Review talk: Non-Relativistic Gravity and Non-Lorentzian Geometry

Beyond Lorentzian Geometry II, Edinburgh, February 6, 2023

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DET FRIE FORSKNINGSRÅD
DANISH COUNCIL FOR INDEPENDENT RESEARCH

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Vetenskapsrådet

based on

2212.11309 (Hartong,NO,Oling) (review paper to appear in Frontiers in Physics issue with many more reviews on NL geometry and applications)

2001.20277 (JHEP), 1807.04765 (PRL) (Hansen,Hartong,NO)

& many other works/authors

NO

Present a review of non-relativistic gravity and its recent developments using Non-Lorentzian geometry

Non-relativistic gravity refers to the study of gravitational forces in a regime where the speeds involved are much less than the speed of light. In this regime, the equations of motion are described by Newton's law of gravitation.

Non-Lorentzian geometry is a mathematical framework that provides an alternative to the traditional Lorentzian geometry used in general relativity. This framework allows for the consideration of non-relativistic gravitational forces in a more general and flexible way, as it relaxes some of the strict assumptions of Lorentzian geometry.

Recent developments in the study of non-relativistic gravity using non-Lorentzian geometry have focused on exploring new ways to describe and understand gravitational forces. This has led to the discovery of new connections between non-relativistic gravity and other areas of physics, such as the study of condensed matter systems and the quantization of gravitational forces.

Another recent development has been the exploration of the relationship between non-relativistic gravity and holography. This has provided new insights into the connection between non-relativistic gravitational forces and the fundamental nature of spacetime and quantum gravity.

Overall, the use of non-Lorentzian geometry in the study of non-relativistic gravity has allowed for a more flexible and general treatment of gravitational forces, and has opened up new avenues for research and discovery in this area.

NO

make a poem on non-relativistic gravity and Newton-Cartan geometry

Non-relativistic gravity, a force that pulls with might,
Described by Newton's law, a brilliant insight.

A world of objects moving slow,
In the absence of speed, a steady flow.

Newton-Cartan geometry, a mathematical delight,
A framework to describe this force, in a geometric light.

Space and time, separate entities,
A more flexible approach, to our tendencies.

The curves of spacetime, no longer just bend,
A new way to view the universe, to a new end.

Recent developments, new discoveries unfold,
In hydrodynamics, phase transitions, and more untold.

A connection to string theory, a glimpse of the divine,
Non-relativistic gravity, a subject worth our time.

So let us ponder and delve into this realm,
And uncover the secrets, of non-relativistic gravity's helm.

Intro

Nature is Lorentz invariant at fundamental level
but appears effectively non-relativistic in many systems:

- condensed matter, biological systems
- gravitational phenomena

original idea of geometrizing Newtonian gravity: Cartan (1923)

gravity = geometry
follows from equivalence principle(s)
local observers in inertial frames see laws of special/Galilean relativity

recent years: revival of research in non-relativistic gravity (NRG)

- richer than Newtonian gravity
- allows for strong field regime (grav. time dilation)

key insights: torsion (no absolute time) & off-shell large c expansions

Intro (ctd)

recent revival:

spurred on by deeper understanding of non-Lorentzian (NL) geometry
& connectionts to field theory, holography, string theory

highlights:

- Newton-Cartan (NC) geometry from gauging Bargmann algebra
  * Andringa,Bergshoeff,Panda,De Roo (2010)
- torsionfull generalization of NC geometry (in context of non-AdS holography)
  * Christensen,Hartong,NO,Rollier/Hartong,Kiritsis,NO (2013/14)
- NR effective field theories couple to NC geometry
  * Son (2013)
- systematic large speed of light expansion of GR
  * Dautcourt (1996),Van den Bleeken(2017),Hansen,Hartong,NO(2018/20)

& many more references/authors (see review paper for many more relevant)

IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **28** (2011) 105011 (12pp) doi:10.1088/0264-9381/28/10/105011

Newtonian gravity and the Bargmann algebra

Roel Andringa[1], Eric Bergshoeff[1], Sudhakar Panda[2] and Mees de Roo[1]

PHYSICAL REVIEW D **89,** 061901(R) (2014)

Torsional Newton-Cartan geometry and Lifshitz holography

Morten H. Christensen,[1,*] Jelle Hartong,[1,†] Niels A. Obers,[1,‡] and B. Rollier[2,§]

PHYSICAL REVIEW D **92,** 066003 (2015)

Schrödinger invariance from Lifshitz isometries in holography and field theory

Jelle Hartong,[1,*] Elias Kiritsis,[2,3,†] and Niels A. Obers[1,‡]

EFT-13-8

Newton-Cartan Geometry and the Quantum Hall Effect

Dam Thanh Son

PAPER

Torsional Newton–Cartan gravity from the large c expansion of general relativity

Dieter Van den Bleeken[1]

Published 23 August 2017 • © 2017 IOP Publishing Ltd

Classical and Quantum Gravity, Volume 34, Number 18

Citation Dieter Van den Bleeken 2017 *Class. Quantum Grav.* **34** 185004

PHYSICAL REVIEW LETTERS **122,** 061106 (2019)

Action Principle for Newtonian Gravity

Dennis Hansen,[1,*] Jelle Hartong,[2,†] and Niels A. Obers[3,‡]

Outline

- History of geometrizing Newtonian gravity
- Basics of torsion-free Newton-Cartan geometry
- Recent history: revival ("gauging the Bargmann algebra")
- Type I torsionsal Newton-Cartan geometry
- Non-relativistic expansion of GR
- Other aspects of NRG
- Discussions/Outlook

Ancient history

- birth of subject: Cartan (1923/24)
- pioneering work on 1/c expansion of GR:
  Friedrichs (1927) [& older work of Weyl (1923)]
  - introduces NC metric $\tau_\mu \tau_\nu$ and co-metric $h^{\mu\nu}$
  - Newton-Cartan metric compatible (non-unique) connection

- further overviews of early aspects of NC geometry:
  * Havas, Malament, MTW

Less ancient: Trautman, Dautcourt, Kunzle, Ehlers

- axiomatic definition of NC gravity a la Trautman (1963-66) (next slide)
- intrinsic (i.e. without reference to 1/c expansion)

1. Spacetime is a four-dimensional differentiable manifold endowed with a symmetric affine connection $\Gamma^{\rho}_{\mu\nu}$.

2. There is a nowhere-vanishing one-form τ_μ and a nowhere vanishing co-metric $h^{\mu\nu}$ of signature $(0,1,1,1)$ such that $h^{\mu\nu}\tau_\mu = 0$.

3. The symmetric affine connection $\Gamma^{\rho}_{\mu\nu}$ is metric-compatible in the sense that

$$\nabla_\mu \tau_\nu = 0 \,, \qquad \nabla_\mu h^{\nu\rho} = 0 \,, \tag{2.1}$$

where ∇_μ is the associated covariant derivative.

4. The Riemann curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$ associated with the affine connection obeys the following two conditions:

$$R_{\kappa\lambda[\mu}{}^{\rho}\tau_{\nu]} = 0 \,, \qquad R_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} = R_{\rho}{}^{\sigma}{}_{\mu}{}^{\nu} \,, \tag{2.2}$$

where the second index has been raised with $h^{\mu\nu}$, so $R_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} = h^{\nu\kappa}R_{\mu\kappa\rho}{}^{\sigma}$. See Appendix A for our conventions for the Riemann tensor

5. Particles in free fall follow geodesics of the affine connection and gravity is described by an equation of the form (in $3+1$ dimensions)

$$R_{\mu\nu} = 4\pi G \rho \tau_\mu \tau_\nu \,, \tag{2.3}$$

where ρ is the mass density, G is Newton's constant and $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$ is the Ricci tensor.

Remarks

- mathematical framework: Dombroski, Horneffer (1964)
- Kunzle (1972):
  - class of torsion-free metric compatible connections
  - 1st Trautman condition implied by asymptotic flatness
- Dautcourt (1964 & 90s):
  - appropriate covariant 1/c expansion of GR reproduces Trautman's formulation
  - realizes option of nontrivial lapse function $\tau = NdT$, but suggests not interesting because of global regularity
- Ehlers (1981 & 90s): frame theory
  - treats Lorentzian and Galilean geometry on equal footing with two parametrs: G and 1/c
- null reduction & Bargmann algebra (details later)
  Duval,Burdet,Kunzle (1985)
  Duval,Gibbons,Horvathy (1991)

Post-Newtonian corrections

- part of motivation of frame theory:
  are solutions to Newtonian gravity extendible to full relativistic ?

 * Rendall (1992): PN corrections not compatible with asymptotic flatness

PN regime corresponding to f. ex. perfect fluid matter source has finite radius of validity (grav. waves dominate in the far regime)

- effect kicks in when going beyond 1st PN correction

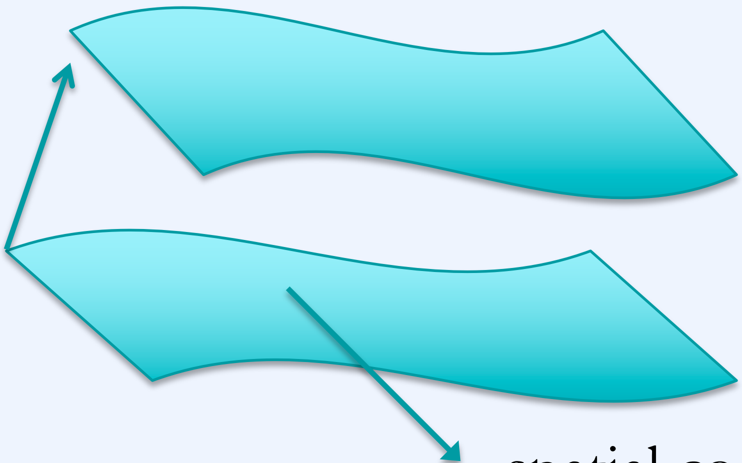
- classic approach: expand EE in powers of 1/c and solve order-by-order
- current PN approach (e.g. Blanchet-Damour/Will-Wiseman):
  mixture of 1/c and G expansion and matching in overlap region

recent covariant 1/c expansion of GR (see later in talk)
"revives" classic approach
See Hartong&Musaeus (in progress) for hybrid approach

Basics of torsion-free NC geometry

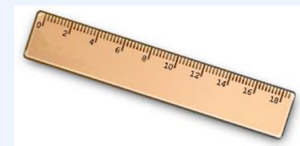
time

τ_μ



$$\tau_\mu h^{\mu\nu} = 0\,.$$

spatial co-metric $h^{\mu\nu}$



$$h_{\mu\rho}h^{\rho\nu} - \tau_\mu v^\nu = \delta_\mu^\nu\,, \qquad v^\mu h_{\mu\nu} = 0\,, \qquad \tau_\mu v^\mu = -1\,, \qquad \tau_\mu h^{\mu\nu} = 0$$

local Galilean boosts $\delta v^\mu = h^{\mu\nu}\lambda_\nu\,, \qquad \delta h_{\mu\nu} = \tau_\mu\lambda_\nu + \tau_\nu\lambda_\mu\,.$ $\quad v^\mu\lambda_\mu = 0,$

spatial vielbeins: $h_{\mu\nu} = \delta_{ab}e_\mu{}^a e_\nu{}^b, \qquad h^{\mu\nu} = \delta^{ab}e^\mu{}_a e^\nu{}_b,$

$a = 1, \ldots, d$ are flat frame indices.

invariant integration measure: $e = \det(\tau_\mu, e_\mu{}^a)$

Torsion-free metric compatible connections

metric compatiblity $\nabla_\mu \tau_\nu = 0\,, \qquad \nabla_\mu h^{\nu\rho} = 0\,,$

$$\Gamma^\rho_{\mu\nu} = \check{\Gamma}^\rho_{\mu\nu} - \frac{1}{2} h^{\rho\sigma} \left(\tau_\mu F_{\nu\sigma} + \tau_\nu F_{\mu\sigma} \right)$$

$$\check{\Gamma}^\rho_{\mu\nu} = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} \right)$$

1st Trautman condition (& reformulation in terms of Ehlers conditions)
→ flat geometry on constant time slices & condition on magnetic part of F (too strong)

2nd Trautaman condition → $F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu$

connection inv. under Gal. boosts if: $\delta m_\mu = \lambda_\mu\,,$

Note: for absolute time $\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu} + \mathcal{O}(c^{-2})\,,$

by expanding GR $g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu + \mathcal{O}(c^{-2})$

Newton-Cartan gravity

test particles follow geodesics of the connection $\bar{\Gamma}^{\rho}_{\mu\nu}$:

$$\ddot{x}^{\mu} + \bar{\Gamma}^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = \ddot{x}^{\mu} + \check{\Gamma}^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - h^{\mu\sigma}\dot{x}^{\rho}F_{\rho\sigma} = 0$$

follows from NR particle acction

- equivalence principle

$$S = m\int d\lambda\left[\frac{h_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{2\tau_{\rho}\dot{x}^{\rho}} - m_{\mu}\dot{x}^{\mu}\right]$$

$m_t = \Phi_N$: Newtonian potential

Trautman's postulate 5:

$$\bar{R}_{\mu\nu} = 8\pi G\frac{d-2}{d-1}\rho\tau_{\mu}\tau_{\nu} \quad , \qquad d\tau = 0$$

after gauge fixing: $\tau = dt\,, \qquad h_{\mu\nu}dx^{\mu}dx^{\nu} = \delta_{ij}dx^{i}dx^{j}\,, \qquad v = -\partial_t\,, \qquad h^{\mu\nu}\partial_{\mu}\partial_{\nu} = \delta^{ij}\partial_i\partial_j\,.$

$\rightarrow$

$$\nabla^2\Phi_{\mathrm{N}} = 8\pi G\frac{d-2}{d-1}\rho$$

Recent revival:

Andringa,Bergshoeff,Panda,de Roo:
- for absolute time: NC gravity = dynamics of geometry obtained by gauging Bargman
- Trautman condition follows from Bianchi identity associated to field strength in gauging procedure

Van den Bleeken:
- can do $1/c$ expansion in full generality, without assuming absolute time
- NR approximations can describe strong gravitational fields

expectation:
- extend gauging methods to nonzero torsion (called type I TNC geometry) to describe NRG with non-absolute time

not the case:
Hansen,Hartong,NO:
- cannot consistently coupled NC geometry with local Bargmann symmetry to matter sources without turning on torsion (even violates hypersurface orthogonality)
- large c expansion of GR reveals a novel avatar: type II TNC geometry (with underlying algebra that follows from Lie algebra expansion of Poincare)

Type I TNC geometry

type I TNC geometry in (d+1) dimesions can be obtained from:

- gauging Bargman algebra
- null reduction of (d+2)-dimensional Loretnzian geometry
- large c limit of (d+1)-dimensional Lorentzian geometry with EM bgr. field

Last two methods also useful to obtain:

- probe acton for a non-relativistic particle in type I bgr.
  (see talk Gerben Oling, as warmup for NR strings)

- action of dynamical type I TNC (this talk)

Gauging the Bargmann algebra

(Galilean algebra is c to infinity limit of Poincare)

$$\underbrace{\overbrace{H, P_a, J_{ab}, G_a}^{\text{Galilean}} \quad N}_{\text{Bargmann}}$$

$$[H, G_a] = P_a$$

$$[P_a, G_b] = 0$$

$$\Downarrow$$

$$[P_a, G_b] = N\delta_{ab}$$

mass generator

Bargmann algebra follows from IW contraction of Poincare x U(1)

Bargmann-valued gauge field:

$$\mathcal{A}_\mu = H\tau_\mu + e_\mu{}^a P_a + G_a \Omega_\mu{}^a + \frac{1}{2}\Omega_\mu{}^{ab} J_{ab} + N m_\mu$$

gauge transformation

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu\,, \Sigma]$$

gauge parameter

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma\,, \qquad \Sigma = G_a \lambda^a + \frac{1}{2} J_{ab}\lambda^{ab} + N\sigma$$

→ reproduce the trafos under local Galilean boosts & rotations & diffeos

of the metric fields: $\tau_\mu,\ e_\mu^a,\ m_\mu$

$$h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$$

Affine connection, torsion, curvature

metric compatibility

& invariance under Gal boosts:

$$\Gamma^{\rho}_{\mu\nu} = \check{\Gamma}^{\rho}_{\mu\nu} + C^{\rho}{}_{\mu\nu}\,,$$

$$\check{\Gamma}^{\rho}_{\mu\nu} = -v^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right)$$

$$\boxed{\bar{\Gamma}^{\rho}_{\mu\nu} = -\hat{v}^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right)}$$

connection has torsion

$$2\check{\Gamma}^{\rho}_{[\mu\nu]} = -v^{\rho}\left(\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}\right)$$

boost invariants

$$\hat{v}^{\mu} = v^{\mu} - h^{\mu\nu}m_{\nu}\,,$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_{\mu}m_{\nu} - \tau_{\nu}m_{\mu}$$

NC = no torsion $\rightarrow$ $d\tau = 0$ absolute time

TTNC = twistless torsion $\rightarrow$ $\tau \wedge d\tau = 0$ preferred foliation equal time slices

TNC no condition on τ_{μ}

- TTNC torsion described by $h^{\mu\nu}a_{\mu}$
- acceleration vector of the foliation: $a_{\mu} = \mathcal{L}_{v}\tau_{\mu}$
- curvature defined in usual way from the connection

Gravity action for type I TNC: from EM

from contraction of Einstein-Maxwell

$$\mathcal{L}_{\rm EM} = \frac{c^3}{16\pi G}\sqrt{-g}R - \frac{1}{4c\,k^2}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$

IW contraction of Poincare x U(1) to Bargmann: at level of fields

$$E_\mu{}^0 = c\tau_\mu + \frac{1}{c}m_\mu\,, \qquad E_\mu{}^a = e_\mu{}^a\,, \qquad A_\mu = \tau_\mu$$

with
c → infinity

relativistic vielbeine   Maxwell field

$$\mathcal{L}_{\rm EM} = e h^{\mu\rho}h^{\nu\sigma}\left(\frac{c^6}{64\pi G}\tau_{\mu\nu}\tau_{\rho\sigma} - \frac{1}{4k^2}F_{\mu\nu}F_{\rho\sigma}\right) + \mathcal{O}(c^4)\,.$$

$$\tau_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$$

can cancel leading order terms
by setting: (cf. probe action)

$$\frac{1}{k^2} = \frac{c^6}{16\pi G}\,, \qquad A_\mu = \tau_\mu\,.$$

remaining part (after rescaling G) is type I TNC action

$$\boxed{\mathcal{L} = \frac{e}{16\pi G}\left(h^{\mu\nu}\check{R}_{\mu\nu} + \frac{1}{2}h^{\mu\nu}a_\mu a_\nu + \frac{1}{2}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}m_{\rho\sigma}\right) \qquad m_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}}$$

Gravity action for type I TNC: from null-reduction

- null reduction $$ds^2 = g_{MN}dx^M dx^N = 2\tau_\mu(du - m_\mu dx^\mu) + h_{\mu\nu}dx^\mu dx^\nu$$

apply to EH:

$$\boxed{\mathcal{L} = \frac{\hat{E}}{16\pi\hat{G}} g^{MN} R_{MN} = \frac{e}{16\pi G}\left(h^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{2}h^{\mu\nu}\hat{a}_\mu\hat{a}_\nu - \frac{1}{2}\hat{\Phi}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}\right)}$$

- can be shown to be exactly previous action (from contraction) in different variables
unique two-derivative action that is Bargmann invariant

at level of EOM: missing equation: $\hat{G}^{uu} = 8\pi\hat{G}\,T^{uu}$ (can be added by hand)

since off-shell we fixed $g_{uu} = 0$

No mass coupling to torsionless type I

- couple previous action (say 1st form) to NR matter

- energy momentum and mass current from variation

$$\delta\mathcal{L}_{\rm mat} = e\left(T^\mu_\tau \delta\tau_\mu + \frac{1}{2}T^{\mu\nu}_h \delta h_{\mu\nu} + T^\mu_m \delta m_\mu\right)$$

variation of gravity part (focussing on m)

$$\delta_m \mathcal{L} = -\frac{e}{8\pi G} G^\mu_m \delta m_\mu \qquad \Longrightarrow \qquad G^\mu_m = \frac{1}{2e}\partial_\nu\left(e h^{\nu\rho} h^{\mu\sigma} \tau_{\rho\sigma}\right)$$

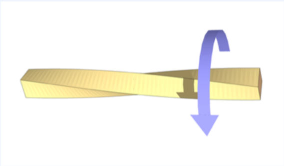
τ_μ projection of the m_μ equation of total action

$$\tau_\mu G^\mu_m = -\frac{1}{4} h^{\mu\rho} h^{\nu\sigma} \tau_{\mu\nu} \tau_{\rho\sigma} = 8\pi G \tau_\mu T^\mu_m = -8\pi G\, \rho\,.$$

Newtonian zero torsion requirement is incompatible with non-zero mass density !

(even worse: do not even get TTNC i.e. foliation of spacetime)

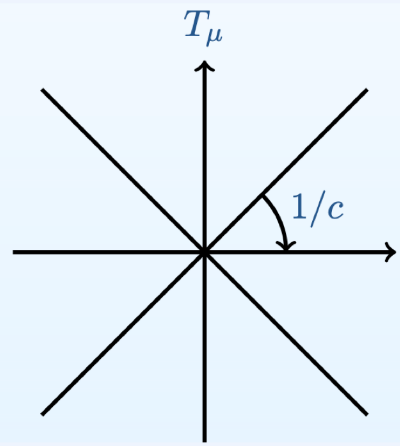
- non-relativistic gravity from 1/c expansion of GR



speed of light dependence in GR

Lorentzian metric: $g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$

T = time-like vielbein
Pi = spatial metric



large speed of light → light-cone opens up

expand in $\sigma = 1/c^2$

small speed of light → light-cone closes up
Carroll: see talks Bagchi/de Boer/Donnay/Vandoren

pre-non-relativistic GR

rewrite GR in terms of T and Pi:

new choice of connection:

$$C^{\rho}_{\mu\nu} = -T^{\rho}\partial_{\mu}T_{\nu} + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_{\mu}\Pi_{\nu\sigma} + \partial_{\nu}\Pi_{\mu\sigma} - \partial_{\sigma}\Pi_{\mu\nu}\right)$$

has torsion: proportional to:

$$T_{\mu\nu} \equiv \partial_{\mu}T_{\nu} - \partial_{\nu}T_{\mu}\,.$$

analogue of metric compatibility

$$\overset{(C)}{\nabla}_{\mu}T_{\nu} = 0\,, \qquad \overset{(C)}{\nabla}_{\mu}\Pi^{\nu\rho} = 0\,,$$

EH Lagrangian in pre-non-relativistic form

$$\tilde{\mathcal{L}} = E\left[\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \sigma\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \sigma^2 T^{\mu}T^{\nu}\overset{(C)}{R}_{\mu\nu}\right]$$

recent progress: understand this in 1st order formulation of GR

Hansen,Hartong,OlingNO(2020)

NR gravity from large c expansion of GR

metric in GR depends on speed of light c:

expand in 1/c

Dautcourt (1996)
van den Bleeken(2017)
Hansen,Hartong,NO(2018,2020)

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} - m_\mu\tau_\nu - m_\nu\tau_\mu + \frac{1}{c^2}\left(\cancel{B_\mu}\tau_\nu + \cancel{B_\nu}\tau_\mu - \Phi_{\mu\nu}\right) + O(c^{-4})$$

- LO and NLO fields define a novel version of NC geometry

LO fields:	$\tau_\mu \quad h_{\mu\nu}$
NLO fields:	$m_\mu \quad \Phi_{\mu\nu}$

\- expand Einstein-Hilbert action of GR:

$$S_{\mathrm{EH}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

Lagrangian expansion

- basic expansion structure (for any type of field)

$$\phi^I(x;\sigma) = \phi^I_{(0)}(x) + \sigma\phi^I_{(2)}(x) + \sigma^2\phi^I_{(4)}(x) + \mathcal{O}(\sigma^3)$$

$\sigma = 1/c^2$

Lagrangian: factor out overall c-power and expand:

$$\mathcal{L}(c^2,\phi,\partial_\mu\phi) = c^N\tilde{\mathcal{L}}(\sigma) = c^N \overset{(-N)}{\mathcal{L}}_{\text{LO}} + c^{N-2}\overset{(2-N)}{\mathcal{L}}_{\text{NLO}} + c^{N-4}\overset{(4-N)}{\mathcal{L}}_{\text{NNLO}} + \mathcal{O}(c^{N-6}),$$

- first terms:

$$\overset{(-N)}{\mathcal{L}}_{\text{LO}} = \tilde{\mathcal{L}}(0) = \overset{(-N)}{\mathcal{L}}_{\text{LO}}(\phi_{(0)},\partial_\mu\phi_{(0)})$$

$$\overset{(2-N)}{\mathcal{L}}_{\text{NLO}} = \left.\frac{\partial\tilde{\mathcal{L}}}{\partial\sigma}\right|_{\sigma=0} + \phi_{(2)}\frac{\delta \overset{(-N)}{\mathcal{L}}_{\text{LO}}}{\delta\phi_{(0)}}$$

$\sigma = 1/c^2$

EOM of the NLO field of NLO Lagrangian
= EOM of LO field of LO Lagrangian
(cascading structure repeats at every order)

→ at particular order: take action at that order and forget about the previous orders

NR expansion of EH Lagrangian

expansion of EH: $$\mathcal{L}_{\mathrm{EH}} = \frac{c^6}{16\pi G}\left[\mathcal{L}_{\mathrm{LO}} + \sigma\mathcal{L}_{\mathrm{NLO}} + \sigma^2\mathcal{L}_{\mathrm{N^2LO}} + O(\sigma^3)\right]$$

$$\mathcal{L}_{\mathrm{LO}} = \frac{e}{4}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}$$

$$\tau_{\mu\nu} = \partial_\mu\tau_\nu - \partial_\nu\tau_\mu$$

EOM enforce TTNC (causality)

$$\mathcal{L}_{\mathrm{NLO}} = eh^{\mu\nu}\check{R}_{\mu\nu} + \frac{\delta\mathcal{L}_{\mathrm{LO}}}{\delta\tau_\mu}m_\mu + \frac{\delta\mathcal{L}_{\mathrm{LO}}}{\delta h_{\mu\nu}}\Phi_{\mu\nu}$$

Galilean gravity

(see also Bergshoeff et al (2017) for 1st order formalism)

Non-relativistic Gravity Action

for EOM of NNLO involving only NLO fields we can use TTNC off-shell (using Lagrange mulitiplier)

$$\mathcal{L} = e\Big[-v^\mu v^\nu \check{R}_{\mu\nu} - 2m_\nu \check{\nabla}_\mu \left(h^{\mu\rho}h^{\nu\sigma} - h^{\mu\nu}h^{\rho\sigma}\right) K_{\rho\sigma} + \Phi h^{\mu\nu}\check{R}_{\mu\nu} + \frac{1}{4}h^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$
$$-\Phi_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\left(\check{R}_{\mu\nu} - \check{\nabla}_\mu a_\nu - a_\mu a_\nu - \frac{1}{2}h_{\mu\nu}h^{\kappa\lambda}\check{R}_{\kappa\lambda} + h_{\mu\nu}e^{-1}\partial_\kappa\left(eh^{\kappa\lambda}a_\lambda\right)\right)\Big], \quad (27)$$

- resulting action (unique 2-derivative action of fields $\tau_\mu,\ h_{\mu\nu},\ m_\mu,\ \Phi_{\mu\nu}$ respecting all invariances)

can be rewritten in manifest (Milne) boost invariant quantities:

Newton's Poisson equation

from large c expansion of relativistic point particle action

expanding also: $$X^\mu = x^\mu + \frac{1}{c^2} y^\mu + \mathcal{O}(c^{-4}) .$$

$$\overset{(-2)}{\mathcal{L}}_{\rm LO} = -m\tau_\mu \dot{x}^\mu$$

couples to NNLO in gravity

EOM of x → no torsion

$$\overset{(0)}{\mathcal{L}}_{\rm NLO} = m\left(\left(\partial_\nu \tau_\mu - \partial_\mu \tau_\nu\right) \dot{x}^\nu y^\mu + \frac{1}{2} \frac{\bar{h}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} \right)$$

same as standard NR particle action on fixed torsionless bgr.

couples to NNNLO in gravity

for NRG gravity action coupled to matter (and absolute time)

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{d-1}\left(-(d-2)\tau_\rho \mathcal{T}^\rho_\tau + h_{\rho\sigma}\mathcal{T}^{\rho\sigma}_h\right)\tau_\mu\tau_\nu$$

→ $$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1} \rho\, \tau_\mu \tau_\nu .$$

$$\rho = m \int d\lambda \delta(x - x(\lambda))/e$$

rho is not a Bargmann mass density but rather leading contribution to energy density

Expansion/transformation of vielbeins

what replaces Poincare ? → easiest to see in vielbein basis

$$T_\mu = \tau_\mu + c^{-2} m_\mu + c^{-4} B_\mu + \mathcal{O}(c^{-6}) \,,$$
$$\mathcal{E}^a_\mu = e^a_\mu + c^{-2} \pi^a_\mu + \mathcal{O}(c^{-4}) \,.$$

relativistic time-like vielbein

relativistic space-like vielbeins

- vielbeins correspond to the gauge connections associated to time/space translations in:

relativistic Poincare gauge connection $A_\mu = T_I \mathcal{A}^I_\mu .$

generators of Poincare

expand the gauge fields $\mathcal{A}^I_\mu = \sum_{n=0}^{\infty} \sigma^n \overset{(2n)}{\mathcal{A}^I_\mu}$

new generators from Lie algebra expansion $T_I^{(n)} \equiv T_I \otimes \sigma^n$

New underlying symmetry algebra

→ Lie algebra expansion from Poincare gives graded algebra

$$\left[H^{(m)}, B_a^{(n)}\right] = P_a^{(m+n)}, \quad \left[P_a^{(m)}, B_b^{(n)}\right] = \delta_{ab} H^{(m+n+1)}, \quad \left[B_a^{(m)}, B_b^{(n)}\right] = -J_{ab}^{(m+n+1)}$$

can be consistently truncated at any level N

generators

level 0 $\{H, P_a, G_a, J_{ab}\}$ (massless Galilean algebra)

level 1 $\{N, T_a, B_a, S_{ab}\}$

N is not central anymore

$$\begin{aligned}
&[H\,, G_a] = P_a\,, \quad [P_a\,, G_b] = N\delta_{ab}\,, \quad [N\,, G_a] = T_a\,, \\
&[H\,, B_a] = T_a\,, \quad [S_{ab}\,, P_c] = 2\delta_{c[a}T_{b]}\,, \quad [G_a\,, G_b] = -S_{ab}\,, \\
&[S_{ab}\,, G_c] = 2\delta_{c[a}B_{b]}\,, \quad [J_{ab}\,, J_{cd}] = 4\delta_{[a[d}J_{c]b]}\,, \\
&[J_{ab}\,, X_c] = 2\delta_{c[a}X_{b]}\,, \quad [J_{ab}\,, S_{cd}] = 4\delta_{[a[d}S_{c]b]}\,. \qquad (18)
\end{aligned}$$

also applied in other works, e.g : Bergshoeff,Izquierdo,Ortin,Romano(2019)/ Gomis,Kleinschmidt,Palmkvist,Salgado-Rebolledo(2019)/ Fontanella/Romano (2020) & …. many other works

Punchline (for large speed of light expansion)

new version of TNC (type II) is what the large c expansion of GR tells us to do !

What does it achieve ?

- while Cartan's original geometry can geometrize EOMS, it cannot be used to define the theory off-shell
  → needed for the action (analogue of EH action for NRG)

- the NRG action can then simply be obtained by doing the (right) expansion of GR

- what replaces Poincare invariance ?
  → new symmetry algebra that follows from Poincare from a well-defined procedure (Lie algebra expansion of Poincare)

(principle which can be used to geometrize any Post-Newtonian order)

Expansion of matter sources

- perform large c expansion of matter Lagrangian & compute the response wrt to the type II TNC fields

→ sourced EOMs for leading order fields (higher order similar)

$$\overset{(2m)}{G}{}_h^{\alpha\beta} = 8\pi G \overset{(2m)}{T}{}_h^{\alpha\beta}\,, \qquad \overset{(2m)}{G}{}_\tau^{\alpha} = 8\pi G \overset{(2m)}{T}{}_\tau^{\alpha}$$

$$\overset{(2n-N)}{T}{}_h^{\alpha\beta} \equiv 2e^{-1}\frac{\delta \overset{(2n-N)}{\mathcal{L}}_{\mathrm{mat},\mathrm{N}^n\mathrm{LO}}}{\delta h_{\alpha\beta}}\,, \qquad \overset{(2n-N)}{T}{}_\tau^{\alpha} \equiv e^{-1}\frac{\delta \overset{(2n-N)}{\mathcal{L}}_{\mathrm{mat},\mathrm{N}^n\mathrm{LO}}}{\delta \tau_{\alpha}}$$

$\overset{(-6)}{T}{}_h^{\alpha\beta} = \overset{(-6)}{T}{}_\tau^{\alpha} = 0$ to avoid spacetimes that violate twisteless torsion (causality)

- verified from large c expansion of (bosonic) matter sources (real/complex scalar field, Maxwell)

weak NR limit of Schwarzschild

Schw with factors of c reinstated

$$ds^2 = -c^2\left(1 - \frac{2Gm}{c^2 r}\right)dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 .$$

weak limit:

m independent of c

$$\begin{aligned}
\tau_\mu dx^\mu &= dt\,, \\
m_\mu dx^\mu &= -\frac{Gm}{r}dt = \Phi dt\,, \\
h_{\mu\nu}dx^\mu dx^\nu &= dr^2 + r^2 d\Omega_2^2\,, \\
\Phi_{\mu\nu}dx^\mu dx^\nu &= \frac{2Gm}{r}dr^2 = -2\Phi dr^2 .
\end{aligned}$$

point mass in flat space with Newtonian potential:

$$\Phi = -v^\mu m_\mu = -Gm/r.$$

absolute time: tau is exact

non-trivial corrections at higher order

strong NR limit of Schwarzschild

$M = m/c^2$

fixed

$$\tau_\mu dx^\mu = \sqrt{1-\frac{2GM}{r}}dt\,,$$
$$m_\mu dx^\mu = 0\,,$$
$$h_{\mu\nu}dx^\mu dx^\nu = \left(1-\frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$
$$\Phi_{\mu\nu}dx^\mu dx^\nu = 0\,.$$

(Van den Bleeken, 2017)

this strong expansion of Schw is not captured by Newtonian gravity:

- still described by NC geometry

different approx. of GR as compared to Post-Newtonian expansion (strong field)

$\tau \wedge d\tau = 0$

tau no longer exact but: hypersurface orthogonal

strong limit captures gravitational time dilaton:
clocks tick faster/slower depending on position on constant time slice

Strong gravity in NRG

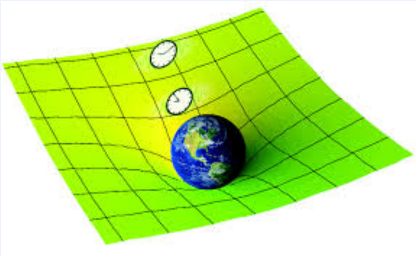
Strong gravity regime:
close to compact object with Schwarzshild radius R_s

warping of time → spacetime with torsion

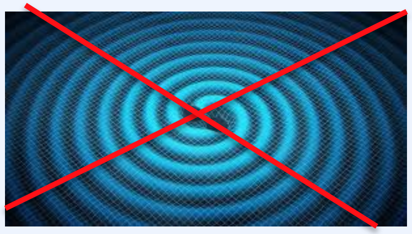
$$\tau_t = -\left(1 - \frac{R_s}{r}\right)$$

NR geodesics pass 3 classical tests of GR:

- precession perihelium
- bending of light
- gravitational redshift



but: no gravitational waves



NR (A)dS

- other example of effect of different scaling with c of parameters

AdS in global coordinates

$$ds^2 = -c^2 \cosh^2 \rho dt^2 + l^2 \left(d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2\right)$$

type II TNC bgr

$$\begin{aligned} \tau_\mu dx^\mu &= \cosh \rho dt \,, \\ h_{\mu\nu} dx^\mu dx^\nu &= l^2 \left(d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2\right) \\ m_\mu &= 0 \,, \\ \Phi_{\mu\nu} &= 0 \,. \end{aligned}$$

coord. trafo

$$r = l \sinh \rho$$

$$ds^2 = -c^2 \left(1 + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega_{d-1}^2$$

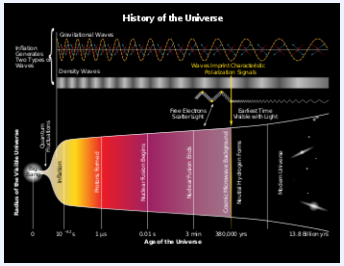
$$l = \frac{c}{H}$$ H = fixed

→ Newton-Hooke

$$\tau = dt \qquad h_{\mu\nu} dx^\mu dx^\nu = d\vec{x} \cdot d\vec{x} \,, \qquad m_\mu dx^\mu = \pm \frac{1}{2} H^2 \vec{x}^2 dt$$

(NR FLRW spacetime)

Further properties of NRG



- cosmology: FRW solutions

- covariant treatment of PN physics

- Newton-Schroedinger theory:

Coupling of non-relativistic field (electron/neutron) to NRG

- well-defined framework to treat PN corrections
- possible useful starting point to further analyze QM effects
(gravitationally induced quantum interference with neutron beams)

→ Towards QM in general fixed background geometry and systematic inclusion of 1/c effects

Have,Hartong,NO,Pikovski (to appear)

Odd powers in 1/c

Ergen, Hamamci , Van den Bleeken (2020)

- allow for sources with odd powers
- certain metrics (in particular coords) exhibit odd powers (e.g. Kerr)
- capture radiation effects

consider:

$$ds^2 = -e^{-\Psi}(cdt + C_i dx^i)^2 + e^{\Psi} k_{ij} dx^i dx^j$$
$$+ \mathcal{O}(c)dt^2 + \mathcal{O}_i(c^0)dtdx^i + \mathcal{O}_{ij}(c^{-1})dx^i dx^j$$

- for time-independent LO fields: solns to stationary EE
- same equations for time-dependent:
  - time dependence sits in integration constants
  - source the next subleading equations

→ expansion around stationary GR solution'
 - illustrated for Kerr

Further developments/extensions

- Connection to the Post-Newtonian expansion
- Carroll expansion of gravity
- Other formulations
- NR gravity models in two and three spacetime dimensions
- Non-relativistic string theory
- Non-relativistic holography
- Supersymmetry
- Generalizations
- Field theory applications & Hydrodynamics

The End

Thank you for your attention !