HIGHER-FORM SYMMETRIES AND TOPOLOGICAL PHASE TRANSITIONS

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© Mildred Thompson “Magnetic Fields”

[2301.09628] Armas, AJ

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MOTIVATION

- **Symmetries** are powerful guiding principle for developing effective theories for physical systems without a detailed understanding of their microscopic constituents.
- Equilibrium phases of matter can be organised according to their symmetries and whether these are **spontaneously** broken or unbroken in the ground state, commonly known as the **Landau paradigm**.
- Symmetries can even be useful when they are only **approximately** respected by the system.
  The canonical example comes from *chiral perturbation theory*, where pions are seen as pseudo-Goldstones of approximate SU(2) chiral symmetry. Other examples: *pinned crystals, pinned charge density waves, pseudo-superfluids* etc.

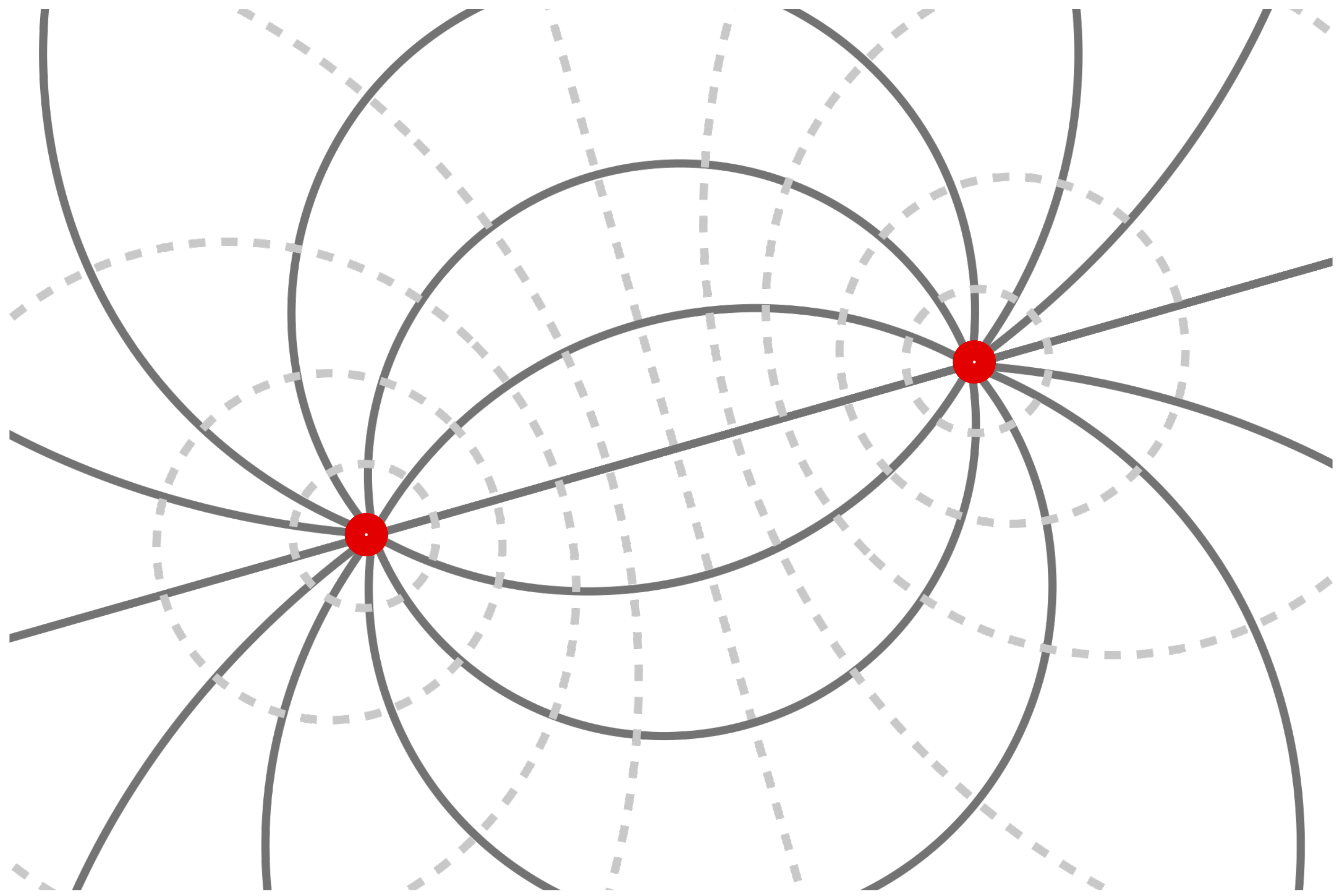
MOTIVATION

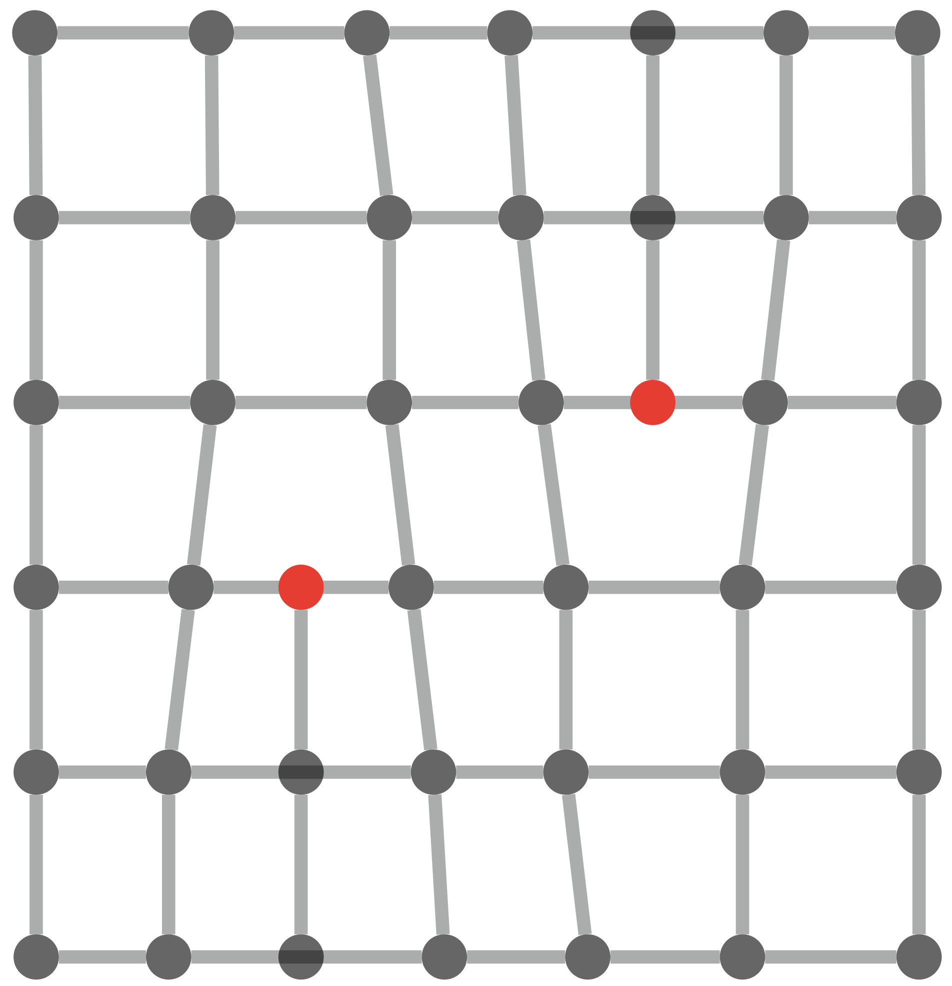
- In recent years, the notion of symmetries has been **generalised** to include higher-form symmetries, higher-group symmetries, subsystem symmetries, and non-invertible symmetries.
- These allow for a **generalised Landau paradigm**, that also include exotic phases of matter, such as *topologically ordered states, spin liquids, fractons, topological insulators,* etc.
- The focus of this talk is **continuous higher-form symmetries**, which concerns higher-dimensional charged objects, such as strings and surfaces.
- These describe **topological order** in many-body systems, such as *equipotential planes* in a superfluid, *lattice planes* in a crystal, *magnetic fields* in a plasma, or *electric fields* in a dielectric fluid.

Gaiotto, Kapustin, Seiberg, Willett [1412.5148]

MOTIVATION

- Explicit breaking of higher-form symmetries describes **topological defects**, such as *superfluid vortices*, *crystal dislocations*, *magnetic monopoles*, or *free charges*.
- Topological defects mediate **topological phase transitions**,[1] wherein a spontaneously broken symmetry gets restored. Examples include *superfluid phase transition*, *melting*, and *plasma phase transition*.





[1]Not to be confused with phase transitions between topologically ordered phases.

HIGHER-FORM SYMMETRIES

and their breaking

0-FORM SYMMETRIES

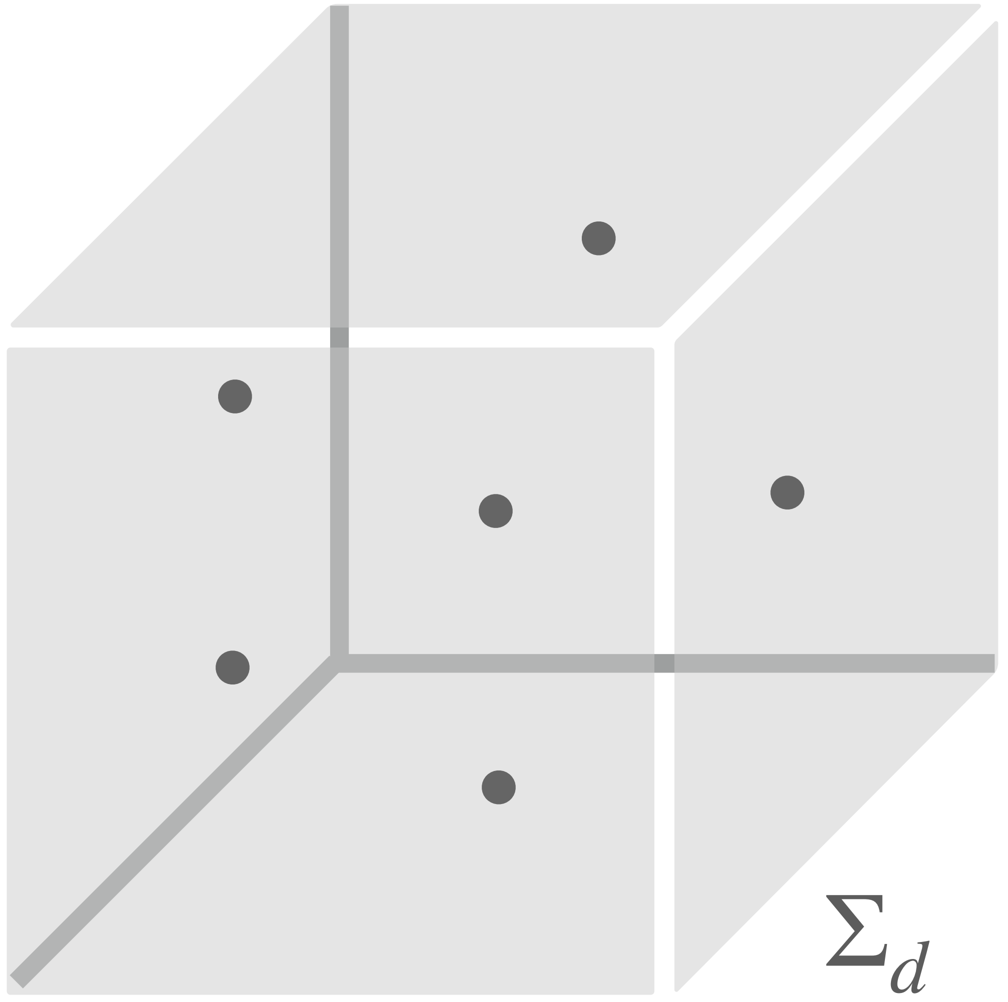
- Continuous 0-form symmetries can be defined by a conservation law

$$\partial_\mu J^\mu = 0 \qquad \Longrightarrow \qquad \partial_t J^t + \partial_i J^i = 0$$

- The total number of charged particles in a volume Σ_d is conserved in time

$$Q[\Sigma_d] = \int \mathrm{d}\Sigma_\mu\, J^\mu = \int \mathrm{d}^d x\, J^t$$

$$\partial_t Q[\Sigma_d] = \int \mathrm{d}^d x\, \partial_t J^t = 0 - \int \mathrm{d}^d x\, \partial_i J^i$$



APPROXIMATE 0-FORM SYMMETRIES

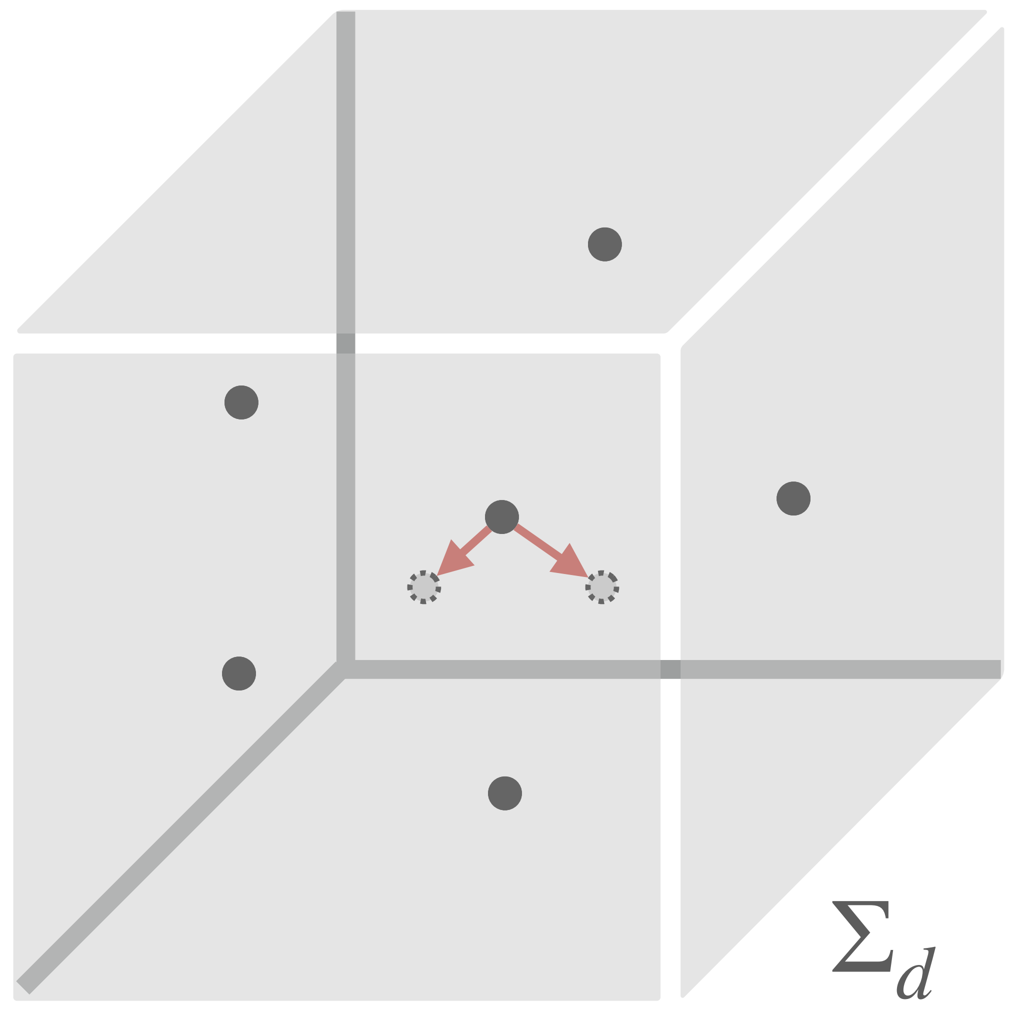
- Approximate 0-form symmetries have weakly violated conservation laws

$$\partial_\mu J^\mu = -\ell L \quad \Longrightarrow \quad \partial_t J^t + \partial_i J^i = -\ell L$$

- The total number of charged particles in a volume Σ_d is only approximately conserved in time

$$Q[\Sigma_d] = \int \mathrm{d}\Sigma_\mu \, J^\mu = \int \mathrm{d}^d x \, J^t$$

$$\partial_t Q[\Sigma_d] = \int \mathrm{d}^d x \, \partial_t J^t = -\ell \int \mathrm{d}^d x \, L - \int \mathrm{d}^d x \, \partial_i J^i$$



1-FORM SYMMETRIES

- A continuous 1-form symmetry can be defined by the conservation laws

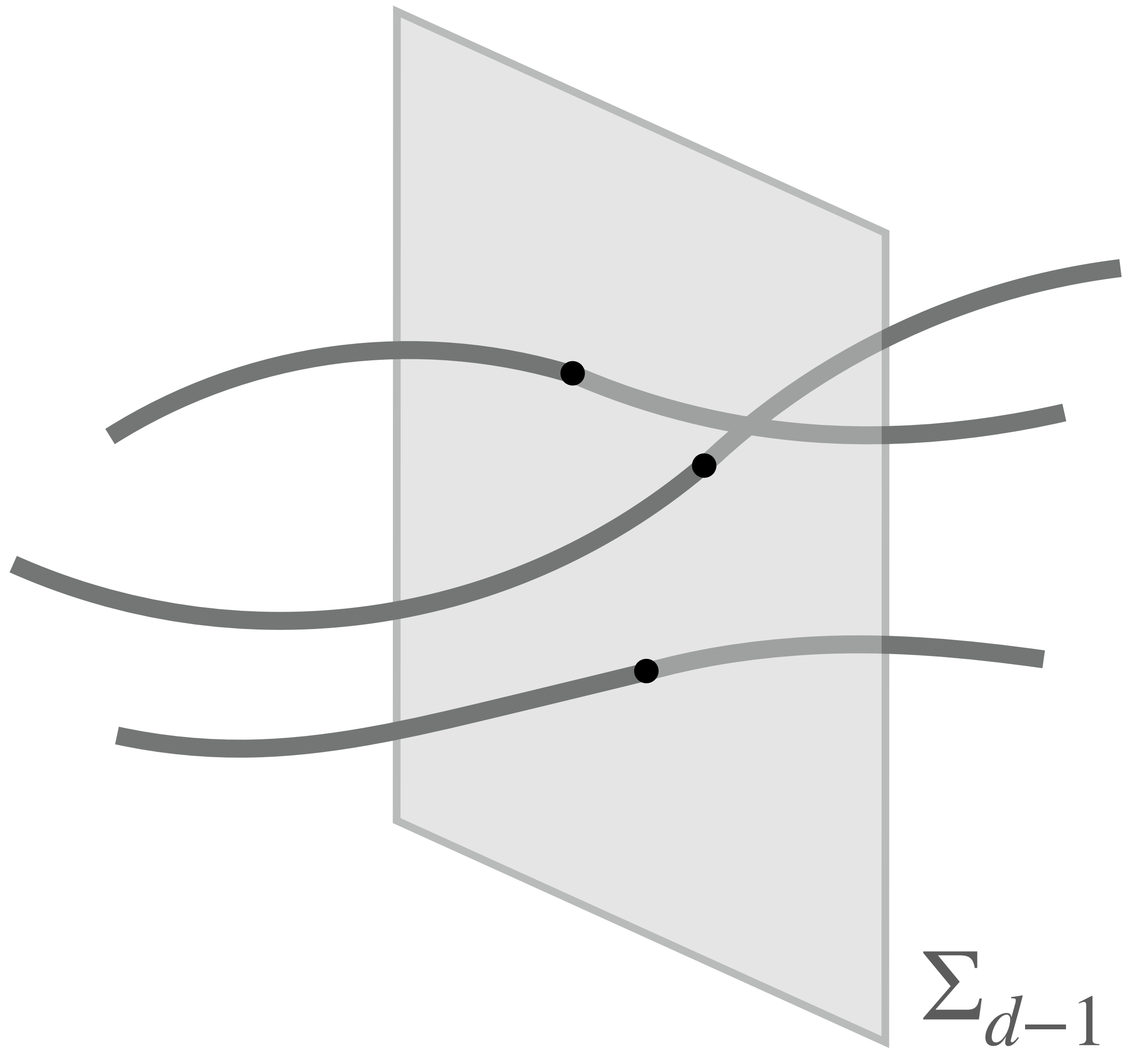
$$\partial_\mu J^{\mu\nu} = 0 \quad \Longrightarrow \quad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= 0 \\ \partial_i J^{ti} &= 0 \end{aligned}$$

- The objects charged under 1-form symmetries are "strings". The total number of strings passing a cross section Σ_{d-1} are conserved under space and time translations

$$Q[\Sigma_{d-1}] = \int \mathrm{d}\Sigma_{\mu\nu}\, J^{\mu\nu} = \int \mathrm{d}^{d-1}x\, J^{tz}$$

$$\partial_t Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_t J^{tz} = 0 \; - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{i_\parallel z}$$

$$\partial_z Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_z J^{tz} = 0 \; - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{t i_\parallel}$$

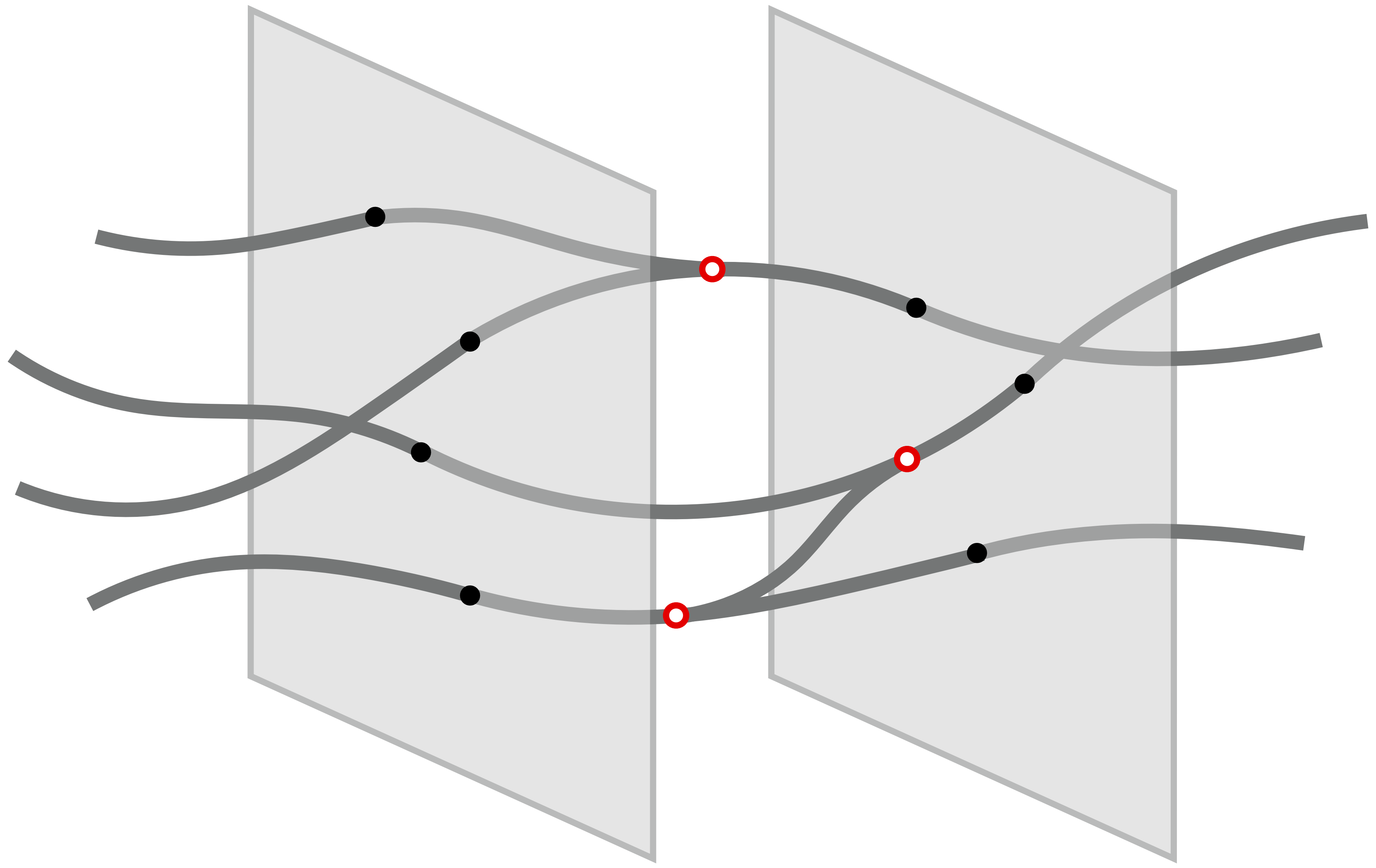


APPROXIMATE 1-FORM SYMMETRIES

- The conservation laws for a 1-form symmetry are

$$\partial_\mu J^{\mu\nu} = -\ell L^\nu \quad \Longrightarrow \quad \begin{aligned} \partial_t J^{ti} + \partial_k J^{ki} &= -\ell L^i \\ \partial_i J^{ti} &= \ell L^t \end{aligned}$$

- The total number of strings passing a cross section Σ_{d-1} are only approximately conserved



$$Q[\Sigma_{d-1}] = \int \mathrm{d}\Sigma_{\mu\nu}\, J^{\mu\nu} = \int \mathrm{d}^{d-1}x\, J^{tz}$$

$$\partial_t Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_t J^{tz} = -\ell \int \mathrm{d}^d x\, L^z - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{i_\parallel z}$$

$$\partial_z Q[\Sigma_{d-1}] = \int \mathrm{d}^{d-1}x\, \partial_z J^{tz} = \ell \int \mathrm{d}^d x\, L^t - \int \mathrm{d}^{d-1}x\, \partial_{i_\parallel} J^{t i_\parallel}$$

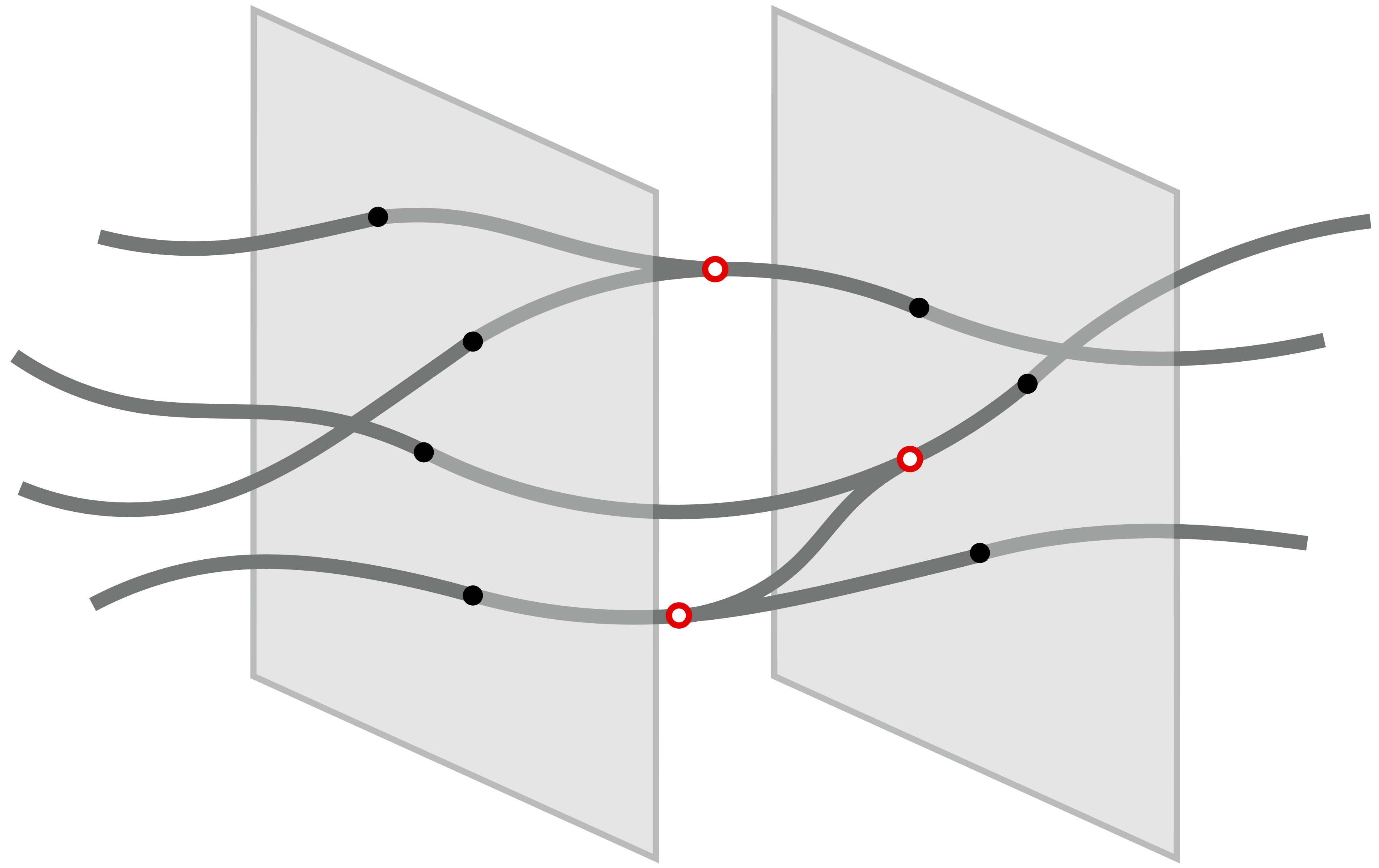
APPROXIMATE 1-FORM SYMMETRIES

- The defects of a 1-form symmetry themselves furnish a 0-form symmetry

$$\partial_\mu L^\mu = 0$$

- The total number of strings passing a cross section Σ_{d-1} are only approximately conserved

$$Q_\ell[\Sigma_d] = \int \mathrm{d}\Sigma_\mu L^\mu = \int \mathrm{d}^{d-1}x\, L^t$$



BACKGROUND FIELDS

- We can introduce a 1-form gauge field A_μ and a background phase Φ to probe an approximate 0-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(J^\mu \, \delta A_\mu + \ell L \, \delta\Phi \right)$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \qquad \Phi \to \Phi - \Lambda$$

- Similarly, we can introduce a 2-form gauge field $A_{\mu\nu}$ and a background phase Φ_μ to probe an approximate 1-form symmetry

$$\delta S[A,\Phi] = \int \mathrm{d}^d x \left(\frac{1}{2} J^{\mu\nu} \, \delta A_{\mu\nu} + \ell L^\mu \, \delta\Phi_\mu \right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \qquad \Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu \Lambda_\ell$$

EXAMPLE: ELECTROMAGNETISM

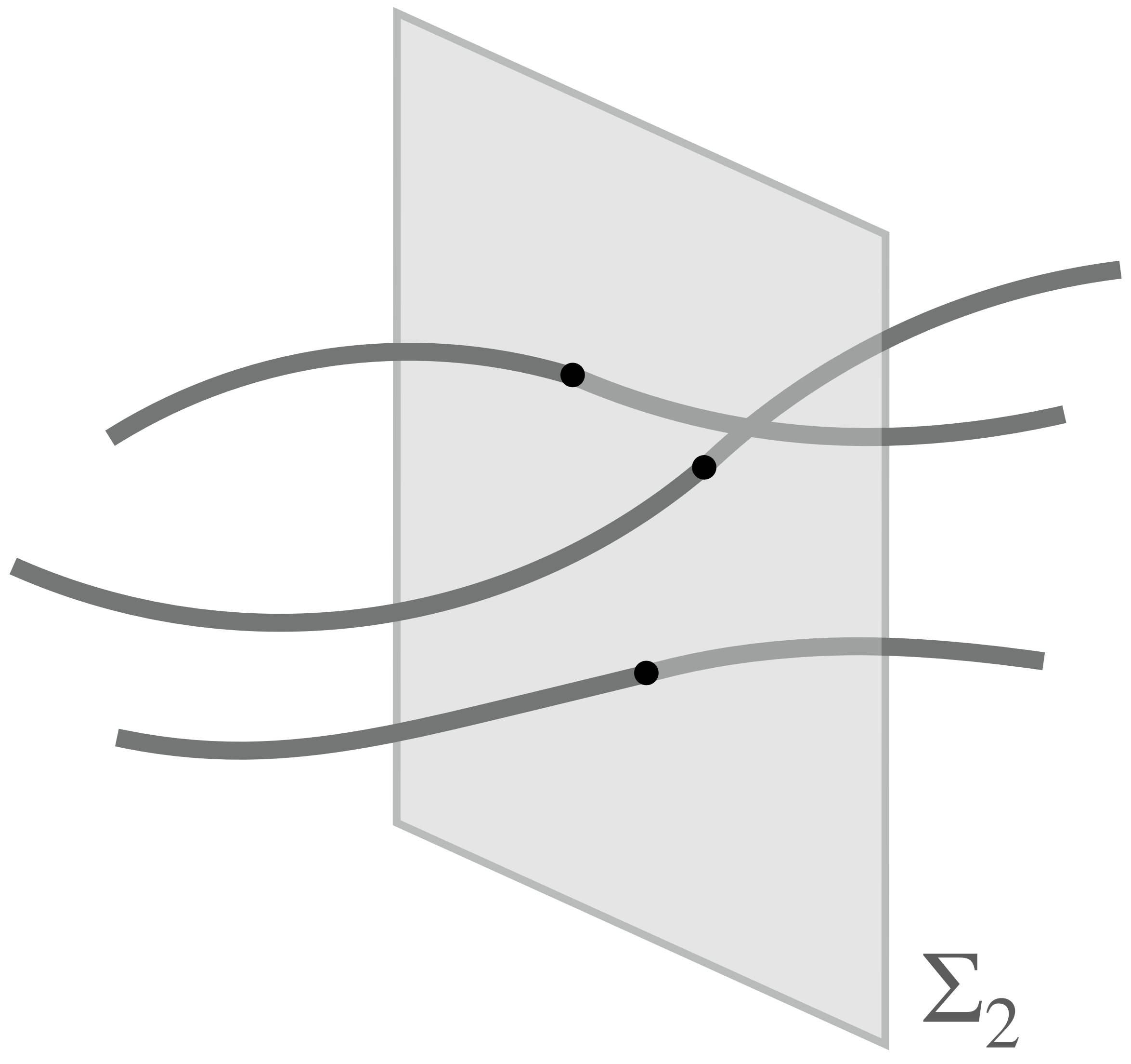
- $d = 3$ electromagnetism in vacuum

$$S = -\int \mathrm{d}^4 x \left(\frac{1}{4} \mathscr{F}^{\mu\nu} \mathscr{F}_{\mu\nu} \right) \qquad \Longrightarrow \qquad \begin{aligned} \partial_\mu \mathscr{F}^{\mu\nu} &= 0 \\ \partial_\mu \star \mathscr{F}^{\mu\nu} &= 0 \end{aligned}$$

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu$$

- It has two 1-form symmetries:

$$\begin{aligned} J^{\mu\nu} &= -\mathscr{F}^{\mu\nu} \\ \tilde{J}^{\mu\nu} &= \star \mathscr{F}^{\mu\nu} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \partial_\mu J^{\mu\nu} &= 0 \\ \partial_\mu \tilde{J}^{\mu\nu} &= 0 \end{aligned}$$



The associated charged objects are electric and magnetic field lines.

- These symmetries also persist in the presence of polarised/dielectric matter

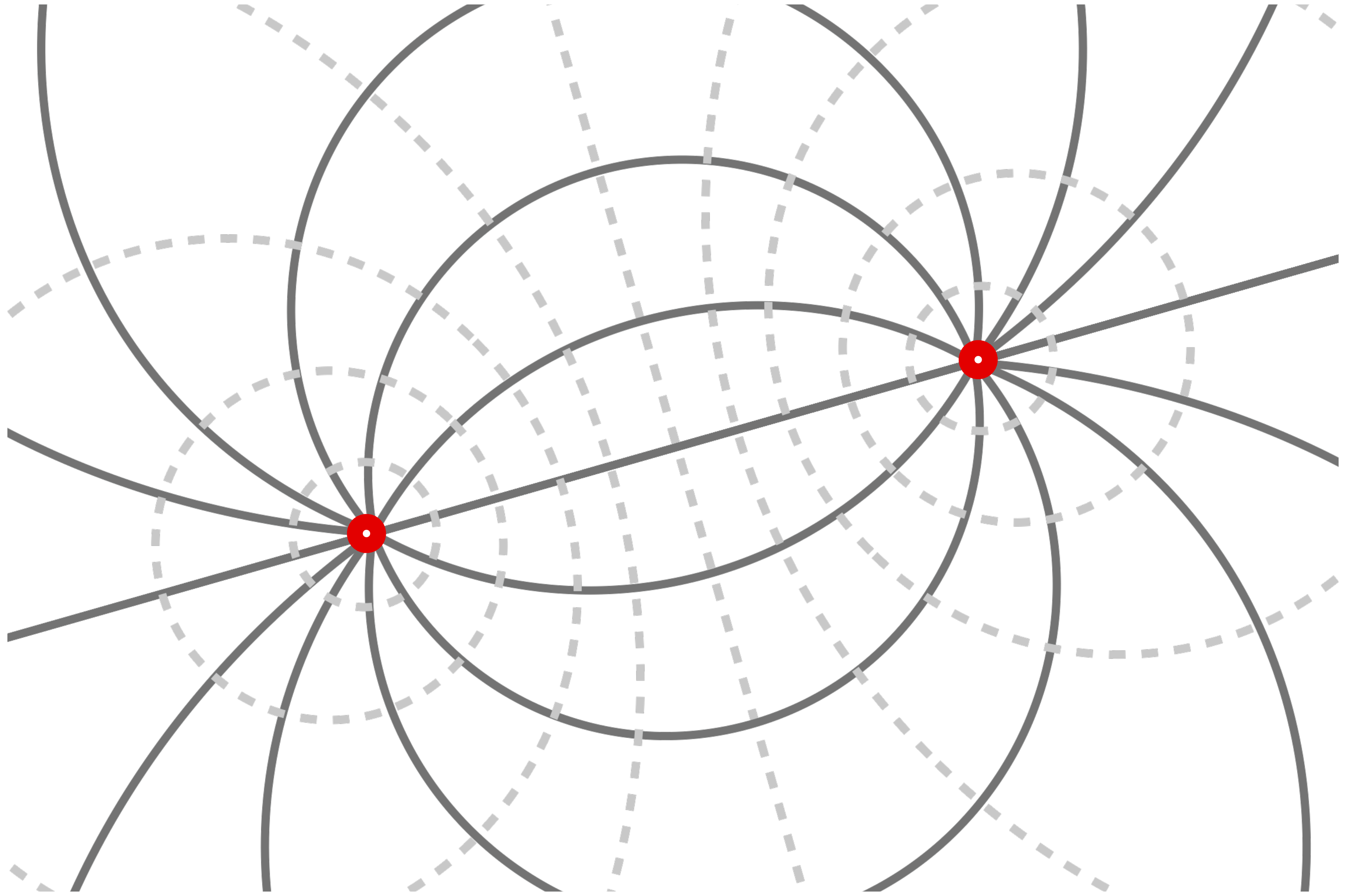
$$J^{\mu\nu} = -\mathscr{F}^{\mu\nu} + \mathscr{M}^{\mu\nu},$$

EXAMPLE: ELECTROMAGNETISM

- In the presence of free electric charges, the electric 1-form symmetry is violated, but the magnetic 1-form symmetry persists.

$$S = -\int d^4x \left(\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + (\partial_\mu + i\ell q\mathcal{A}_\mu)\Psi^* (\partial^\mu - i\ell q\mathcal{A}^\mu)\Psi + V(\Psi^*\Psi) \right)$$

$$\begin{aligned} J^{\mu\nu} &= -\mathcal{F}^{\mu\nu} \\ \tilde{J}^{\mu\nu} &= \star\mathcal{F}^{\mu\nu} \\ L^\mu &= iq\left(\Psi^*\partial_\mu\Psi - \partial_\mu\Psi^*\Psi\right) \\ &\quad +2\ell q^2\mathcal{A}_\mu\Psi^*\Psi \end{aligned} \implies \begin{aligned} \partial_\mu J^{\mu\nu} &= -\ell L^\nu \\ \partial_\mu \tilde{J}^{\mu\nu} &= 0 \end{aligned}$$



- Similarly, breaking of the magnetic 1-form symmetry amounts to the introduction of magnetic monopoles.

EXAMPLE: ELECTROMAGNETISM

- We can manifest the electric 1-form symmetry via

$$S[A,\Phi] = -\int \mathrm{d}^4x \left(\frac{1}{4}\xi^{\mu\nu}\xi_{\mu\nu} + (\partial_\mu + i\ell\psi_\mu)\Psi^* (\partial^\mu - i\ell\psi^\mu)\Psi + V(\Psi^*\Psi) \right)$$

$$\xi_{\mu\nu} = \mathscr{F}_{\mu\nu} + A_{\mu\nu}, \qquad \psi_\mu = \ell\left(\mathscr{A}_\mu - \Phi_\mu\right)$$

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \qquad \Phi_\mu \to \Phi_\mu - \Lambda_\mu + \partial_\mu\Lambda_\ell, \qquad \mathscr{A}_\mu \to \mathscr{A}_\mu - \Lambda_\mu, \qquad \Psi \to \mathrm{e}^{-i\ell\Lambda_\ell}\Psi$$

- We can manifest the magnetic 1-form symmetry via[1]

$$S[\tilde{A}] = -\int \mathrm{d}^4x \left(\frac{1}{4}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + (\partial_\mu + i\ell q\mathscr{A}_\mu)\Psi^* (\partial^\mu - i\ell q\mathscr{A}^\mu)\Psi + V(\Psi^*\Psi) - \frac{1}{2}\star\mathscr{F}^{\mu\nu}\tilde{A}_{\mu\nu} \right)$$

$$\tilde{A}_{\mu\nu} \to \tilde{A}_{\mu\nu} + \partial_\mu\tilde{\Lambda}_\nu - \partial_\nu\tilde{\Lambda}_\mu, \qquad \tilde{\mathscr{A}}_\mu \to \tilde{\mathscr{A}}_\mu - \tilde{\Lambda}_\nu$$

$$\star\mathscr{F}_{\mu\nu} = \partial_\mu\tilde{\mathscr{A}}_\nu - \partial_\nu\tilde{\mathscr{A}}_\mu$$

[1] Both symmetries cannot be gauged together on account of a mixed 't Hooft anomaly.

HIGHER-FORM SYMMETRIES

- In general spatial dimensions d, electromagnetism has an electric 1-form and magnetic $(d-2)$-form symmetry. The defects are free electric charges and magnetic monopoles.
- Electromagnetism can be viewed as a 1-form or $(d-2)$-form superfluid.[1]
- Ordinary 0-form superfluids have a $(d-1)$-form symmetry, with the defects being vortices.[2]

$$J^{\mu\nu\ldots} = \epsilon^{\lambda\mu\nu}\partial_\lambda\phi$$

- Crystals also have a $(d-1)$-form symmetry, with the defects being dislocations.[3]

$$J^{I\mu\nu\ldots} = \epsilon^{\lambda\mu\nu}\partial_\lambda\phi^I$$

[1] Hofman, Iqbal [1802.09512]; Armas, AJ [1808.01939, 1811.04913]
[2] Delacrétaz, Hofman, Mathys [1908.06977]
[3] Grozdanov, Poovuttikul [1801.03199]; Armas, AJ [1908.01175]; Armas, Heumen, AJ, Lier [2211.02117]

HIGHER-FORM FLUIDS

with approximate higher-form symmetry

THERMAL EQUILIBRIUM

- Many-body systems at thermal equilibrium can be characterised by their **thermal partition function**.
- Thermal partition function is a functional of background fields and can be used to obtain equilibrium values of (approximately) conserved densities and fluxes.
- For systems with spontaneously unbroken symmetries, the thermal partition function is a "local" functional of background fields.
- For systems with spontaneously broken symmetries, the thermal partition function is "non-local", and is given by a functional integral over the time-independent configurations of the Goldstone fields.

THERMAL EQUILIBRIUM: 0-FORM HYDROSTATICS

- For a 0-form symmetry, a thermal ensemble with constant charge density is described by a thermal partition function

$$\mathcal{Z}[A] = \exp \int \mathrm{d}^d x \left(\frac{1}{2} \chi \mu^2 + \ldots \right) \qquad \mu = \mu_0 + A_t$$

$$\Longrightarrow \qquad J^t = n = \chi\mu + \ldots, \qquad J^i = 0$$

- This works because A_t is invariant under time-independent gauge transformations

$$A_t \to A_t + \partial_t \Lambda$$

This is no longer true for higher-form symmetries

$$A_{ti} \to A_{ti} + \partial_t \Lambda_i - \partial_i \Lambda_t$$

So, it is not possible to construct "local" partition functions with nonzero higher-form density.

THERMAL EQUILIBRIUM: 1-FORM HYDROSTATICS

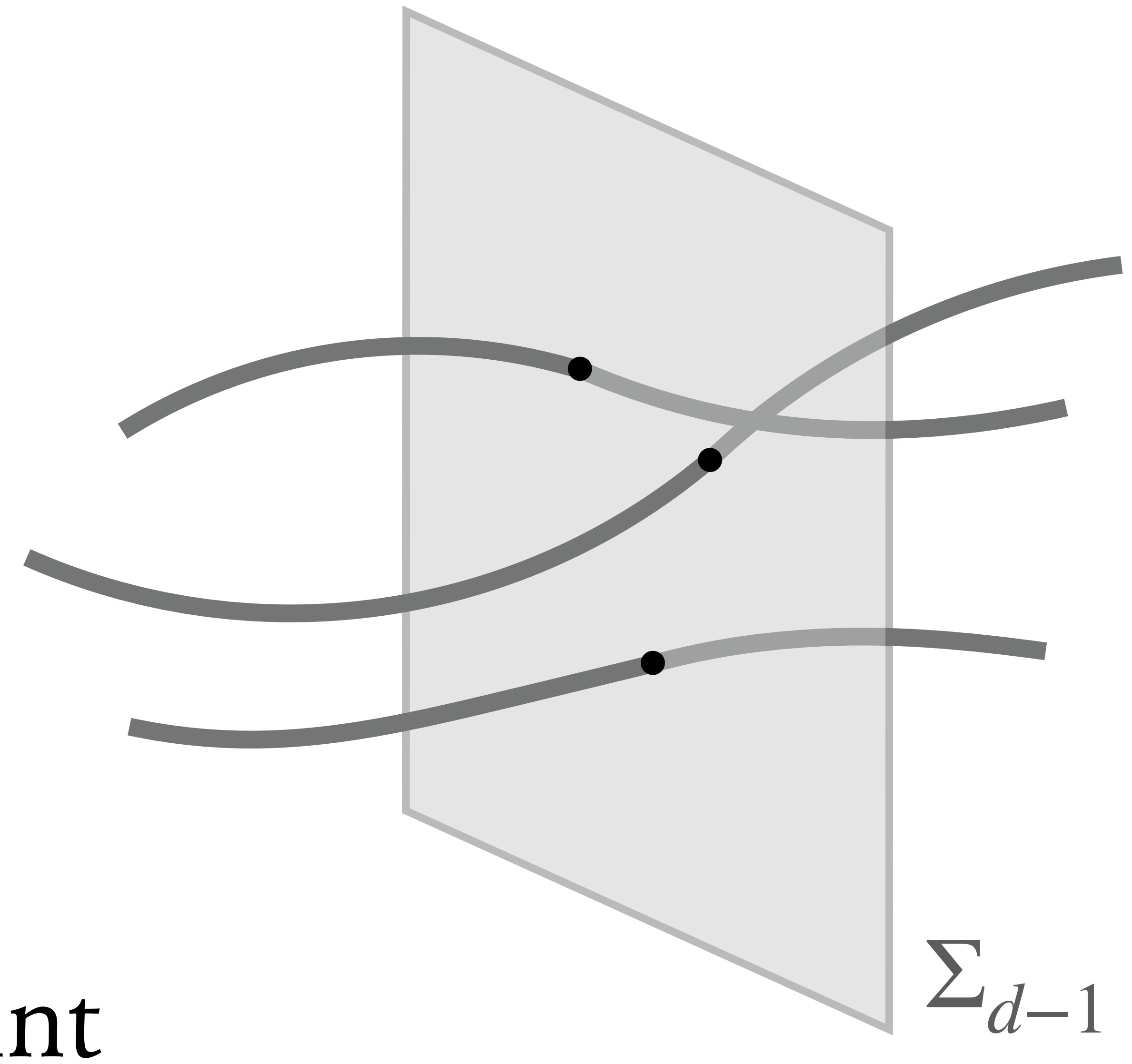
- We need to partially-spontaneously break the higher-form symmetry in the time-direction

$$\varphi \to \varphi - \Lambda_t \qquad\qquad \mu_i = -\partial_i \varphi + A_{ti}$$

This allows us to construct a "non-local" partition function

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^3 x \left(\frac{1}{2} \chi \mu_i \mu^i + \dots \right)$$

$$\implies \qquad J^{ti} = n^i = \chi \mu^i, \qquad J^{ij} = 0$$



- Classical configuration equation of φ implements the Gauss constraint

$$\partial_i J^{ti} = 0 \implies \partial_i \partial^i \varphi = 0 \qquad\qquad \varphi = -\mu_0 z \implies n_i = \chi \mu_0 \, \delta_i^z$$

Armas, AJ [1808.01939, 1811.04913]

EXPLICITLY-BROKEN SYMMETRIES

- The partition function can also depend on Φ_μ through the "defect chemical potential"

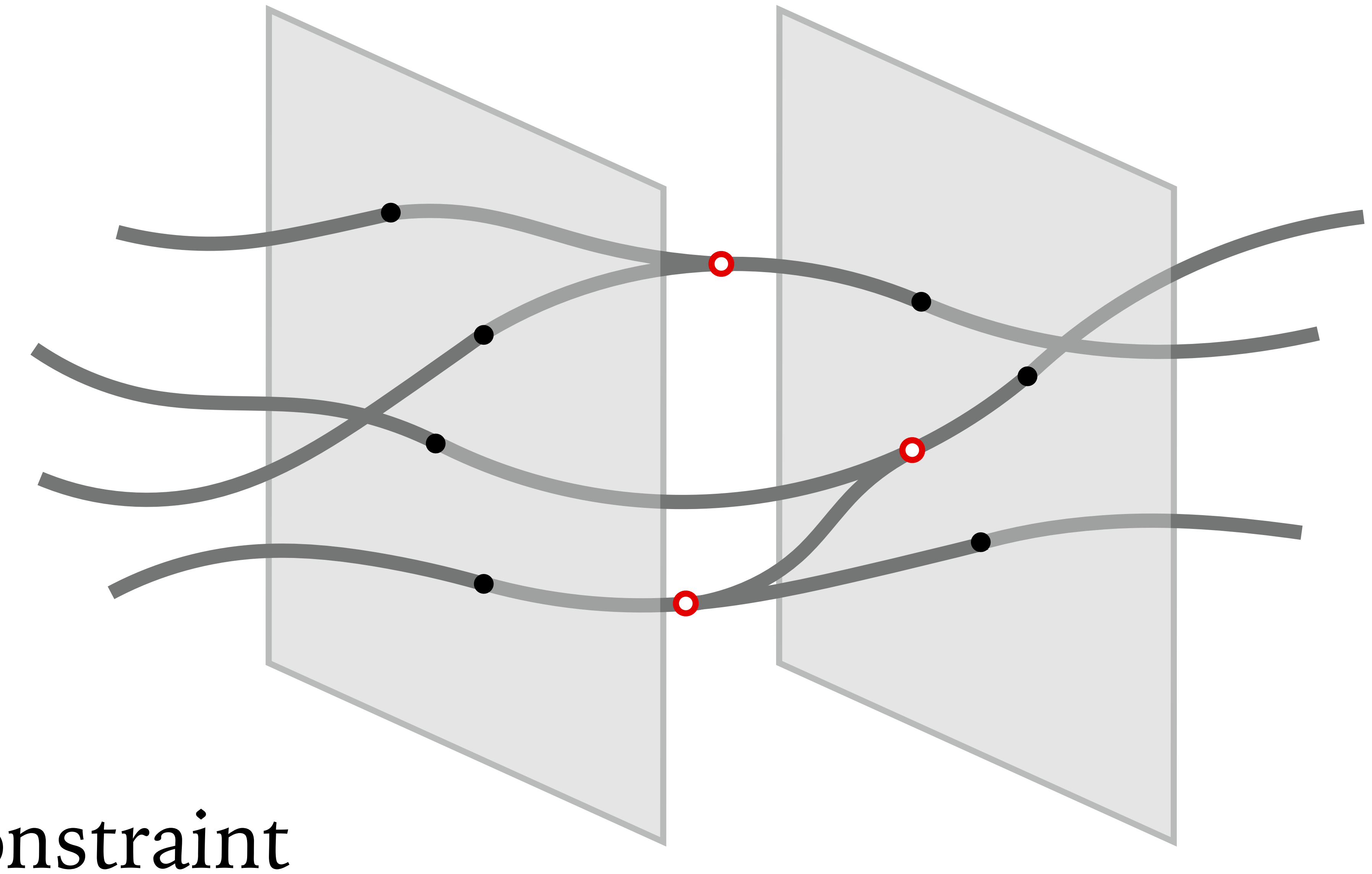
$$\mu_\ell = -\ell\left(\varphi - \Phi_t\right) \qquad\qquad \Phi_t \to \Phi_t + \partial_t\Lambda_\ell$$

- The partition function takes the form

$$\mathscr{Z}[A] = \int \mathscr{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i + \frac{1}{2}\chi_\ell\mu_\ell^2 + \ldots\right)$$

$$\Longrightarrow \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = 0$$

$$L^t = n_\ell = \chi_\ell\mu_\ell \qquad L^i = 0$$

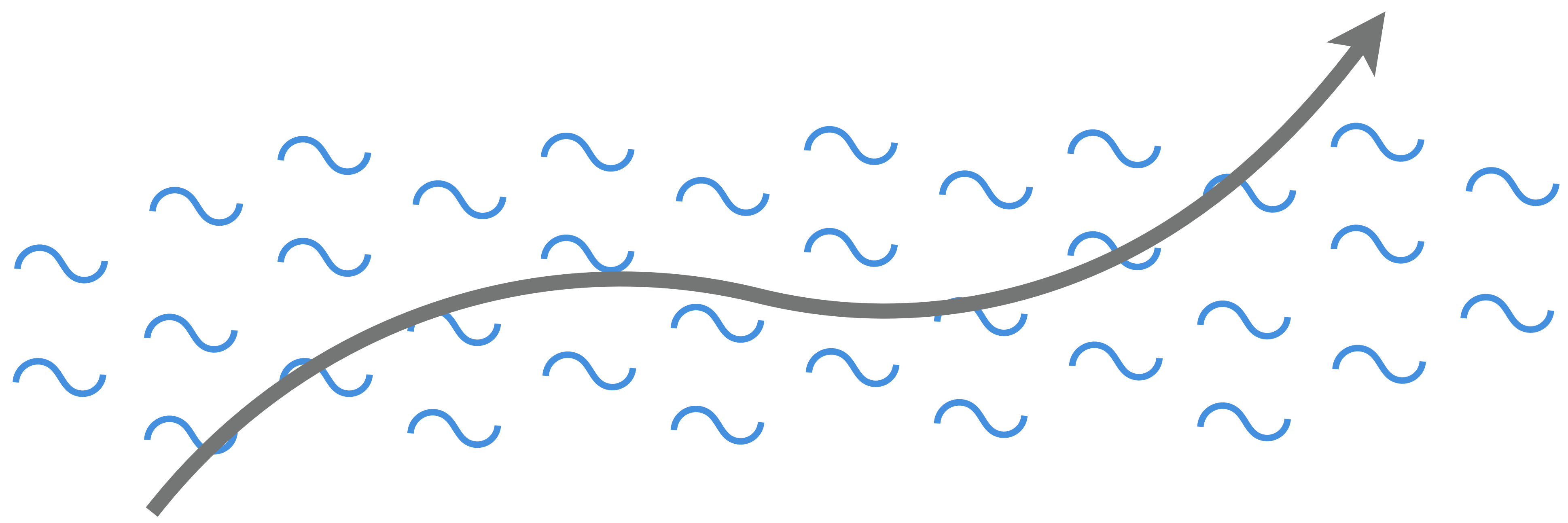


- Classical configuration equation for φ imposes the Gauss constraint

$$\partial_i J^{ti} = \ell L^t \Longrightarrow \partial_i\partial^i\varphi = k_0^2\varphi \qquad\qquad \frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2\chi_\ell}} \quad \text{“Debye length”}$$

HYDRODYNAMICS

- **Hydrodynamics** is a framework to capture perturbative departures of a many-body system from thermal equilibrium.
- The relevant hydrodynamic degrees of freedom are a set of symmetry parameters corresponding to each global symmetry (conserved charge) of the system.
- Additionally, we need to add massless Goldstone fields for each spontaneously broken global symmetry.



0-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the conservation laws

$$\nabla_\mu J^\mu = -\ell L, \qquad \nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho + \Xi^\mu L$$

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$$

$$\Xi_\mu = \partial_\mu \Phi + A_\mu$$

we have allowed for violation of the 0-form symmetry.

- The hydrodynamic fields are a set of symmetry parameters Λ_β, β^μ, transforming as

$$\delta\Lambda_\beta = \pounds_\chi \Lambda_\beta - \pounds_\beta \Lambda, \qquad \delta\beta^\mu = \pounds_\chi \beta^\mu$$

- These can be used to define gauge-invariant hydrodynamic fields μ, T, u^μ as

$$\frac{\mu}{T} = \Lambda_\beta + \beta^\mu A_\mu, \qquad \frac{u^\mu}{T} = \beta^\mu$$

0-FORM HYDRODYNAMICS

- Hydrodynamics is characterised by its constitutive relations

$$J^\mu[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}], \qquad L[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}], \qquad T^{\mu\nu}[\mu, T, u^\mu; A_\mu, \Phi, g_{\mu\nu}]$$

The constitutive relations are required to satisfy the second law of thermodynamics.

- For example, at first order in derivatives, we find the constitutive relations

$$J^\mu = n\, u^\mu - \sigma\, P^{\mu\nu}\left(T\partial_\nu \frac{\mu}{T} + u^\lambda F_{\lambda\nu}\right) \qquad P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$L = -\,\ell\,\sigma_\ell \left(u^\mu \Xi_\mu - \mu\right)$$

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\, g^{\mu\nu} - \eta\, P^{\mu\rho}P^{\nu\sigma}\left(2\nabla_{(\rho}u_{\sigma)} - \frac{2}{d}P_{\rho\sigma}\nabla_\lambda u^\lambda\right) - \zeta\, P^{\mu\nu}\nabla_\lambda u^\lambda$$

- The coefficients follow the constraints

$$\delta p = s\delta T + n\delta\mu, \qquad \epsilon = Ts + \mu n - p, \qquad \sigma, \sigma_\ell, \eta, \zeta \geq 0$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the fluid sound and shear modes

$$u_\parallel, T: \quad \omega = \pm v_s k - \frac{i}{2} D_\pi^\parallel k^2 + \ldots \qquad v_s^2 = \frac{\partial p}{\partial \epsilon}$$

$$u_\perp: \quad \omega = -iD_\pi^\perp k^2 + \ldots \qquad D_\pi^\parallel = \frac{\zeta + 2\frac{d-1}{d}\eta}{\epsilon + p}, \qquad D_\pi^\perp = \frac{\eta}{\epsilon + p}$$

- The charge fluctuations give rise to a diffusive mode

$$\mu: \quad \omega = -iD_n k^2 - i\Gamma \qquad D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}$$

Charge fluctuations are damped due to explicit symmetry breaking.

1-FORM HYDRODYNAMICS

- The hydrodynamic description is based on the conservation laws

$$F_{\mu\nu\rho} = 3\partial_{[\mu}A_{\nu\rho]}$$

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu, \qquad \nabla_\mu L^\mu = 0, \qquad \nabla_\mu T^{\mu\nu} = \frac{1}{2}F^{\nu\rho\sigma}J_{\rho\sigma} + \ell \Xi^{\nu\rho}L_\rho$$

$$\Xi_{\mu\nu} = 2\partial_{[\mu}\Phi_{\nu]} + A_{\mu\nu}$$

- The hydrodynamic fields are a set of symmetry parameters Λ^β_μ, Λ^β_ℓ, β^μ, transforming as

$$\delta\Lambda^\beta_\mu = \pounds_\chi \Lambda^\beta_\mu - \pounds_\beta \Lambda_\mu, \qquad \delta\Lambda^\beta_\ell = \pounds_\chi \Lambda^\beta_\ell - \pounds_\beta \Lambda_\ell, \qquad \delta\beta^\mu = \pounds_\chi \beta^\mu$$

We also need the temporal Goldstone field φ transforming as

$$\delta\varphi = \pounds_\chi \varphi - \beta^\mu \Lambda_\mu$$

- These can be used to define the covariant hydrodynamic fields

$$\frac{\mu_\mu}{T} = \Lambda^\beta_\mu + \beta^\lambda A_{\lambda\mu} - \partial_\mu \varphi, \qquad \frac{\mu_\ell}{T} = -\ell\left(\varphi - \beta^\mu \Phi_\mu - \Lambda^\beta_\ell\right), \qquad \frac{u^\mu}{T} = \beta^\mu$$

Grozdanov, Hofman, Iqbal [1610.07392]; Armas, AJ [1808.01939, 1811.04913, 2301.09628]

JOSEPHSON EQUATION FOR TEMPORAL GOLDSTONE

- The dynamics of φ is governed by a Josephson equation of the form

$$\pounds_\beta \varphi = \beta^\mu \Lambda^\beta_\mu + \ldots$$

- We can absorb possible corrections to this equation by redefining Λ^β_μ. This implies

$$u^\mu \mu_\mu = u^\mu \Lambda^\beta_\mu - u^\mu \partial_\mu \varphi = 0$$

This still leaves redefinition freedom in the spatial components of Λ^β_μ.

GAUGE REDUNDANCY

- There is a gauge redundancy in the description that can be obtained by setting

$$\Lambda_\mu = \partial_\mu \lambda, \qquad \Lambda_\ell = \lambda$$

which leaves the background fields $A_{\mu\nu}, \Phi_\mu$ invariant.

- The dynamical fields transform as

$$\delta_\lambda \Lambda^\beta_\mu = -\partial_\mu £_\beta \lambda, \qquad \delta_\lambda \Lambda^\beta_\ell = -£_\beta \lambda, \qquad \delta_\lambda \beta^\mu = 0$$

$$\delta_\lambda \varphi = -£_\beta \lambda$$

The physical hydrodynamic fields $\mu_\mu, \mu_\ell, T, u^\mu$ are invariant under these gauge transformations.

1-FORM HYDRODYNAMICS

- The constitutive relations are a straight-forward generalisation of the 0-form case

$$J^{\mu\nu} = 2u^{[\mu}n^{\nu]} - \sigma P^{\mu\rho}P^{\nu\sigma}\left(2T\partial_{[\rho}\frac{\mu_{\sigma]}}{T} + u^\lambda F_{\lambda\rho\sigma}\right)$$

$$L^\mu = n_\ell u^\mu - \sigma_\ell P^{\mu\nu}\left(T\partial_\nu\frac{\mu_\ell}{T} + \ell u^\lambda \Xi_{\lambda\nu} - \ell\mu_\nu\right)$$

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\, g^{\mu\nu} - n^\mu\mu^\nu - \eta P^{\mu\rho}P^{\nu\sigma}\left(2\nabla_{(\rho}u_{\sigma)} - \frac{2}{d}P_{\rho\sigma}\nabla_\lambda u^\lambda\right) - \zeta P^{\mu\nu}\nabla_\lambda u^\lambda$$

Note that σ_ℓ now behaves like a conductivity for the defect flux.

- The constraints are given as

$$\delta p = s\delta T + n^\mu \delta\mu_\mu + n_\ell\delta\mu_\ell, \qquad \epsilon = Ts + \mu_\mu n^\mu + \mu_\ell n_\ell - p, \qquad \sigma, \sigma_\ell, \eta, \zeta \geq 0$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same fluid sound and shear modes.

- The charge fluctuations give rise to two diffusive modes

$$\mu_\perp : \quad \omega = -iD_n k^2 - i\Gamma$$

$$\mu_\parallel : \quad \omega = -iD_\ell k^2 - i\Gamma$$

$$D_n = \frac{\sigma}{\chi}, \qquad \Gamma = \frac{\ell^2 \sigma_\ell}{\chi}$$

$$D_\ell = \frac{\sigma_\ell}{\chi_\ell}$$

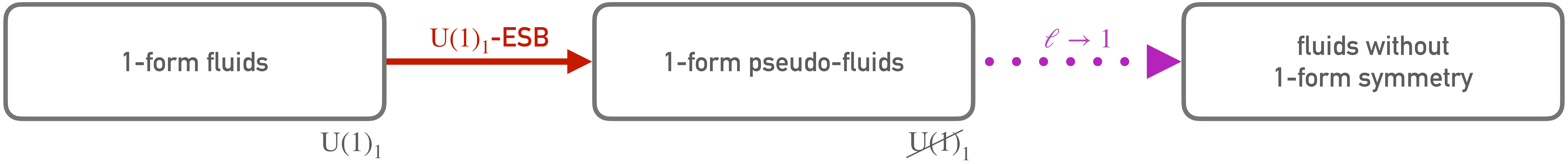
- The $\mu_\parallel$ mode obeys a damping-attenuation relation

$$\Gamma = D_\ell k_0^2$$

$$k_0^2 = \frac{\ell^2 \chi_\ell}{\chi}$$

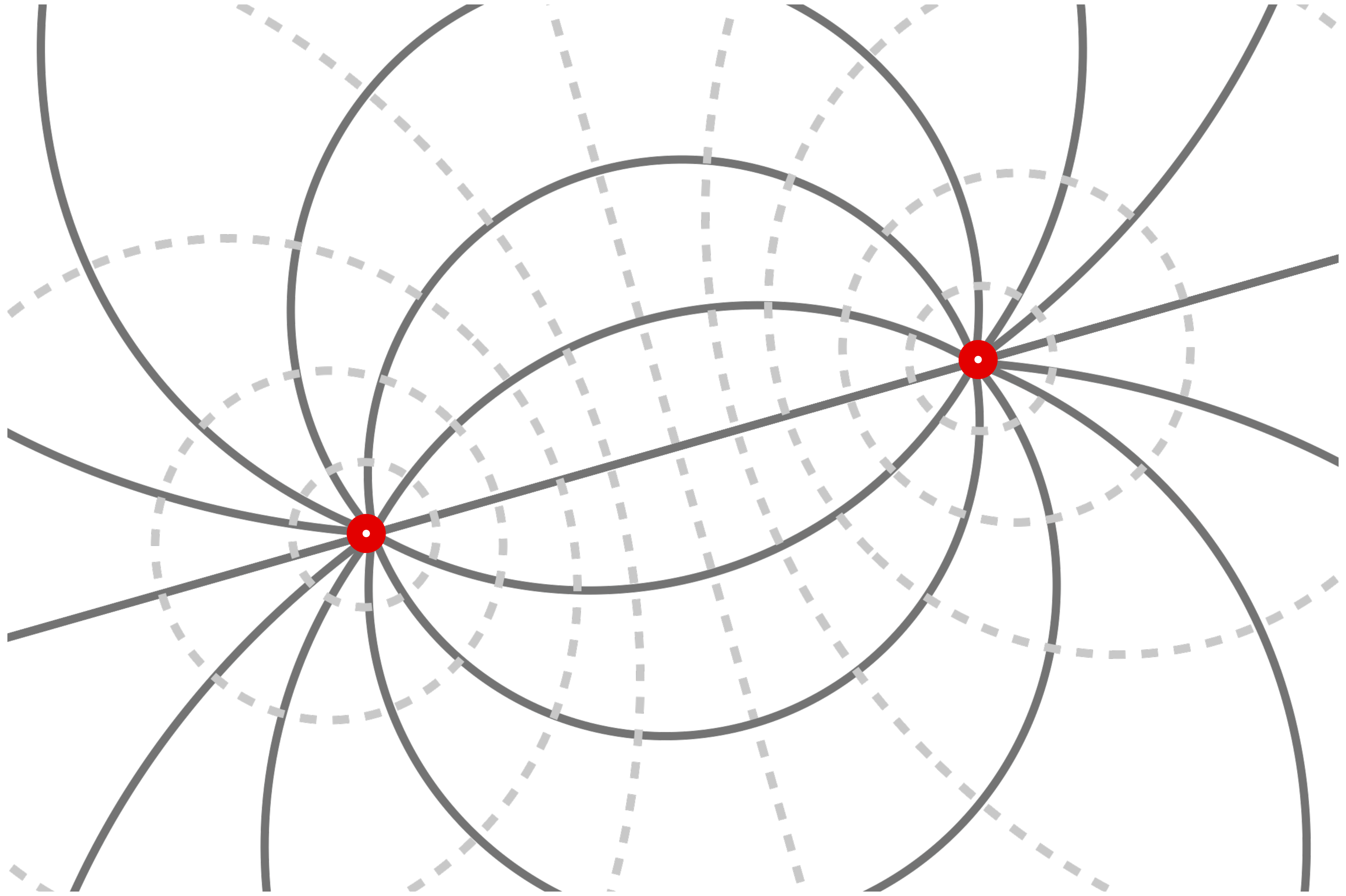
TOPOLOGICAL PHASE TRANSITIONS

- If we increase the strength of defects, i.e. increase ℓ, the charge fluctuations gap out and we are left with a fluid without 1-form symmetry.



$$\nabla_\mu \left(u^\mu n^\nu - u^\nu n^\mu + \ldots\right) = \ell n_\ell u^\mu + \Gamma n^\mu + \ldots$$

- The same discussion also applies to the phase transition of 0-form superfluids to 0-form fluids, mediated by vortices.



- Optical correlation functions (Coulomb/relaxed phase)

$$\mathrm{Re}\frac{i}{\omega}G^R_{J^{xy}J^{xy}}(\omega) = \chi D_n$$

$$\mathrm{Re}\frac{i}{\omega}G^R_{\ell L^x \ell L^x}(\omega) = \chi \Gamma \frac{\omega^2/\Gamma^2}{1+\omega^2/\Gamma^2}$$

HIGHER-FORM SUPERFLUIDS

with approximate higher-form symmetry

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1-FORM SUPERFLUIDS IN EQUILIBRIUM

- In the superfluid phase, the higher-form symmetry is completely spontaneously-broken

$$\varphi \to \varphi - \Lambda_t \qquad\qquad \mu_i = -\partial_i\varphi + A_{ti}$$

$$\phi_i \to \phi_i - \Lambda_i \qquad\qquad \xi_{ij} = \partial_i\phi_j - \partial_j\phi_i + A_{ij}$$

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi\mathcal{D}\phi_i \exp\int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i - \frac{1}{4\tilde{\chi}}\xi_{ij}\xi^{ij} + \ldots\right)$$

$$\Longrightarrow \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}}\xi^{ij}$$

- The configuration equations imply

$$\partial_i J^{ti} = 0 \implies \partial_i\partial^i\phi = 0 \qquad\qquad \varphi = -\mu_0 z \implies n_i = \chi\mu_0\,\delta_i^z$$

$$\partial_t J^{ti} + \partial_k J^{ki} = 0 \implies \partial_k\partial^k\phi^i - \partial^i\partial_k\phi^k = 0 \qquad\qquad \phi^i = (0,\,\tilde{\mu}_0 x,\,0) \implies \xi_{ij} = \tilde{\mu}_0\,\varepsilon_{ijz}$$

Armas, AJ [1811.04913]

1-FORM PSEUDO-SUPERFLUIDS IN EQUILIBRIUM

- In the presence of explicit symmetry breaking, the 1-form superfluid can exist in two phases depending on the 0-form defect symmetry being spontaneously broken or not.
- In the **relaxed/Coulomb phase**, the 0-form defect symmetry is spontaneously unbroken and we can only construct the "defect chemical potential"

$$\nabla_\mu J^{\mu\nu} = \ell L^\nu$$
$$\nabla_\mu L^\mu = 0$$

$$\mu_\ell = -\ell\left(\varphi - \Phi_t\right)$$

- In the **pinned/Higgs phase**, the 0-form defect symmetry is spontaneously broken

$$\phi_\ell \to \phi_\ell - \Lambda_\ell$$

This allows us to also construct the phase misalignment vector

$$\psi_i = \ell\left(\phi_i - \Phi_i - \partial_i\phi_\ell\right)$$

In the language of Higgs mechanism, the 1-form phase ϕ_i can eat 0-form phase ϕ_ℓ to become massive.

HIGHER-FORM PSEUDO-SUPERFLUIDS IN EQUILIBRIUM

- The partition function takes the form

$$\mathcal{Z}[A] = \int \mathcal{D}\varphi \exp \int \mathrm{d}^3x \left(\frac{1}{2}\chi\,\mu_i\mu^i - \frac{1}{4\tilde{\chi}}\xi_{ij}\xi^{ij} + \frac{1}{2}\chi_\ell\mu_\ell^2 - \frac{m^2}{2}\psi_i\psi^i + \ldots \right)$$

$$\implies \qquad J^{ti} = n^i = \chi\,\mu^i, \qquad J^{ij} = -\frac{1}{\tilde{\chi}}\xi^{ij}$$

$$L^t = n_\ell = \chi_\ell\mu_\ell \qquad L^i = m^2\psi^i$$

- Classical configuration equation for φ, ϕ_i impose the conservation equations in equilibrium

$$\partial_i J^{ti} = \ell L^t \implies \partial_i\partial^i\varphi = k_0^2\varphi$$

$$\partial_t J^{ti} + \partial_k J^{ki} = -\ell L^i \implies \partial_k\partial^k\phi^i - \partial^i\partial_k\phi^k = k_{0\phi}^2\phi^i$$

$$\frac{1}{k_0} = \sqrt{\frac{\chi}{\ell^2\chi_\ell}} \qquad \frac{1}{k_{0\phi}} = \sqrt{\frac{1}{\ell^2 m^2\tilde{\chi}}}$$

"Debye length" *"London depth"*

1-FORM PSEUDO-SUPERFLUID DYNAMICS

- The conservation equations for a 1-form superfluid remain the same

$$\nabla_\mu J^{\mu\nu} = -\ell L^\nu, \qquad \nabla_\mu L^\mu = 0, \qquad \nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho + \Xi^\mu L$$

Hydrodynamic fields are Λ^β_μ, Λ^β_ℓ, β^μ and ϕ_μ, ϕ_ℓ.

$$\frac{u^\mu}{T} = \beta^\mu$$

- We have a Josephson equation for ϕ_μ

$$\beta^\mu \phi_\mu = \varphi$$

$$\pounds_\beta \phi_\mu = \Lambda^\beta_\mu + \ldots \qquad \Longrightarrow \qquad u^\mu \xi_{\mu\nu} = \mu_\nu + \ldots$$

$$\xi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$$

$$\frac{\mu_\mu}{T} = \Lambda^\beta_\mu + \beta^\lambda A_{\lambda\mu} - \partial_\mu \varphi$$

- We also have a Josephson equation for ϕ_ℓ in the pinned/Higgs phase

$$\pounds_\beta \phi_\ell = \Lambda^\beta_\ell + \ldots \qquad \Longrightarrow \qquad u^\mu \psi_\mu = -\mu_\ell + \ldots$$

$$\frac{\mu_\ell}{T} = -\ell \left(\varphi - \beta^\mu \Phi_\mu - \Lambda^\beta_\ell\right)$$

$$\psi_\mu = -\ell\left(\phi_\mu - \Phi_\mu\right)$$

LINEARISED FLUCTUATIONS

- Let us assume that we are fluctuating around $\mu_\mu = \xi_{\mu\nu} = 0$ state.
  In this limit, energy and momentum fluctuations decouple from charge fluctuations, and propagate via the same fluid sound and shear modes.

- The transverse charge and Goldstone fluctuations give rise to the photon mode in the Coulomb/relaxed phase

$$\mu_\perp, \phi_\perp : \quad \left(i\omega - D_n k^2 - \Gamma\right)\left(i\omega - \tilde{D}_n k^2\right) + v_\perp^2 k^2 = 0 \qquad v_\perp^2 = \frac{\lambda^2}{\chi\tilde{\chi}}$$

- In the pinned/Higgs phase, you get additional pinning and relaxation effects

$$\mu_\perp, \phi_\perp : \quad \left(i\omega - D_n k^2 - \Gamma\right)\left(i\omega - \tilde{D}_n k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_\perp^2 k^2 = 0 \qquad \omega_0^2 = v_\perp^2 k_{0\phi}^2$$

$$\tilde{\Omega} = \tilde{D}_n k_{0\phi}^2 \qquad k_{0\phi}^2 = \ell^2 m^2 \tilde{\chi}$$

- Similar results hold for q-form superfluids with vortices or impurities for any q.

LINEARISED FLUCTUATIONS

- The longitudinal charge mode in the relaxed/Coulomb phase returns the previous damped diffusive mode

$$\mu_{\|}: \quad i\omega = D_\ell k^2 + \Gamma$$

- For pinned/Higgs phase, this couples to longitudinal Goldstone fluctuation and gives

$$\mu_{\|}, \phi_{\|}: \quad (i\omega - D_\ell k^2 - \Gamma)\left(i\omega - \tilde{D}_\psi k^2 - \tilde{\Omega}\right) + \omega_0^2 + v_{\|}^2 k^2 = 0$$

$$\Gamma = D_\ell k_0^2$$

$$k_0^2 = \frac{\ell^2 \chi_\ell}{\chi}$$

TOPOLOGICAL PHASE TRANSITIONS

- We can implement topological phase transitions by increasing the strength of ℓ, thereby increasing the strength of explicit symmetry breaking.
- The product of the phase transition depends on if we are starting form the relaxed/Coulomb phase of pinned/Higgs phase.
- In the relaxed/Coulomb phase, we arrive at a $(d-2)$-form fluid. In the context of electromagnetism, this describes magnetohydrodynamics with conserved magnetic field lines.

$$\omega = -i\left(\tilde{D}_n + \frac{v_\perp^2}{\Gamma}\right)k^2$$

- In the pinned/Higgs phase, we arrive at a neutral fluid. In the context of electromagnetism, this describes expulsion of all electromagnetic fields inside a superconductor.

TOPOLOGICAL PHASE TRANSITIONS

- Optical correlation functions (Coulomb/relaxed phase)

$$\mathrm{Re}\frac{i}{\omega}G^R_{J^{xy}J^{xy}}(\omega) = \chi D_n$$

$$\mathrm{Re}\frac{i}{\omega}G^R_{\xi^{tx}\xi^{tx}}(\omega) = \tilde{\chi}\tilde{D}_n + \frac{\lambda^2}{\chi\Gamma}\frac{1}{1+\omega^2/\Gamma^2}$$

$$\mathrm{Re}\frac{i}{\omega}G^R_{\ell L^x \ell L^x}(\omega) = \chi\Gamma\frac{\omega^2/\Gamma^2}{1+\omega^2/\Gamma^2}$$

HIGHER-FORM SUPERFLUIDS

- It is possible to also keep the "magnetic" $(d-2)$-form symmetry of a 1-form superfluid manifest by coupling the system to a $(d-1)$-form gauge field and accounting for the mixed anomaly.
- Explicit breaking of the "magnetic" $(d-2)$-form symmetry give rise to vortices in the 1-form superfluid. Only relaxed/Coulomb phase can admit vortices.
- Extension to q-form superfluids is straight-forward.

OUTLOOK

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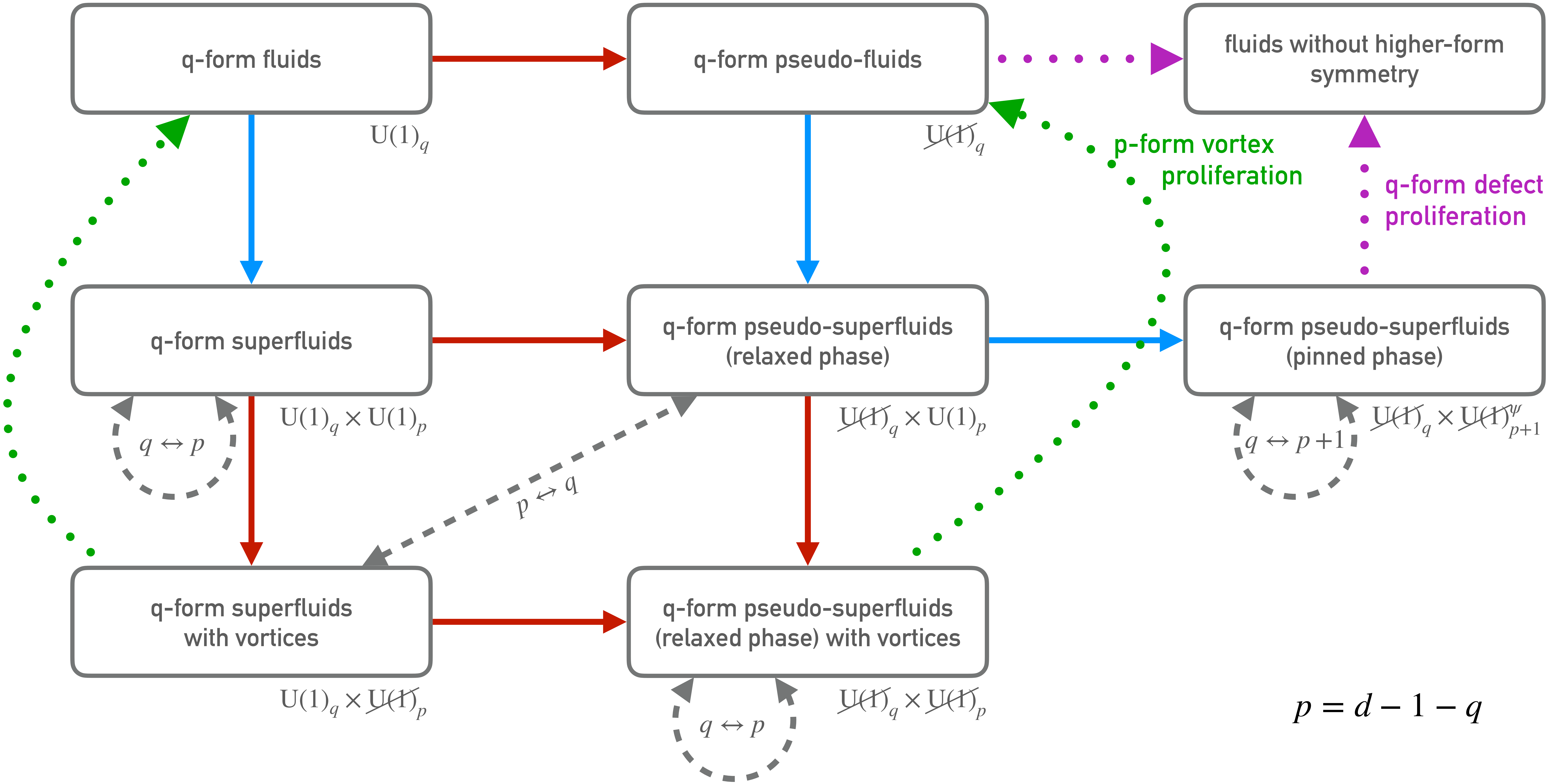
OUTLOOK

- Higher-form symmetries can be used to classify phases of matter with topological order.
- Breaking of continuous higher-form symmetries is associated with topological defects, which mediate topological phase transitions.
- A hydrodynamic theory with approximate higher-form symmetries provides a model for dynamical phase transitions based on symmetries.
- Further applications include emergent magnetic monopoles in spin ice, plasma phase transitions, melting phase transition in higher-dimensions, superfluid and superconductor phase transitions.


q-form fluids
q-form pseudo-fluids
fluids without higher-form symmetry
$U(1)_q$
$\cancel{U(1)}_q$
p-form vortex proliferation
q-form defect proliferation
q-form superfluids
q-form pseudo-superfluids (relaxed phase)
q-form pseudo-superfluids (pinned phase)
$U(1)_q \times U(1)_p$
$q \leftrightarrow p$
$p \leftrightarrow q$
$\cancel{U(1)}_q \times U(1)_p$
$\cancel{U(1)}_q \times \cancel{U(1)}^{\psi}_{p+1}$
$q \leftrightarrow p+1$
q-form superfluids with vortices
q-form pseudo-superfluids (relaxed phase) with vortices
$U(1)_q \times \cancel{U(1)}_p$
$q \leftrightarrow p$
$\cancel{U(1)}_q \times \cancel{U(1)}_p$

$$p = d - 1 - q$$