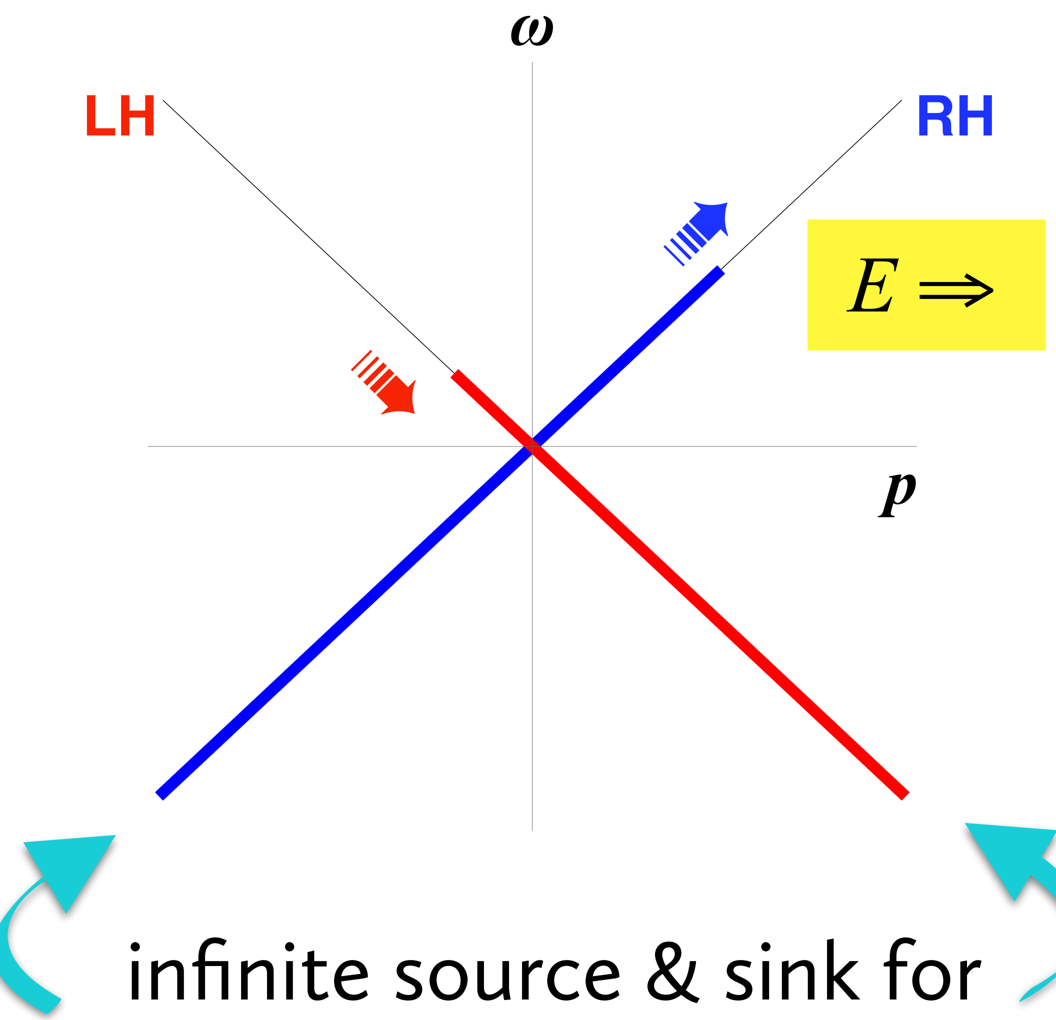
One could introduce mirror fermions to allow mass terms...but how to decouple them?

At the perturbative level, even dimensional regularization is problematic: chirality only exists in even spacetime dimensions

The anomaly is key to representing chiral symmetry on the lattice (global or gauged)

One way of looking at anomalies:

massless Dirac fermions in an electric field E, 1+1 dim



infinite source & sink for fermions

$$dp = qE\,dt$$

$$dn_R = +\frac{dp}{2\pi} \qquad dn_L = -\frac{dp}{2\pi}$$

$$\frac{dn_5}{dt} = \frac{qE}{\pi}$$

$$\partial_\mu j_5^\mu = \frac{qE}{\pi}$$

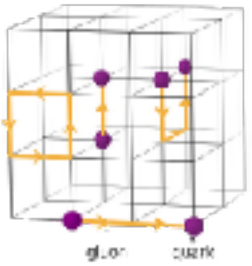
d=1+1 anomaly

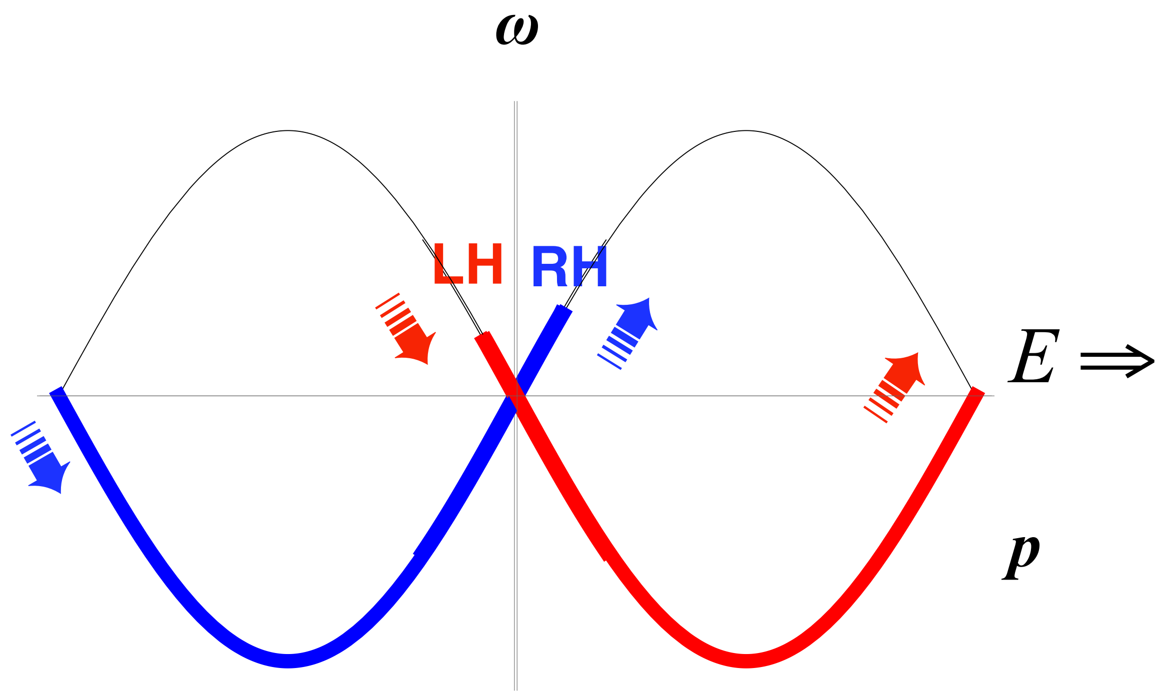
quantum violation of a classical U(1)$_A$ symmetry

In the continuum, the Dirac sea is filled…but is a Hilbert Hotel which always has room for more

Not so on the lattice:

Can reproduce continuum physics for long wavelength modes…





…but **no** anomalies in a system with a finite number of degrees of freedom

$$\partial_\mu j_5^\mu = 0$$

<u>anomalous symmetry</u> in the continuum

must be

<u>explicitly broken symmetry</u> on the lattice

The Nielsen-Ninomiya Theorem:

The Euclidian fermion action:

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4}\,\overline{\Psi}_{-\mathbf{p}}\tilde{D}(\mathbf{p})\Psi(\mathbf{p})$$

cannot have a kinetic operator D satisfying all four of the following properties simultaneously:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ; $\Leftarrow$ regulated, local
2. $D(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$; $\Leftarrow$ Dirac @ long wavelength
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$; $\Leftarrow$ No doubling of flavors
4. $\cancel{\{\Gamma, \tilde{D}(\mathbf{p})\} = 0}$. $\Leftarrow$ respects a chiral symmetry

Advances in the 1990s showed us how to break global chiral symmetry in just the right way for QCD…but will be problematic when chiral symmetry is gauged!

How **Wilson fermions** reproduce the $U(1)_A$ anomaly in QCD:

Karsten, Smit 1980

Lattice covariant derivatives

$$\mathcal{L} = \bar\psi\left(\not{D} + m + aD^2\right)\psi$$

violates chiral symmetry

- Wilson fermions eliminate doublers by giving them a big mass
- Price paid: mass & Wilson terms explicitly break the (global) chiral flavor symmetries
- fine tune $m \sim 1/a$ to continuum limit...find some of the chiral symmetry breaking does not decouple & correct anomalous Ward identities are found
- **Lost**: the benefits of chiral symmetry - multiplicative mass renormalization, non-mixing of operators...

Domain Wall Fermions

Consider fermions in Euclidian 5d with a 4d gauge field:

Compact extra dimension: $-L < s < L \ , \ \psi(-L) = \psi(L)$

$$\text{mass} = \Lambda\epsilon(s)$$

mass flips sign at s=0,±L

$$S_5 = \int d^4x \oint ds\ \bar{\Psi}\left[\not{D}_4 + \gamma_5\partial_s - \Lambda\epsilon(s)\right]\Psi$$

Expand Ψ in terms of 4d fields:

Define: $Q \equiv -\partial_s - \Lambda\epsilon(s)$

Solve: $Qf_n(s) = \mu_n b_n(s) \ , \quad Q^\dagger b_n(s) = \mu_n f_n(s)$

Expand: $$\Psi(x,s) = \sum_{n=0}^{\infty}\left[P_-\psi_n(x)b_n(s) + P_+\psi_n(x)f_n(s)\right]$$

4d Dirac spinors

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

$$
\begin{aligned}
S_5 &= \int d^4x \oint ds\; \bar{\Psi}\left[\not{D}_4 + \gamma_5 \partial_s - \Lambda \epsilon(s) \right] \Psi \\
&= \sum_{n=0}^{\infty} \int d^4x\, \bar{\psi}_n \left[\not{D}_4 - \mu_n \right] \psi_n
\end{aligned}
$$

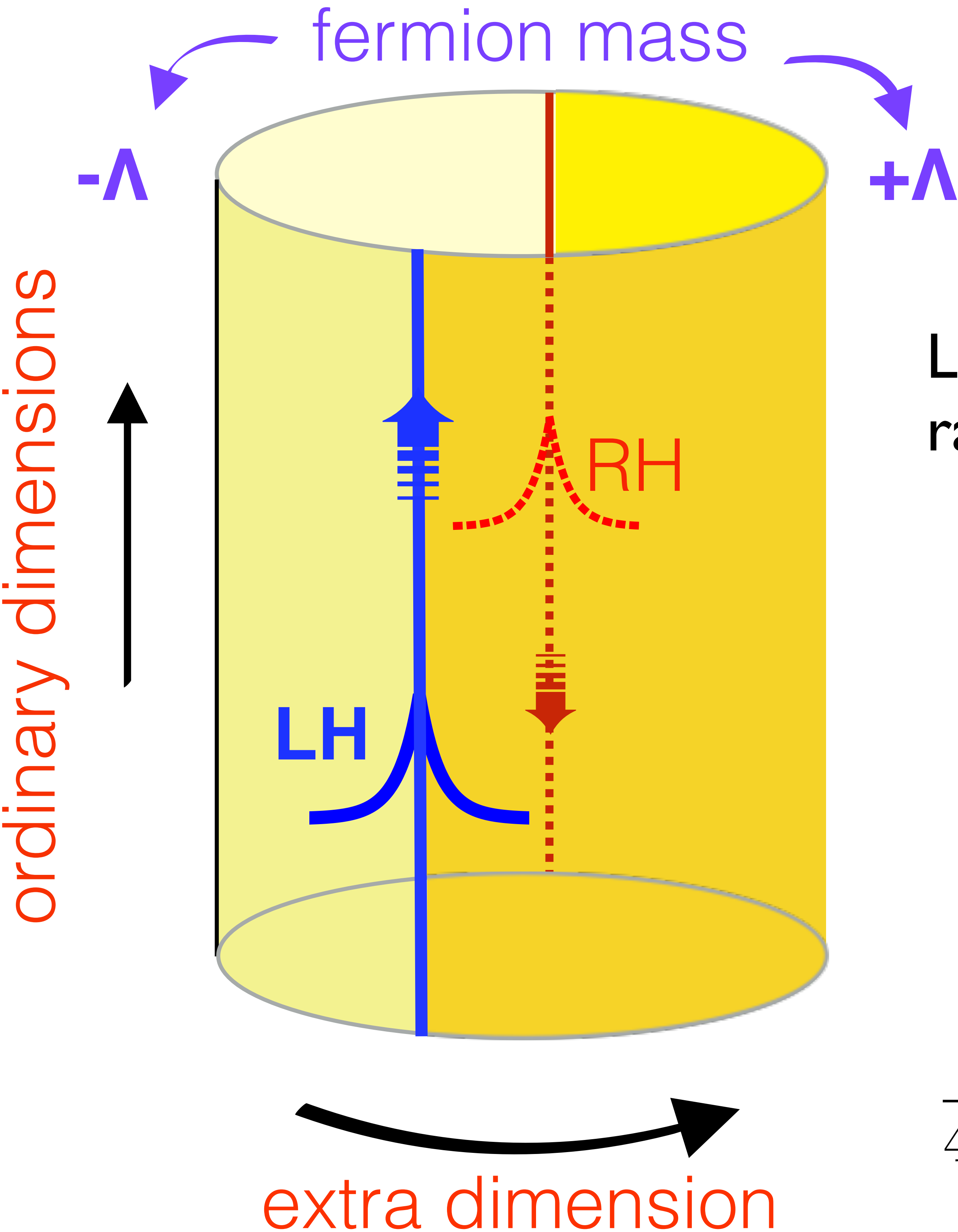
Special zero mode solutions (μ_0=0):

$$f_0 \propto e^{-\int^s ds' \, \Lambda \epsilon(s')} \qquad b_0 \propto e^{+\int^s ds' \, \Lambda \epsilon(s')} \qquad \mu_0 = 0 \;, \quad \mu_{1,2,\ldots} \geq \Lambda$$

Localized at s=0   Localized at s=±L

Looks like a 4d theory with one massless fermion and an infinite tower of heavy states.

The massless fermion has a **chiral symmetry**
So what?
In 5d, no chiral symmetry!
LH and RH parts of the massless fermion are physically separated in s!



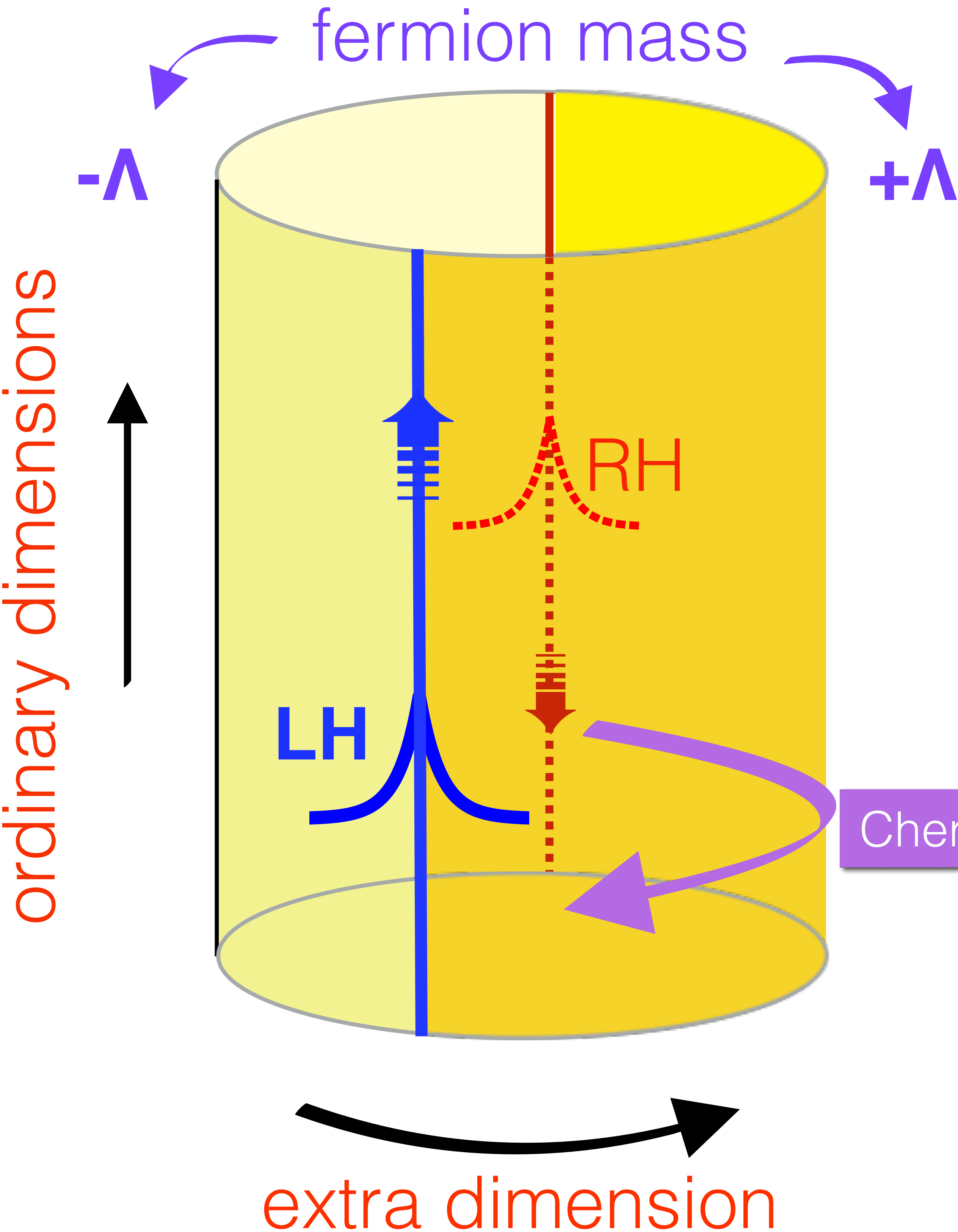
Light fermion mass stable against radiative corrections. E.g. add:

$$\int d^5x\,\frac{\alpha}{4\pi}\Lambda\bar{\Psi}\Psi$$

To O(α) the mass induced for the surface modes is:

$$\underbrace{\frac{\alpha}{4\pi}\Lambda\oint ds\, f_0(s)b_0(s)}_{\text{wavefunction overlap}} \simeq \frac{\alpha}{4\pi}L\Lambda^2 e^{-L\Lambda} \xrightarrow[L\to\infty]{} 0$$

In large L limit, surface modes remain massless, even with radiative corrections



But how do anomalies work (eg, U(1)$_A$)?

For effective 4d theory, must have:

$$\partial_\mu j_5^\mu \propto F\tilde{F} \ , \quad \partial_\mu j^\mu = 0$$

…but no U(1) anomalies in 5d theory!

Callan-Harvey (1984):

- integrating out heavy modes induces Chern-Simons term in bulk
- allows fermion current between defects

Callan Harvey argument:

Integrating out heavy fermion modes induces CS term.
(For now: 5d background gauge fields)

$$\frac{m(s)}{|m(s)|}\epsilon_{abdce}A_a\partial_b A_c\partial_d A_e$$

Differentiate w.r.t. A_5 to get J_5

$$J_5 \propto \frac{m(s)}{|m(s)|}F\tilde{F} \implies \partial_5 J_5 \propto \left[\delta(s)-\delta(s-L)\right]F\tilde{F}$$

Bulk current explains anomalous disappearance of charge on one defect and reappearance on the other

Can implement on the lattice without doublers: DBK 1992

$$S_5 = \bar{\Psi}\left[\partial_s\gamma_5 + \not{D}_4 - \Lambda\epsilon(s) + \frac{1}{\Lambda}\underbrace{(\partial_s\gamma_5 + \not{D}_4)^2}_{\text{Wilson term}}\right]\Psi$$

Λ = (lattice spacing)$^{-1}$

Nielsen- Ninomiya?

⇒ chiral symmetry is explicitly broken

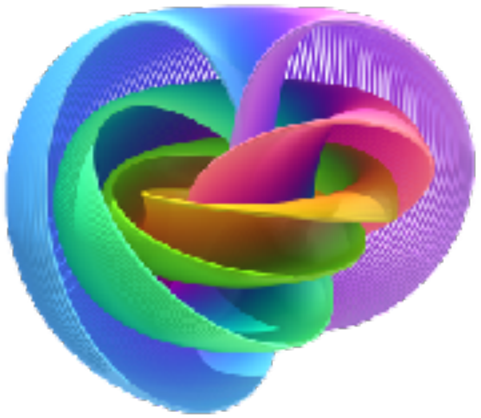
Only relevant effect of explicit chiral symmetry breaking?

⇒ the $U(1)_A$ anomaly

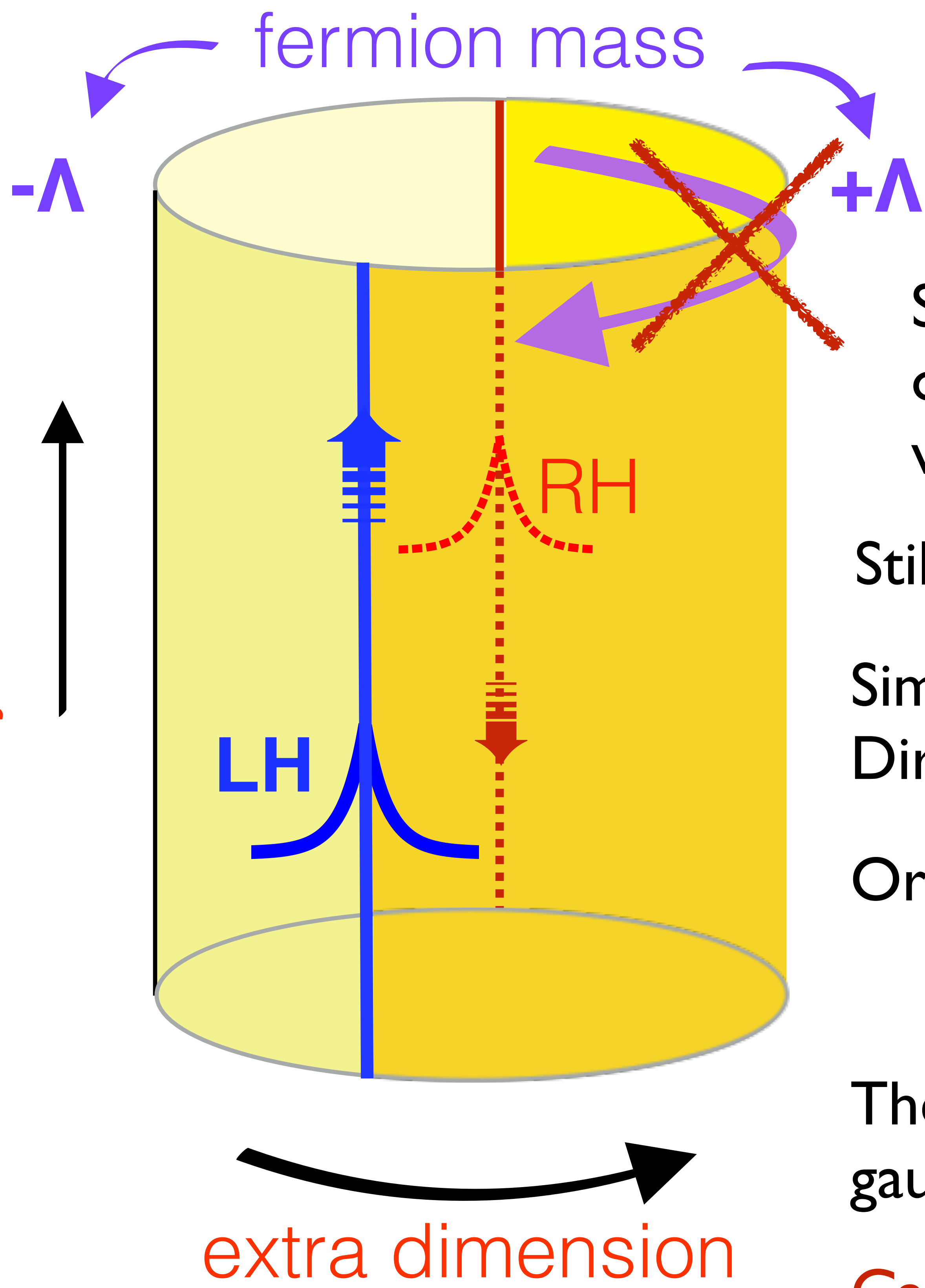
In general lattice theory can have various numbers of surface modes

The number of surface modes is determined by the topology of the bulk fermion dispersion relation in momentum space

Golterman, Jansen, DBK 1992



So far: only discussed global chiral symmetries



Suppose we arrange charges and chiralities of zeromodes so that the bulk CS current vanishes for the gauge current

Still have CS current for *global* symmetry charges

Simplest: U(1) gauge theory with massless Dirac fermion at each defect

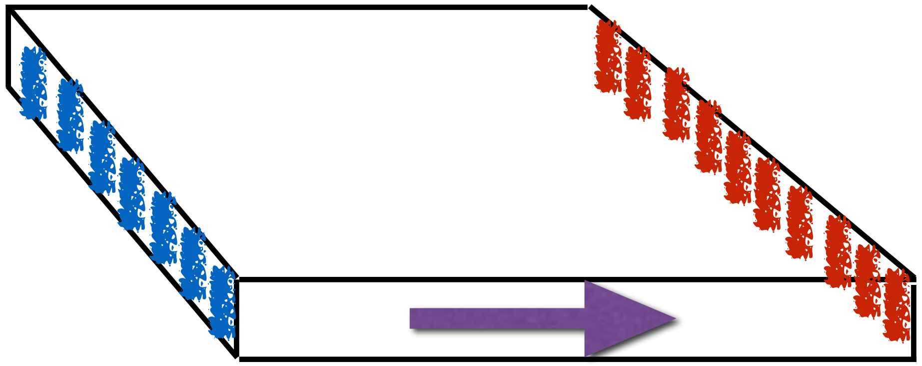
Or: 5*+10 for SU(5) gauge theory

Then each defect looks like a healthy (chiral) gauge theory by itself

Can we localize gauge fields on one defect?

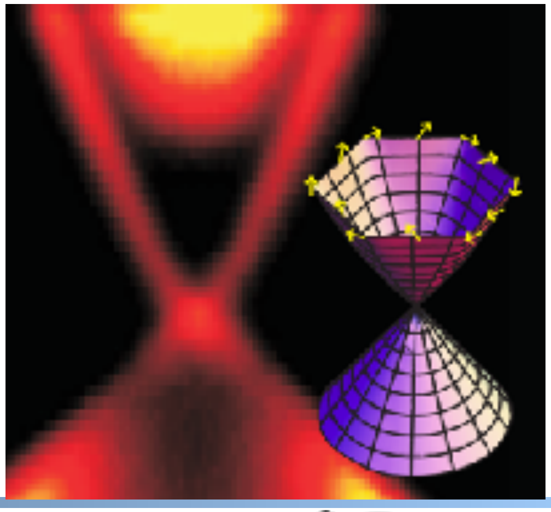
Aside: translator for communication between particle & condensed matter physicists:

Material with topologically determined surface modes, but mass gap in the bulk = "topological insulator"



Quantized Chern-Simons current in the bulk = "integer quantum Hall current"

Dirac fermion surface modes with only bulk $U(1)_A$ current = "Quantum spin Hall effect" (Kane-Mele model, 2005)



Two main routes toward trying to put chiral gauge theories on the lattice:

Start with Dirac fermions = LH chiral fermions + RH mirror fermions and:

Break the gauge symmetry explicitly to give the mirror fermions mass

Make mirror fermions decouple in a gauge invariant way

Either way must fail if the LH chiral fermions are in an anomalous rep of gauge symmetry, since then the continuum theory does not exist.

Two main routes toward trying to put chiral gauge theories on the lattice:

Start with Dirac fermions = LH chiral fermions + RH mirror fermions and:

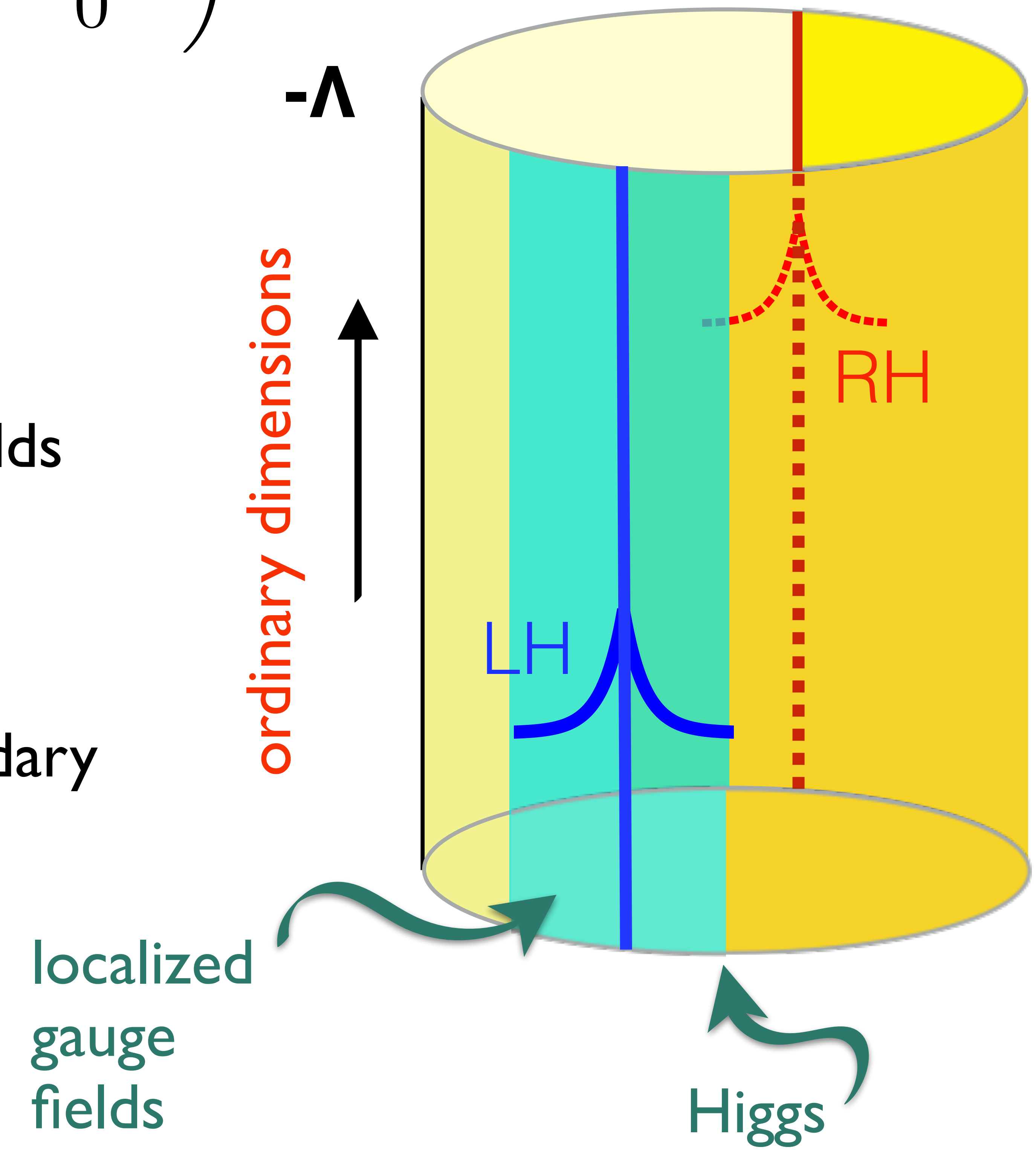
This talk:

- Use domain wall fermions to separate LH and RH modes in 5th dimension
- include 4d gauge fields that depend on extra dimension so that RH mirror modes don't couple

Make mirror fermions decouple in a gauge invariant way

Motivation: $\mathcal{D}_\chi = \begin{pmatrix} 0 & D_\mu \sigma_\mu \\ \partial_\mu \bar{\sigma}_\mu & 0 \end{pmatrix}$

Alvarez-Gaume et al. (1984,1986)



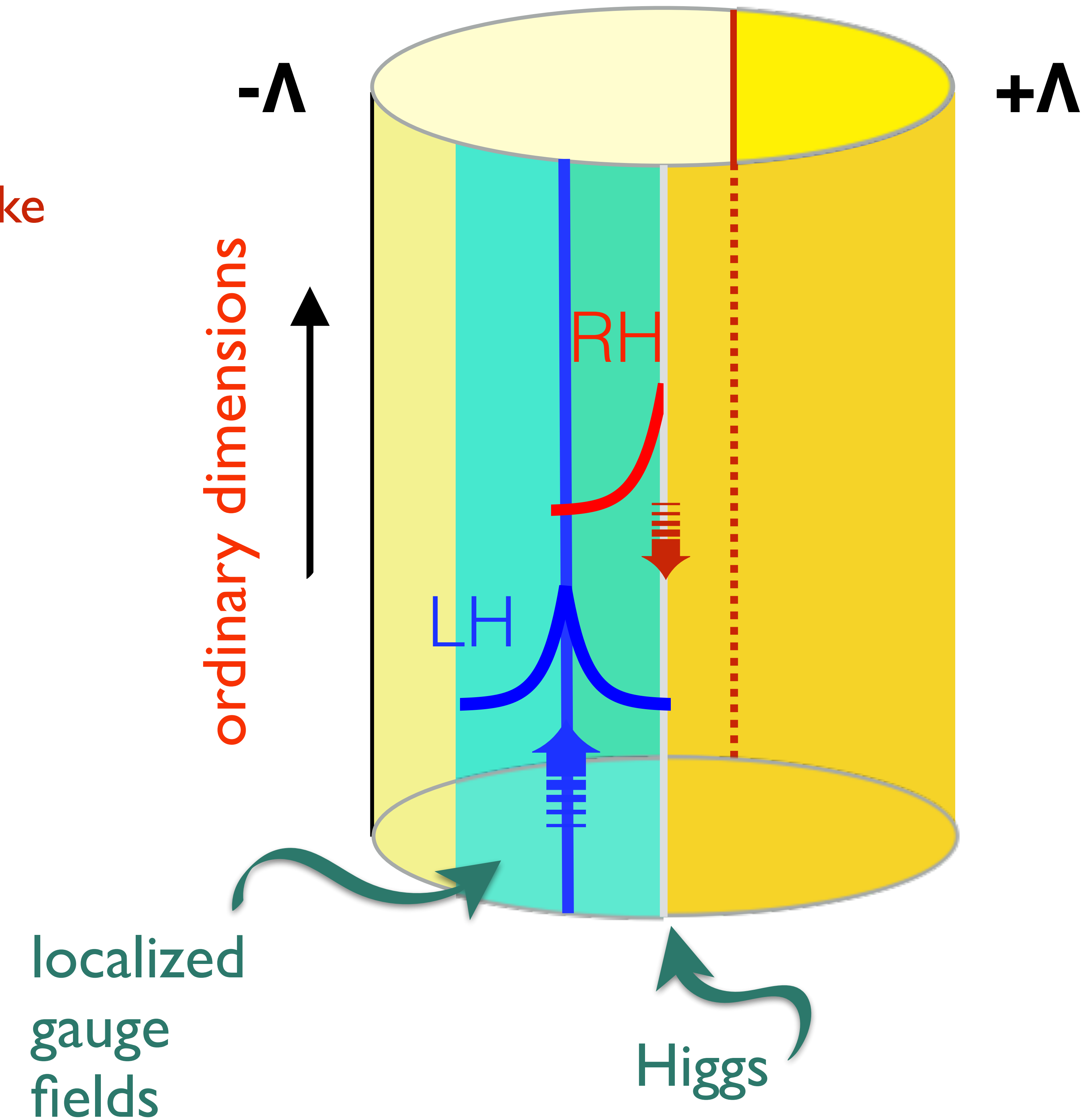
1[st] attempt: localizing gauge fields near one domain wall

Requires “Higgs” field at boundary to maintain gauge invariance

Never works.

One finds a Dirac fermion & vector-like gauge theory.

Golterman, Jansen, Vink 1993

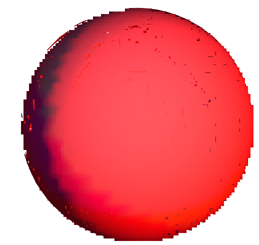


New proposal: "localize" gauge fields using gradient flow

Dorota Grabowska, D.B.K.

- Phys.Rev.Lett. 116 211602 (2016) [arXiv:1511.03649]
- Phys.Rev. D94 (2016) no.11, 114504 [arXiv:1610.02151]

Gradient flow smooths out fields by evolving them classically in an extra dimension via a heat equation



4d world

$\bar{A}_\mu(x,t)$ lives in 5d bulk

t

$$\frac{\partial \bar{A}_\mu(x,t)}{\partial t} = -D_\nu \bar{F}_{\mu\nu}$$ covariant flow eq.

$$\bar{A}_\mu(x,0) = A_\mu(x)$$ boundary condition

$A_\mu(x)$ lives on 4d boundary of 5d world

Gradient flow (continuum version):

4d world

$\bar{A}_\mu(x,t)$ lives in 5d bulk

t

$$\frac{\partial \bar{A}_\mu(x,t)}{\partial t} = -D_\nu \bar{F}_{\mu\nu}$$ covariant flow eq.

$$\bar{A}_\mu(x,0) = A_\mu(x)$$ boundary condition

$A_\mu(x)$ lives on 4d boundary of 5d world

2d/3d U(1) example:

$$A_\mu \equiv \partial_\mu \omega + \epsilon_{\mu\nu}\partial_\nu \lambda \quad \Rightarrow \quad \partial_t \bar{\omega} = 0\ , \quad \partial_t \bar{\lambda} = \Box \bar{\lambda}$$

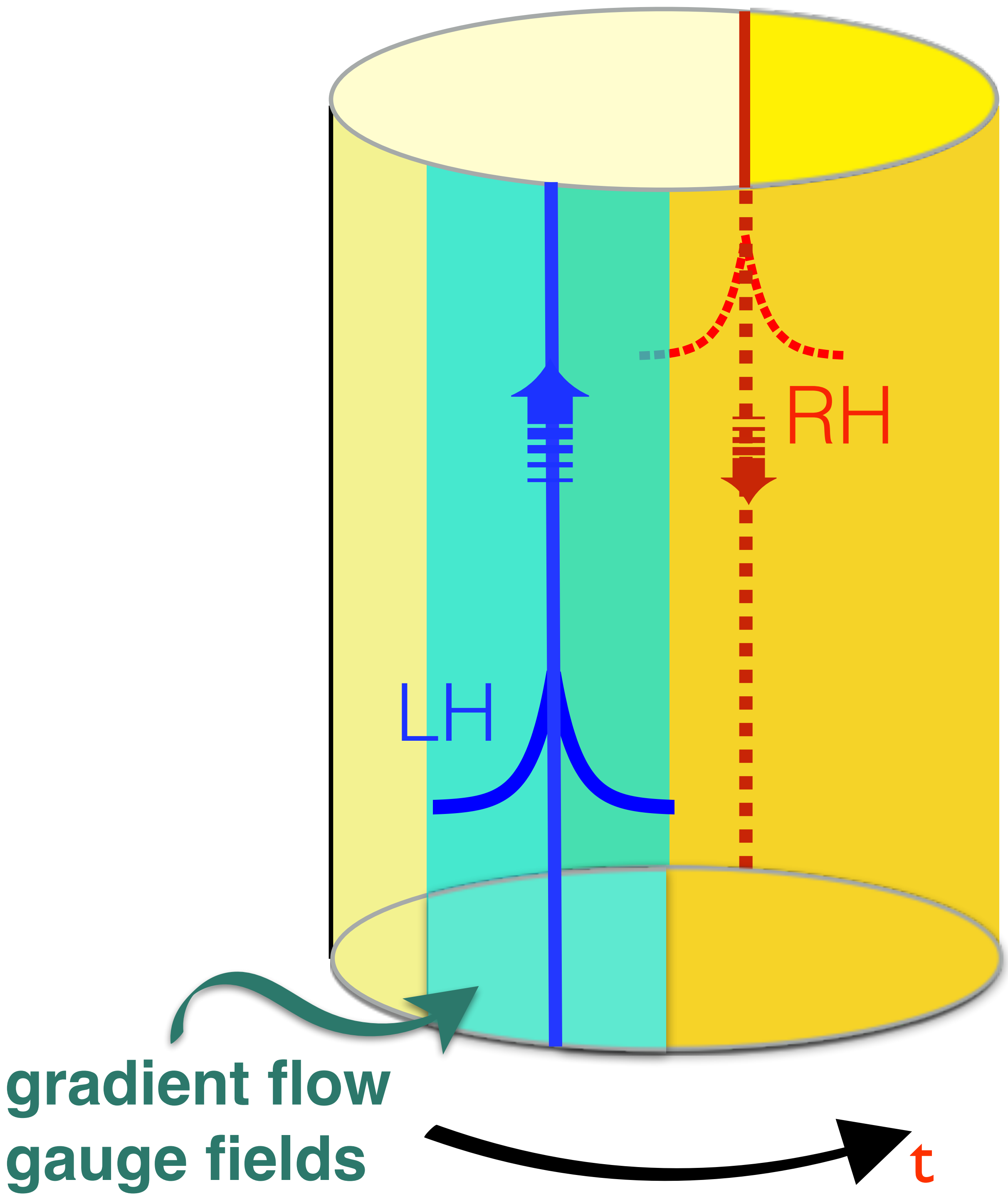
Evolution in *t* damps out high momentum modes in physical degree of freedom only

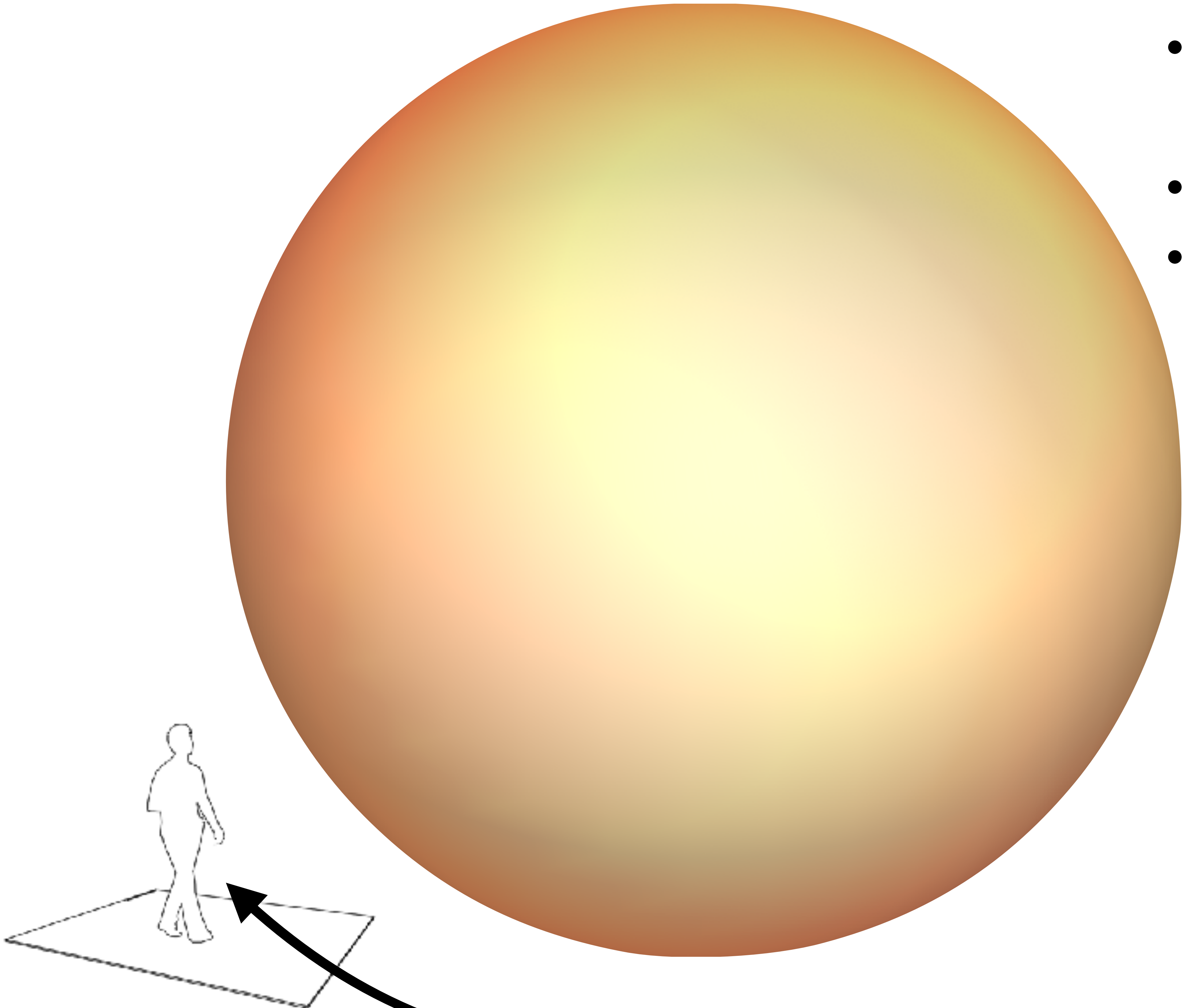
$$\bar{\lambda}(p,t) = \lambda(p) e^{-p^2 t}$$

This will allow λ(p) to be localized near t=0 while maintaining gauge invariance

Combining gradient flow gauge fields with domain wall fermions:

- quantum gauge field $A_\mu(x)$ lives at defect at s=0 where LH fermions live
- gauge field $\mathbf{A}_\mu(x,s)$ defined as solution to gradient flow equation with BC: $\mathbf{A}_\mu(x,0) = A_\mu(x)$
- flow equation is symmetric on both sides of defect
- RH mirror fermions behave as if with very soft form factor… "Fluff"…and decouple from gauge bosons
- gauge invariance maintained





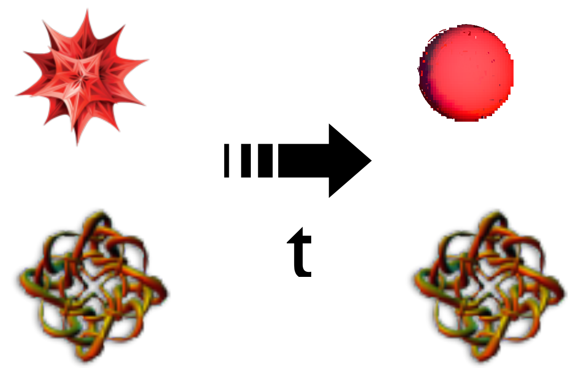
- Mirror top quark (fluff)
- mass = 170 GeV
- couples only to radio waves?

- lattice gauge theorist

Decoupling mirror fermions as soft fluff in a gauge invariant way:

- Can show that this could only lead to a local 4d quantum field theory if the fermion representation has no gauge anomalies
- ...but $\exp(-p^2 t)$ form factors are a problem in Minkowski spacetime!
- gradient flow doesn't damp out instantons, which can induce interactions with fluff

$$\frac{\partial \bar{A}_\mu(x,t)}{\partial t} = -D_\nu \bar{F}_{\mu\nu}$$



Suggests taking L → ∞ limit first, before lattice spacing —> 0

...gradient flow like a projection operator A → A★

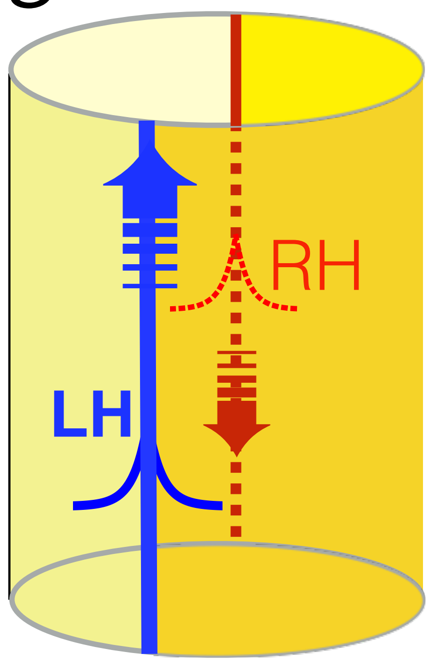
We now have one weird theory in L —> ∞ limit !

Ordinary chiral gauge theory + mirror fermions

- mirror fermions have infinitely soft form factors (very nonlocal interactions) so they decouple from gauge bosons
- ...but still couple to gauge field topology
- ...but not in an extensive way (configuration with 201 instantons + 200 anti-instantons flows to 1 instanton)

Is $L \to \infty$ limit even practical? Yes, using "the overlap operator"

Remarkable 4D effective theory exists for surface modes for ordinary ordinary domain wall fermions (vector-like gauge theories) in L —> ∞ limit:



$$S_4 = \int d^4x\, \bar\psi \mathcal{D}_V \psi$$

$$\mathcal{D}_V = 1 + \gamma_5 \epsilon \qquad \epsilon \equiv \epsilon(H_w) = \frac{H_w}{\sqrt{H_w^2}}$$

$$\gamma_5 H_w = \left[\tfrac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \tfrac{1}{2}\nabla_\mu \nabla^*_\mu - m\right]$$

properties:

$$\{\mathcal{D}_V^{-1}, \gamma_5\} = a\gamma_5$$

just right for $U(1)_A$ anomaly!

$$\lim_{a\to 0} \mathcal{D}_V = \frac{1}{am}\begin{pmatrix} 0 & D_\mu \sigma_\mu \\ D_\mu \bar\sigma_\mu & 0 \end{pmatrix}$$

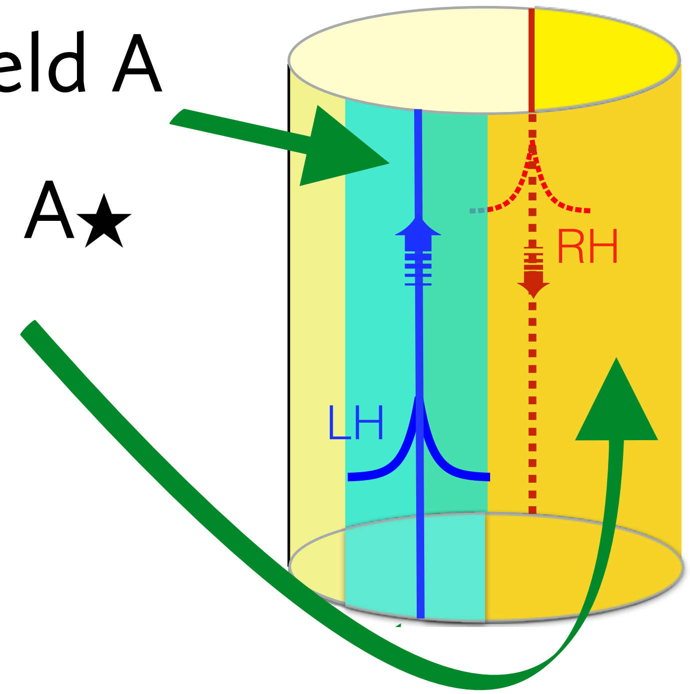
desired continuum limit!

Neuberger, Narayanan 1993-1998

Follow same procedure for system with gauge field that changes in the extra dimension & take the L → ∞ limit:

gauge field A

gauge field A★



$$S_4 = \int d^4x\, \bar\psi \mathcal{D}_\chi \psi$$

$$\mathcal{D}_\chi = 1 + \gamma_5 \left[1 - (1-\epsilon_\star)\frac{1}{\epsilon\,\epsilon_\star + 1}(1-\epsilon)\right]$$

$$\epsilon \equiv \epsilon(H_w[A]) \qquad \epsilon_\star \equiv \epsilon(H_w[A_\star])$$

properties:

$$\{\mathcal{D}_V^{-1}, \gamma_5\} = a\gamma_5$$

just right for anomaly!

$$\lim_{a\to 0} \mathcal{D}_V = \frac{1}{am}\begin{pmatrix} 0 & D_\mu\sigma_\mu \\ D_\mu\bar\sigma_\mu & 0 \end{pmatrix}$$

right continuum limit!

Neuberger, Narayanan 1993-1998

Can construct a gauge invariant overlap operator

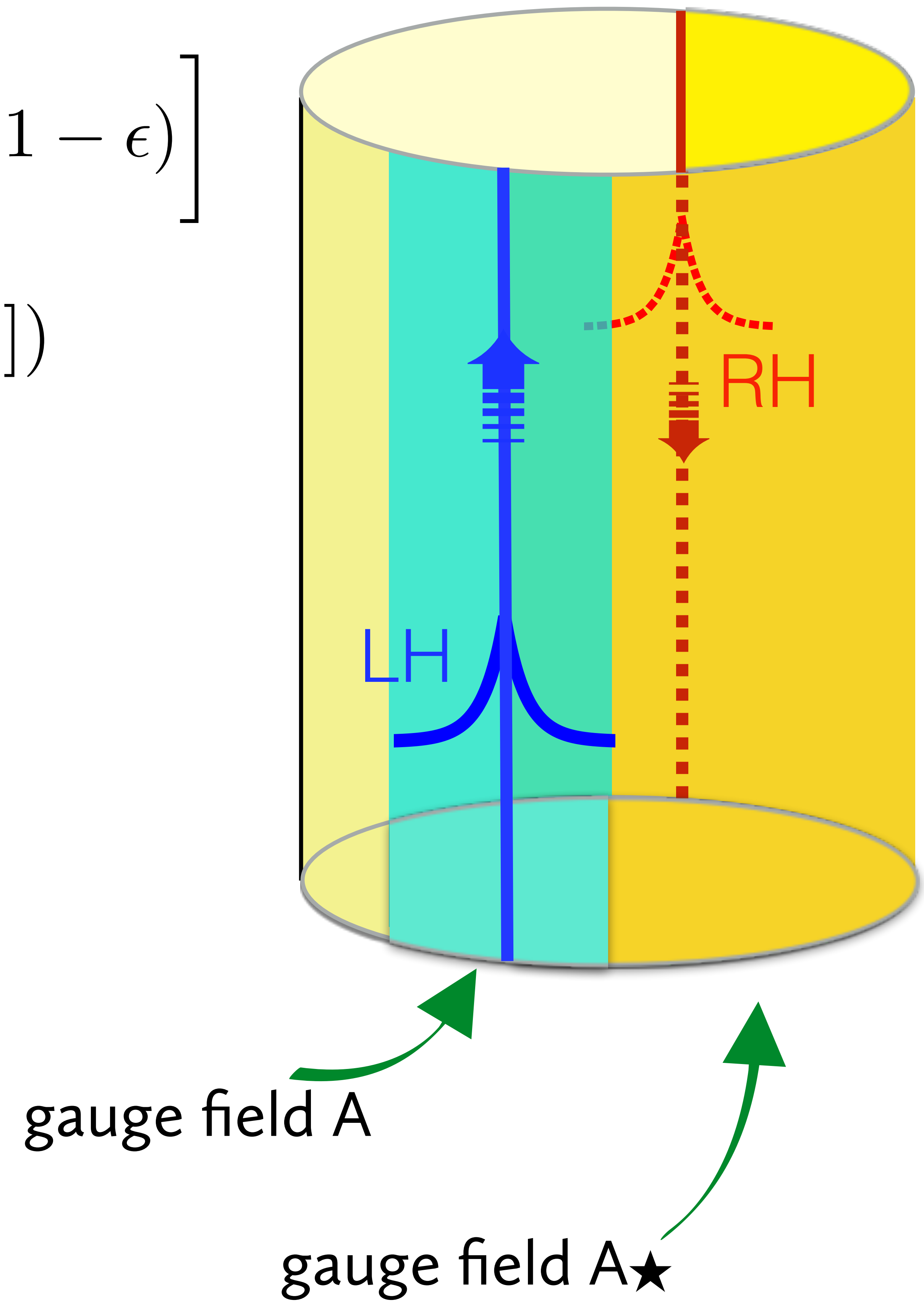
(DG, DBK, to appear):

$$\mathcal{D}_\chi = 1 + \gamma_5 \left[1 - (1-\epsilon_\star)\,\frac{1}{\epsilon\,\epsilon_\star + 1}\,(1-\epsilon) \right]$$

$$\epsilon \equiv \epsilon(H_w[A]) \qquad \epsilon_\star \equiv \epsilon(H_w[A_\star])$$

- Obeys Ginsparg-Wilson eq. - $U(1)_A$
- Has continuum limit:

$$\mathcal{D}_\chi \to \begin{pmatrix} 0 & \sigma_\mu D_\mu(A) \\ \bar{\sigma}_\mu D_\mu(A_\star) & 0 \end{pmatrix}$$



$$\mathcal{D}_\chi = 1 + \gamma_5 \left[1 - (1-\epsilon_\star) \frac{1}{\epsilon\, \epsilon_\star + 1} (1-\epsilon) \right]$$

$$\mathcal{D}_\chi \to \begin{pmatrix} 0 & \sigma_\mu D_\mu(A) \\ \bar{\sigma}_\mu D_\mu(A_\star) & 0 \end{pmatrix}$$

Chiral overlap operator

- Looks like a LH Weyl fermion interacting with gauge field A,
- RH Weyl fermion interacting with gauge field A★

(This approximation of a sudden jump from A to A★ is an oversimplification of gradient flow, but tractable analytically)

Not clear this approach will work for regulating chiral gauge theories on the lattice…but if it does, should we take fluff seriously?

Could fluff be real in the SM?

- Mirror fermions we cannot transfer momentum to
- Possible similarly nonstandard gravitational interactions
- Fluff does see global gauge field topology…solution to strong CP problem?

- What could go wrong throwing away locality?