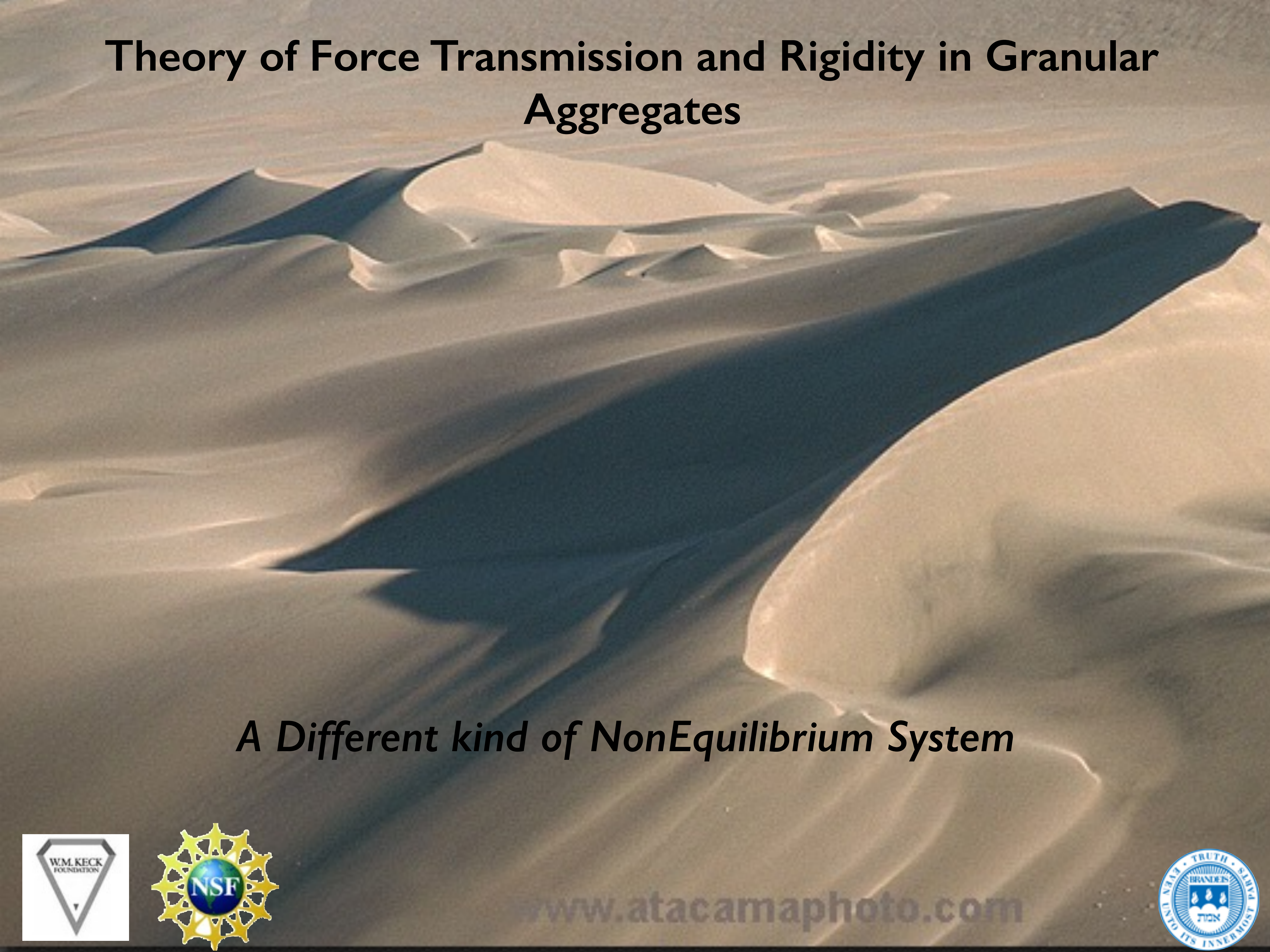
Theory of Force Transmission and Rigidity in Granular Aggregates



A Different kind of NonEquilibrium System

W.M. KECK FOUNDATION

NSF

BRANDEIS TRUTH EVEN UNTO ITS INNERMOST PARTS

Dapeng Bi

Sumantra Sarkar

Kabir Ramola

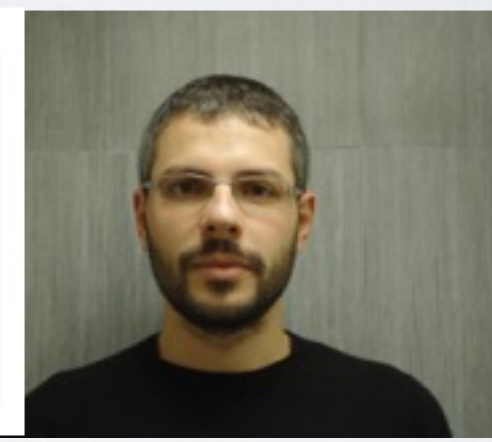
Jetin Thomas

Bob Behringer,

Jie Ren

Dong Wang

Jeff Morris

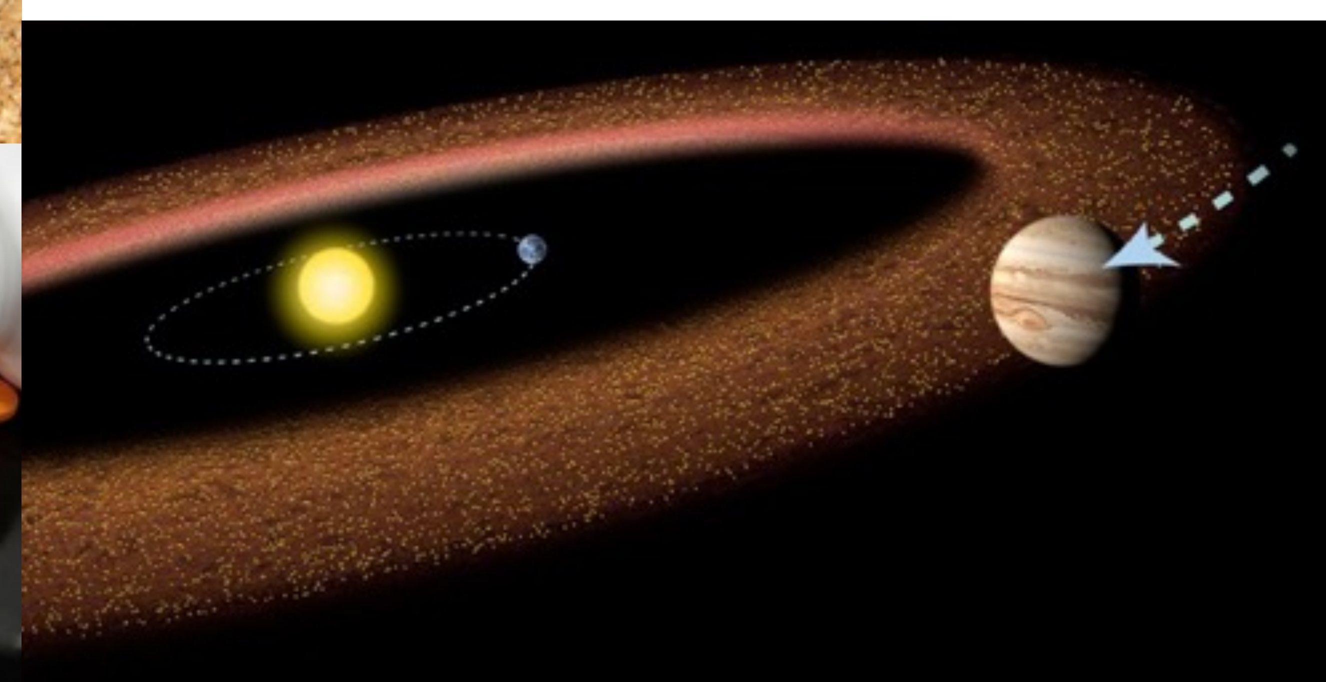


Romain Mari

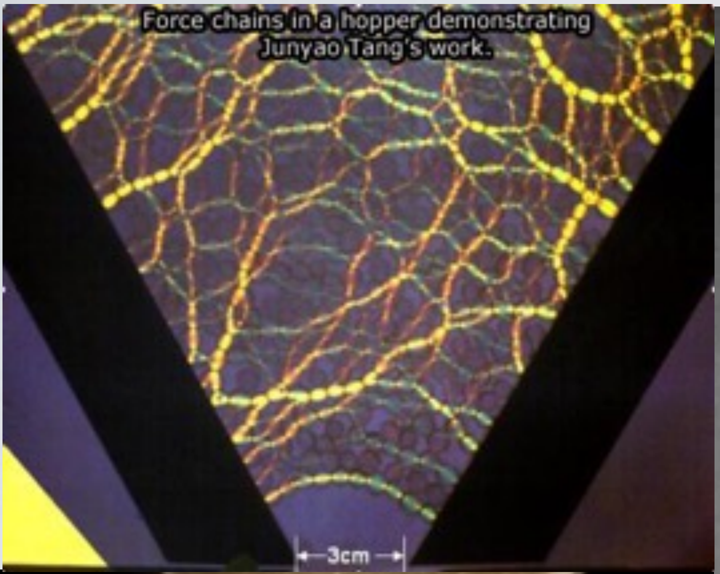
Ryohei Seto

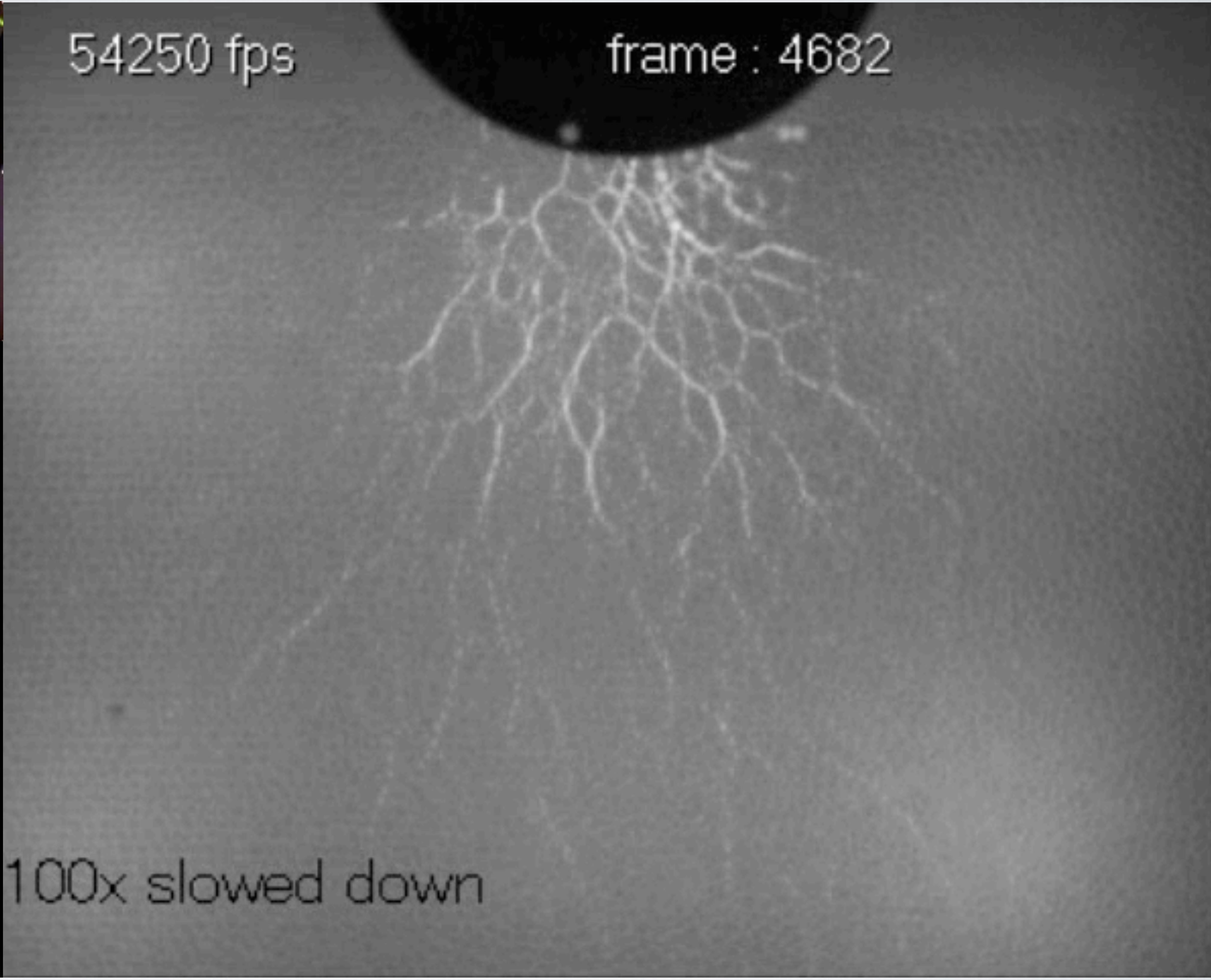
GRANULAR WORLD

- Collection of macroscopic objects
- Purely repulsive, contact interactions. No thermal fluctuations to restore broken contacts
- Friction: Forces are independent degrees of freedom
- States controlled by driving at the boundaries or body forces: shear, gravity
- Non-ergodic in the extreme sense: stays in one configuration unless driven

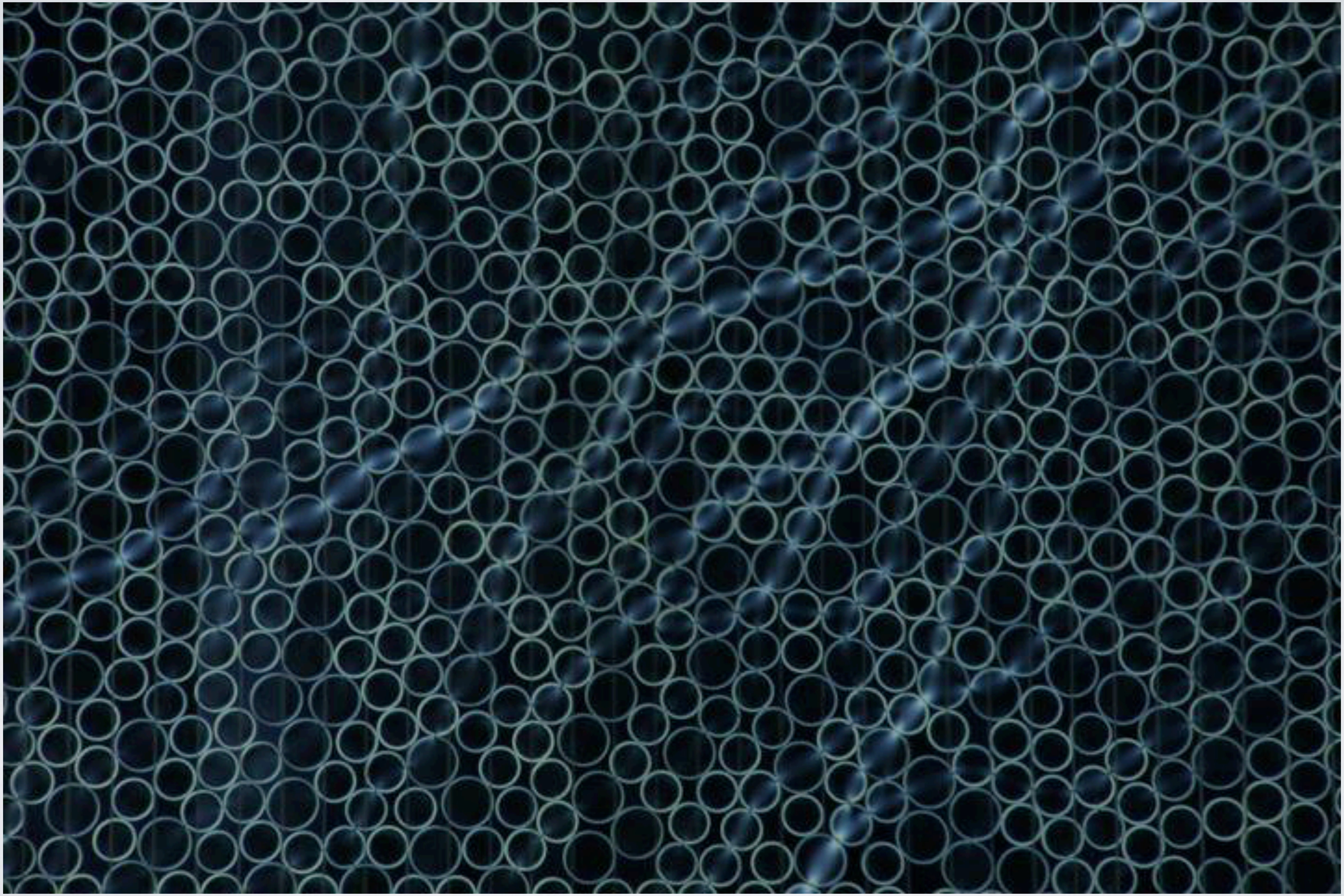


LOOKING INSIDE SAND





Stress Metric



Statics of Granular Media: Constraints of mechanical equilibrium determine collective behavior

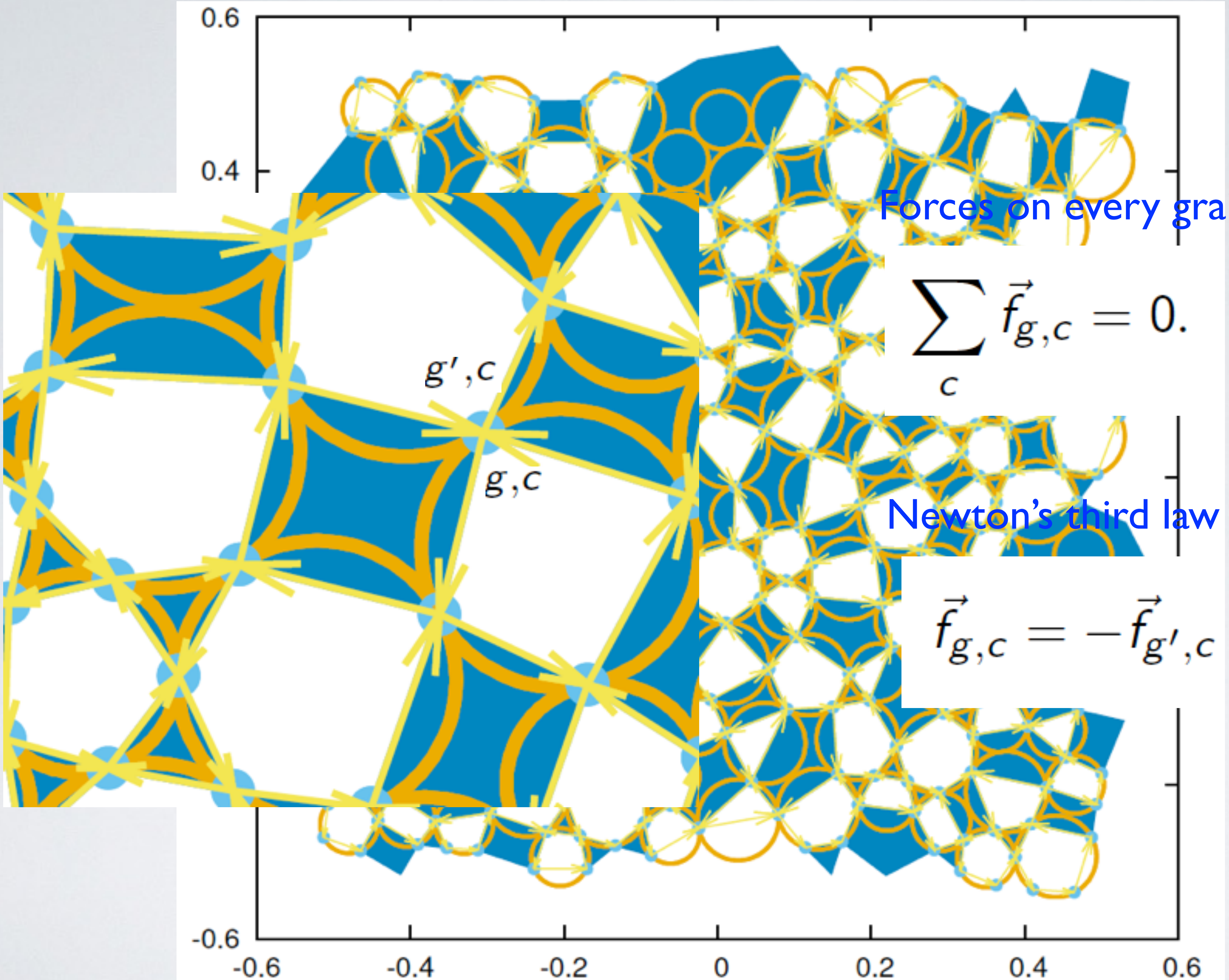
Local force & torque balance satisfied for every grain

Friction law on each contact $f_t \leq \mu f_N$

Positivity of all forces $f_N \geq 0$

Imposed stresses determine sum of stresses over all grains

Imposing force balance (2D)



Forces on every grain sum to zero:

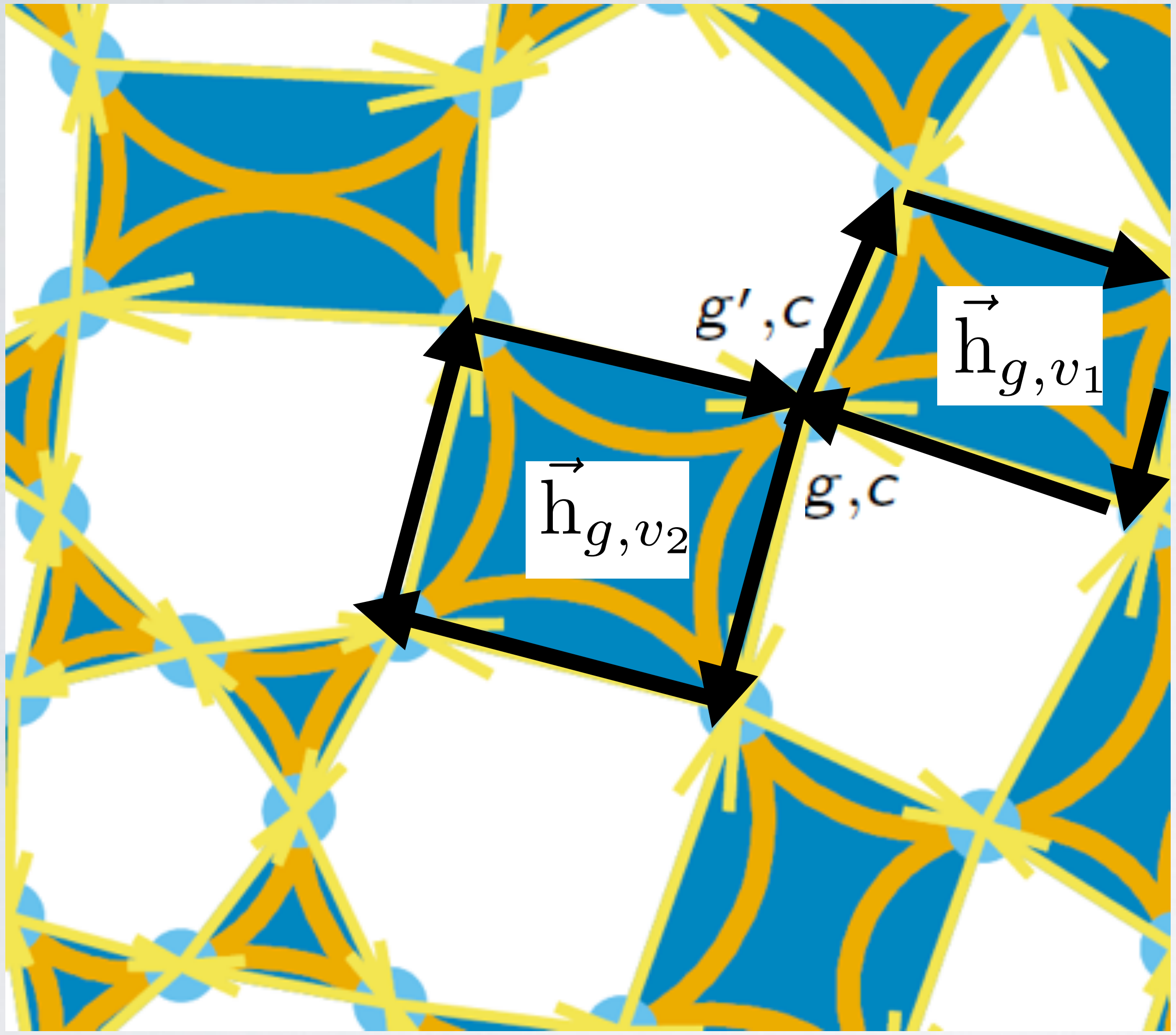
$$\sum_c \vec{f}_{g,c} = 0.$$

Newton's third law dictates:

$$\vec{f}_{g,c} = -\vec{f}_{g',c}$$

Height Representation

Ball & Blumenfeld, 2003



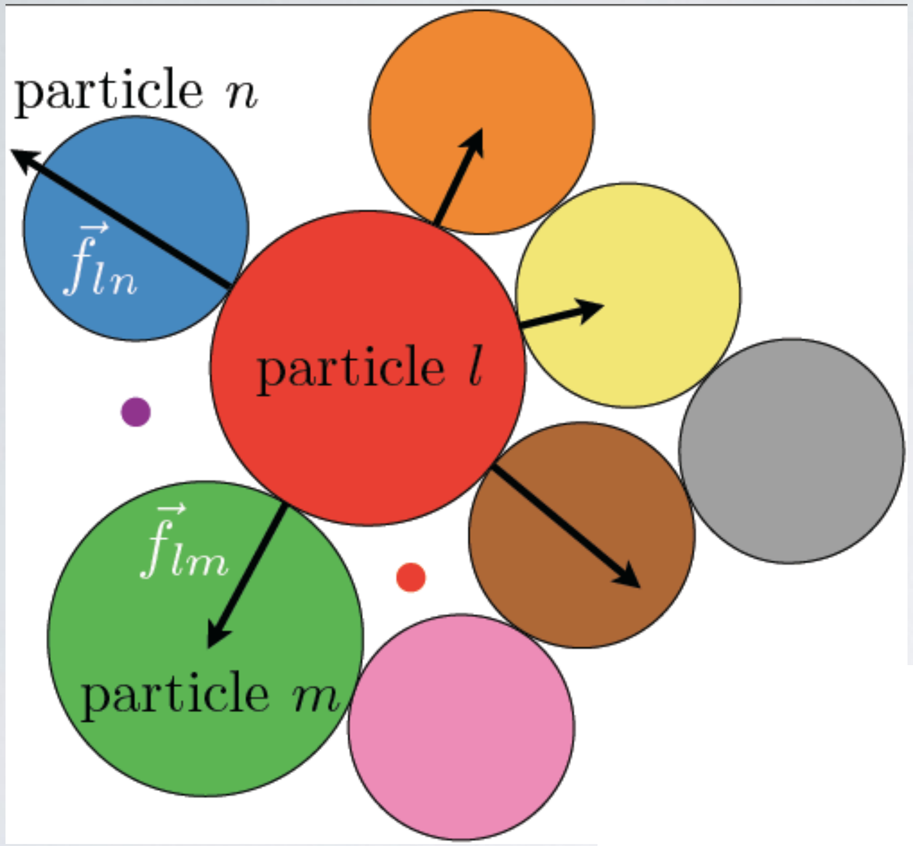
$$
\begin{aligned}
\vec{f}_{g,c_1} &= \vec{h}_{g,v_1} - \vec{h}_{g,v_2},\\
\vec{f}_{g,c_2} &= \vec{h}_{g,v_2} - \vec{h}_{g,v_3},\\
\vec{f}_{g,c_3} &= \vec{h}_{g,v_3} - \vec{h}_{g,v_4},\\
\underbrace{\vec{f}_{g,c_4}}_{0} &= \underbrace{\vec{h}_{g,v_4} - \vec{h}_{g,v_1}}_{0}.
\end{aligned}
$$

The conditions of mechanical equilibrium ensure the uniqueness of the height representation

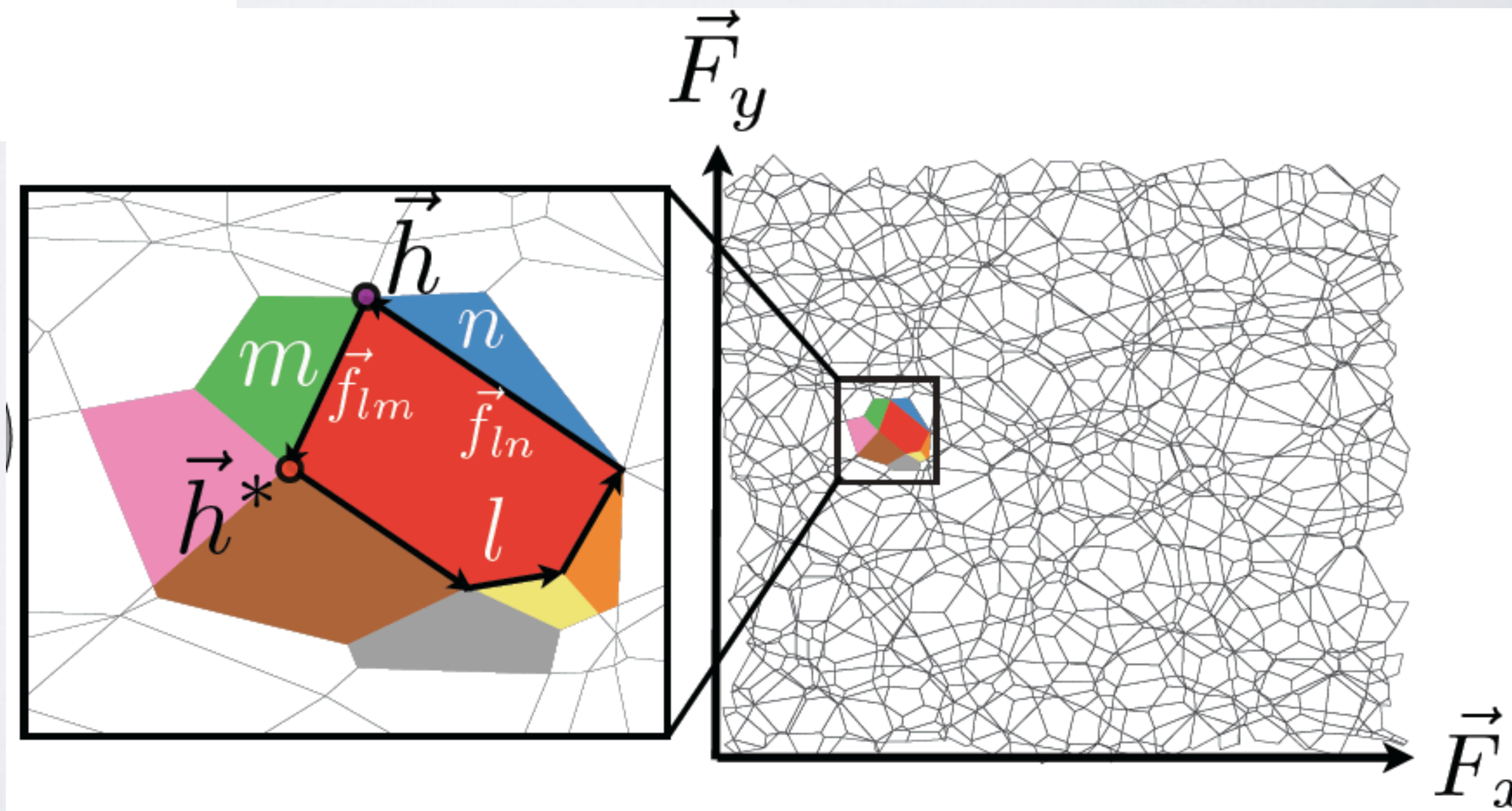
$$\nabla \cdot \hat{\sigma} = 0 \rightarrow \text{Vector potential}$$

The heights live on a random network

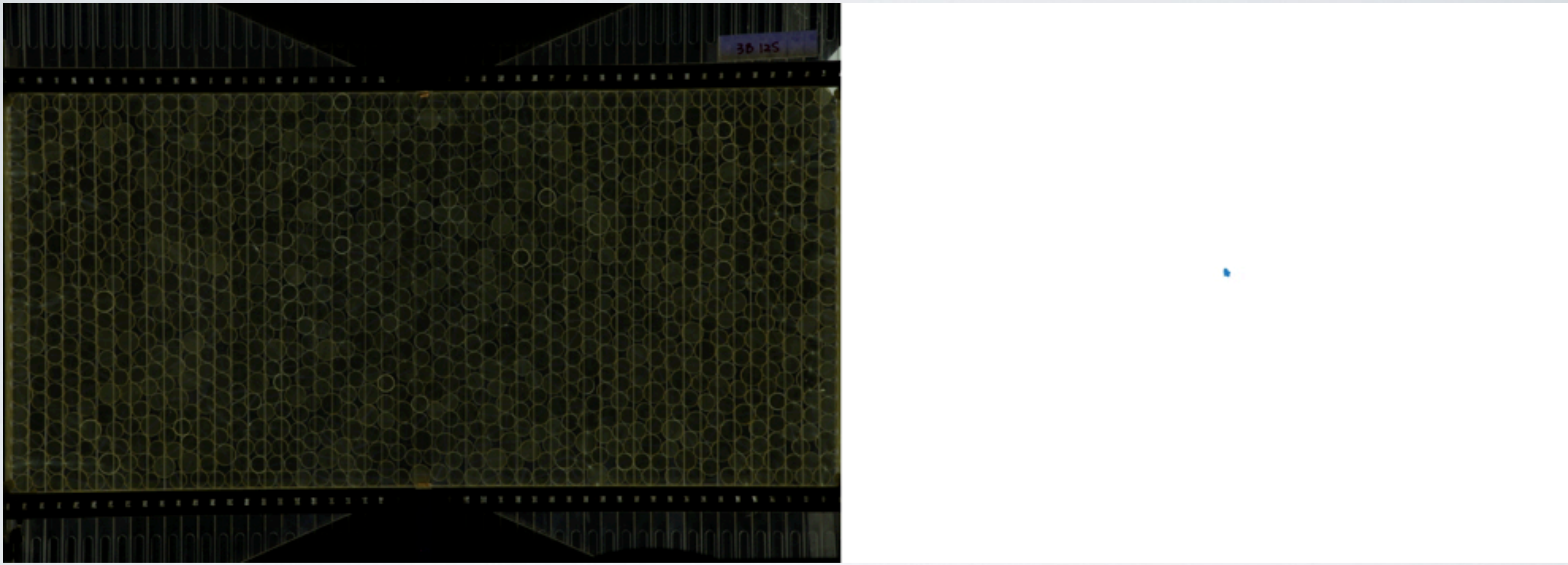
Force Tilings



For systems where all normal forces are repulsive, we have a single sheet



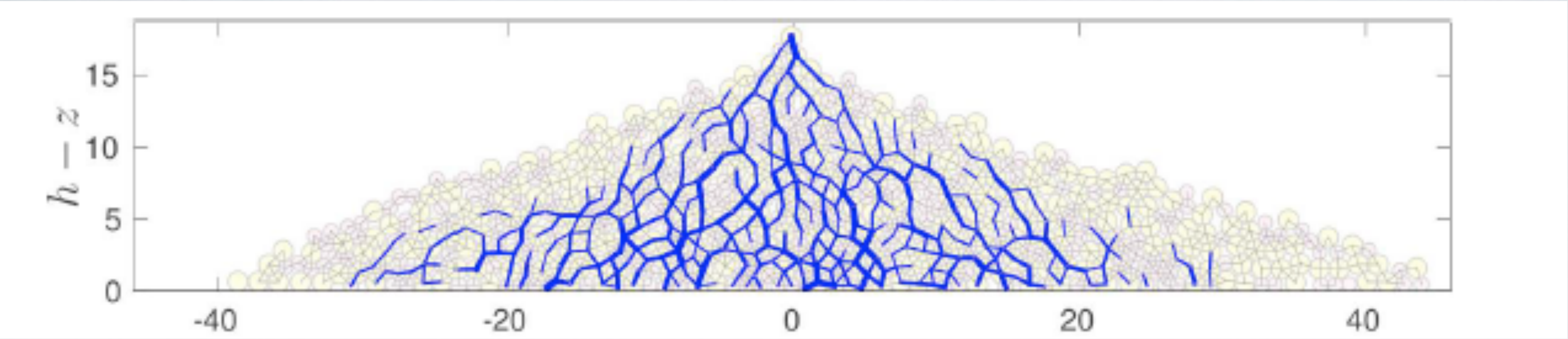
TWO REPRESENTATIONS



http://www.aps.org/meetings/march/vpr/2015/videogallery/index.cfm

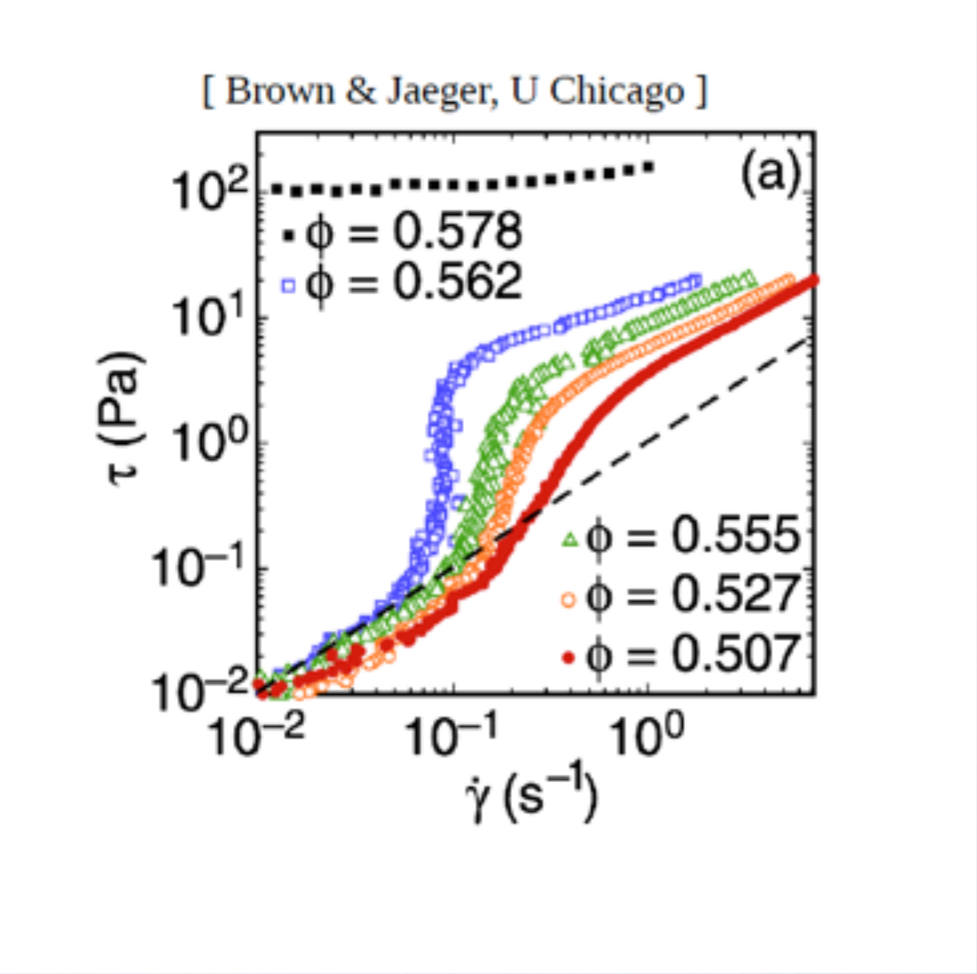
Two problems:

•Stress Transmission in Static Granular Aggregates

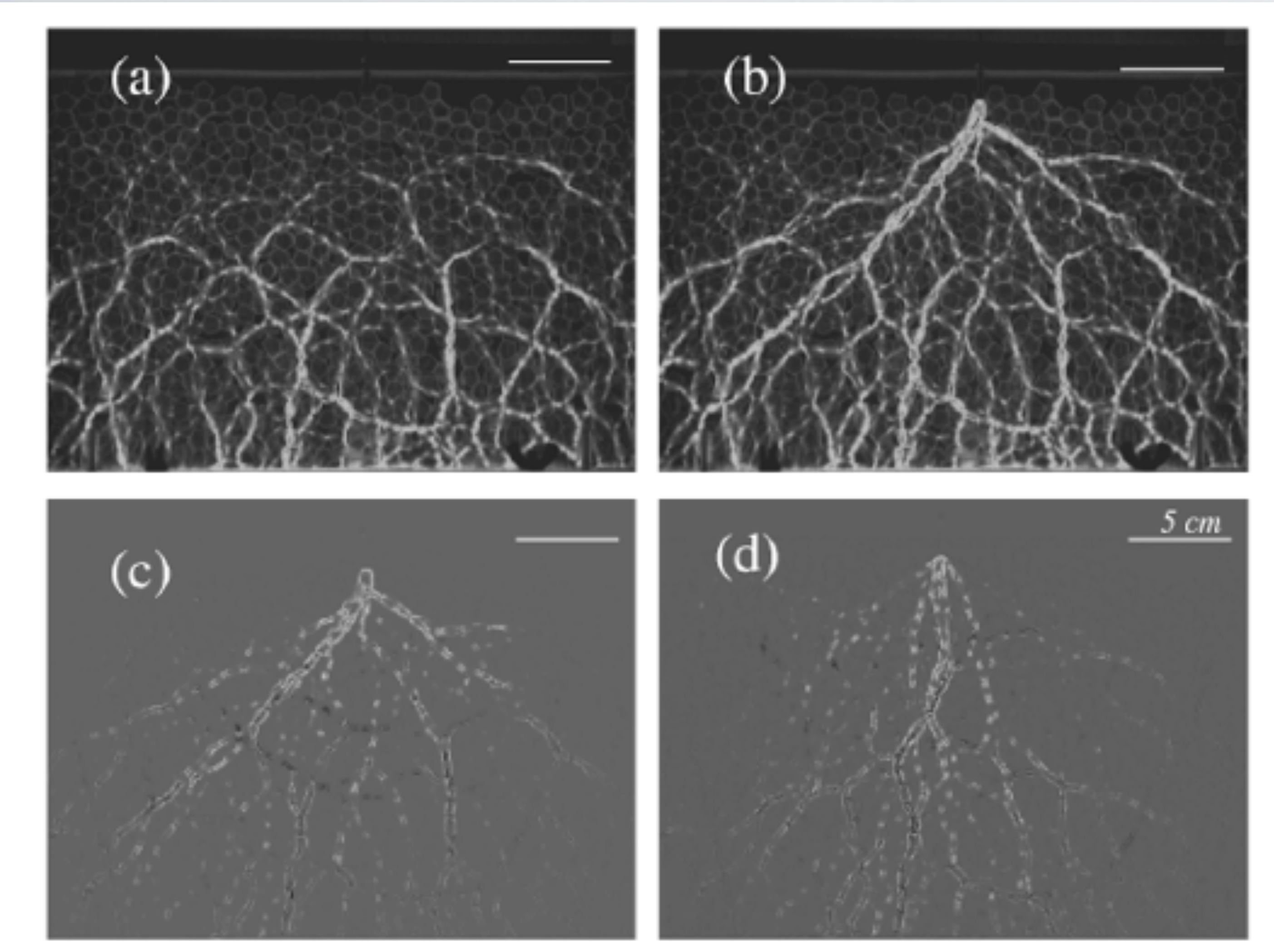


Procaccia group: Numerical Simulations (2016)

•Discontinuous Shear Thickening in Dense Suspensions

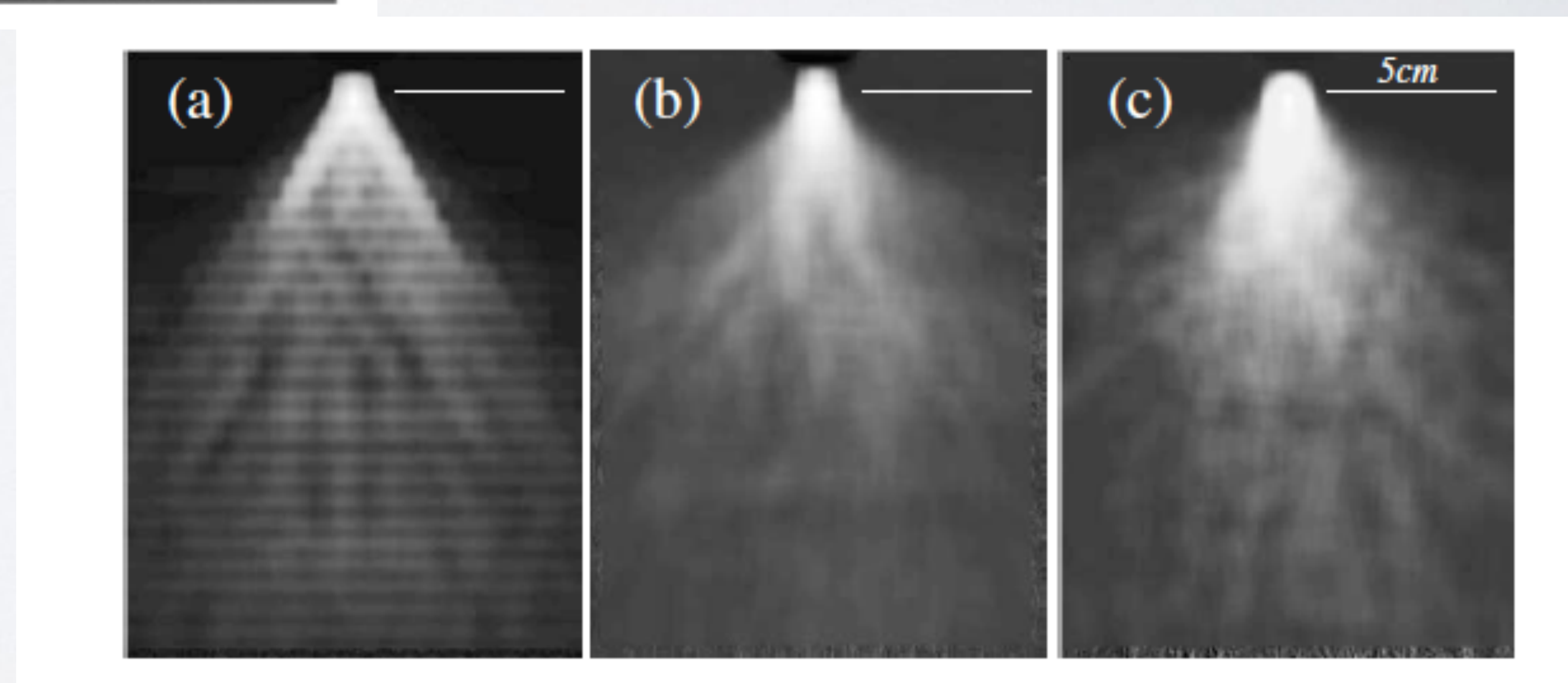


How do granular materials respond to applied forces?



“Forces are carried primarily by a tenuous network that is a fraction of the total number of grains” Geng et al, PRL (2001)

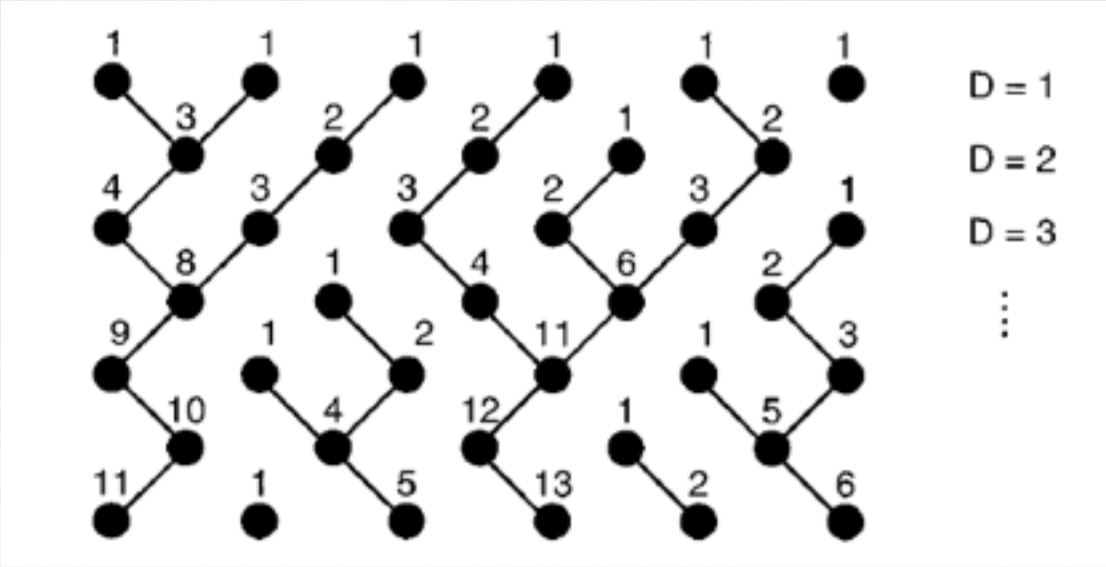
Ensemble averaged patterns are sensitive to nature of underlying spatial disorder



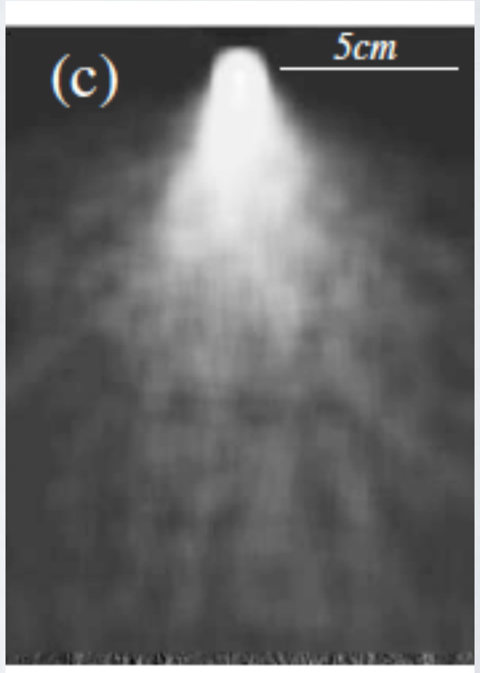
Theoretical Models

q-model (Coppersmith et al (1996)): Scalar force balance on an ordered network. Disorder incorporated at contacts: how forces get transmitted at contacts. In continuum, reduces to the diffusion equation.

Broad distribution of forces.



In response to a localized force at the top of a pile, the pressure profile at the bottom has a peak with width proportional to the square root of the height.



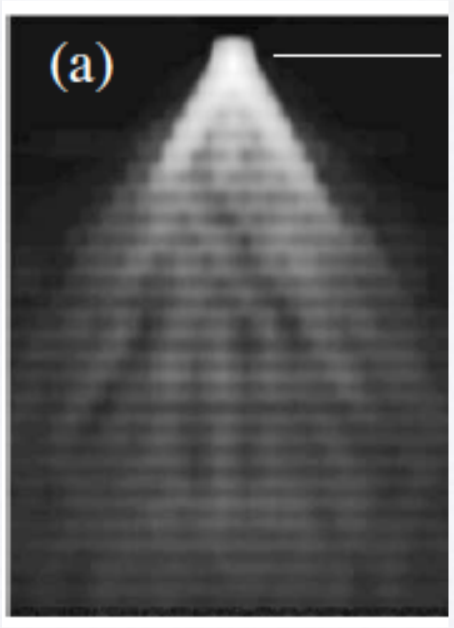
Theoretical Models

Missing stress-geometry equation: no well defined strain field/compatibility relations

Continuum models with prescribed constitutive law relating stress components. determined by history of preparation. For example,

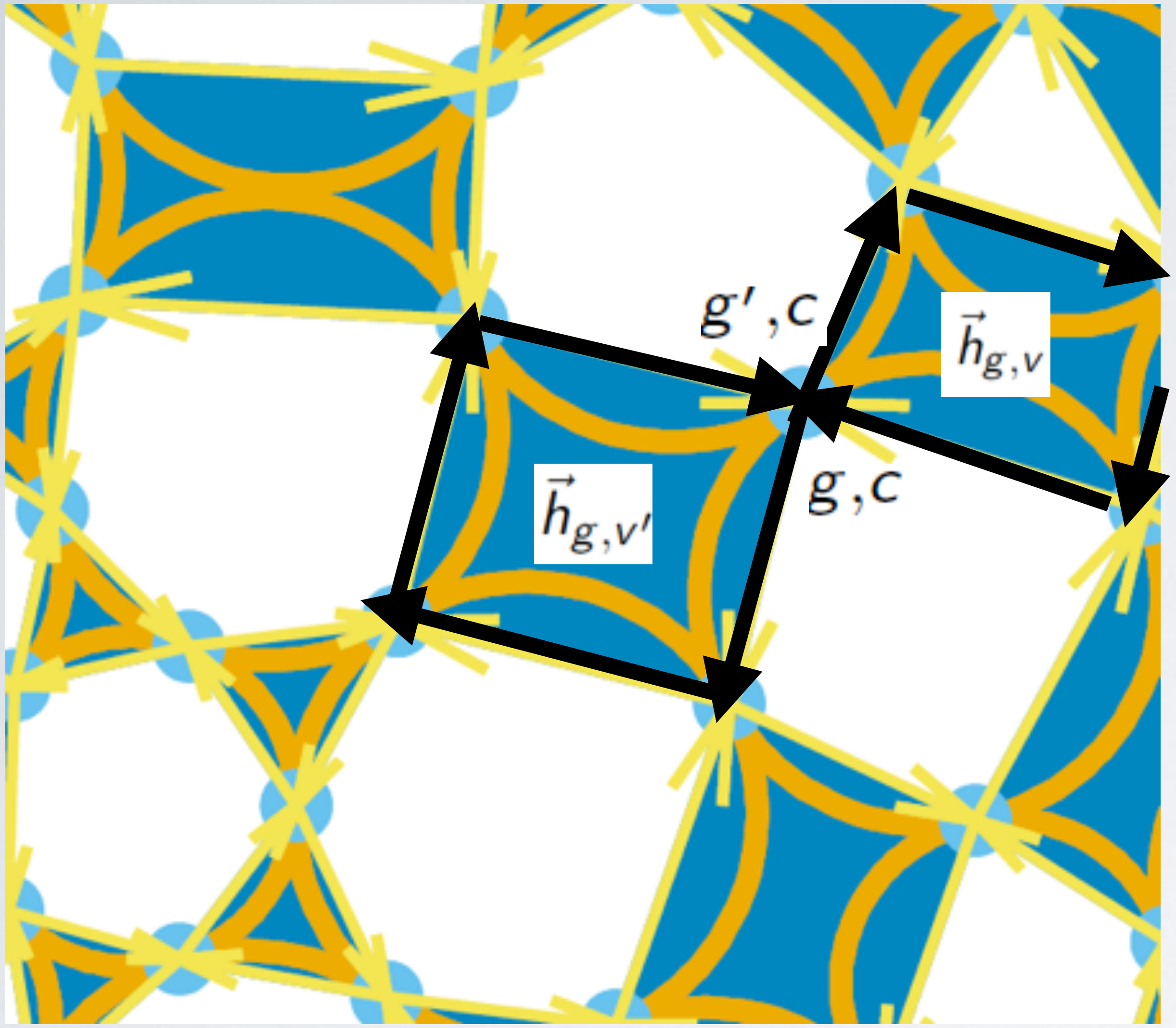
$$\sigma_{zz} = c_0^2 \sigma_{xx}$$

More elaborate closure relations: (Review: J.-P. Bouchaud Les Houches Lectures)



Stresses propagate/ get transmitted along lines

Force Response of a network to a perturbation



$$
\begin{aligned}
\vec{f}_{g\ ,c_1} &\neq \vec{h}_{g,v_1} - \vec{h}_{g,v_2},\\
\vec{f}_{g\ ,c_2} &\neq \vec{h}_{g,v_2} - \vec{h}_{g,v_3},\\
\vec{f}_{g\ ,c_3} &\neq \vec{h}_{g,v_3} - \vec{h}_{g,v_4},\\
\underbrace{\vec{f}_{g\ ,c_4}}_{0} &\neq \underbrace{\vec{h}_{g,v_4} - \vec{h}_{g,v_1}}_{0}.
\end{aligned}
$$

$$
\begin{aligned}
\vec{f}_{g\ ,c_1} &= \vec{h}_{g,v_1} - \vec{h}_{g,v_2} + \vec{\phi}_{g_1} - \vec{\phi}_{g_0},\\
\vec{f}_{g\ ,c_2} &= \vec{h}_{g,v_2} - \vec{h}_{g,v_3} + \vec{\phi}_{g_2} - \vec{\phi}_{g_0},\\
\vec{f}_{g\ ,c_3} &= \vec{h}_{g,v_3} - \vec{h}_{g,v_4} + \vec{\phi}_{g_3} - \vec{\phi}_{g_0},\\
\underbrace{\vec{f}_{g\ ,c_4}}_{-\vec{f}_{g}^{body}} &= \underbrace{\vec{h}_{g,v_4} - \vec{h}_{g,v_1}}_{0} + \underbrace{\vec{\phi}_{g_4} - \vec{\phi}_{g_0}}_{\Box^2\vec{\phi}_0}.
\end{aligned}
$$

Geometry of contact network represented by the network Laplacian

Random matrix: diagonal elements contain the number of contacts, otherwise the adjacency matrix

Framework

$\Box^2|\vec{\phi}\rangle = -|\vec{f}_{body}\rangle$ Equation defining the auxiliary fields

Given a contact network and a set of body forces, solution is unique

If the solution violates torque balance/static friction condition, network will rearrange

Disorder of contact network represented by network Laplacian

Diffusion on a random network: Localization ?

Eigenfunction expansion $\Box^2 = \sum_{i=1}^{N} \lambda_i |\lambda_i\rangle\langle\lambda_i|$

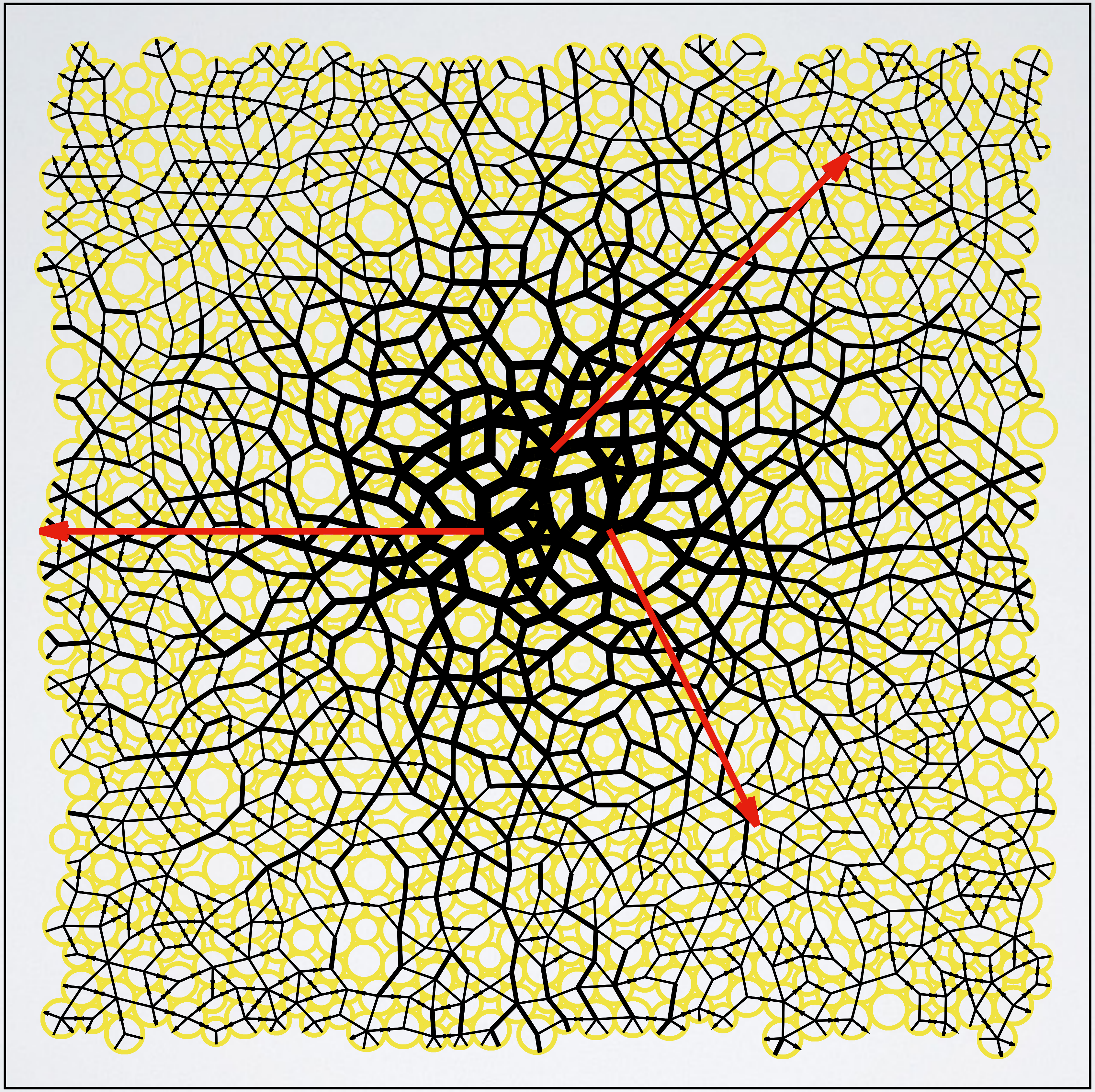
One zero mode $\lambda_1 = 0 \;, |\lambda_1\rangle = (1\;1\;1\ldots 1)$

Localized ? $|\vec{\phi}\rangle = \sum_{i=1}^{N} \frac{1}{\lambda_i} \langle\lambda_i|\vec{f}_{body}\rangle|\lambda_i\rangle$

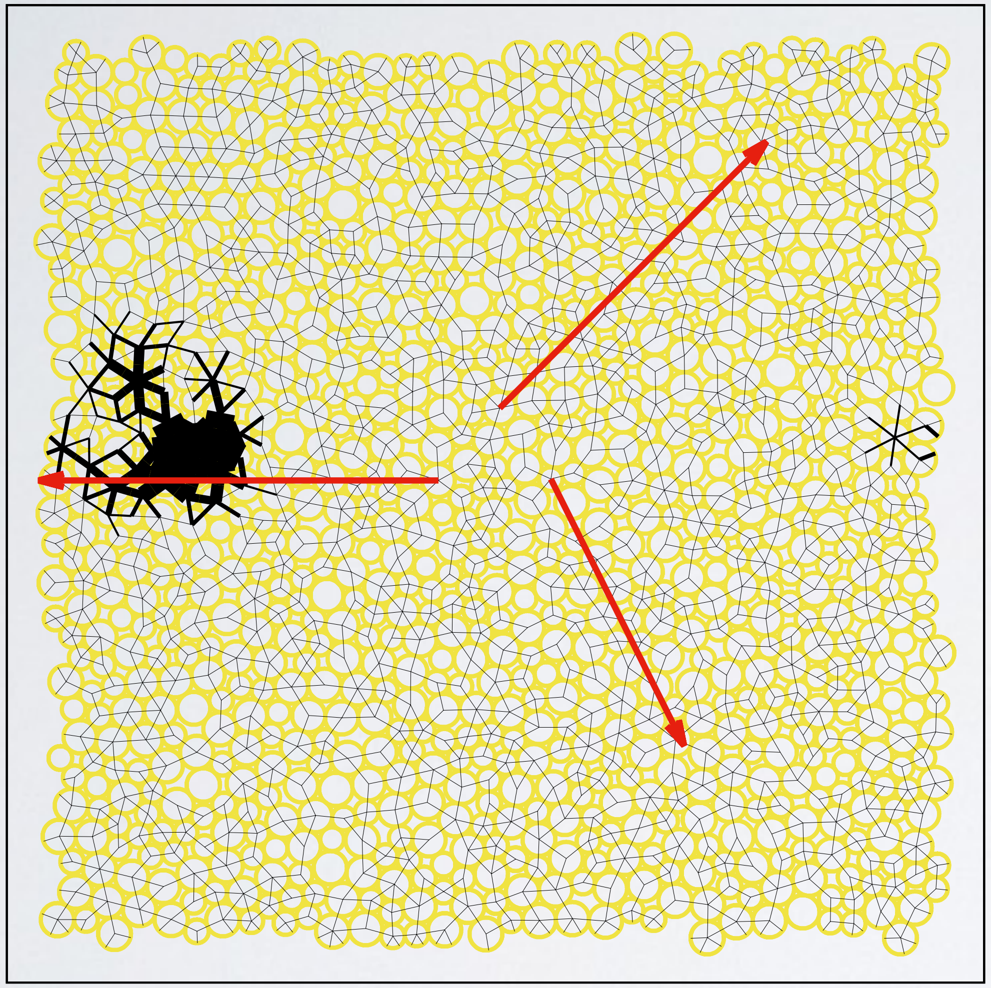
Different from q model: On a given contact network, how the force perturbation gets distributed among contacts is completely determined by the constraints of force balance

Constitutive Law determined by statistical properties of the ensemble of Laplacians

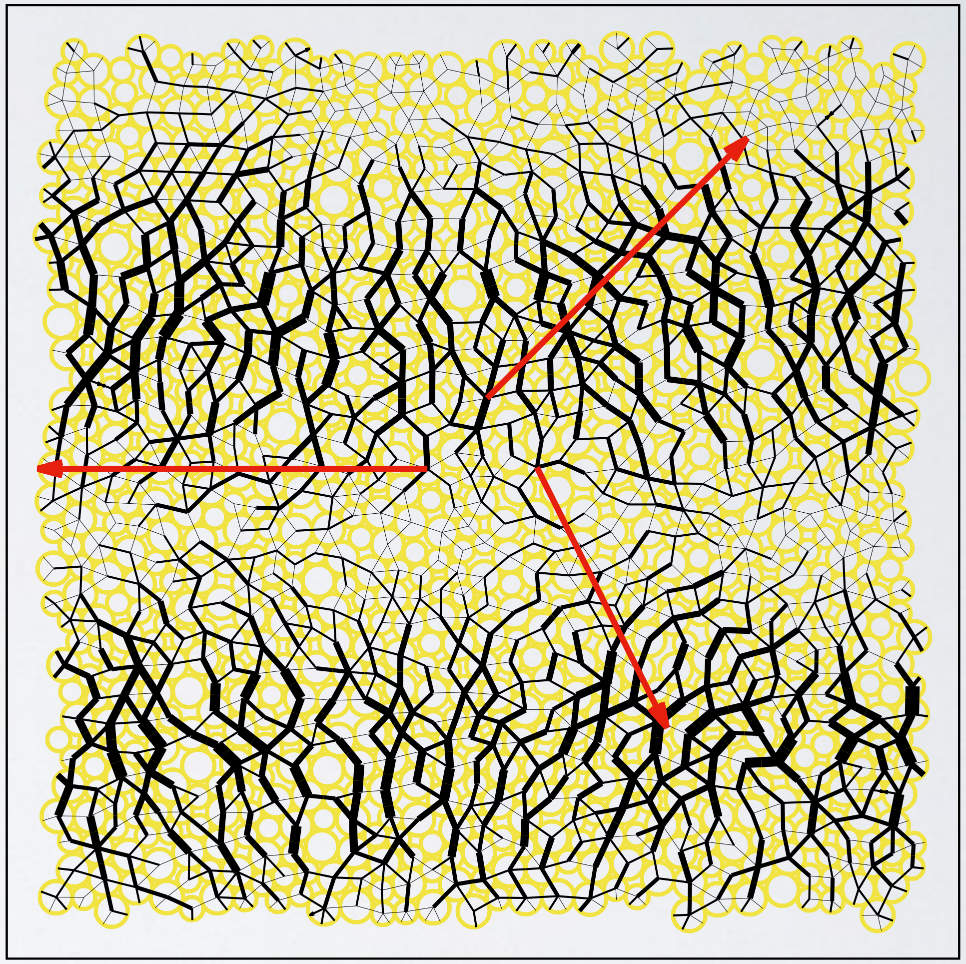
Response of a frictionless granular solid



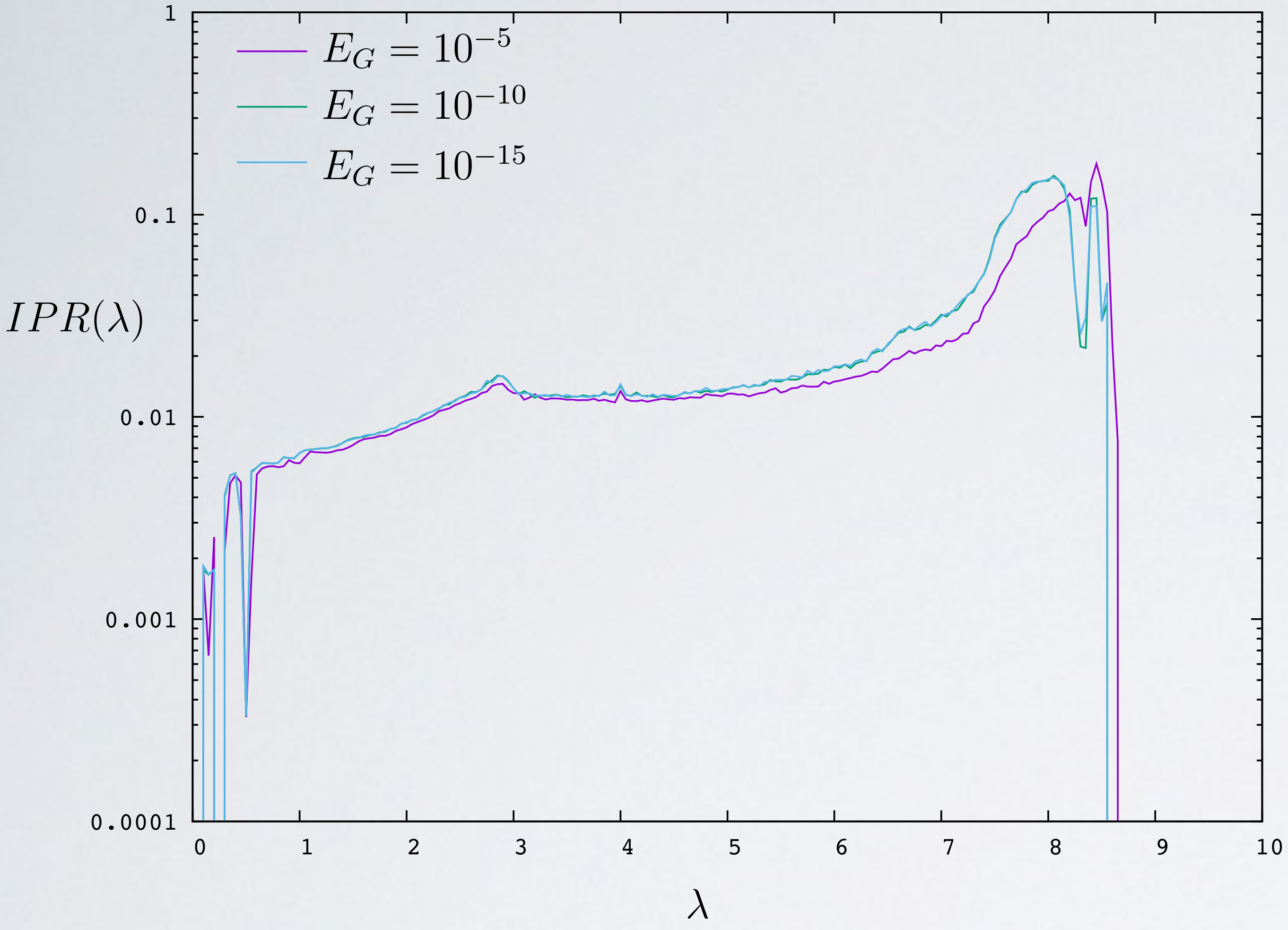
Highest eigenvalue: strongly localized



Lowest eigenvalue: delocalized

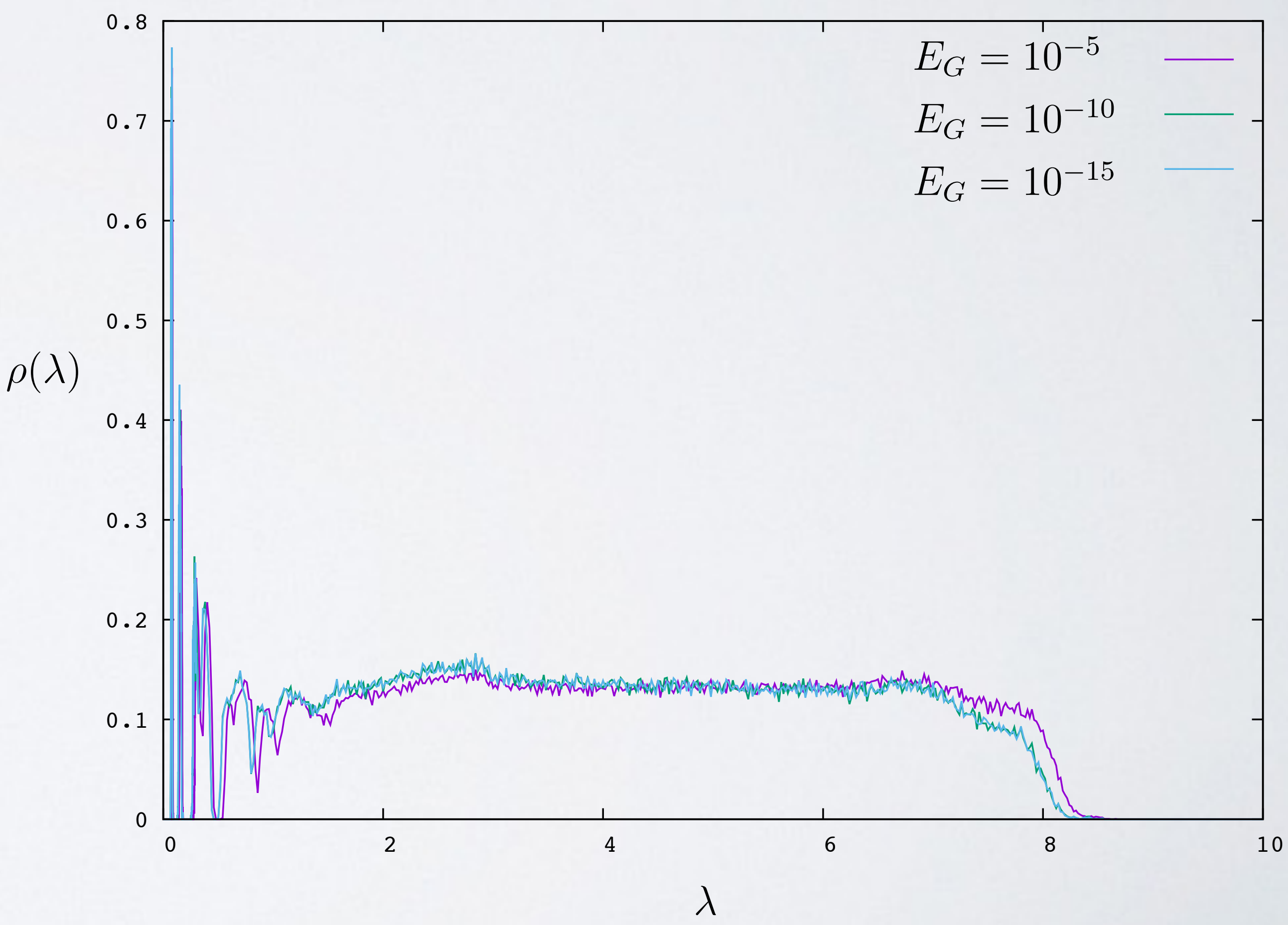


Ensemble average: Spectral properties



Inverse Participation Ratio

Density of States



Stress Localization

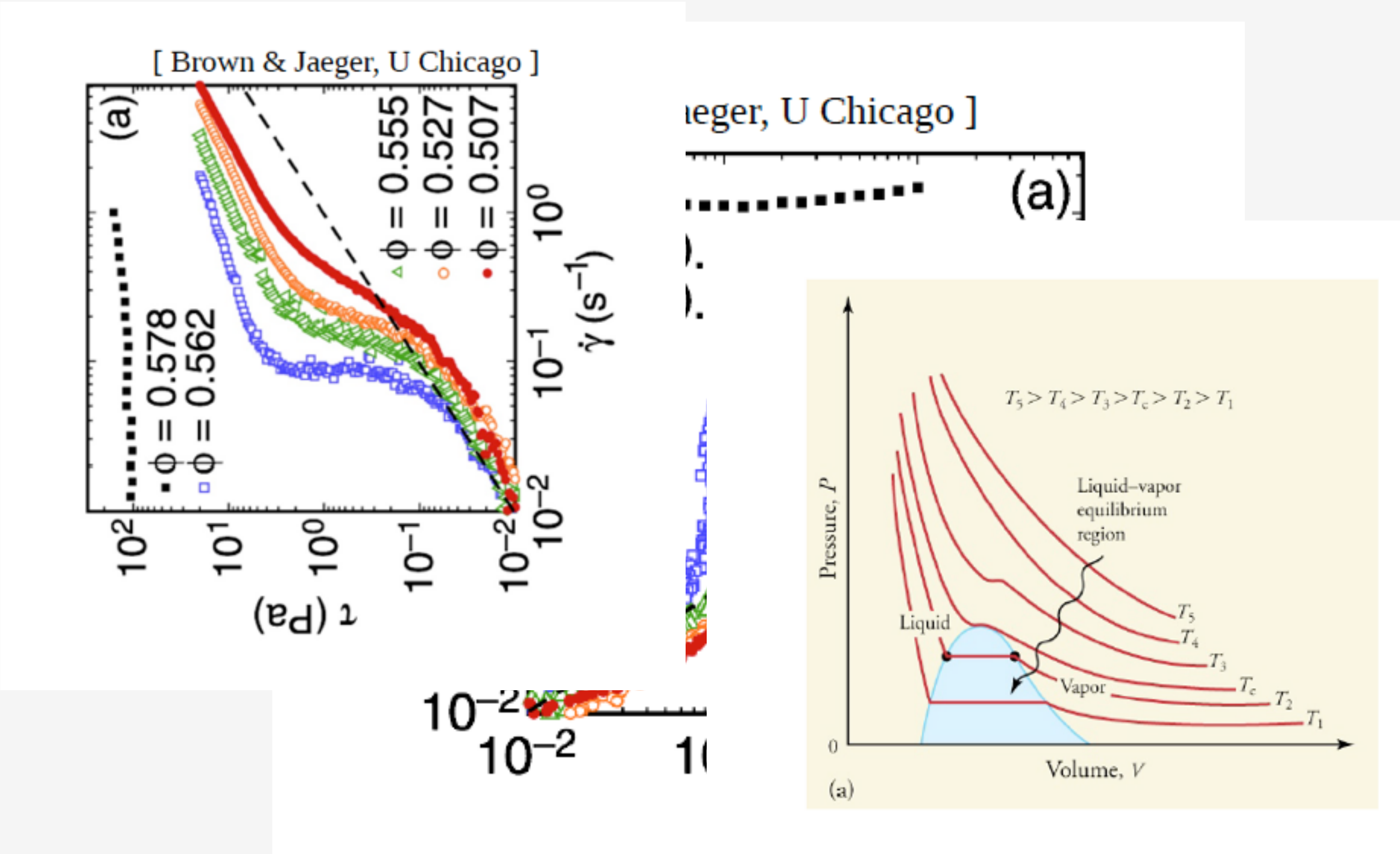
- Maps granular response problem to the localization problem

"Absence of Diffusion in Certain Random Lattices"

P.W. Anderson (1958)

- Theory relates response to the disorder in the underlying network
- Random Matrix Ensembles: Characterizing Jammed Networks
- Many Body Localization: Strongly Interacting System

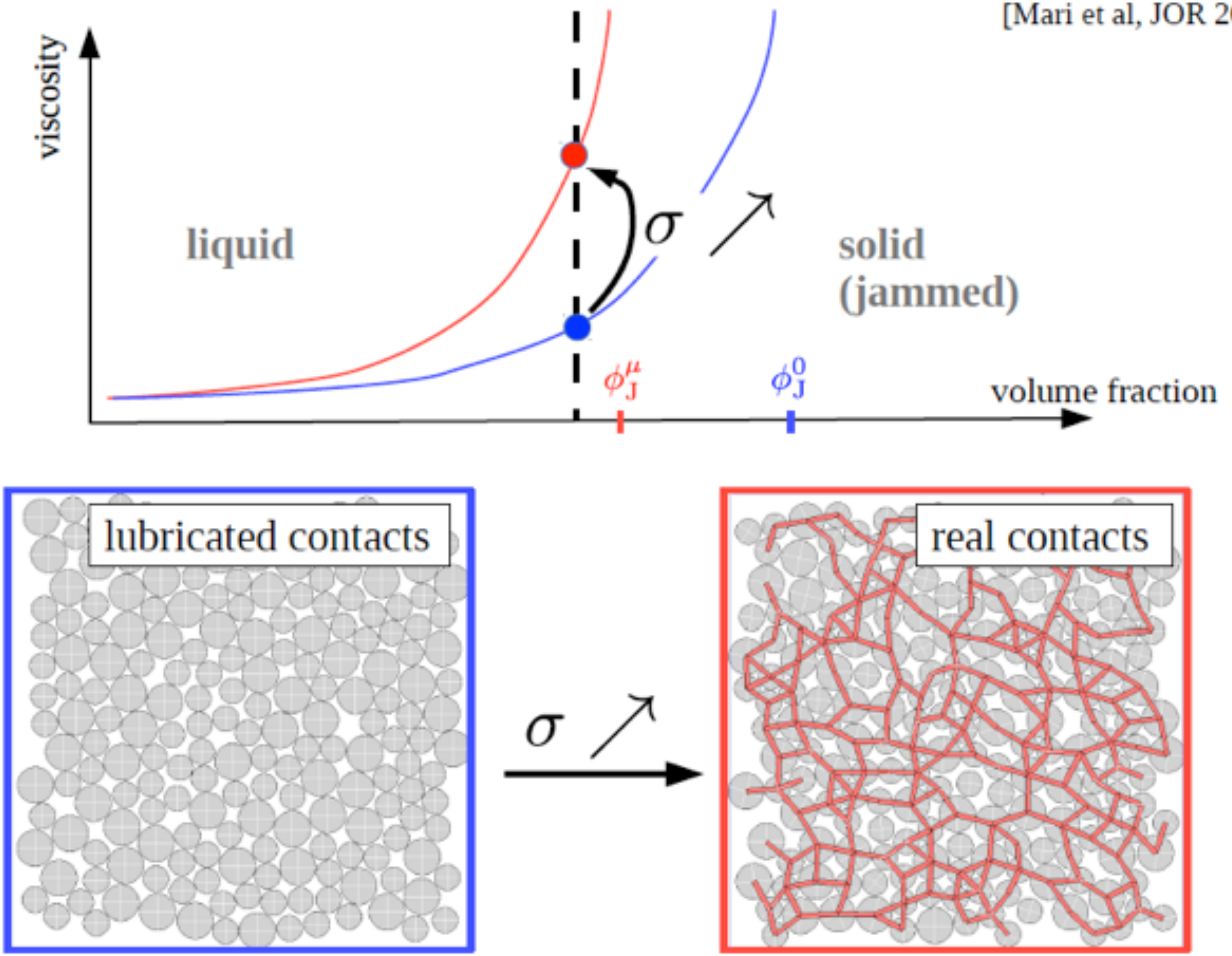
Discontinuous Shear Thickening



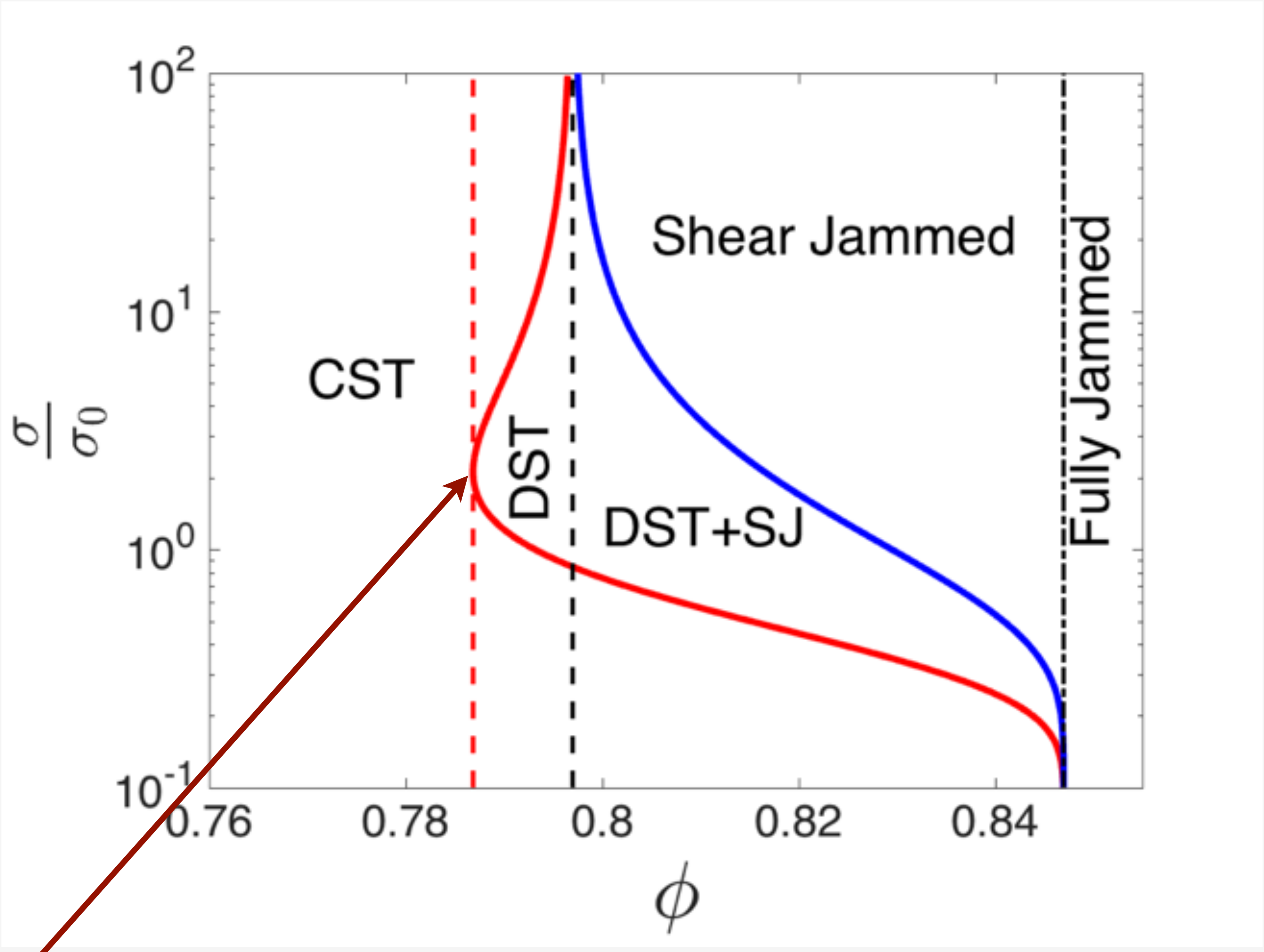
Shear thickening as a (out-of-equilibrium) phase transition?

A Thickening Scenario

[Fernandez et al, PRL 2013]
[Seto et al, PRL 2013]
[Heussinger, PRE 2013]
[Wyart and Cates PRL 2013]
[Mari et al, JOR 2014]



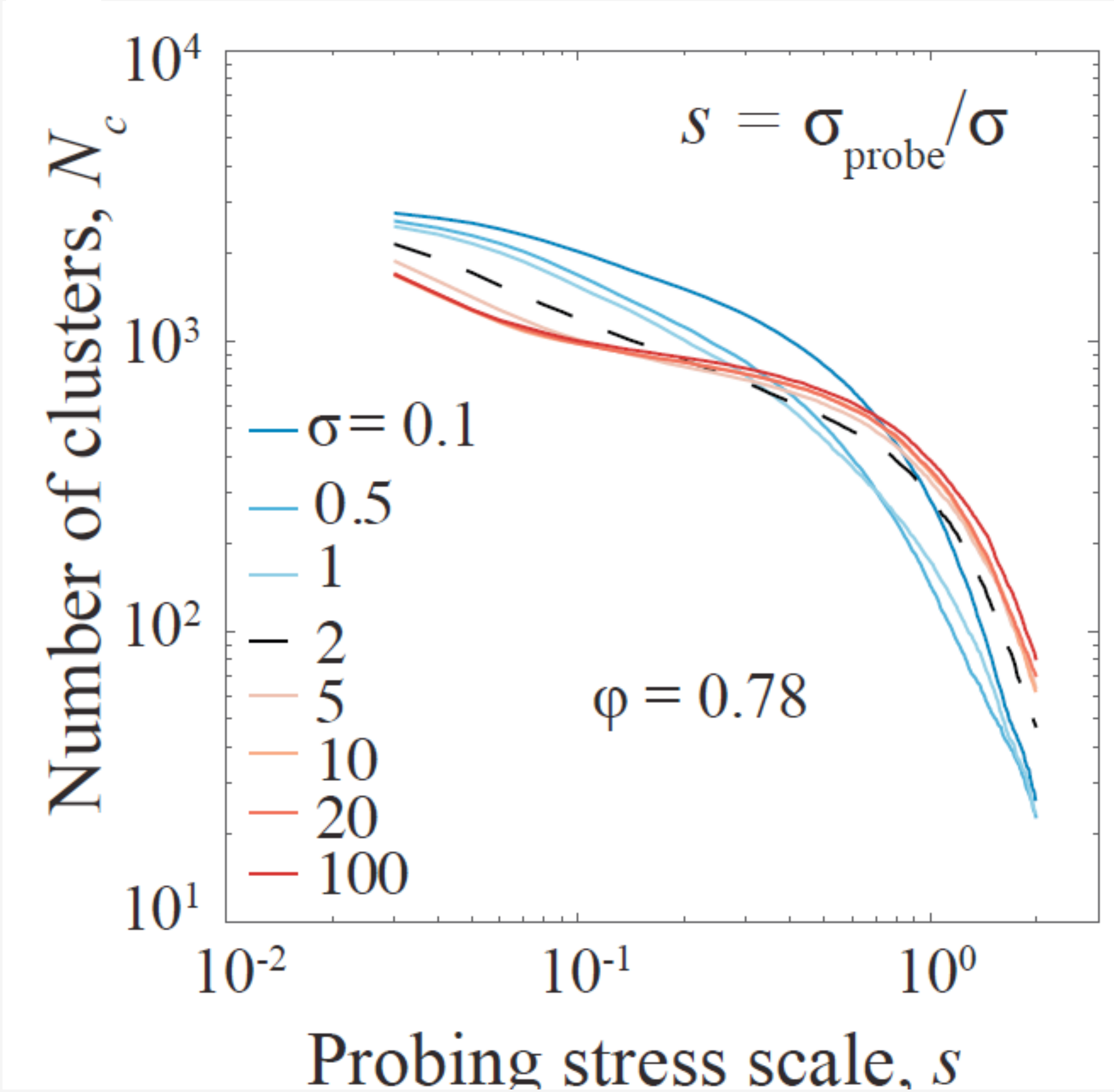
Phase Diagram from Rheology: Abhi Singh & Jeff Morris



Is this a critical point ?

Clustering of points

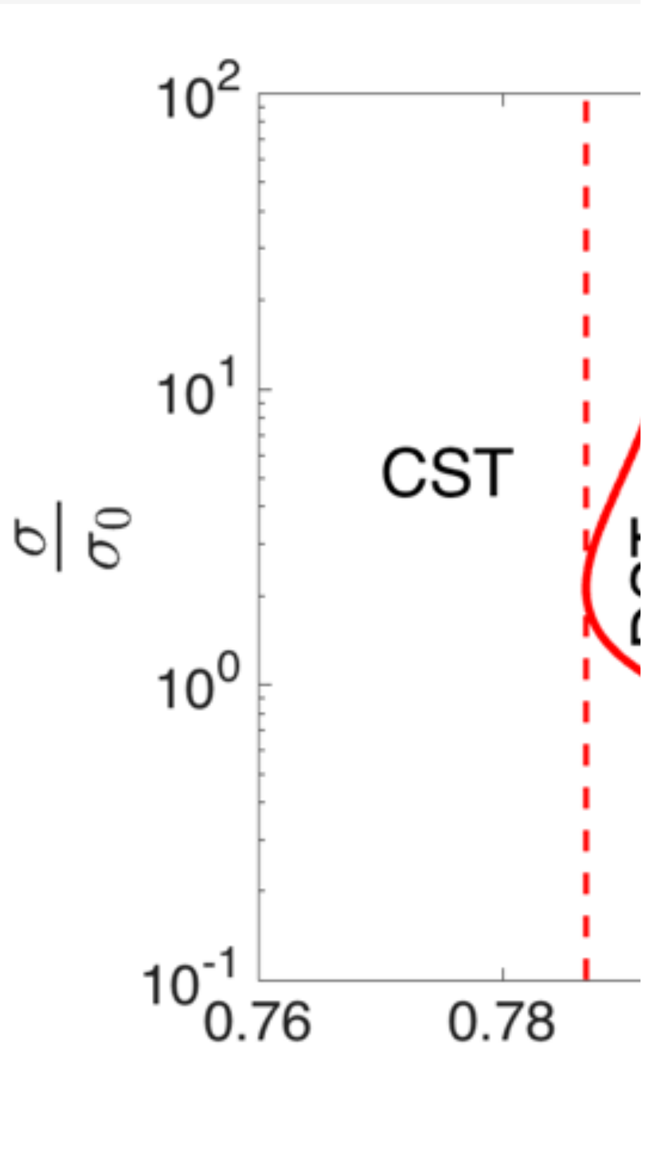
Clustering/Clumping of points

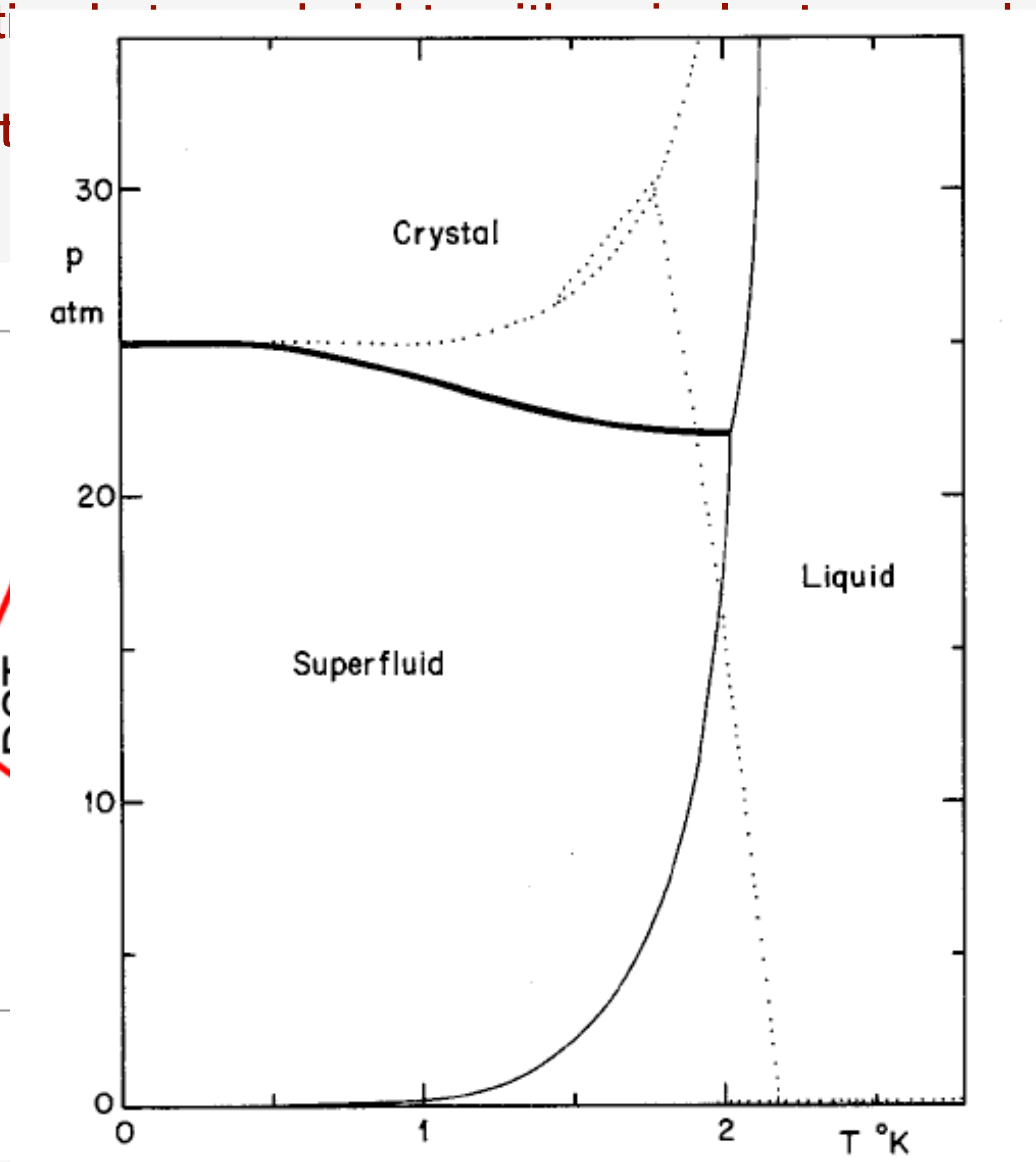


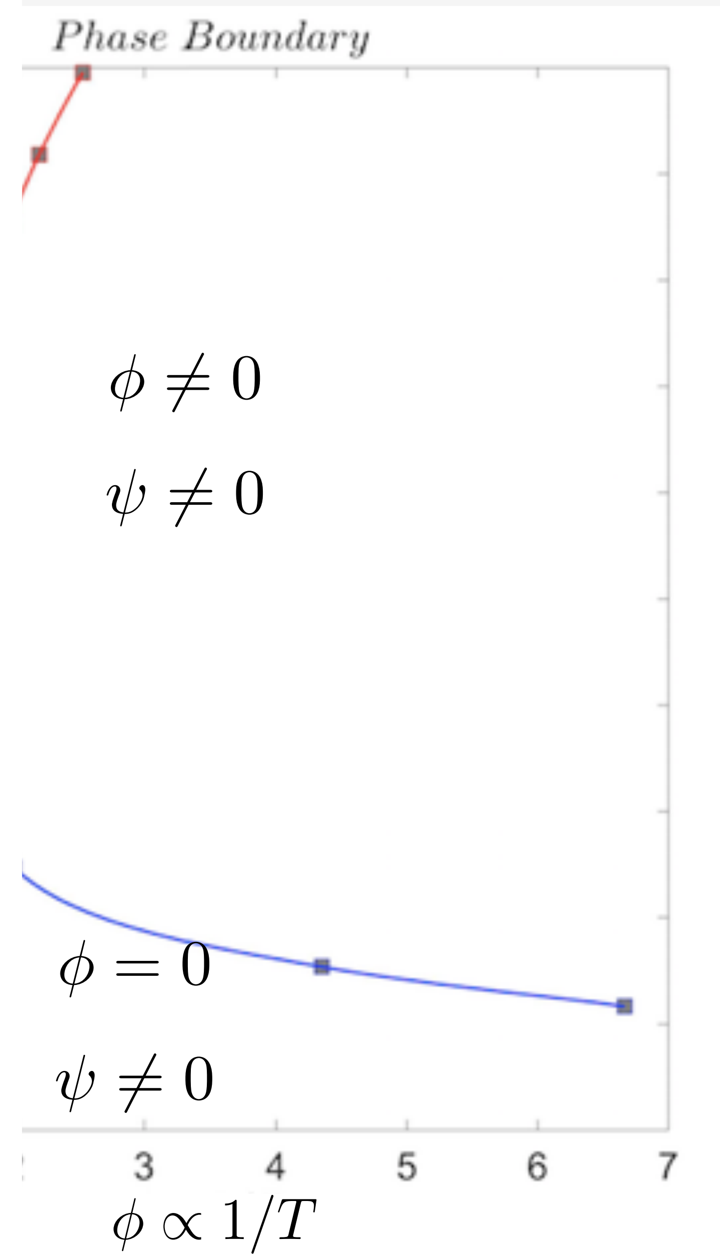
$s = \sigma_{probe}/\sigma$
Number of clusters, N_c
Probing stress scale, s
10^4
10^3
10^2
10^1
10^{-2}
10^{-1}
10^0
$\sigma = 0.1$
0.5
1
2
5
10
20
100
$\varphi = 0.78$

“Equibrium” Model

- Repulsive interacti
- Simplest: If height







A tricritical point ?

Statistical Mechanics of Granular Media

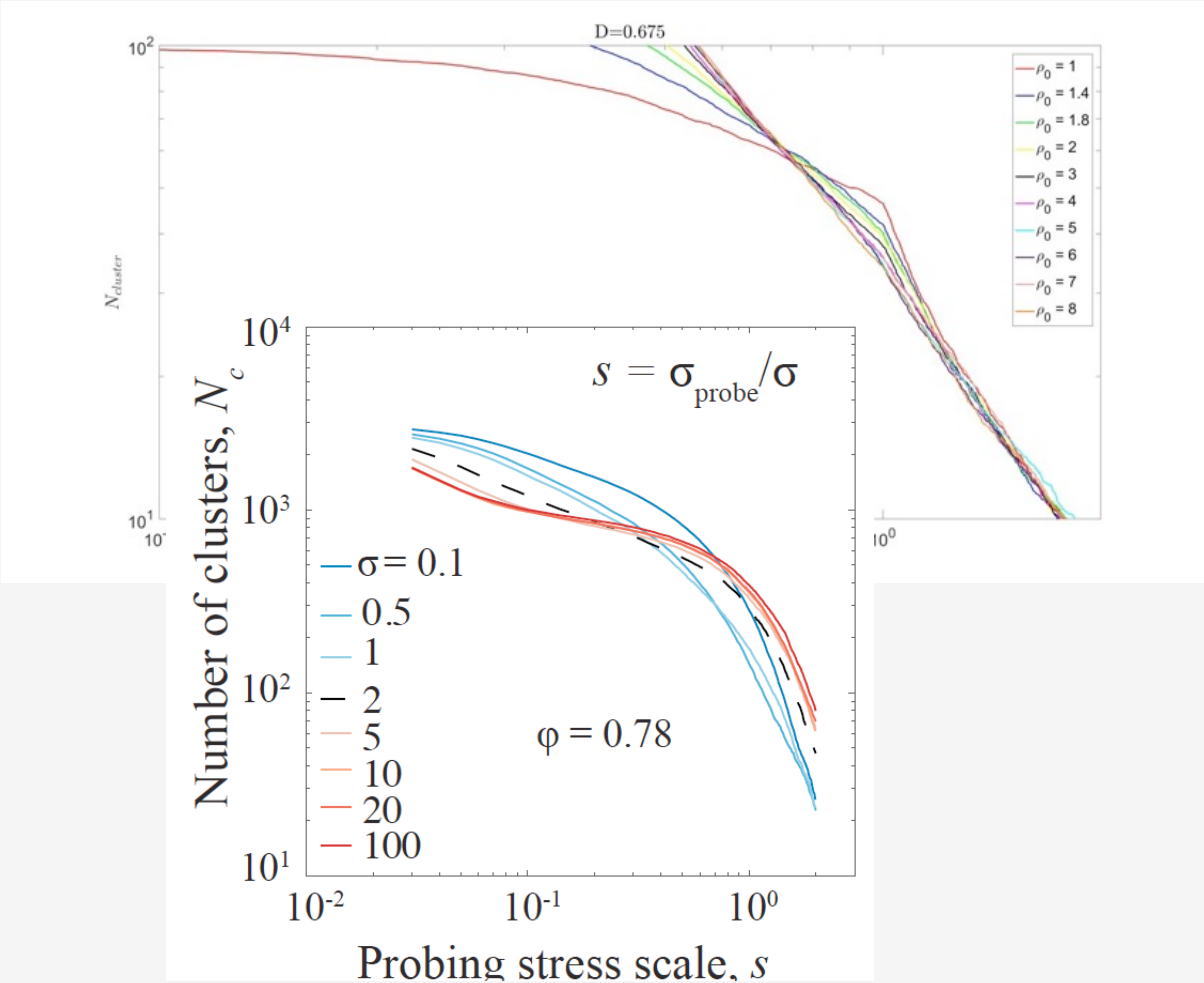
- Dual Networks: Contacts and Force Tilings

- History Dependence:

Including forces in defining microstates takes away that indeterminacy

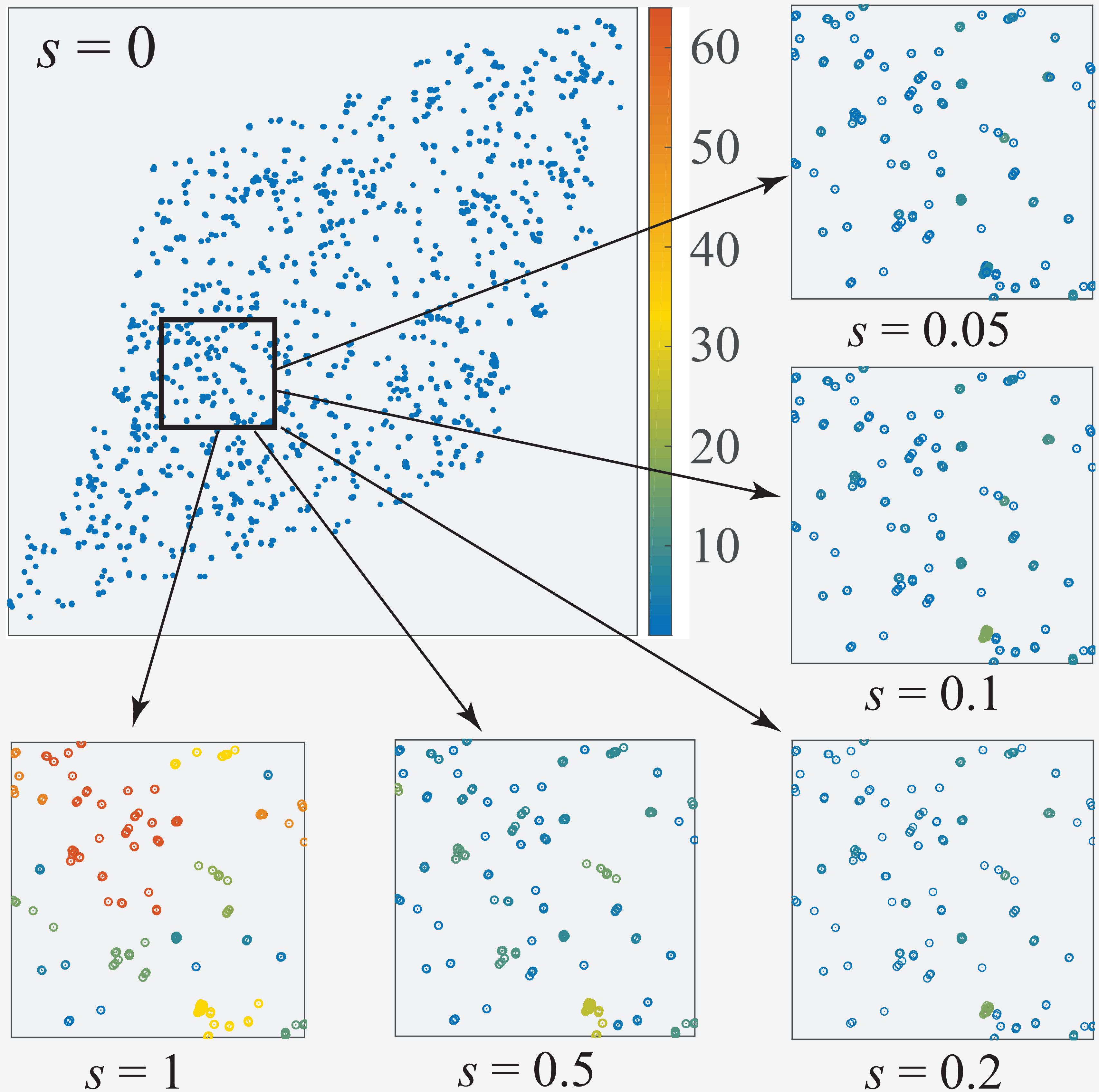
Contact Networks are random but can characterize ensembles

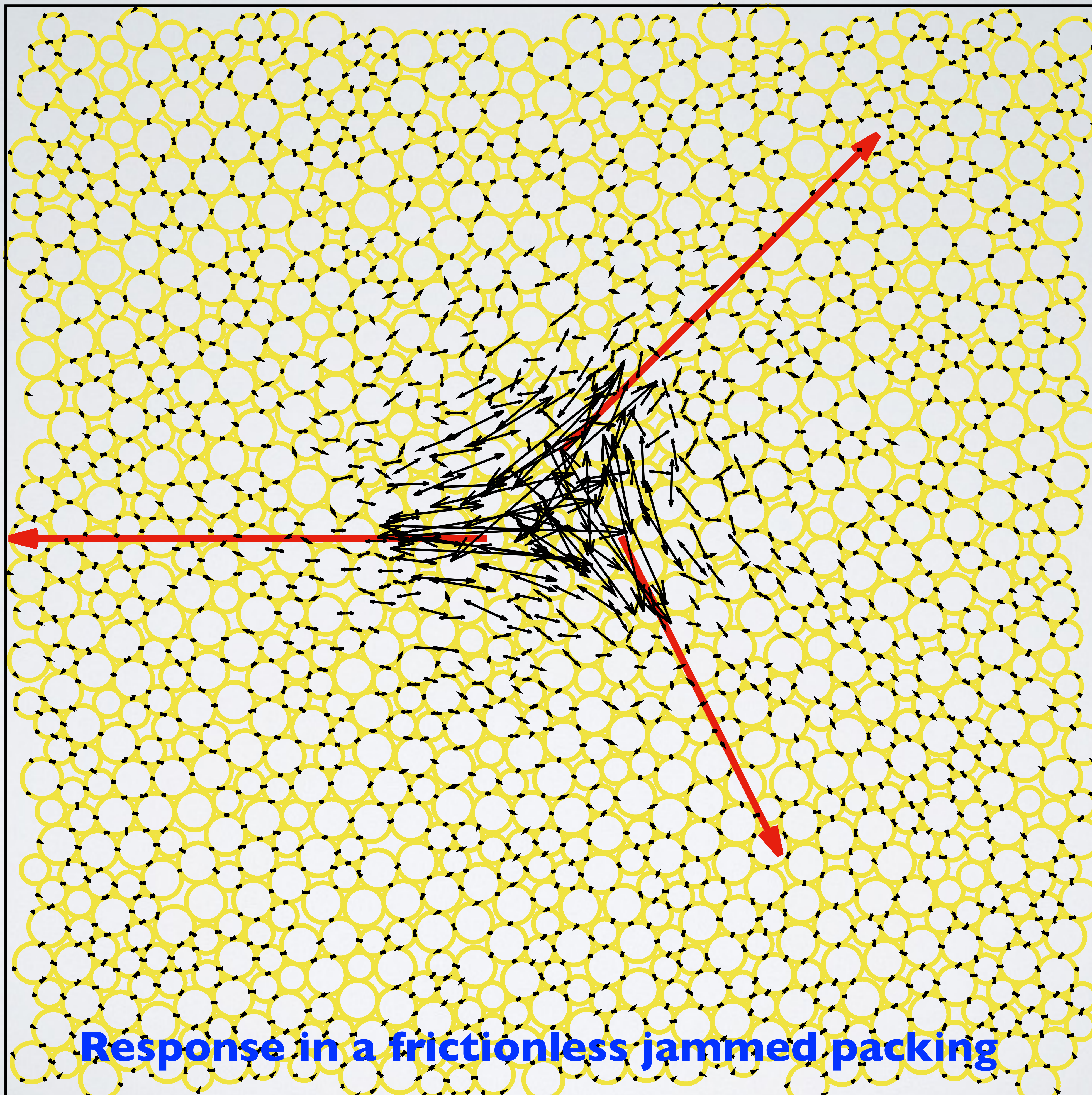
- Pattern formation in height fields: Distinguish phases



D=0.675
10^2
10^1
$N_{cluster}$
10^0
$\rho_0 = 1$
$\rho_0 = 1.4$
$\rho_0 = 1.8$
$\rho_0 = 2$
$\rho_0 = 3$
$\rho_0 = 4$
$\rho_0 = 5$
$\rho_0 = 6$
$\rho_0 = 7$
$\rho_0 = 8$
$s = \sigma_{probe}/\sigma$
Number of clusters, N_c
10^4
10^3
10^2
10^1
σ = 0.1
0.5
1
2
5
10
20
100
φ = 0.78
10^{-2}
10^{-1}
10^0
Probing stress scale, s

Cluster Analysis




Response in a frictionless jammed packing