Theory of Force Transmission and Rigidity in Granular Aggregates

A Different kind of NonEquilibrium System
GRANULAR WORLD

- Collection of macroscopic objects
- Purely repulsive, contact interactions. No thermal fluctuations to restore broken contacts
- Friction: Forces are independent degrees of freedom
- States controlled by driving at the boundaries or body forces: shear, gravity
- Non-ergodic in the extreme sense: stays in one configuration unless driven
LOOKING INSIDE SAND

54250 fps
frame: 4682

100x slowed down
Stress Metric
Statics of Granular Media: Constraints of mechanical equilibrium determine collective behavior

Local force & torque balance satisfied for every grain

Friction law on each contact

\[ f_t \leq \mu f_N \]

Positivity of all forces

\[ f_N \geq 0 \]

Imposed stresses determine sum of stresses over all grains
Imposing force balance (2D)

Forces on every grain sum to zero:

\[ \sum_c \vec{f}_{g,c} = 0. \]

Newton’s third law dictates:

\[ \vec{f}_{g,c} = -\vec{f}_{g',c} \]
The conditions of mechanical equilibrium ensure the uniqueness of the height representation

\[ \nabla \cdot \hat{\sigma} = 0 \rightarrow \text{Vector potential} \]

The heights live on a random network
For systems where all normal forces are repulsive, we have a single sheet.
TWO REPRESENTATIONS

http://www.aps.org/meetings/march/vpr/2015/videogallery/index.cfm
Two problems:

• **Stress Transmission in Static Granular Aggregates**

Procaccia group: Numerical Simulations (2016)

• **Discontinuous Shear Thickening in Dense Suspensions**
How do granular materials respond to applied forces?

“Forces are carried primarily by a tenuous network that is a fraction of the total number of grains”  Geng et al, PRL (2001)

Ensemble averaged patterns are sensitive to nature of underlying spatial disorder
Theoretical Models


In response to a localized force at the top of a pile, the pressure profile at the bottom has a peak with width proportional to the square root of the height.
Theoretical Models

Missing stress-geometry equation: no well defined strain field/compatibility relations

Continuum models with prescribed constitutive law relating stress components. determined by history of preparation. For example,

\[ \sigma_{zz} = c_0^2 \sigma_{xx} \]

More elaborate closure relations: (Review: J.-P. Bouchaud Les Houches Lectures)

Stresses propagate/ get transmitted along lines
Geometry of contact network represented by the network Laplacian

Random matrix: diagonal elements contain the number of contacts, otherwise the adjacency matrix
Framework

\[ \Box^2 |\tilde{\phi}\rangle = -|\vec{f}_{body}\rangle \]

Equation defining the auxiliary fields

Given a contact network and a set of body forces, solution is unique

If the solution violates torque balance/static friction condition, network will rearrange

Disorder of contact network represented by network Laplacian

Diffusion on a random network: Localization?

Eigenfunction expansion

\[ \Box^2 = \sum_{i=1}^{N} \lambda_i |\lambda_i\rangle \langle \lambda_i| \]

One zero mode

\[ \lambda_1 = 0 , |\lambda_1\rangle = (1 \ 1 \ 1 \ldots \ 1) \]

Localized?

\[ |\tilde{\phi}\rangle = \sum_{i=1}^{N} \frac{1}{\lambda_i} \langle \lambda_i | \vec{f}_{body} \rangle |\lambda_i\rangle \]

Different from q model: On a given contact network, how the force perturbation gets distributed among contacts is completely determined by the constraints of force balance

Constitutive Law determined by statistical properties of the ensemble of Laplacians
Response of a frictionless granular solid
Highest eigenvalue: strongly localized

Lowest eigenvalue: delocalized
Ensemble average: Spectral properties

**Inverse Participation Ratio**

\[ IPR(\lambda) \]

\[ E_G = 10^{-5} \]
\[ E_G = 10^{-10} \]
\[ E_G = 10^{-15} \]

**Density of States**

\[ \rho(\lambda) \]
Stress Localization

• Maps granular response problem to the localization problem
  “Absence of Diffusion in Certain Random Lattices”
  P.W. Anderson (1958)

• Theory relates response to the disorder in the underlying network

• Random Matrix Ensembles: Characterizing Jammed Networks

• Many Body Localization: Strongly Interacting System
Discontinuous Shear Thickening

Shear thickening as a (out-of-equilibrium) phase transition?
A Thickening Scenario

![Diagram showing the transition from liquid to solid (jammed) with lubricated contacts and real contacts.](image-url)
Phase Diagram from Rheology: Abhi Singh & Jeff Morris

Is this a critical point?
Clustering of points

<table>
<thead>
<tr>
<th>Stress</th>
<th>Clustering/Clumping of points</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
“Equilibrium” Model

- Repulsive interaction between heights with a single stress scale
- Simplest: If height points are closer than this scale, they repel

\[
\phi = 0 \\
\psi \neq 0
\]

\[
\phi \propto 1/T
\]

A tricritical point?
Statistical Mechanics of Granular Media

• Dual Networks: Contacts and Force Tilings

• History Dependence:
Including forces in defining microstates takes away that indeterminacy
Contact Networks are random but can characterize ensembles

• Pattern formation in height fields: Distinguish phases
Cluster Analysis

$s = 0$

$s = 0.05$

$s = 0.1$

$s = 1$

$s = 0.5$

$s = 0.2$
Response in a frictionless jammed packing