Random tensor networks and holographic duality

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Outline

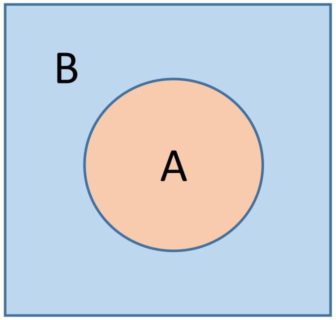
- Quantum entanglement and tensor networks
- Holographic duality
- Random tensor networks (RTN)
  - Properties of a fixed graph
  - RTN on all graphs as *geometry coherent states*

Reference

- Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, Zhao Yang, arxiv: 1601.01694, JHEP 11 (2016) 009
- XLQ, Zhao Yang, Yi-Zhuang You, in preparation

Quantum entanglement

- "spooky action at a distance" (Einstein). Consequence of quantum linear superposition principle.
- Simplest bipartite entanglement
- $|\Psi_{AB}\rangle = \sum_n \sqrt{p_n} |n_A\rangle |n_B\rangle$
- Entanglement entropy $S_E = -\sum_n p_n \log p_n$
- Example: EPR pair $p_n = \frac{1}{N}, S_E = \log N$



$|EPR\rangle =$

Tensor networks

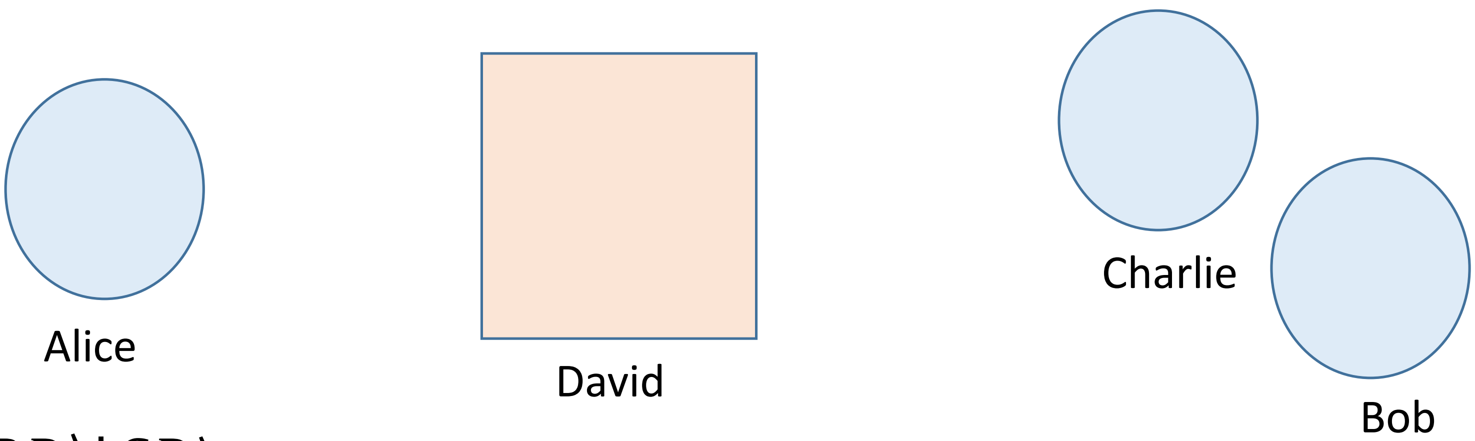
- Building many-body entangled states from few-qubit building blocks.
- Tensor contraction just like in Feynman diagrams

$T_{1\mu\nu\tau}|\mu\nu\tau\rangle = |V_1\rangle$ $T_{2\alpha\beta\gamma}|\alpha\beta\gamma\rangle = |V_2\rangle$ $|L\rangle = g^{\mu\beta}|\mu\beta\rangle$

$$|\Psi\rangle = T_1^{\mu\nu\tau} T_{2\ \mu}^{\ \alpha\gamma} |\mu\nu\alpha\gamma\rangle$$

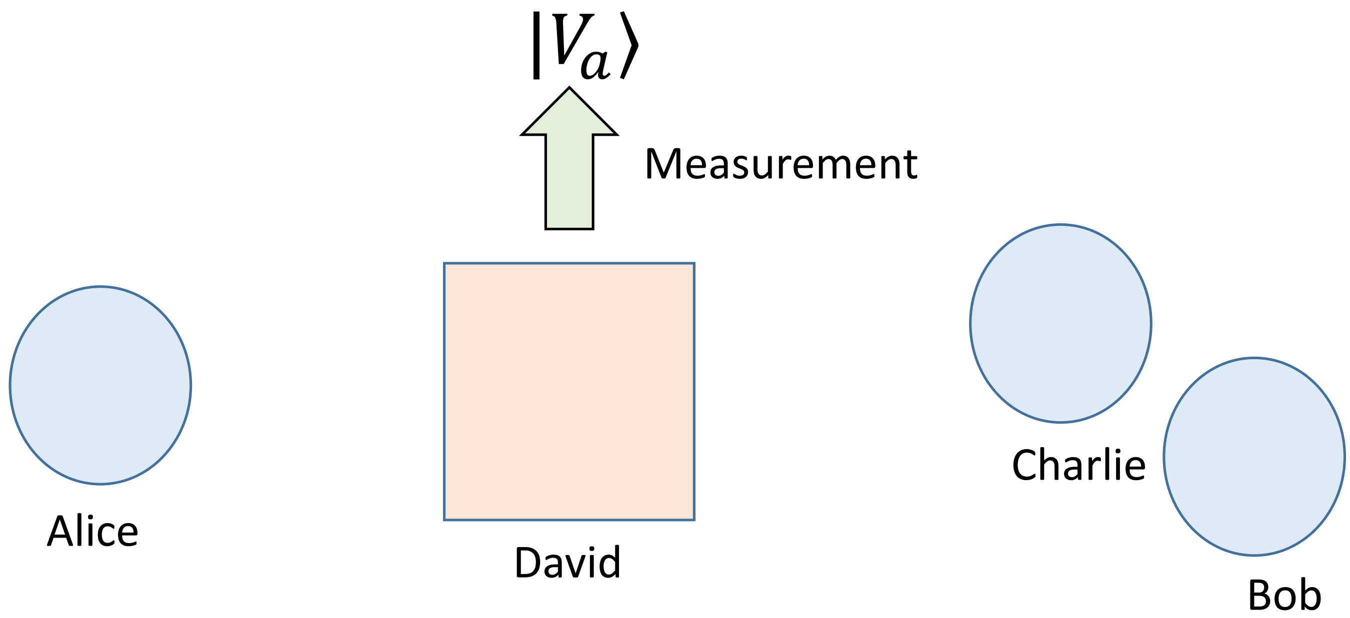
Tensor networks: Physical interpretation

- Projected Entangled Pair States (PEPS) F. Verstraete, J.I. Cirac, 04'
- 1. Prepare and distribute EPR pairs. Alice, Bob and Charlie are all entangled with David, but not with each other.



$$|AD\rangle|BD\rangle|CD\rangle$$

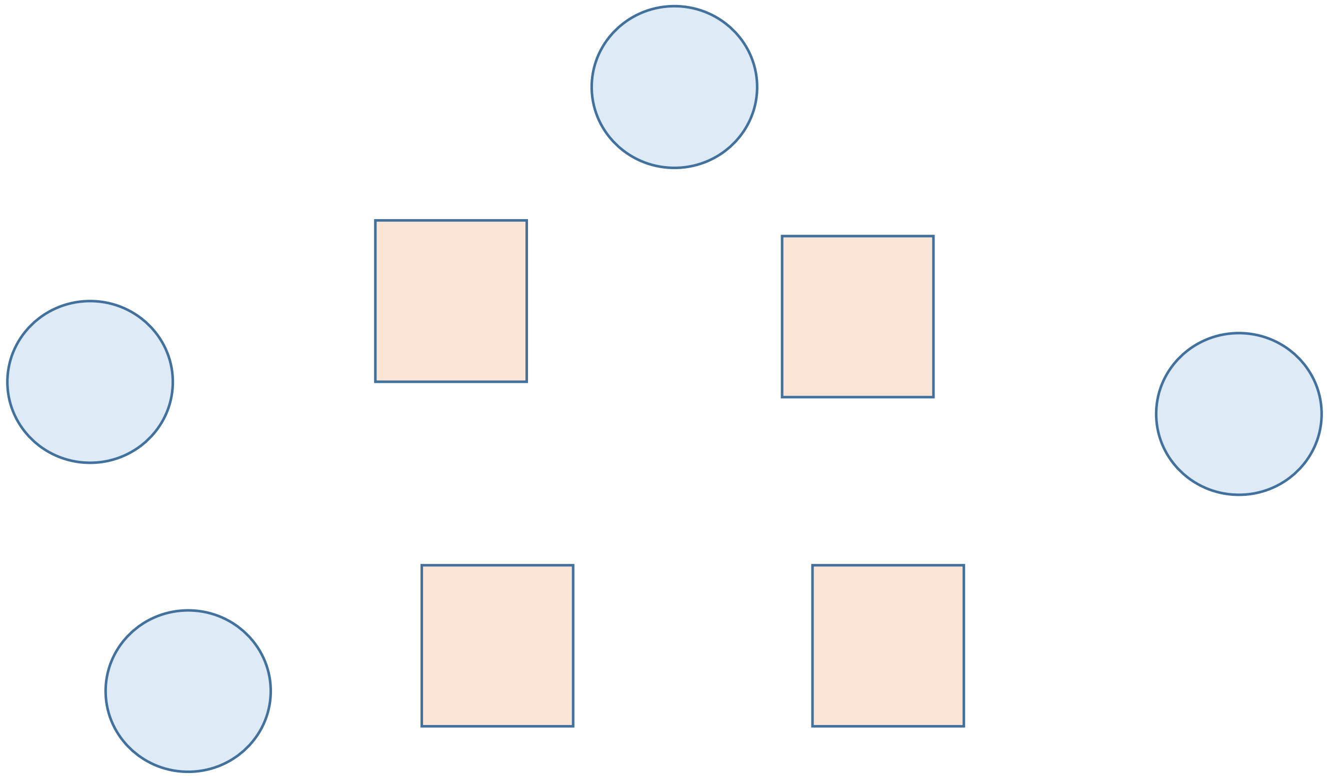
- 2. David measures the qubits he has.
- $\rho_D \to |V_a\rangle\langle V_a|$ with probability $p_a = \langle V_a|\rho_D|V_a\rangle$



- For a given output a, David now has a pure state, but Alice, Bob and Charlie are entangled. (Entanglement of assistance, D.P. DiVincenzo et al. '99)
- $|\Psi_{ABC}\rangle = \langle V_a|AD\rangle|BD\rangle|CD\rangle$

Tensor networks: Physical interpretation

- More generally, measurements occur on multiple parties, creating a complicated entangled state of the remaining parties that are not measured.



Why are tensor networks interesting states?

- Entanglement structure encoded in geometry (and the vertex tensors $|V_x\rangle$)
- For example, for any region A, $S_A \leq \log D_{min}(A)$
- or $S_A \leq |\gamma_A| \log D$. $|\gamma_A|$ is the minimal surface area bounding A region.

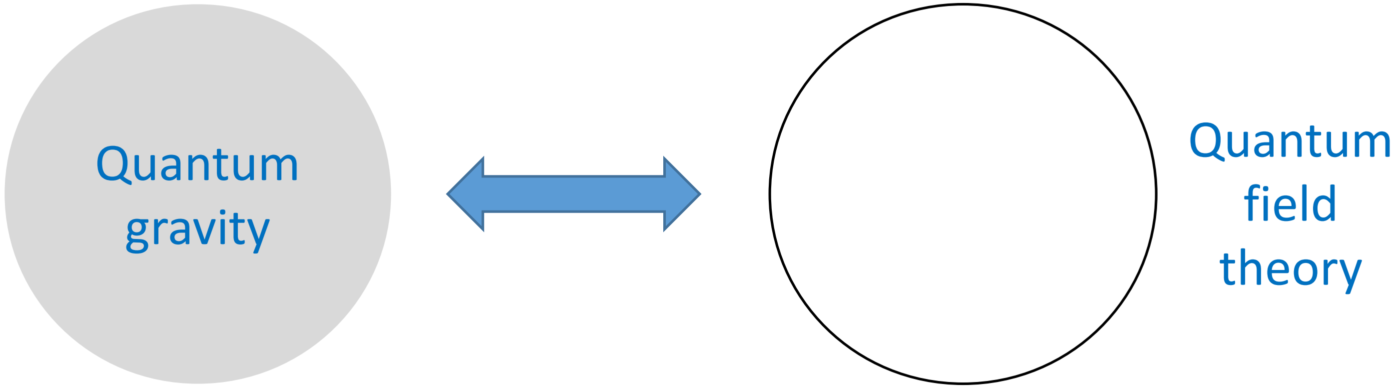
$$|\Psi\rangle = \sum_{a=1}^{D} |\Psi_a^A\rangle |\Psi_a^B\rangle$$

$$\Rightarrow rank(\rho_A) \leq D, S_A \leq \log D$$

Tensor networks in condensed matter

- Density matrix renormalization group (DMRG) (S. White '92) (1D non-critical states)
- Multi-scale entanglement renormalization ansatz (MERA) (G Vidal '07) (critical states)
- 2D PEPS (Cirac, Verstraete, Wen, Levin, Gu et al) (2D, gapped or critical)

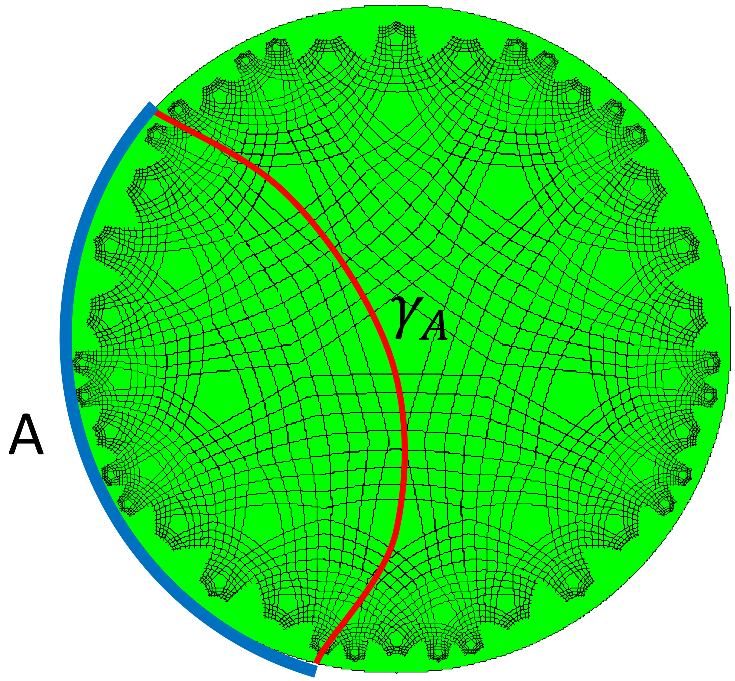
Holographic duality

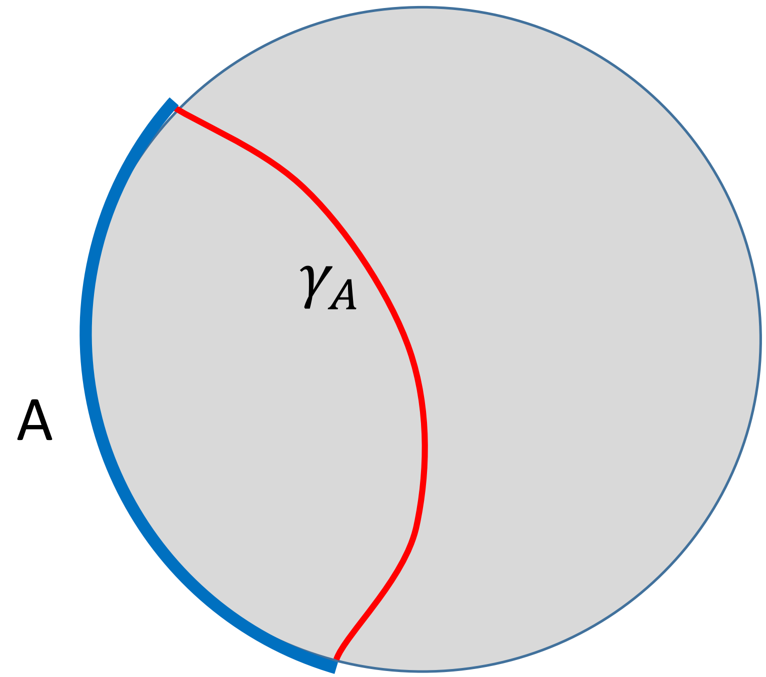


AdS	CFT
Black hole	Thermal state
$Z = \int D\phi e^{-S}$	$Z = \int D\psi e^{-S}$
Boundary condition of ϕ	$\langle O \rangle$ local operators
Classical equation of motion	RG flow

(Maldacena ’97, Witten ’98, Gubser, Klebanov & Polyakov ’98)

Area measures entanglement





Minimal surface area

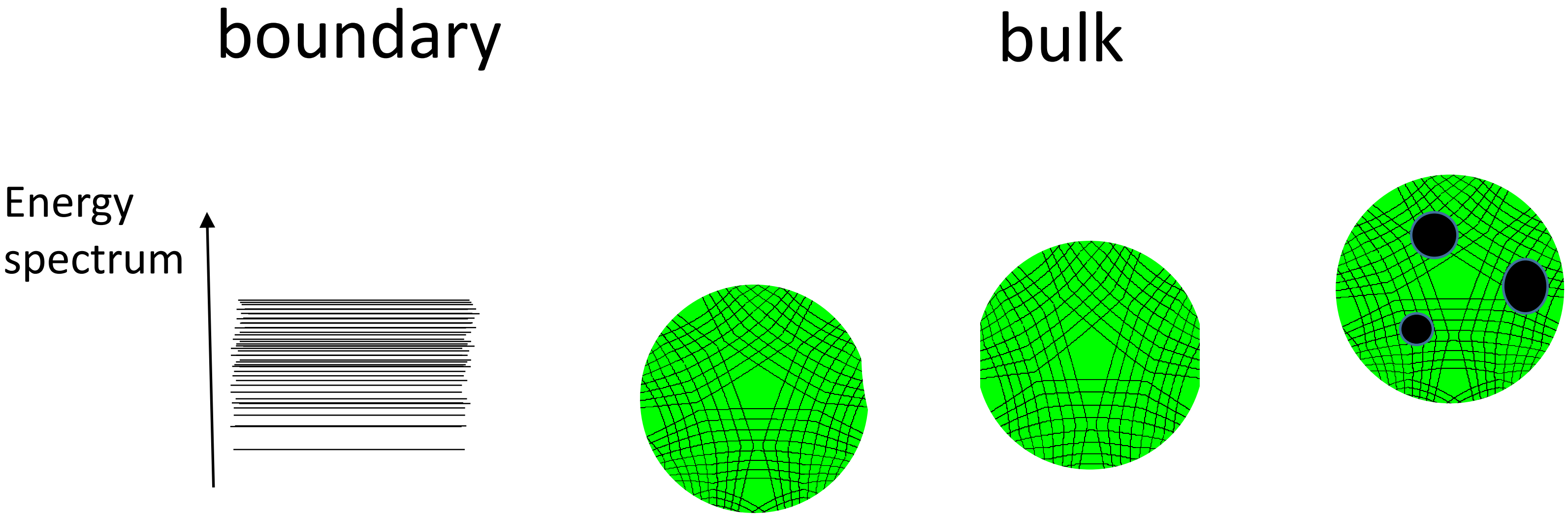
$$\frac{1}{4G_N}|\gamma_A|$$

Entanglement entropy

$$S_A$$

Ryu&Takayanagi '06

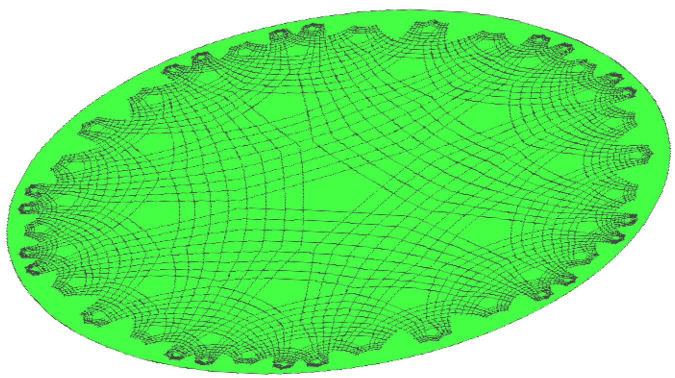
Tensor networks and holography

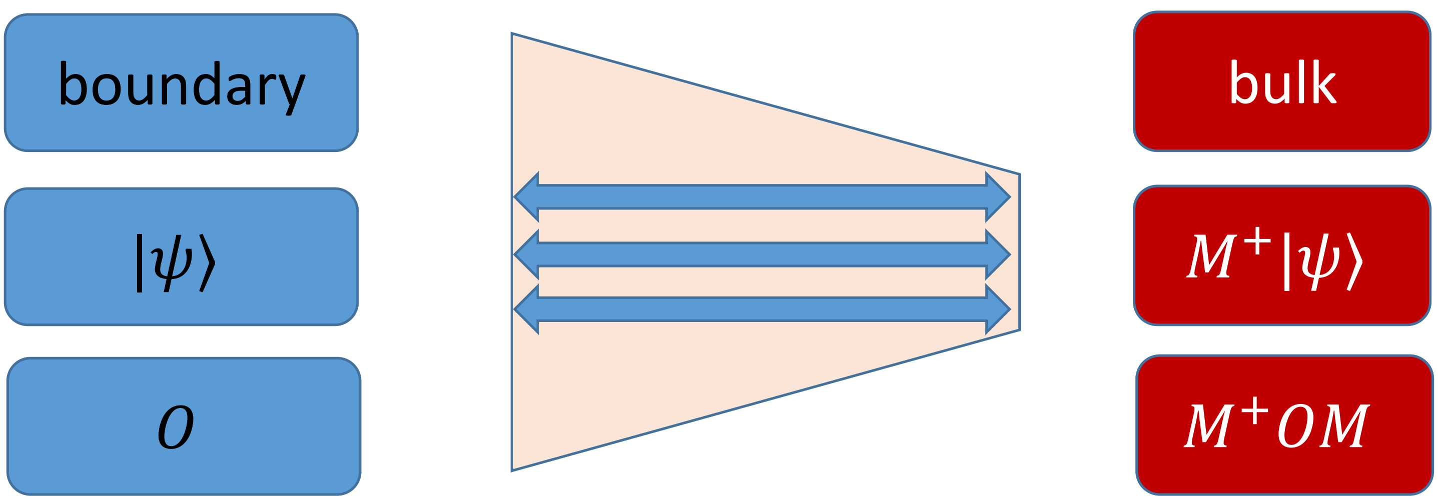


- TNW in holography: geometry emerges from the entanglement structure of quantum states (Swingle '09)
- Various tensor network related proposals (Nozaki et al '12, XLQ '13, Hartman&Maldacena '13, Maldacena & Susskind '13, Czech et al '14-15, Pastawski et al '15, Yang et al '15)
- Goal: an explicit holographic mapping between bulk and boundary

Tensor network holographic mapping

- Tensor networks with bulk and boundary indices provide a possible definition of holographic mapping.
- The task is to find suitable tensor networks





Random tensor networks

- Entanglement quantities are hard to compute for a given tensor networks.
- Random average greatly simplifies entanglement calculations.
- A random tensor $V_{\mu\nu\tau}$ corresponds to a (Haar) random state in the Hilbert space $|V\rangle = V_{\mu\nu\tau}|\mu\rangle|\nu\rangle|\tau\rangle$.
- Random tensor network state

$$|\Psi\rangle = \prod_x \langle V_x| \, |P\rangle$$

Random vertex states

Link states. EPR pairs or more general

- The link state can be EPR pairs $|\mathrm{P}\rangle = \prod_{xy} |L_{xy}\rangle$ but can also be more general

Entanglement properties of random tensor networks

- $\rho = |\Psi\rangle\langle\Psi| = tr(\prod_x |V_x\rangle\langle V_x|\, \rho_P)$ is a linear function of $|V_x\rangle\langle V_x|$.
- Renyi entropies $S_A = \frac{1}{1-n}\log\frac{tr(\rho_A^n)}{tr(\rho)^n}$
- For any quantity that is polynomial in ρ, such as $tr(\rho_A^n)$, the random average can be easily obtained.
- For example, second Renyi $tr\left(\rho_A^2\right) = tr(\rho \otimes \rho X_A)$ $= tr(\rho_P \otimes \rho_P [X_A \otimes \prod_x |V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|])$
- Random average $\overline{|V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|} = \frac{1}{D_x^2 + D_x}(I_x + X_x)$
- Random average ⇔ sum over an Ising variable at each x

Summary of the key results

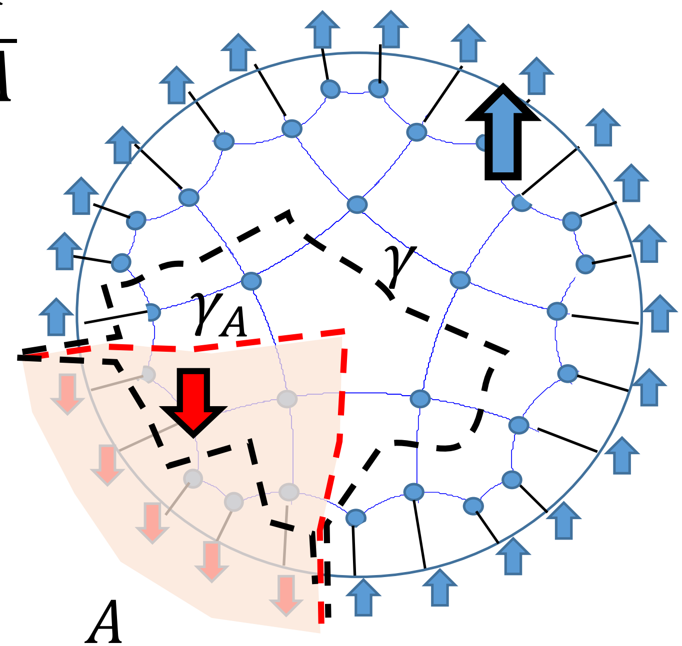
- For a random tensor network
- $\overline{tr(\rho_A^2)} = Z_A = \sum_{\{\sigma_x=\pm 1\}} e^{-\mathcal{A}[\{\sigma_x\}]}$
- $\mathcal{A}[\{\sigma_x\}] = S(\{\sigma_x = -1\}; \rho_P)$ "the second Renyi entropy of $\sigma_x = -1$ domain for state $\rho_P = |P\rangle\langle P|$"
- Boundary condition: spin $\downarrow$ in A and $\uparrow$ elsewhere

Random
average

- The second Renyi entropy $S_A \simeq -\log \frac{Z_A}{Z_\emptyset}$ is the "cost of free energy" of flipping spins in A from $\uparrow$ to $\downarrow$.

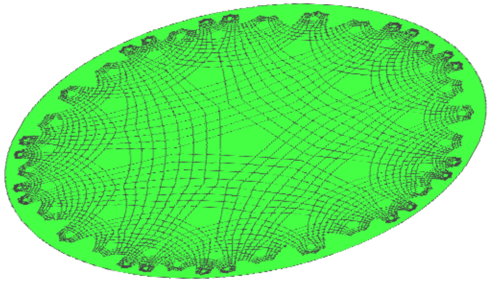
RT formula

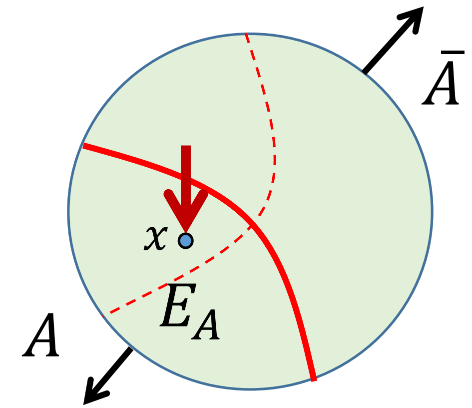
- If $|P\rangle = \prod_{xy} |L_{xy}\rangle$ consists of maximally entangled EPR pairs with rank D,
- $\mathcal{A}[\{\sigma_x\}] = -\frac{1}{2} \log D \sum_{xy} \sigma_x \sigma_y$
- Boundary cond. $\sigma_x = \begin{cases} -1, x \in A \\ +1, x \in \bar{A} \end{cases}$
- The action is proportional to the domain wall area.
- $\overline{tr(\rho_A^2)} = \sum_{\gamma \sim A} e^{-\log D |\gamma|}$
- $D \to \infty \Rightarrow$
  low T limit of Ising model
- $\overline{S} \simeq -\log \overline{tr(\rho_A^2)} \simeq \log D \, |\gamma_A|$ (RT formula)

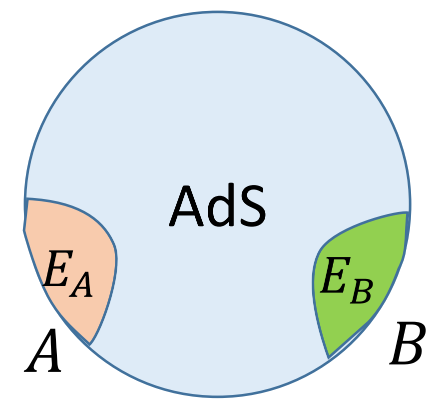


Other properties of RTN

- The random average technique applies to more general networks
- Other properties of RTN:
- Higher Renyi entropies
- RT formula with quantum correction (agree with Faulkner, Lewkowycz & Maldacena'13) $S_A \simeq \log D\,|\gamma_A| + S(\mathrm{E}_A, |\Psi_b\rangle\langle\Psi_b|)$.
- RT formula for higher Renyi entropies
- Quantum error correction properties (Almheiri, Dong, Harlow '14) ϕ_x can be reconstructed on boundary region A if $x \in E_A$.
- Scaling dimension of operators
- $\langle O_A O_B \rangle \propto \langle \phi_x \phi_y \rangle_{bulk}$



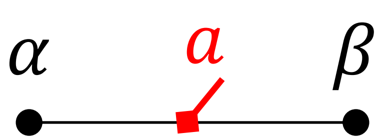




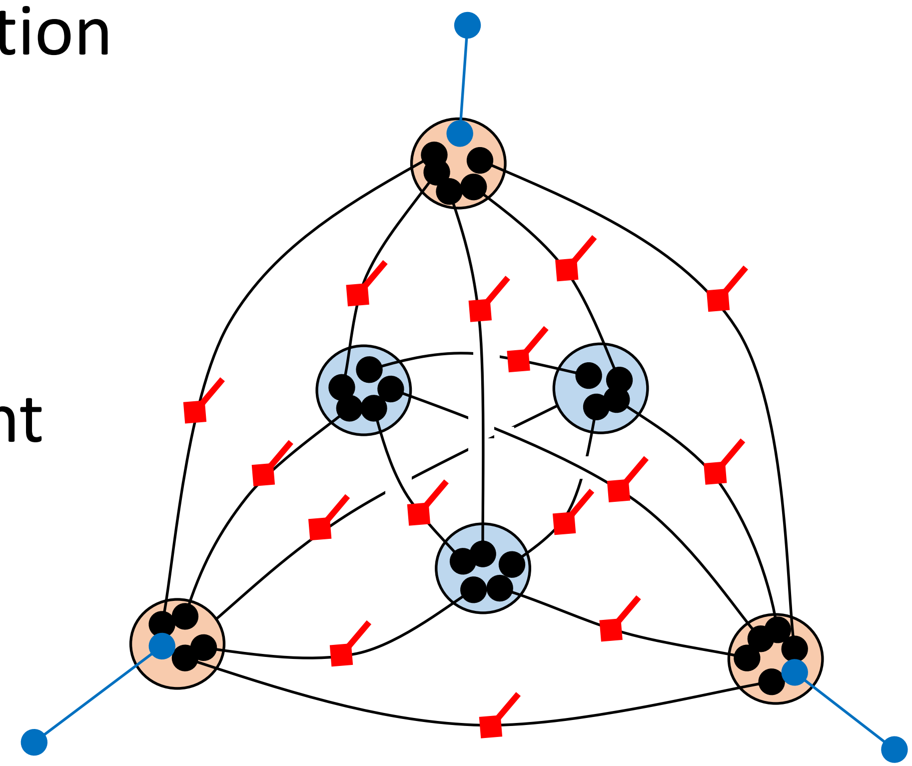
(For more details see our paper 1601.01694)

Superposition of geometries

- RTN represent ``ansatz states” with various holographic properties
- To describe quantum gravity, we need to allow superposition of geometries
- RTN with geometry fluctuation can be defined by considering link qudits
- $|a\rangle = L^{a}_{\alpha\beta}|\alpha\rangle|\beta\rangle$
- a controls the entanglement of this link

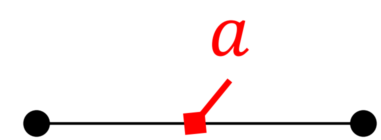


$= L^{a}_{\alpha\beta}$

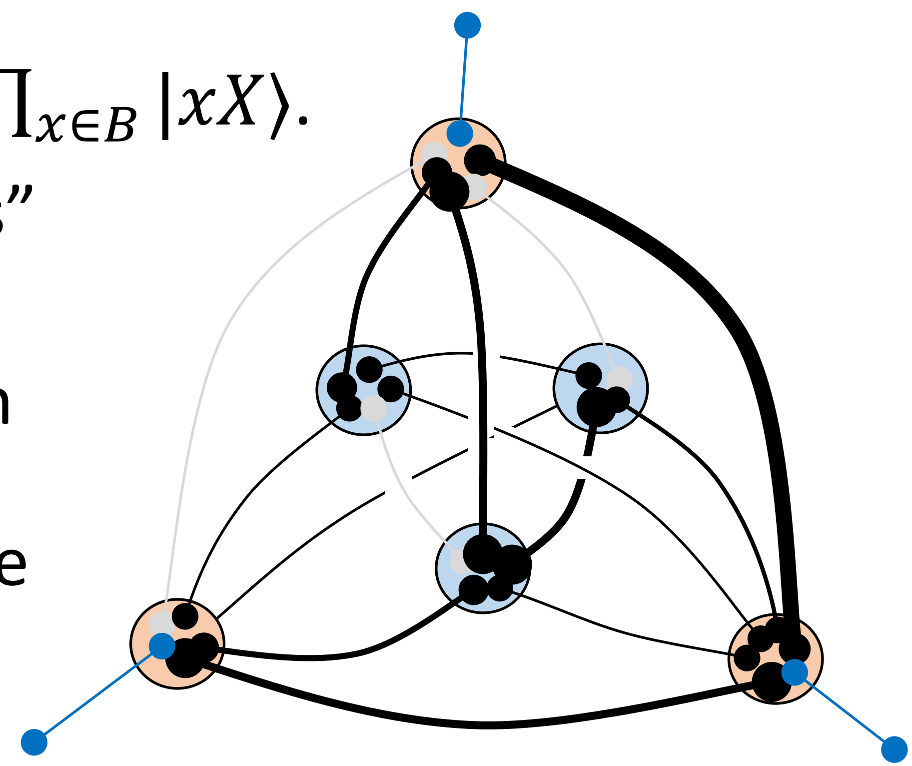


Geometry coherent states

- S_a increases with $a = 0,1,2,\ldots,D_L - 1$



- $\langle a|b\rangle = \delta_{ab}$. $|a=0\rangle = |0\rangle|0\rangle$ corresponds to a disconnected link
- Random tensors map each weighted graph a_{xy} to a boundary state $|\Psi[\mathrm{a}]\rangle = \prod_x \langle V_x| \prod_{xy} |a_{xy}\rangle \prod_{x\in B} |xX\rangle$.
- $|\Psi[a]\rangle$ are ``geometry states" satisfying RT formula.
- Question: Do $|\Psi[a]\rangle$ form an (over-)complete basis?
  Short answer: Yes. $|\Psi[a]\rangle$ are *``geometry coherent states"*



Boundary-to-bulk isometry

- With enough number of bulk vertices, $|\Psi[a]\rangle$ is an overcomplete basis satisfying $\sum_a |\Psi[a]\rangle\langle\Psi[a]| = \mathbb{I}$
- Boundary-to-bulk isometry
- Random average ➔ Ising model on the complete graph
- $\mathcal{A} = -\frac{J}{4}\sum_{xy} s_x s_y - \frac{h}{2}\sum_x s_x + \frac{1}{2}\log D \sum_x s_x.$
- Isometry condition

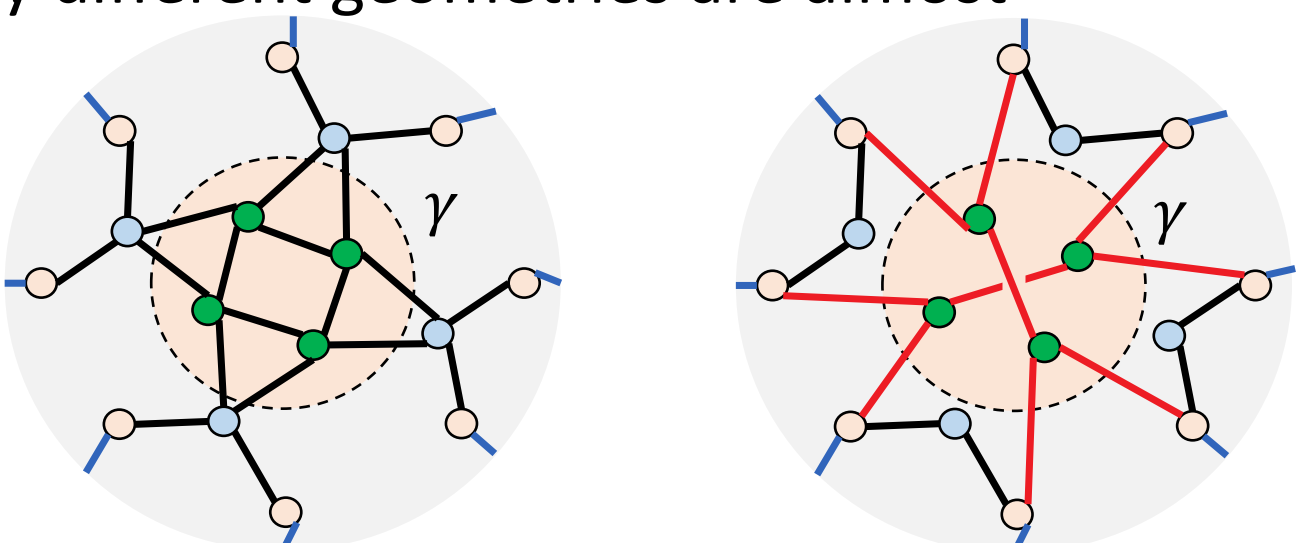
$$\log D_L \frac{V(V-1)}{2} > V_B \log D, \qquad JV_b > 2\log D - (V-1)\log D_L$$

dim(bulk)> dim(boundary)

Bound on mutual information J of each link

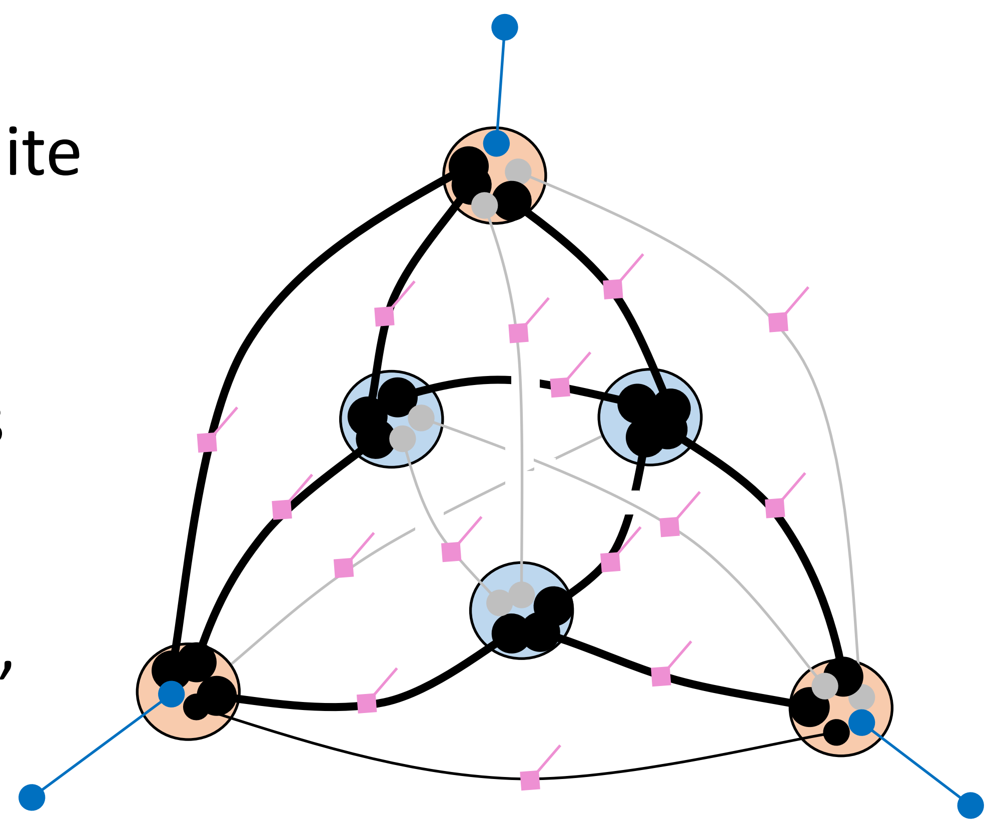
Classical geometries

- $|\Psi[a]\rangle$ are not orthogonal
- $C_{ab} = \langle\Psi[a]|\Psi[b]\rangle$
- $\overline{|C_{ab}|^2}$ can be studied by the random average technique
- $\overline{|C_{ab}|^2} \leq e^{-\frac{1}{2}(|\gamma|_a + |\gamma|_b)}$
- γ: minimal area surface enclosing the region $a \neq b$.
- Macroscopically different geometries are almost orthogonal



Small fluctuations

- If we take D_L to be large, we can define small fluctuation around a classical geometry.
- $|\Psi[a_0 + \delta a]\rangle$
- $|\delta a| \leq \Lambda$
- In the limit $D_L \to \infty$, finite Λ, there is an isometry from bulk to boundary.
- Emergent local degrees of freedom
- The small fluctuations form a "code subspace" $\mathbb{H}_a$.



Comparison with boson coherent states

- Boson coherent state of a superfluid $|\phi(x)\rangle = e^{\int d^d x \phi(x) b^+(x)}|0\rangle$
- Overcomplete basis $\int D\phi |\phi\rangle\langle\phi| = \mathbb{I}$
- Overlap $|\langle\phi|\phi'\rangle| = \exp\left(-\int d^d x |\phi(x) -\right.$

Summary and open questions

- Random tensor networks form a basis of "geometry coherent states" with holographic properties.
- A generic boundary states is mapped to a superposition of geometries $\sum_a \phi_a |\Psi[a]\rangle$
- The basis is overcomplete but different classical geometries are almost orthogonal.
- Small fluctuations are mapped to boundary isometrically, with error correction properties. They are local bulk quantum fields.
- Open question:
  - Optimization of geometry for a given boundary state.
  - Einstein equation from boundary dynamics?