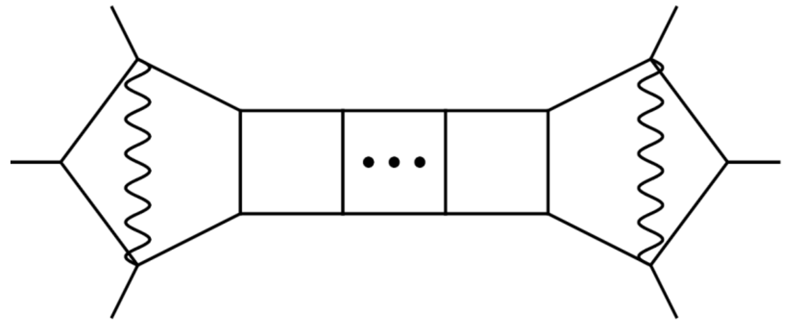
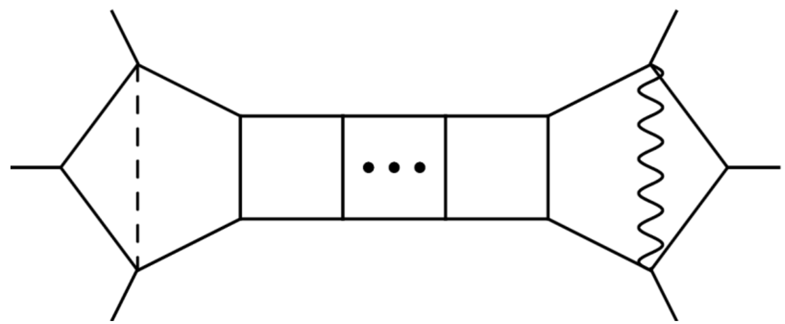
Double penta-ladders to all orders



$\Omega^{(L)}$



$\tilde{\Omega}^{(L)}$

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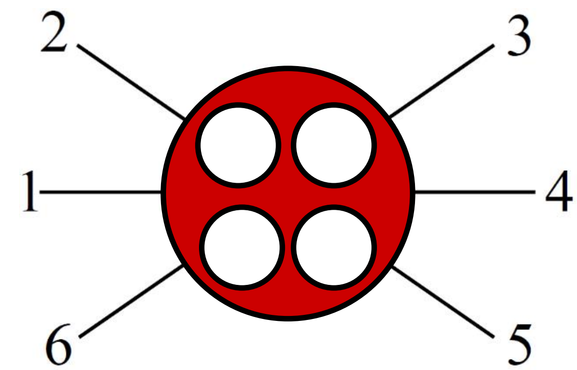
13 April 2017

Hexagon (heptagon) function bootstrap

LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Drummond, Papathanasiou, Spradlin, 1412.3763;
LD, Drummond, Harrington, McLeod, Papathanasiou,, Spradlin, 1612.08976



Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly,
without ever peeking inside the loops.

Works to at least 6 loops (MHV) at 6 points, 4 loops (MHV) at 7 points.

First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Orienting the space of functions

- We don't need to know which functions correspond to which integrals, but it does help organize the full space more efficiently.
- The double pentaladders, or Ω functions, together with their coproducts, define an interesting subspace, which we now understand quite well to all loop orders, as well as at finite coupling.

Hexagon symbol letters

- Momentum twistors Z_i^A, $i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$u = \frac{\langle 6123 \rangle \langle 3456 \rangle}{\langle 6134 \rangle \langle 2356 \rangle} \qquad 1-u = \frac{\langle 6135 \rangle \langle 2346 \rangle}{\langle 6134 \rangle \langle 2356 \rangle} \qquad y_u = \frac{\langle 1345 \rangle \langle 2456 \rangle \langle 1236 \rangle}{\langle 1235 \rangle \langle 3456 \rangle \langle 1246 \rangle}$$

+ cyclic

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \qquad v = \frac{s_{23} s_{56}}{s_{234} s_{123}} \qquad w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

Discrete symmetries

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-} \,, \ \ldots \quad \text{where} \quad \begin{aligned} z_\pm &= \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big] \\ \Delta &= (1 - u - v - w)^2 - 4uvw \end{aligned}$$

- Function space graded by parity:

$$\begin{aligned} i\sqrt{\Delta} &\leftrightarrow -i\sqrt{\Delta} \\ z_+ &\leftrightarrow z_- \\ y_i &\leftrightarrow 1/y_i \\ u_i &\leftrightarrow u_i \end{aligned}$$

- Also a (dihedral) S_3 symmetry permuting u, v, w
- Broken for Ω system to Z_2 flip: $u \leftarrow\rightarrow v$

Hexagon functions include:

1. HPLs [Remiddi, Vermaseren (1999)]: One variable, symbol letters $\{u, 1\text{-}u\}$. Near-collinear limit, lines $(u,u,1)$, $(u,1,1)$
2. Cyclotomic Polylogarithms [Ablinger, Blumlein, Schneider, 1105.6063]: One variable, letters $\{y_u, 1+y_u, 1+y_u+y_u^2\}$. Line (u,u,u).
3. SVHPLs [F. Brown, 2004]: Two variables, letters $\{z, 1\text{-}z, \bar{z}, 1\text{-}\bar{z}\}$. First entry/single-valuedness constraint. Multi-Regge limit.
4. Ω functions. All three variables, all symbol letters,

$$\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

branch-cut condition(s),…

Coproduct notation

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n transcendental functions f.
- Define by derivatives:

$$d\, f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d \ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ coproduct component}$$

are also pure functions, weight n-1

Duhr, Gangl, Rhodes, 1110.0458

- Iterate: $d\, f^{s_k} \Rightarrow f^{s_j, s_k} \equiv \{n-2,1,1\} \text{ component}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

- Symbol = {1,1,…,1} component (maximally iterated)

Single-valued multiple polylogarithms (SVMPLs or SVHPLs)

Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 527

- Controls hexagon functions in multi-Regge limit
- Symbol letters: $\mathcal{S} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- Also require function to be real analytic in

$$(z, \bar{z}) \in \mathbb{C} - \{0, 1\}$$

- Constrains first entry of the symbol to be

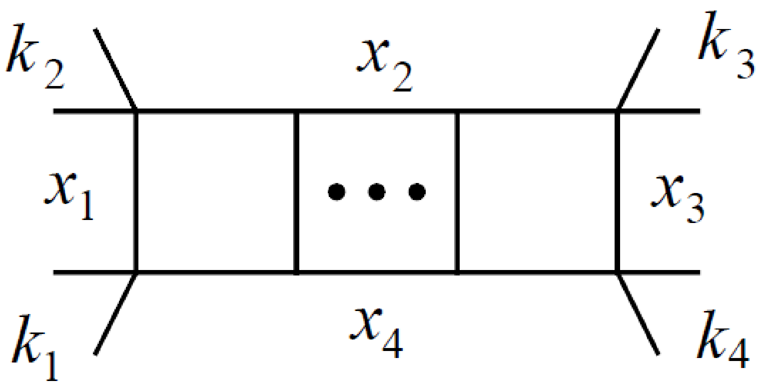
$$z\bar{z} \leftrightarrow \ln|z|^2 \quad \text{or} \quad (1-z)(1-\bar{z}) \leftrightarrow \ln|1-z|^2$$

- One SVMPL for each MPL
- Powerful constraint: $4^n \rightarrow 2^n$ functions at wt. n

$$\mathcal{L}_{\vec{w}}\,, \quad w_i \in \{0, 1\}$$

A well-known ladder lies in this space

Usyukina, Davydychev, Phys. Lett. B305 (1993) 136;
Broadhurst, Phys. Lett. B307 (1993) 132;
Broadhurst, Davydychev, arXiv:1007.0237 [hep-th]



$$\Phi^{(L)}(u,v)$$

$$u = \frac{k_1^2\, k_3^2}{s\, t} = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} \qquad v = \frac{k_2^2\, k_4^2}{s\, t} = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}$$

$$u = \frac{1}{(1-z)\,(1-\bar{z})} \qquad v = \frac{z\,\bar{z}}{(1-z)\,(1-\bar{z})}$$

Ladder in terms of SVMPLs

$$\frac{(z-\bar{z})\ \Phi^{(L)}(z,\bar{z})}{(1-z)(1-\bar{z})} = \sum_{j=L}^{2L} \frac{j!}{L!\ (j-L)!\ (2L-j)!} \times \ln^{2L-j}\Big(\frac{1}{|z|^2}\Big)\ \Big[\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})\Big]$$

$$= (-1)^L[\mathcal{L}_{\underbrace{0,\ldots,0,1}_{L},\underbrace{0,0,\ldots,0}_{L}} - \mathcal{L}_{\underbrace{0,\ldots,0,0}_{L},\underbrace{1,0,\ldots,0}_{L}}]$$

- Taking z derivative gives factor of $1/z$, clips 0 from left
- Taking $\bar{z}$ derivative gives factor of $1/\bar{z}$, clips 0 from right
- Until "1" is on left [or right], then get $1/(1-z)$ * $\mathcal{L}_{0,\ldots,0}$

The Φ space

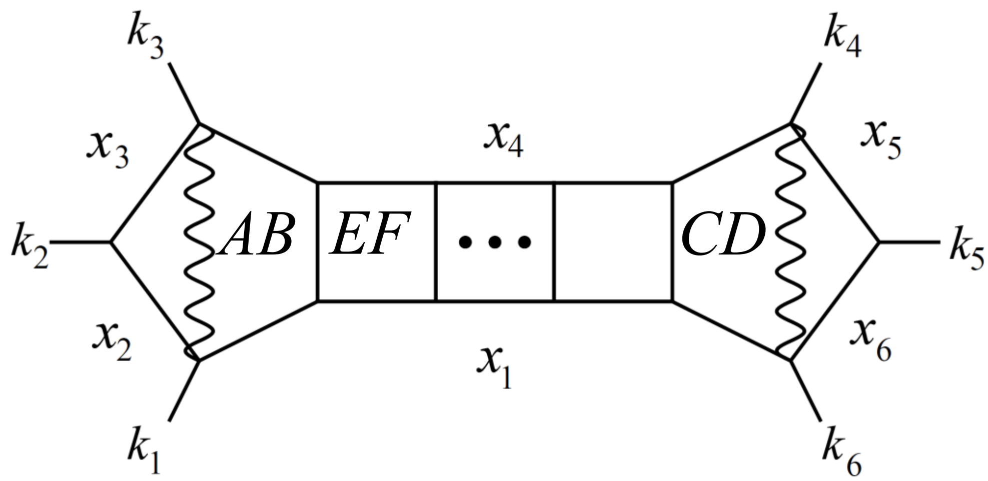
- Dimension of the space of iterated $\{n,1,\ldots,1\}$ coproducts from weight n from 0 to $2L$ at fixed loop order L:
  $1,2,3,\ldots,L,L+1,L,\ldots,3,2,1$
- Completion ($L \to \infty$ limit) has dimension $n+1$ at weight n:
  $1,2,3,4,5,\ldots$
- This is just the subspace of depth ≤1 SVMPLs:

$$\mathcal{L}_{0,\ldots,0} \qquad \mathcal{L}_{1,0\ldots,0}\,,\ \mathcal{L}_{0,1,0,\ldots,0}\,,\ldots,\ \mathcal{L}_{0,\ldots,0,1}$$

- Obviously closed under taking coproducts
- Next, describe the analog for the Ω functions

Ω definition

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka, 1008.2958



$$\Omega^{(\mathrm{L})}(u,v,w)$$

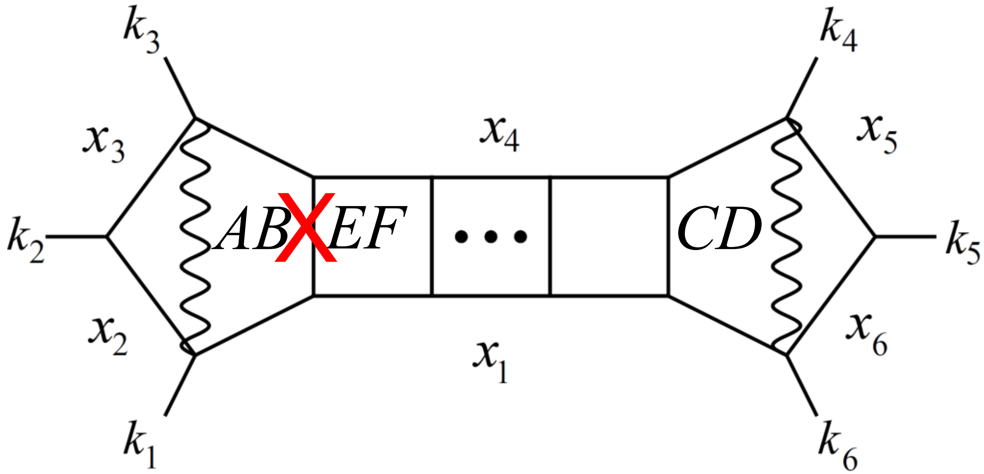
$$\Omega^{(2)} = \int \frac{d^4x_{AB}}{\pi^2}\frac{d^4x_{CD}}{\pi^2}\frac{\langle AB13\rangle\langle CD46\rangle\langle 5612\rangle\langle 2345\rangle\langle 6134\rangle}{(\langle AB61\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle)\langle ABCD\rangle(\langle CD34\rangle\langle CD45\rangle\langle CD56\rangle\langle CD61\rangle)}$$

- For $\Omega^{(\mathrm{L})}$, insert $\displaystyle\int \frac{d^4x_{EF}}{\pi^2}\frac{\langle ABCD\rangle\langle 6134\rangle}{\langle ABEF\rangle\langle EF61\rangle\langle EF34\rangle\langle EFCD\rangle}$, etc.

A differential equation

Drummond, Henn, Trnka, 1010.3679

- Appropriate 2nd order differential operator removes a rung:



$$\Omega^{(L)}(u,v,w)$$

$$\frac{\langle 2345\rangle\langle 3456\rangle}{\langle 1245\rangle} Z_2 \cdot \frac{\partial}{\partial Z_3} \left(\frac{1}{\langle 3456\rangle} Z_1 \cdot \frac{\partial}{\partial Z_2} \Omega^{(L)}(u,v,w) \right) = \Omega^{(L-1)}(u,v,w)$$

- In terms of projectively invariant variables:

$$y_w \partial_{y_w} \left[\sqrt{\Delta}\, \partial_w \Omega^{(L)} \right] = \Omega^{(L-1)}$$

1st order differential equations

LD, Drummond, Henn, 1104.2787

- Split $y_w \partial_{y_w} \left[\sqrt{\Delta} \partial_w \Omega^{(L)} \right] = \Omega^{(L-1)}$ into two 1st order differential eqs. by defining an "odd ladder integral":

$$\sqrt{\Delta} \partial_w \Omega^{(L)} = \mathcal{O}^{(L-1)}, \qquad y_w \partial_{y_w} \mathcal{O}^{(L)} = \Omega^{(L)}$$

- $\Omega^{(L)}$ has weight $2L$, is parity even, flip even (u ←→ v)
- $\mathcal{O}^{(L)}$ has weight $2L+1$, is parity odd, flip even

- We have a basis of all (Steinmann satisfying) hexagon functions through weight 10.
- Easy to solve differential equations within this relatively small function space, through 5 or 6 loops, and examine the functions' coproducts.

Coproduct Relations

- Some empirical, some from structure of differential equations.

$$\Omega^u + \Omega^{1-u} = \Omega^v + \Omega^{1-v} = \Omega^w = \Omega^{1-w} = \Omega^{y_w} = 0$$

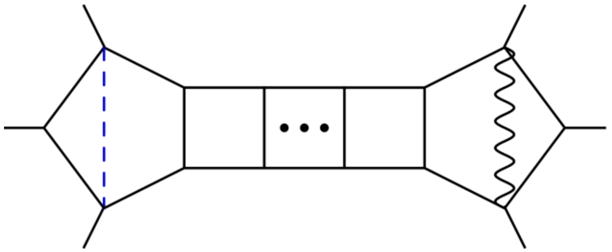
$$[\Omega^{(L)}]^{y_u} = [\Omega^{(L)}]^{y_v} = \frac{1}{2}\mathcal{O}^{(L-1)}$$

$$\mathcal{O}^u = -\mathcal{O}^{1-u} = -\mathcal{O}^v = \mathcal{O}^{1-v} = -\widetilde{\Omega}_{\rm o}$$

$$\mathcal{O}^w = \mathcal{O}^{1-w} = 0$$

$$\mathcal{O}^{y_u} = \mathcal{O}^{y_v} = \Omega + \widetilde{\Omega}_{\rm e}\,, \qquad \mathcal{O}^{y_w} = \Omega$$

where $\widetilde{\Omega} = \widetilde{\Omega}_{\rm e} + \widetilde{\Omega}_{\rm o}$
has an opposite chirality numerator



Dimension of Ω space

weight	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathcal{O}^{(L)}$ coproducts, $L \to \infty$	1	3	5	7	9	11	13	15	17	19	21	23	25	27
P even	1	3	5	6	8	9	11	12	14	15	17	18	20	21
P even, no y	1	3	5	6	6	6	6	6	6	6	6	6	6	6
P even, has y	0	0	0	0	2	3	5	6	8	9	11	12	14	15
P even, has y, flip $+$	0	0	0	0	2	1	4	2	6	3	8	4	10	5
P even, has y, flip $-$	0	0	0	0	0	2	1	4	2	6	3	8	4	10
P odd	0	0	0	1	1	2	2	3	3	4	4	5	5	6
P odd, flip $+$	0	0	0	1	0	2	0	3	0	4	0	5	0	6
P odd, flip $-$	0	0	0	0	1	0	2	0	3	0	4	0	5	0

- This counting, together with final-entry properties, reveals a canonical solution to the integrability conditions.

Trivial no-y functions

- 6 for any weight N. Just made out of HPL's & ln's

$$
\begin{aligned}
\kappa_1^{(N)} &= H_N\left(1-\frac{1}{u}\right), \\
\kappa_2^{(N)} &= H_N\left(1-\frac{1}{v}\right), \\
\kappa_3^{(N)} &= \ln\frac{v}{w}\, H_{N-1}\left(1-\frac{1}{u}\right) + \sum_{i=1}^{N-2} H_{i,N-i}\left(1-\frac{1}{u}\right) \\
\kappa_4^{(N)} &= \ln\frac{u}{w}\, H_{N-1}\left(1-\frac{1}{v}\right) + \sum_{i=1}^{N-2} H_{i,N-i}\left(1-\frac{1}{v}\right) \\
\kappa_5^{(N)} &= \sum_{i=0}^{\lfloor N/2 \rfloor} \frac{1}{(N-2i)!} \ln^{N-2i}\frac{u}{v}\; \Omega^{(i)}(1,1,w), \\
\kappa_6^{(N)} &= -(1-w)\frac{\partial}{\partial w}\kappa_5^{(N+1)}.
\end{aligned}
$$

see below for definition

P-odd functions

- Change letters from

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

to

$$\mathcal{S}' = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}$$

where

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad \& \text{ cyclic}$$

- New letters much better for exposing Steinmann properties,…

- P-odd functions $\tilde{o}_i^{(N)}$ have only 4 final entries:

$$\frac{m_u}{m_v}, y_u, y_v, y_w$$

P-odd functions (cont.)

- P-odd functions $\tilde{o}_i^{(N)}$ have only 4 final entries:

$$\frac{m_u}{m_v}, y_u, y_v, y_w$$

- For N even, the number of P-odd functions is the same as one weight lower. They are connected by the unique P-even, flip-odd letter in the set of final entries: $\frac{m_u}{m_v}$
- So we require:

$$d\tilde{o}_i^{(N)} = \frac{1}{2} d\ln(m_u/m_v)\ \tilde{o}_i^{(N-1)} + \ldots$$

- For N odd, we use y_u/y_v to "line up" one extra function:

$$\begin{aligned} d\tilde{o}_i^{(N)} &= \frac{1}{2}\, d\ln(m_u/m_v)\ \tilde{o}_i^{(N-1)} + d\ln(y_u/y_v) \times 0 + \ldots, \qquad i = 1, 2, \ldots, n_o - 1 \\ d\tilde{o}_{n_o}^{(N)} &= d\ln(m_u/m_v) \times 0 + d\ln(y_u/y_v)\ \kappa_6^{(N-1)} + \ldots, \quad N \text{ odd} \end{aligned}$$

P-even functions

- Put the 6 no-y functions first: $o_i^{(N)} = \kappa_i^{(N)}, \quad i = 1, 2, \ldots, 6$
- Then the "big" set, which numbers: $n_e^b(N) \;=\; 2 \times \left\lfloor \frac{N-2}{2} \right\rfloor$
- Finally the "small set", which numbers: $n_e^s(N) \;=\; \left\lfloor \frac{N-3}{2} \right\rfloor$
- The number of "big" P-even functions is twice the number of P-odd ones at 1 weight lower. There are two P-odd letters to connect them: $y_u y_v$ and y_w
- Require: $$do_{6+i}^{(N)} \;=\; d\ln(y_u y_v)\; \tilde{o}_i^{(N-1)} + d\ln y_w \times 0 \;+\; \ldots, \qquad i = 1, 2, \ldots, n_e^b/2$$ $$do_{6+n_e^b/2+i}^{(N)} \;=\; d\ln(y_u y_v) \times 0 + d\ln y_w\; \tilde{o}_i^{(N-1)} \;+\; \ldots, \qquad i = 1, 2, \ldots, n_e^b/2$$
- Plus other conditions to get a unique solution
- "Small" set: $$do_{6+n_e^b+i}^{(N)} = d\ln c\; o_{6+i}^{(N-1)} + \sum_{i=1}^{3} d\ln y_i \times 0 \;+\; \ldots, \qquad i = 1, 2, \ldots, n_e^s$$

The line $(u,v,w) = (1,1,w)$

- **On this line, all Ω functions become an integer multiple of either ($weight = 2k = L$):**

$$\Omega^{(L)}(1,1,w) = (-1)^{L-1} H_{2^L}(1-w) + \sum_{m=1}^{L} (-1)^{L-m}(2-4m)\zeta_{2m} H_{2^{L-m}}(1-w)$$

$$2^L \equiv \underbrace{2,2,\ldots,2}_{L\text{ times}} \equiv \underbrace{0,1,0,1,\ldots,0,1}_{L\text{ times}}$$

- **Or ($weight = 2k+1 = L\text{-}1$):** $$(1-w)\frac{d}{dw}\Omega^{(L)}(1,1,w)$$

The line $(u,v,w) = (1,1,w)$ (cont.)

- This fact fixes the integration constants at $(1,1,1)$ needed for the P-even functions.
- **Also, the integer coefficients of proportionality turn out to describe the "coproduct tables"**
- Key formula is for the P-odd, odd-weight case:

$$\tilde{o}_i^{(2k+1)}(1,1,w) = \tilde{d}_{k,i} \times (1-w)\frac{d}{dw}\Omega^{(2k+2)}(1,1,w)$$

$$\tilde{d}_{k,i} = (-1)^{k-i}\frac{(2k)!}{(2i-1)!(2k+2-2i)!}\mathrm{Ge}_{k+1-i}$$

The Genocchi numbers

- Related to the Bernoulli numbers:

$$\mathrm{Ge}_n = (-1)^{n+1} 2(4^n - 1) B_{2n}$$
$$= 1, 1, 3, 17, 155, 2073, 38227, 929569, \ldots$$

- Generating function:

$$t \tan(t/2) = \sum_{n=1}^{\infty} \frac{\mathrm{Ge}_n}{(2n)!} t^{2n}$$

- Recursive definition:

$$\mathrm{Ge}_n = -\sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j \frac{n!}{(2j)!(n-2j)!} \mathrm{Ge}_{n-j}$$

The line $(u,v,w) = (1,1,w)$ (cont.)

- P-odd, even-weight functions all odd under $u \leftrightarrow v$
- So they vanish on $(1,1,w)$:

$$\tilde{o}_i^{(2k)}(1,1,w) \;=\; 0$$

- P-even functions can be expressed in terms of $\tilde{d}_{k,i}$
- Even-weight case:

$$o_i^{(2k)}(1,1,w) = d'_{k,i} \times \Omega^{(2k)}(1,1,w)$$

$$d'_{k,i} = \begin{cases} 1, & i = 5 \\ 2\,\tilde{d}_{k-1,i-6}, & 7 \le i \le k+5 \\ -2\,\tilde{d}_{k-1,i-k-5}, & k+6 \le i \le 2k+3 \\ -2\,\tilde{d}_{k-1,k-1} + 1, & i = 2k+4 \\ 0, & \text{otherwise} \end{cases}$$

Finding the Ω's

- Besides the 1st order coproduct relations for Ω, $\mathcal{O}$, and similar ones for $\widetilde{\Omega}_{\rm e}$, $\widetilde{\Omega}_{\rm o}$, there are 2nd order coproduct relations:

$$
\begin{aligned}
X[\Omega] &= 0 \\
X[\widetilde{\Omega}_{\rm e}^{(L)}] &= 2\widetilde{\Omega}_{\rm e}^{(L-1)} \\
X[\widetilde{\Omega}_{\rm o}^{(L)}] &= 2\widetilde{\Omega}_{\rm o}^{(L-1)} \\
X[\mathcal{O}^{(L)}] &= 2\mathcal{O}^{(L-1)}
\end{aligned}
$$

Finding the Ω's (cont.)

- Here

$$
\begin{aligned}
X[F] \;=\; & F^{a,a} + F^{b,b} + F^{c,c} + F^{a,m_u} - F^{m_u,a} - F^{b,m_u} + F^{m_u,b} - F^{c,m_u} + F^{m_u,c} \\
& + F^{b,m_v} - F^{m_v,b} - F^{c,m_v} + F^{m_v,c} - F^{a,m_v} + F^{m_v,a} \\
& - F^{c,m_w} + F^{m_w,c} + F^{a,m_w} - F^{m_w,a} + F^{b,m_w} - F^{m_w,b} \\
& + F^{y_u,y_u} + F^{y_v,y_v} - F^{y_u,y_v} - F^{y_v,y_u} + F^{y_u,y_w} + F^{y_v,y_w} + F^{y_w,y_u} + F^{y_w,y_v} \\
& - 3\, F^{y_w,y_w}
\end{aligned}
$$

- Same operator has very nice action on MHV amplitude → simple combination of MHV and NMHV amplitudes at one lower loop.

LD, von Hippel, 1408.1505

Caron-Huot, LD, von Hippel, McLeod,1609.00669

Finding the Ω's (cont.)

- First and second order coproduct relations suffice to uniquely determine all four integrals, loop order by loop order.
- Use coproduct tables to compute limits onto lines $(1,u,u)$, $(1,u,1)$, check vs. alternative evaluations to high orders.

Results for Integrals

- Let $$c_{L,i} = \frac{(-1)^{L+1}\ (2i-1)\ [2(L-i)]!}{L!\ (L-2i+1)!}$$

$$\Omega^{(L)} = -c_{L,1}\,(o_1^{(2L)} + o_2^{(2L)}) + \frac{1}{2}\sum_{i=1}^{\lfloor L/2 \rfloor} c_{L-1,i}\, o_{6+i}^{(2L)}$$

$$\mathcal{O}^{(L)} = \sum_{i=1}^{\lfloor (L+1)/2 \rfloor} c_{L,i}\, \tilde{o}_i^{(2L+1)}$$

- Similar formulae for $\widetilde{\Omega}_{\rm e}^{(L)}$, $\widetilde{\Omega}_{\rm o}^{(L)}$

Finite coupling

- An isometry of the Ω integral allows the differential equation to be solved at finite coupling in terms of hypergeomtric functions
- Let $\Omega(u,v,w,g^2) \propto r^{i\nu/2} F(x,y,g^2)$

  where
$$
\begin{aligned}
x &= \sqrt{\frac{(1-u)(1-v)}{uvy_u y_v}} = 1 + \frac{1-u-v-w+\sqrt{\Delta}}{2uv} \\
y &= \sqrt{\frac{y_u y_v (1-u)(1-v)}{uv}} = 1 + \frac{1-u-v-w-\sqrt{\Delta}}{2uv} \\
r &= \frac{u(1-v)}{v(1-u)}
\end{aligned}
$$

→
$$
\begin{aligned}
\left[(1-x)(x\partial_x)^2 + \frac{1}{4}(1-x)\nu^2 - xg^2\right] F(x,y,g^2) &= 0 \\
\left[(1-y)(y\partial_y)^2 + \frac{1}{4}(1-y)\nu^2 - yg^2\right] F(x,y,g^2) &= 0
\end{aligned}
$$

Finite coupling (cont.)

$$\rightarrow \quad \Omega(u,v,w,g^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2i} \left(\frac{u(1-v)}{(1-u)v}\right)^{i\nu/2} \frac{F^j_{+\nu}(x)F^j_{+\nu}(y) - F^j_{-\nu}(x)F^j_{-\nu}(y)}{\sinh(\pi\nu)}$$

$$F^j_\nu(x) = \frac{\Gamma(1+\frac{i\nu+j}{2})\Gamma(1+\frac{i\nu-j}{2})}{\Gamma(1+i\nu)} x^{i\nu/2} {}_2F_1\left(\frac{i\nu+j}{2}, \frac{i\nu-j}{2}, 1+i\nu, x\right)$$

$$j(\nu^2) \equiv i\sqrt{\nu^2+4g^2}$$

Zhukowsky variable

- Straightforward to expand around (1,1,1)
- Or sum up in terms of 2dHPLs on $(1,v,w)$ or limits thereof, to compare with the all orders perturbative results

Summary & Outlook

- The double pentaladders, or Ω functions, together with their coproducts, define an interesting subspace of hexagon functions, which we now understand well to all loop orders, as well as at finite coupling.
- Look for more subspaces between Ω and full (Steinmann) hexagon function space (Simon).
- Try to extend to similar 7 & 8 point integrals.