

Geometry of Super Null Infinity

Noémie Parrini

Ongoing work with N. Boulanger and Y. Herfray

Workshop Beyond Lorentzian Geometry II



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- 2 The super case : Orbits exploration
- 3 Chirality in super Minkowski space
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Context

- Null Infinity is the boundary of asymptotically flat spacetimes
- Asymptotically flat spacetimes are relevant
- Asymptotic group of symmetries of these space is BMS [BMS 62]
- More recently, it was shown that these symmetries are equivalent to the conformal Carrollian symmetries (symmetries of Null Infinity) [Duval, Gibbons, Horvathy]

Goal of the work

Generalize this in a superspace formulation, with a Carrollian approach :

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Several works already investigated the subject [AGS 86][Henneaux et al.]

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Generalize this in a superspace formulation, with a Carrollian approach :
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- We would like to propose a definition for *asymptotically flat superspaces*, for which the boundary would carry a *superconformal Carrollian geometry*
- This would give a geometrical realization of the *super BMS group*, as the conformal Carrollian symmetry group of super Null Infinity
- Starting point : super Minkowski space, through the study of homogeneous spaces

Klein Geometry and homogeneous spaces

Homogeneous space

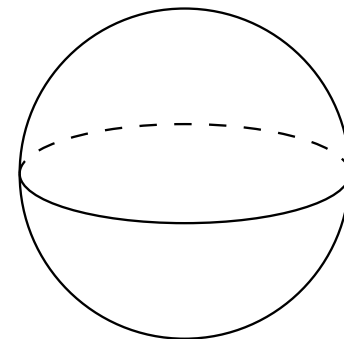
Space M with a transitive action of a Lie group G

"All points look the same"

→ $M \simeq G/H$, where H is the stabilizer of one point $x \in G$

Examples :

- $S^2 \simeq \frac{SO(3)}{SO(2)}$
- $M^{1,3} \simeq \frac{ISO(1,3)}{SO(1,3)}$



Klein pair

The pair (G, H) is a Klein geometry

Example : Conformally compactified Minkowski

Conformally compactified Minkowski $\overline{\mathbb{M}}^{1,3}$ is a homogeneous space for the conformal group

$$\overline{\mathbb{M}}^{1,3} = \frac{SO(2,4)}{\mathbb{R}^4 \rtimes (\mathbb{R} \times SO(1,3))}$$

We can choose $ISO(1,3) \subset SO(2,4)$ and break the conformal invariance by imposing to stabilize the preferred degenerate direction called null

infinity tractor $I' = \begin{bmatrix} 1 \\ 0^{AA'} \\ 0 \end{bmatrix}$

→ Split of $\overline{\mathbb{M}}^{1,3}$ into orbits of Poincaré

Orbit decomposition

Three orbits (subspaces invariant under the action of Poincaré)

$$\overline{\mathbb{M}}^{1,3} = \mathbb{M}^{1,3} \sqcup \mathcal{I} \sqcup \{I\}$$

Because Poincaré acts transitively on each of these subspaces, **they are homogeneous spaces for $ISO(1, 3)$** :

$$\overline{\mathbb{M}}^{1,3} = \frac{ISO(1, 3)}{SO(1, 3)} \sqcup \frac{ISO(1, 3)}{\mathbb{R}^3 \rtimes (\mathbb{R} \times ISO(2))} \sqcup \frac{ISO(1, 3)}{ISO(1, 3)}$$

Conformal Carrollian geometry

$$\mathcal{I} = \frac{\text{ISO}(1, 3)}{\mathbb{R}^3 \rtimes (\mathbb{R} \times \text{ISO}(2))} \simeq \mathbb{R} \times S^2$$

Conformal boundary of asymptotically flat spacetime is null, n^μ normal vector at \mathcal{I} such that

$$n^b h_{ab} = 0$$

where the degenerate metric h_{ab} is the induced conformal metric from the conformal manifold \overline{M} .

$\Rightarrow (n^a, h_{ab})$ constitutes a conformal Carrollian geometry

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Super Minkowski space

Goal : generalize this to the supersymmetric case (naive generalization)

$$\begin{array}{lll} \overline{\mathbb{M}}^4 & \rightarrow & \text{super compactified Minkowski space } \overline{\mathbb{M}}^{4|2\mathcal{N}} \\ \text{SU}(2, 2) \twoheadrightarrow \text{SO}(2, 4) & \rightarrow & \text{super conformal group } \text{SU}(2, 2|\mathcal{N}) \\ \text{SU}(2, 2) \circlearrowleft \overline{\mathbb{M}}^4 & \rightarrow & \text{SU}(2, 2|\mathcal{N}) \circlearrowleft \overline{\mathbb{M}}^{4|2\mathcal{N}} \end{array}$$

$\Rightarrow \overline{\mathbb{M}}^{4|2\mathcal{N}}$ (real) is an homogeneous space for the superconformal group

e.g. [Man97]

Question : Orbit decomposition of $\overline{\mathbb{M}}^{4|2\mathcal{N}}$ for super Poincaré group ?

Grassmannian definition of compactified Minkowski

Definition that generalizes to the susy case : (complexified)

$$\overline{M}^4 := \text{Gr}(2, \mathbb{C}^4)$$

$$= \{ \text{span}(Z^{\alpha 1}, Z^{\alpha 2}) \mid Z^{\alpha b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}^b \end{bmatrix} \in \mathbb{C}^4 \text{ for } b = 1, 2 \}$$

↓

$$\overline{M}_\ell^{4|2\mathcal{N}} := \text{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}})$$

$$= \{ \text{span}(Z^{\hat{\alpha} 1}, Z^{\hat{\alpha} 2}) \mid Z^{\hat{\alpha} b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}^b \\ \theta^{lb} \end{bmatrix} \in \mathbb{C}^{4|\mathcal{N}} \text{ for } b = 1, 2 \}$$

We can change the basis of the plane :

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{01} & \omega^{02} \\ \omega^{11} & \omega^{12} \\ \pi_{0',1} & \pi_{0',2} \\ \pi_{1',1} & \pi_{1',2} \\ \theta'^1 & \theta'^2 \end{bmatrix} \sim \begin{bmatrix} \omega^{01} & \omega^{02} \\ \omega^{11} & \omega^{12} \\ \pi_{0',1} & \pi_{0',2} \\ \pi_{1',1} & \pi_{1',2} \\ \theta'^1 & \theta'^2 \end{bmatrix} \cdot M$$

where $M \in \text{GL}(2, \mathbb{C})$.

+ Natural action of $\text{SL}(4|\mathcal{N}, \mathbb{C})$ (complexified super conformal group)

Orbit decomposition

Choice of a preferred super null direction $I^{\alpha b} = \begin{bmatrix} 1^{Ab} \\ 0_{A' b} \\ 0^{Ib} \end{bmatrix}$

\iff Choice of $\text{ISO}(1, 3|2\mathcal{N})_{\mathbb{C}} \subset \text{SU}(2, 2|2\mathcal{N})_{\mathbb{C}}$

Result of the decomposition : more orbits !

$$\overline{M}_{\ell}^{4|2\mathcal{N}} = M_{\ell}^{4|2\mathcal{N}} \sqcup \mathcal{I}_{\ell}^{(3|\mathcal{N})} \sqcup \mathcal{O}_{\ell} \sqcup \mathcal{H} \sqcup \{I\}$$

\hookrightarrow Each of these is an homogeneous space for super Poincaré

Coordinates

$$Z^{\hat{\alpha}b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}{}^b \\ \theta^{lb} \end{bmatrix} \in \overline{M}_\ell^{4|2\mathcal{N}}$$

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$$\det \pi \neq 0$$

$$\det \pi = 0$$

Coordinates

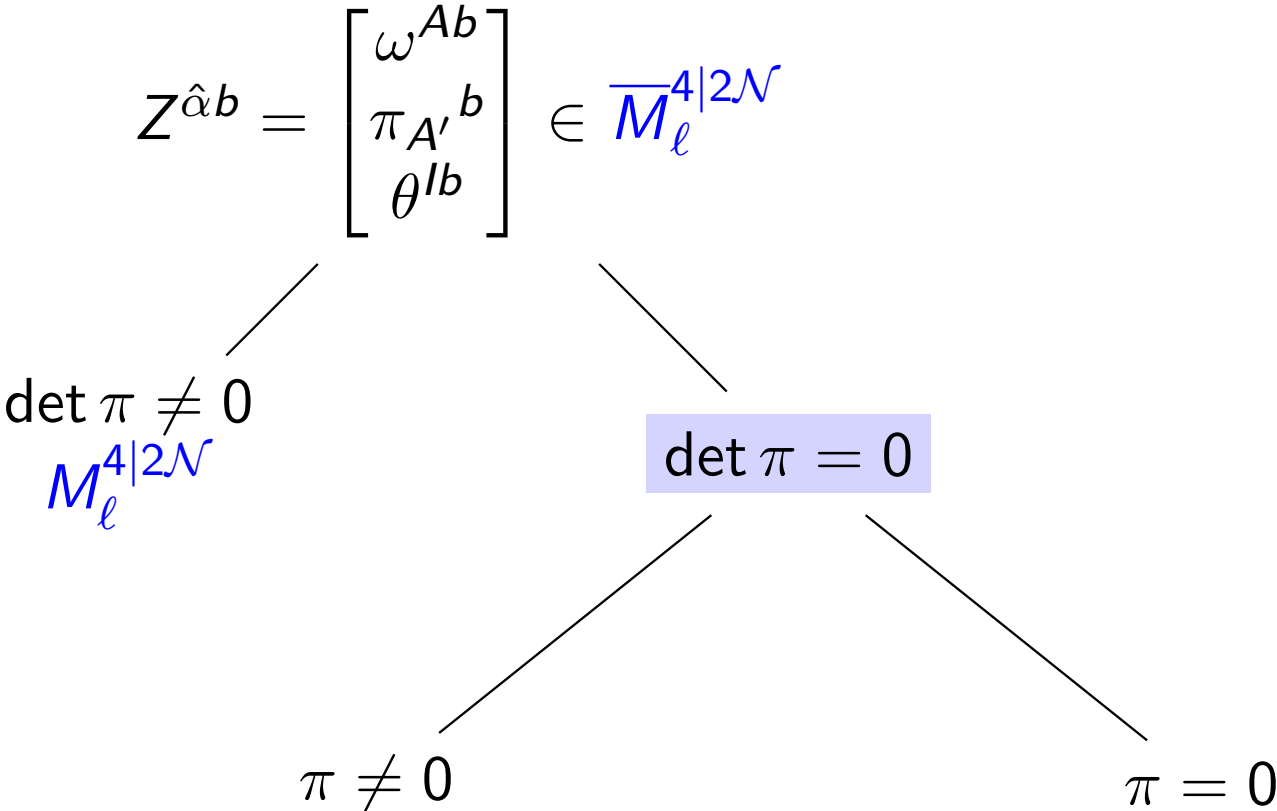
$$Z^{\hat{a}b} = \begin{bmatrix} \omega^{Ab} \\ \pi_{A'}^b \\ \theta^{lb} \end{bmatrix} \in \overline{M}_\ell^{4|2\mathcal{N}}$$

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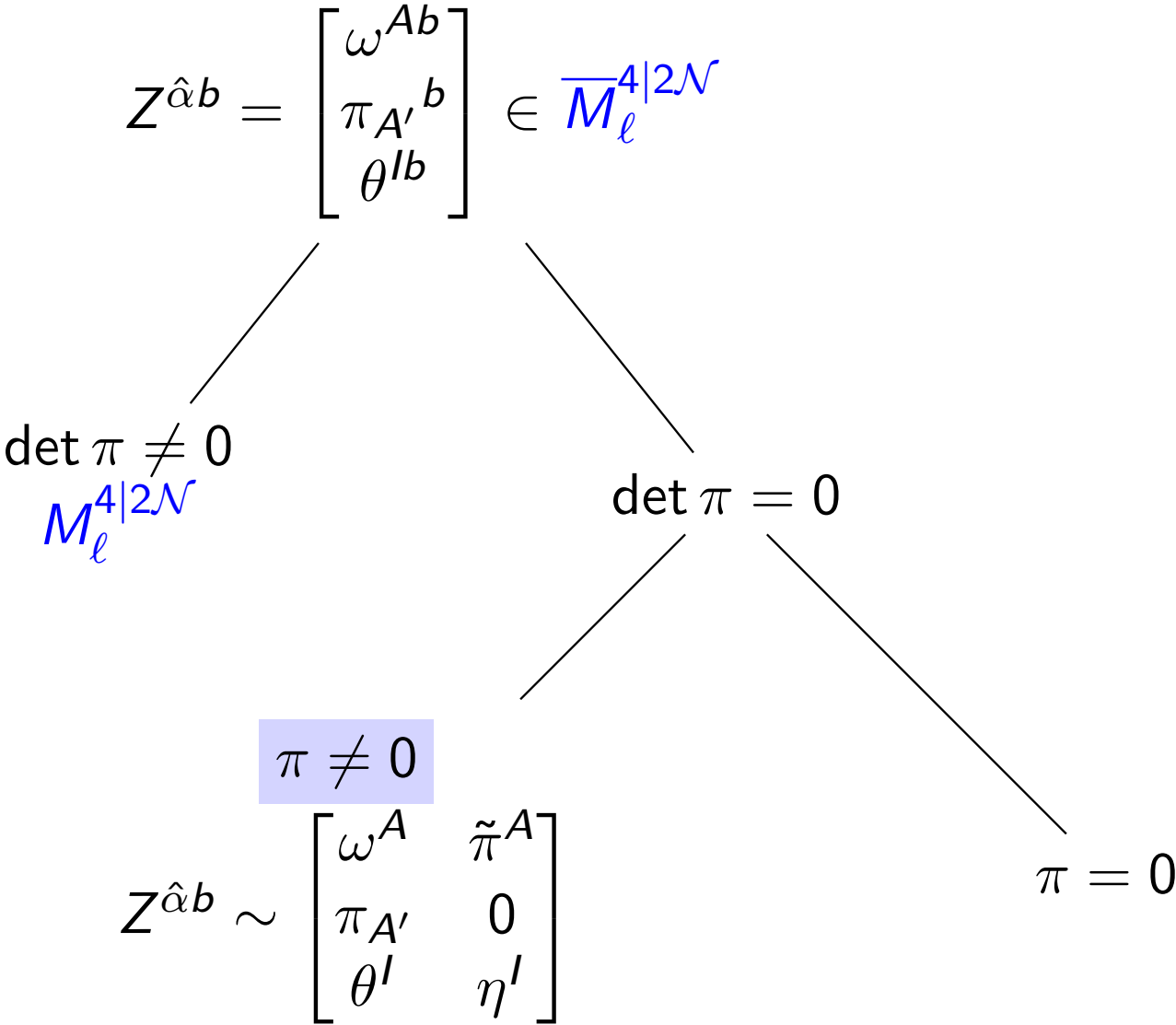
$$Z^{\hat{a}b} \sim \begin{bmatrix} iX_\ell^{AA'} \\ \delta_{A'}^b \\ \theta^{lb} \end{bmatrix}$$

$$\det \pi = 0$$

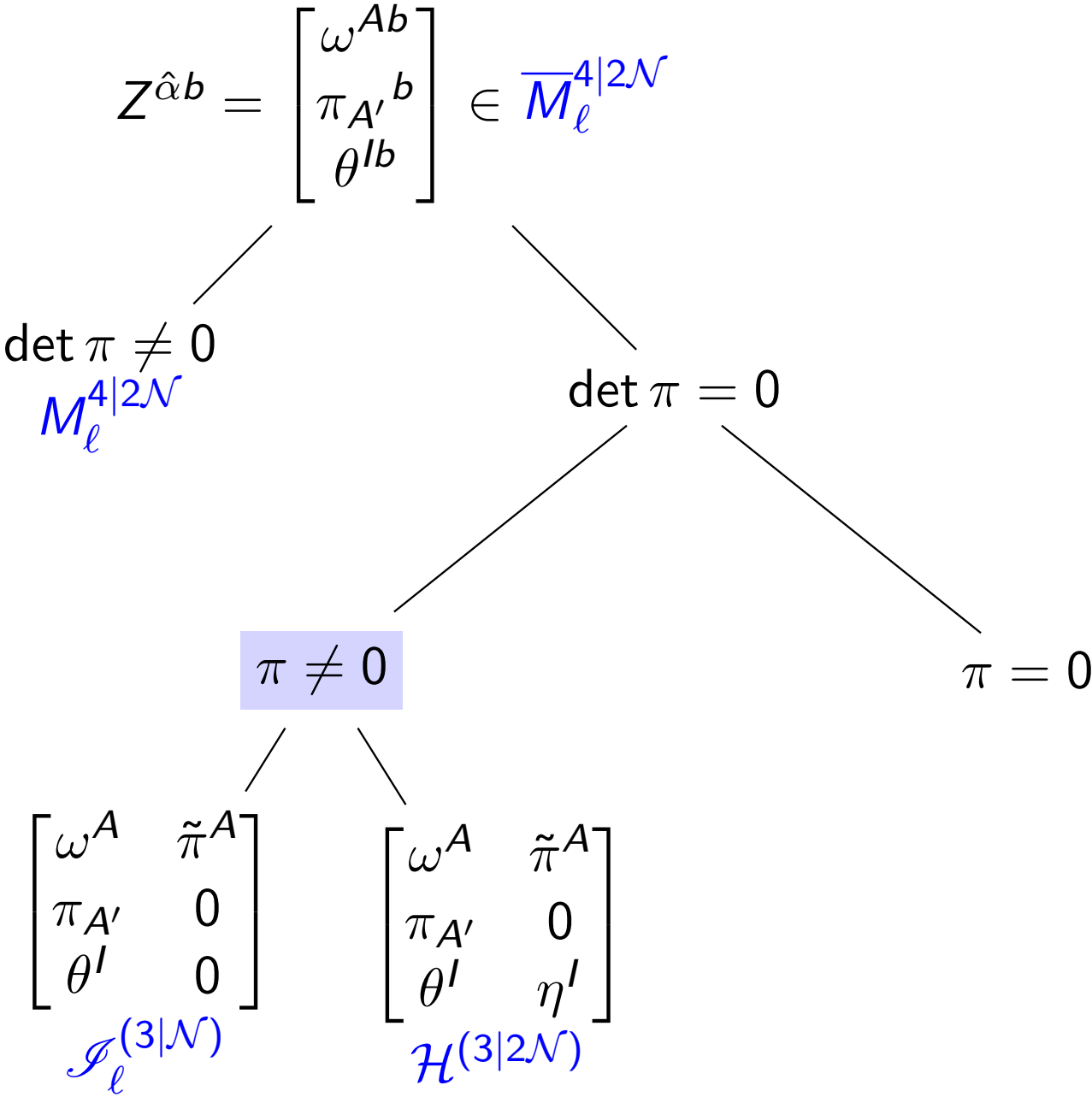
Coordinates



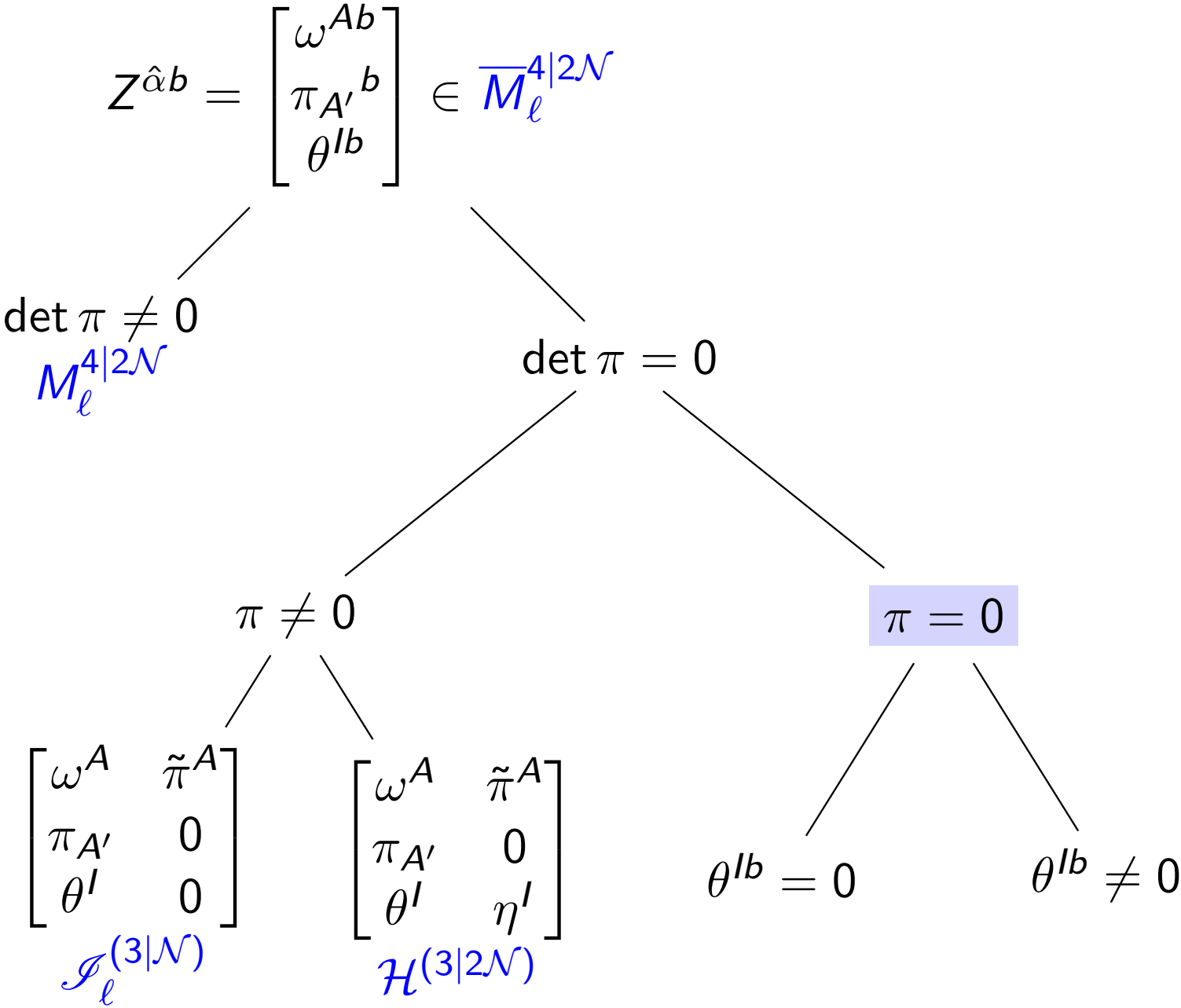
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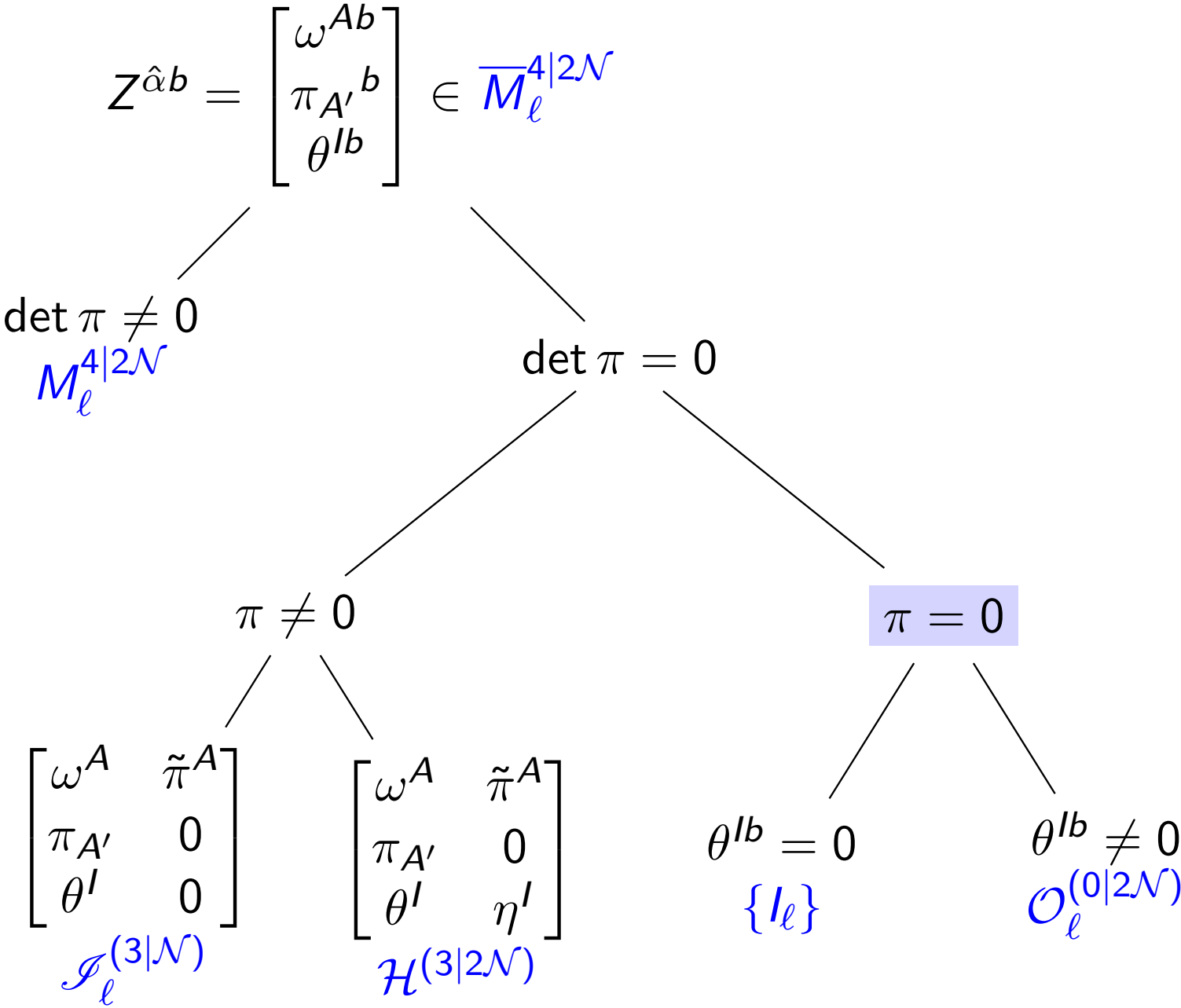
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Coordinates



Coordinates



Coordinates on these orbits ? (reality conditions)

- On $M_\ell^{1,3|2\mathcal{N}} \simeq \frac{\text{ISO}(1,3|\mathcal{N})}{\text{SO}(1,3) \times \text{SU}(\mathcal{N})}$ we find coordinates $(X_\ell^{AA'}, \theta^{A'}{}_I)$ such that one can write

$$X_\ell^{AA'} = X^{AA'} + \frac{i}{2} \theta^{A'}{}_I \bar{\theta}^A{}_J \delta^{IJ},$$

for a real $X^{AA'}$

Chiral left coordinates appear naturally !

Coordinates on these orbits ? (reality conditions)

- On $M_\ell^{1,3|2\mathcal{N}} \simeq \frac{\text{ISO}(1,3|\mathcal{N})}{\text{SO}(1,3) \times \text{SU}(\mathcal{N})}$ we find coordinates $(X_\ell^{AA'}, \theta^{A'}{}_I)$ such that one can write

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Chiral left coordinates appear naturally !

- On $\mathcal{I}_\ell^{(3|\mathcal{N})} \simeq \frac{\text{ISO}(1,3|\mathcal{N})}{\mathbb{R}^3 \times (\mathbb{R}^{0|\mathcal{N}} \times (\text{ISO}(2) \times \mathbb{R} \times \text{SU}(\mathcal{N})))}$ we find coordinates $([\pi^A], u_\ell = -i\omega^A \tilde{\pi}_A, \theta_I)$ such that one can write

$$u_\ell = u + \frac{i}{2} \theta_I \bar{\theta}^I \delta^{IJ}$$

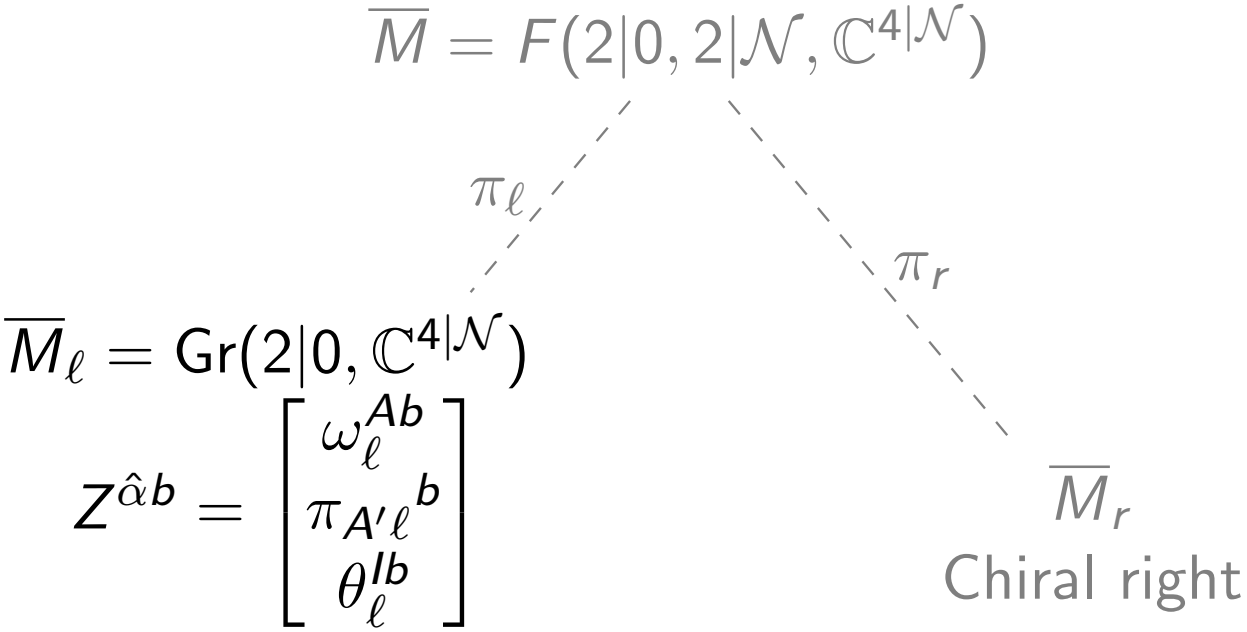
for a real u

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Chiral structure

So far, we have been looking only at one half of the history ! Complexify and [Man97] :



Flag variety

Let $0 \leq d_0 < d_1 < \dots < d_k \leq \dim V$

$$F(d_0, d_1, \dots, d_k, V) := \{V_0 \subset V_1 \subset \dots \subset V_k \text{ sub v.s. of } V \mid \dim(V_i) = d_i\}$$

Flag variety

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$$F(d_0, d_1, \dots, d_k, V) := \{V_0 \subset V_1 \subset \dots \subset V_k \text{ sub v.s. of } V \mid \dim(V_i) = d_i\}$$

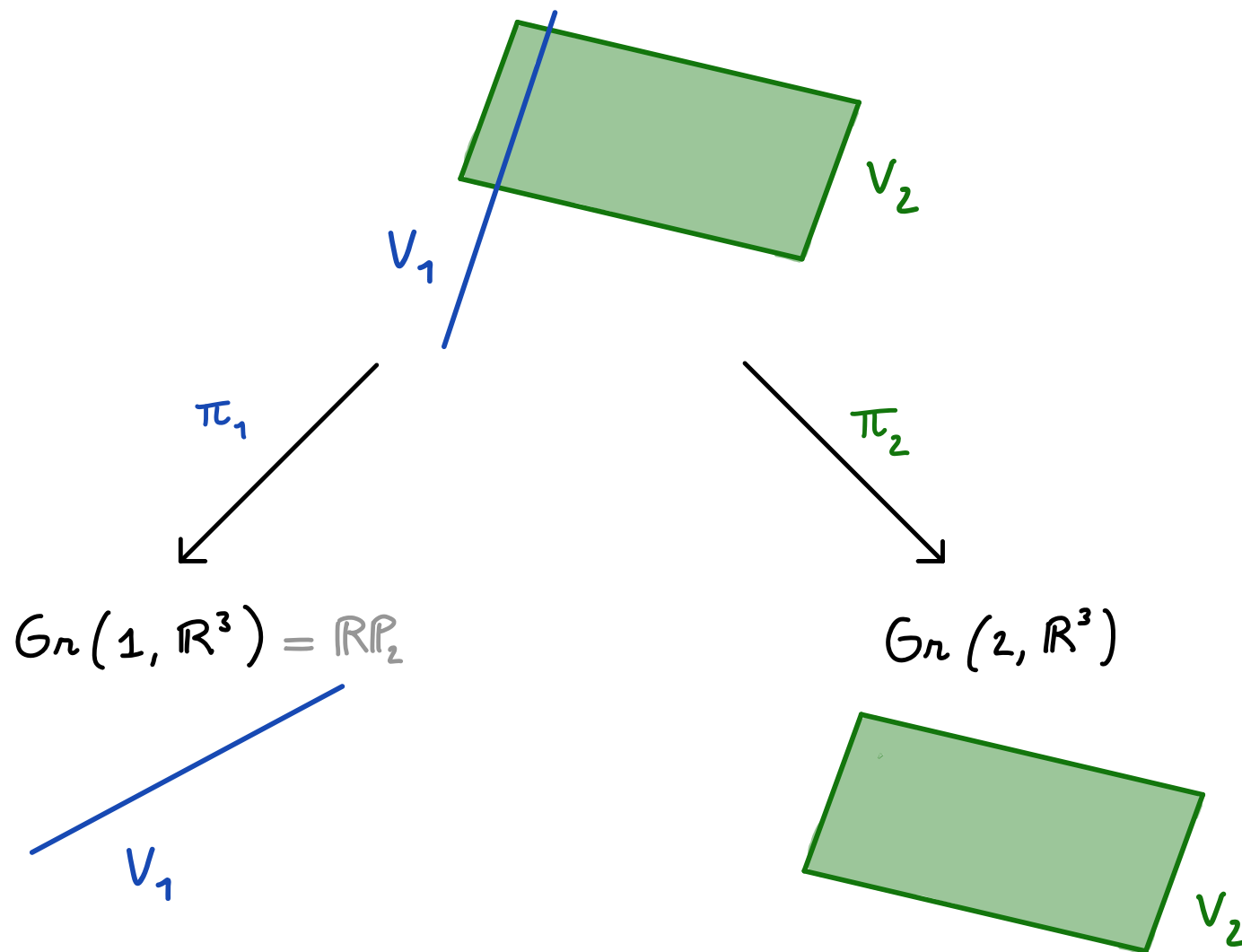
in particular

$$F(k, V) = \text{Gr}(k, V)$$

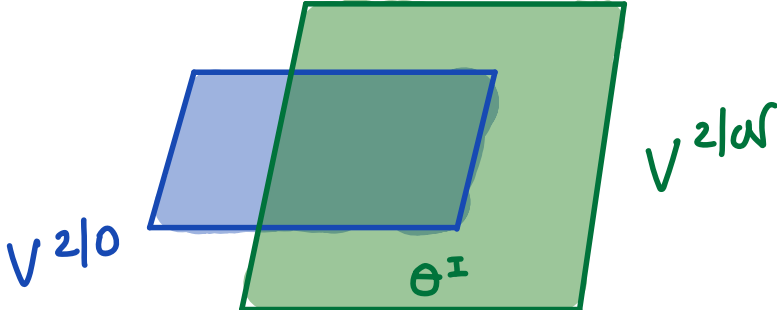
$$F(1, V) = \mathbb{P}_{\dim V - 1}(V)$$

Example

$$F(1, 2, \mathbb{R}^3) := \{V_1 \subset V_2 \text{ sub v.s. of } \mathbb{R}^3 \mid \dim(V_i) = i\}$$



Chiral structure



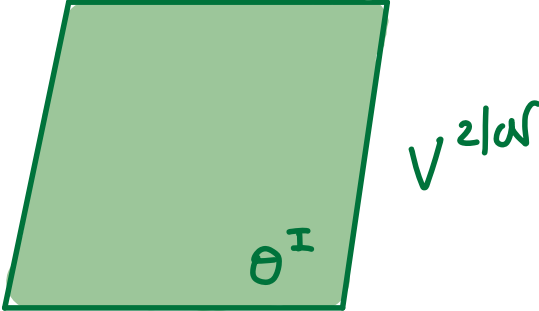
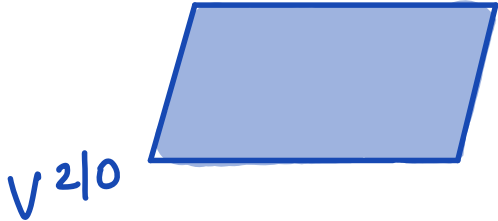
$$\bar{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$$

π_ℓ

π_r

$$\bar{M}_\ell = \text{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}})$$

$$\bar{M}_r = \text{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}})$$



[Man97]

Chiral structure

Full complexified compactified super Minkowski space

$$\begin{array}{ccc}
 \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) & & \\
 \swarrow \pi_\ell & & \searrow \pi_r \\
 \overline{M}_\ell = \text{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r = \text{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \\
 & & \simeq \text{Gr}(2|0, (\mathbb{C}^{4|\mathcal{N}})^*) \\
 Z^{\hat{\alpha}b} = \begin{bmatrix} \omega_\ell^{Ab} \\ \pi_{A'\ell}{}^b \\ \theta_\ell^{lb} \end{bmatrix} & & \tilde{Z}_{\hat{\alpha}}{}^b = \begin{bmatrix} \pi_{Ar}{}^b \\ \omega_r^{A'b} \\ \theta_{lr}{}^b \end{bmatrix}^t \\
 \text{Chiral left} & & \text{Chiral right}
 \end{array}$$

[Man97]

Chiral structure

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 \end{array}$$

Chiral left

Chiral right

[Man97] Flag condition : $\forall a, b, Z^{\hat{\alpha}a} \tilde{Z}_{\hat{\alpha}}{}^b = 0$

Chiral right

Two important orbits :

- In chiral right super Minkowski space M_R , the subspace of \overline{M}_R where $\det \pi \neq 0$, it exists $X^{A'A} \in \mathbb{R}$ such that we can write

$$X_r^{A'A} = X^{A'A} - \frac{i}{2} \theta^{IA'} \delta_{IJ'} \bar{\theta}^{J'B}.$$

Chiral right

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- In chiral right super Minkowski space M_R , the subspace of \overline{M}_R where $\det \pi \neq 0$, it exists $X^{A'A} \in \mathbb{R}$ such that we can write

$$X_r^{A'A} = X^{A'A} - \frac{i}{2} \theta^{IA'} \delta_{IJ'} \bar{\theta}^{J'B}.$$

- On the chiral right super Null Infinity \mathcal{I}_r it exists $u \in \mathbb{R}$ such that we can write

$$u_- = u - \frac{i}{2} \theta_I \bar{\theta}_{\bar{J}} \delta^{I\bar{J}}.$$

Non chiral

- Non chiral super Minkowski space is defined as

$$\{(Z^{\hat{\alpha}b}, \tilde{Z}_{\hat{\alpha}b}) \mid Z^{\hat{\alpha}b} \in M_L \text{ and } \tilde{Z}_{\hat{\alpha}b} \in M_R\}$$

→ Flag condition imposes $x_{\ell}^{AA'} - x_r^{AA'} = 2i\theta_{\ell}^{A'}\theta_{rA'}$

Non chiral

- Non chiral super Minkowski space is defined as

$$\{(Z^{\hat{a}b}, \tilde{Z}_{\hat{a}b}) \mid Z^{\hat{a}b} \in M_L \text{ and } \tilde{Z}_{\hat{a}b} \in M_R\}$$

→ Flag condition imposes $x_\ell^{AA'} - x_r^{AA'} = 2i\theta_\ell^{A'}\theta_r^A$

- The obvious candidate for non chiral super null infinity is

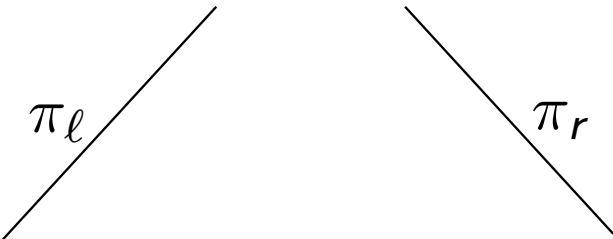
$$\{(Z^{\hat{a}b}, \tilde{Z}_{\hat{a}b}) \mid Z^{\hat{a}b} \in \mathcal{S}_L^{(\mathcal{N})} \text{ and } \tilde{Z}_{\hat{a}b} \in \mathcal{S}_R^{(\mathcal{N})}\}$$

→ Flag condition imposes $\tilde{\pi}_\ell^A \propto \pi_r^A$, $\tilde{\pi}_r^A \propto \pi_\ell^A$ and $u_\ell - u_r = i\delta_{IJ}\theta_I^J\theta_r^I$

Non chiral super Minkowski (big cell)

Super Minkowski

$$M \quad (x^{AA'} := \frac{1}{2}(x_\ell^{AA'} + x_r^{A'A}), \theta_\ell^{lb}, \theta_{lr}^b)$$



$$M_\ell \quad (x_\ell^{AA'} = x^{AA'} + \frac{i}{2}\theta_\ell^{lA}\theta_{lr}^{A'}, \theta_\ell^{lb})$$

Chiral left

$$M_r \quad (x_r^{A'A} = x^{AA'} - \frac{i}{2}\theta_\ell^{lA}\theta_{lr}^{A'}, \theta_{lr}^b)$$

Chiral right

Non chiral super Null Infinity

Super Null Infinity

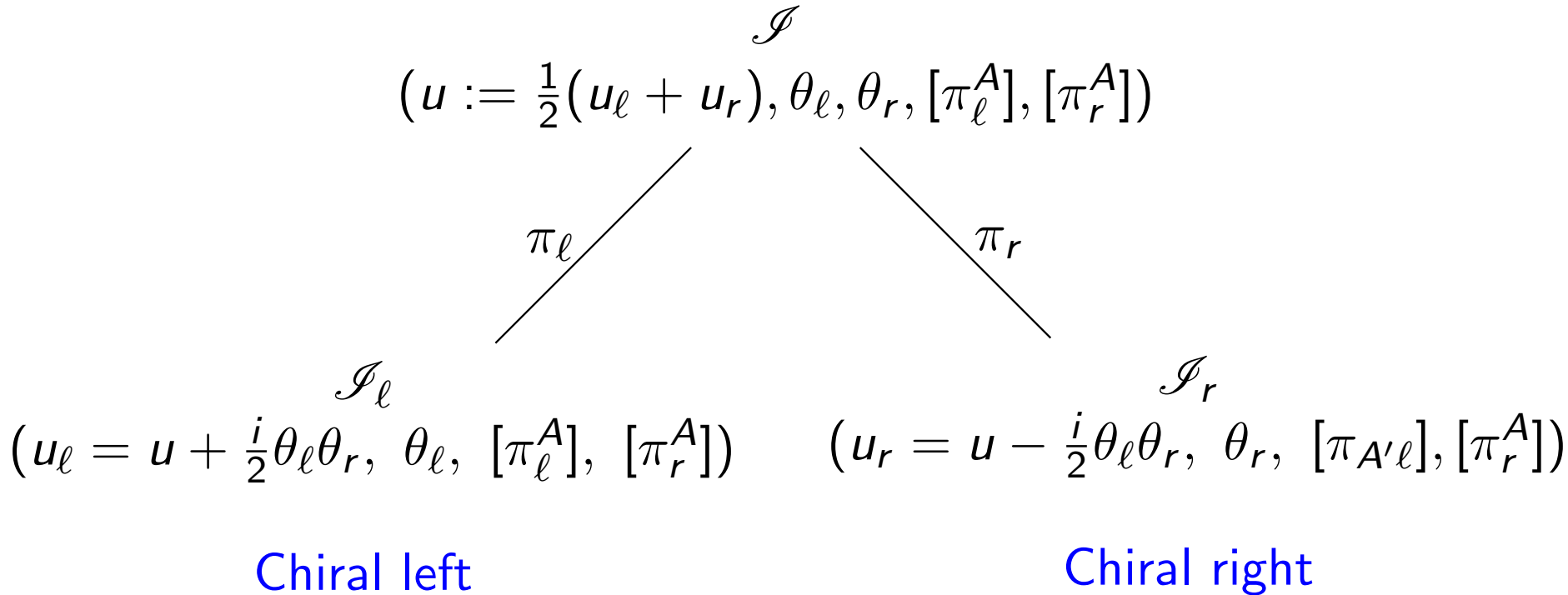
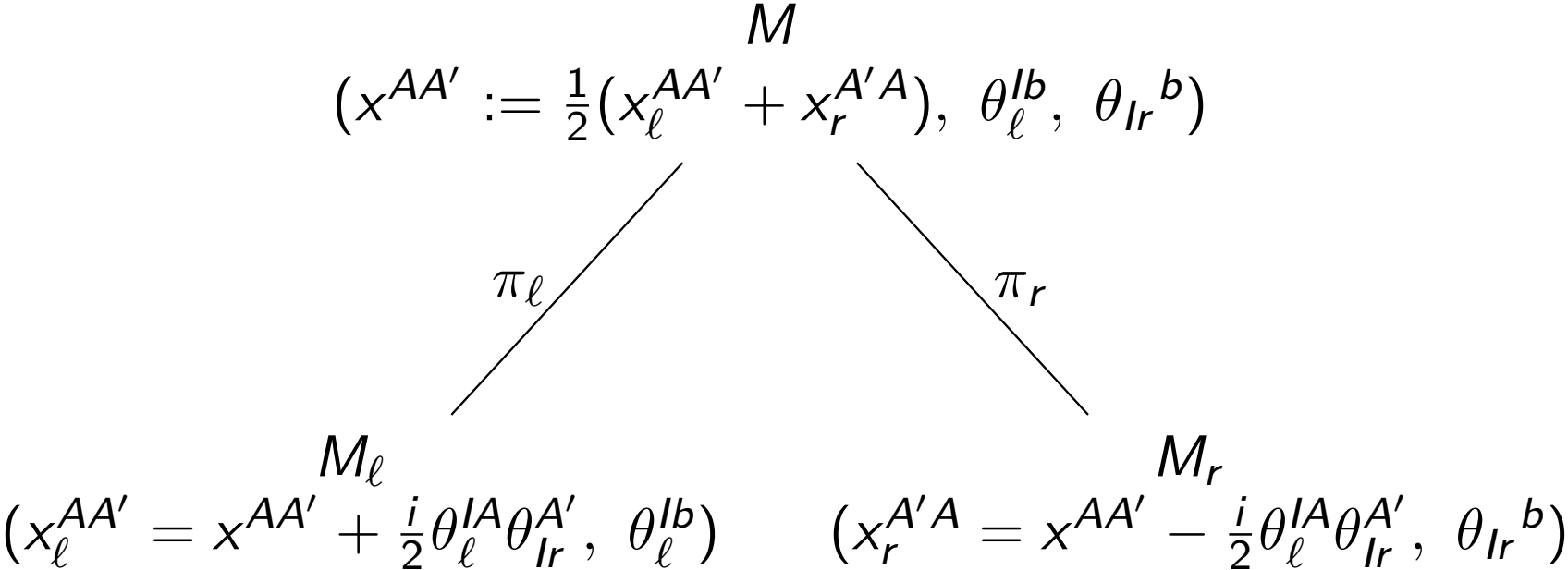


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Distributions

For super Minkowski :



Distributions

For super Minkowski :

Two integrable distributions $D_\ell, D_r \subset TM$ associated to the two fibrations
Basis of the sections given by :

$$D_{\ell|b} = \partial_{\theta_\ell^{|b}} + i\theta_{r|}^b \partial_{x^{AA'}} \longrightarrow \text{s.t. } D_\ell(x_r, \theta_r) = 0$$

$$D_{r|}^b = \partial_{\theta_{r|}^b} + i\theta_l^{|b} \partial_{x^{AA'}} \longrightarrow \text{s.t. } D_r(x_\ell, \theta_\ell) = 0$$

\Rightarrow Covariant derivatives

Distributions

$$\begin{array}{ccc} \overline{M} = F(2|0, 2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) & & \\ \pi_\ell \swarrow & & \searrow \pi_r \\ \overline{M}_\ell = \text{Gr}(2|0, \mathbb{C}^{4|\mathcal{N}}) & & \overline{M}_r = \text{Gr}(2|\mathcal{N}, \mathbb{C}^{4|\mathcal{N}}) \end{array}$$

Distributions ("covariant derivatives") already there in the homogeneous superconformal model !

Superconformal geometry

In fact, they are part of the superconformal geometry :

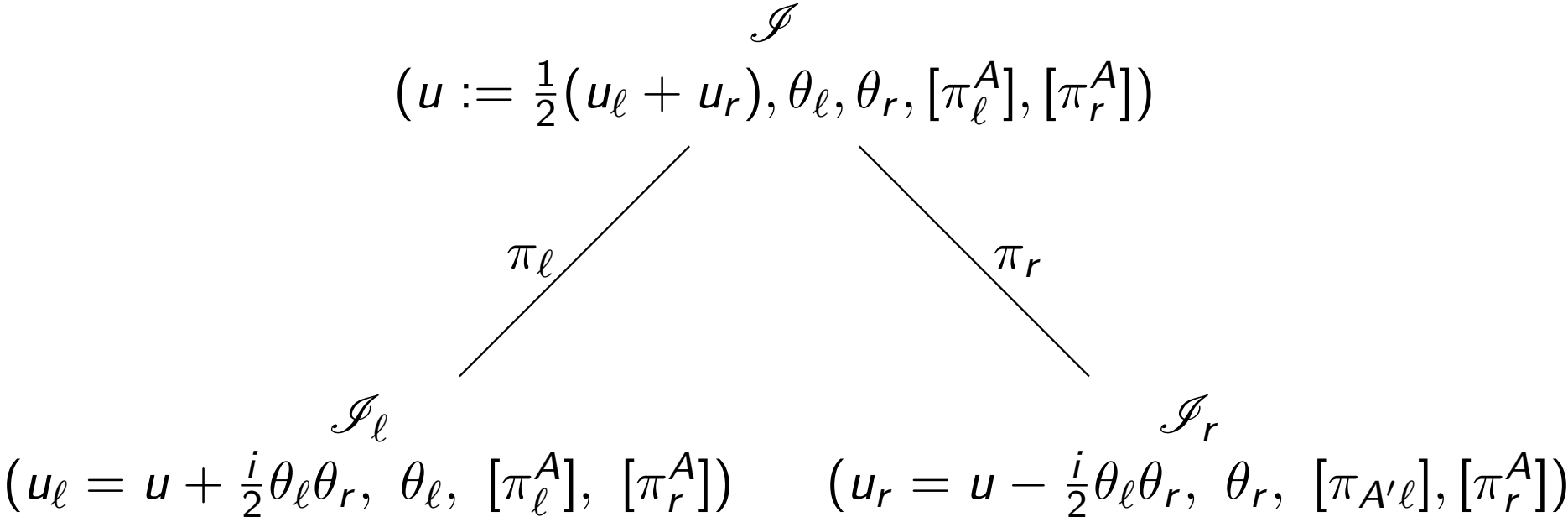
Superconformal geometry [Mer92, Man97]

A superconformal geometry on a complex supermanifold $M^{4|4}$ is a pair of integrable $(0|2)$ -dimensional distributions $D_\ell, D_r \subset TM$ s.t.

- Their sum is direct in TM
- $\varphi : D_\ell \otimes D_r \rightarrow TM / (D_\ell \oplus D_r) : (X \otimes Y) \mapsto [X, Y] \text{ mod } (D_\ell \oplus D_r)$ is an isomorphism

Distributions

For super Null Infinity : works in the same way



Distributions

For super Null Infinity : works in the same way

Two integrable distributions $D_\ell, D_r \subset T\mathcal{I}$ associated to the two fibrations

$$D_\ell = \partial_{\theta_\ell} - i\theta_r \partial_u \quad \rightarrow \text{s.t. } D_\ell(u_r, \theta_r) = 0$$

$$D_r = \partial_{\theta_r} - i\theta_\ell \partial_u \quad \rightarrow \text{s.t. } D_r(u_\ell, \theta_\ell) = 0$$

Super conformal Carrollian : curved setup

Based on the flat model, we propose to extend this definition for $\mathcal{I}^{3|1}$, as a model for a *super conformal Carrollian geometry* : the data of (D_ℓ, D_r, n) , with some compatibility conditions

Results :

Super conformal Carrollian : curved setup

Based on the flat model, we propose to extend this definition for $\mathcal{I}^{3|1}$, as a model for a *super conformal Carrollian geometry* : the data of (D_ℓ, D_r, n) , with some compatibility conditions

Results :

- We can always write :
 - ▶ $D_\ell = \partial_{\theta_\ell} - i\theta_r \partial_u$
 - ▶ $D_r = \partial_{\theta_r} - i\theta_\ell \partial_u$
 - ▶ $n = \partial_u$
- The symmetry group of these is the *super BMS group* of Awada-Gibbons-Shaw [AGS 86]

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Conclusion

Short summary

- Mostly in the flat case
- Chirality : the non chiral super Minkowski space is a flag variety
- Identify two remarkable orbits inside the compactification : super Minkowski and **super \mathcal{I}**
- For non chiral super Minkowski and **non chiral super \mathcal{I}** we have the same structure of double fibration, characterized by two distributions
- This structure is the basis of the definition of superconformal geometry

Conclusion

Short summary



- Mostly in the flat case
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- Identify two remarkable orbits inside the compactification : super Minkowski and **super \mathcal{I}**
- For non chiral super Minkowski and **non chiral super \mathcal{I}** we have the same structure of double fibration, characterized by two distributions
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Still in progress ...

- Give precise definitions to treat the general asymptotically flat case
- Application to self dual supergravity

Thank you for your attention !

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-  S. A. Merkulov, *Simple supergravity, supersymmetric nonlinear gravitons and supertwistor theory*, *Class. Quant. Grav.* **9** (1992), 2369–2393.