High-energy amplitudes and evolution of Wilson lines

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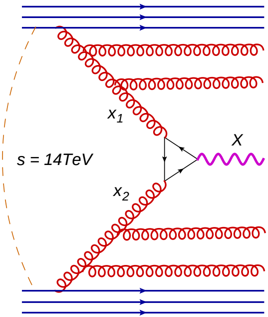
JLAB & ODU

Regge Limit and Friends   Edinburgh, 10 April 2017

Outline

1. High-energy scattering and Wilson lines
   - High-energy scattering and Wilson lines.
   - Light-ray vs Wilson-line operator expansion.
   - Evolution equation for color dipoles.
   - Leading order: BK equation.
2. NLO high-energy amplitudes
   - Conformal composite dipoles and NLO BK kernel in $\mathcal{N} = 4$.
   - NLO amplitude in $\mathcal{N} = 4$ SYM
   - Photon impact factor.
   - NLO BK kernel in QCD.
   - rcBK.
   - Conclusions
3. Two applications:
   - QCD structure constants in the BFKL limit
   - Rapidity evolution of gluon TMDs

BFKL in particle production



Collinear factorization (LLA(Q^2)):

$$\sigma_{pp\to X} = \int_0^1 dx_1 dx_2 D_g(x_1, m_X) D_g(x_2, m_X)\sigma_{gg\to X}$$

sum of the logs $\left(\alpha_s \ln \frac{m_X^2}{m_N^2}\right)^n, \quad \ln \frac{s}{m_X^2} \sim 1$

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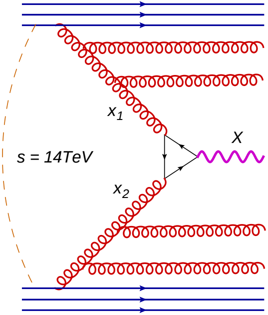

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LLA(x): k_T-factorization

$$\sigma_{pp\to X} = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg\to X}$$

- sum of the logs $\left(\alpha_s \ln x_i\right)^n$, $\ln \frac{m_X^2}{m_N^2} \sim 1$

Much less understood theoretically.

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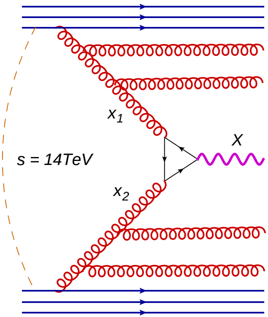

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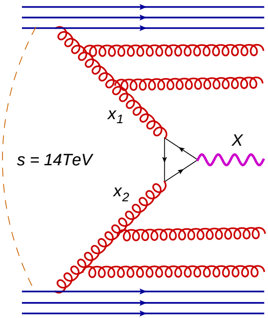
For Higgs production in the central rapidity region $x_{1.2} \sim \frac{m_H}{\sqrt{s}} \simeq 0.01$ and we know from DIS experiments that at such x_B the DGLAP formalism works pretty well ⇒ no need for BFKL resummation

BFKL in particle production



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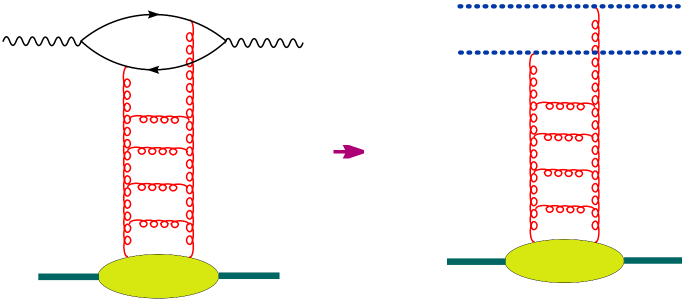
Much less understood theoretically.

For $m_X \sim 10$GeV (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well $\Rightarrow$ some kind of BFKL resummation may help.

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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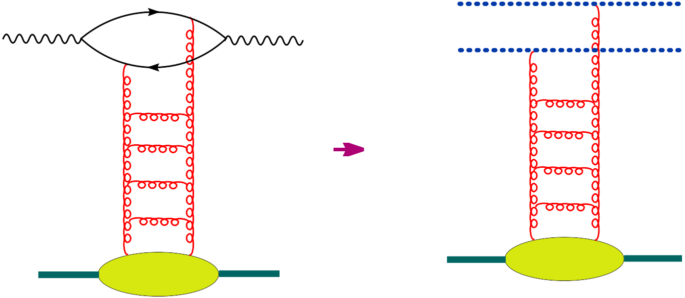
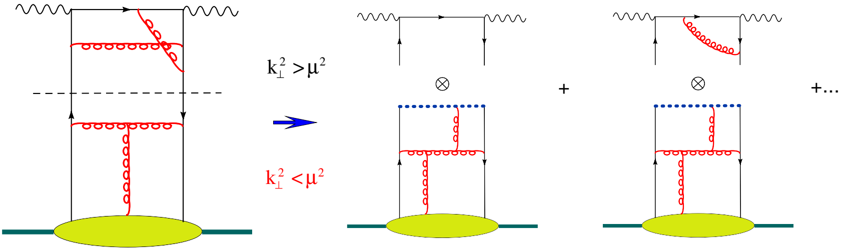


$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

Formally, ➔ means the operator expansion in Wilson lines

Light-cone expansion and DGLAP evolution in the NLO



μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

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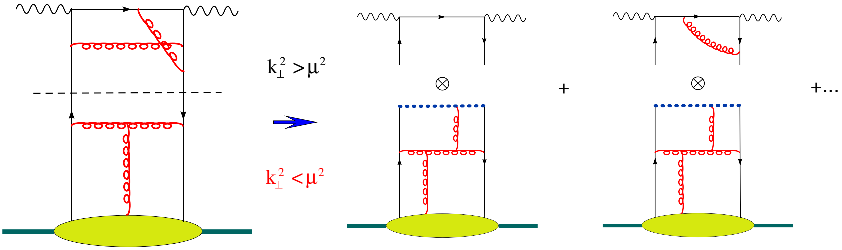

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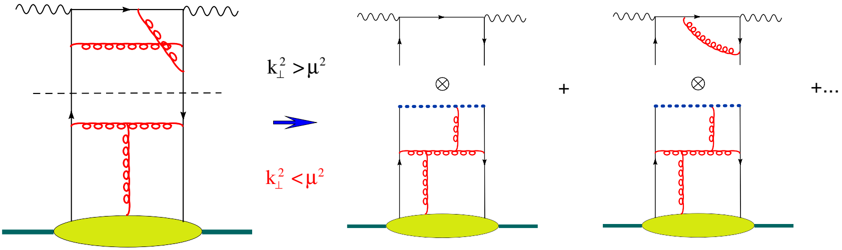
OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(0)\} \ = \ \frac{x_\xi}{2\pi^2 x^4}\Big[1+\frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,0]\psi(0) + O(\frac{1}{x^2})$$

$$[x,y] \ \equiv \ Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

Light-cone expansion and DGLAP evolution in the NLO



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Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \ = \ K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

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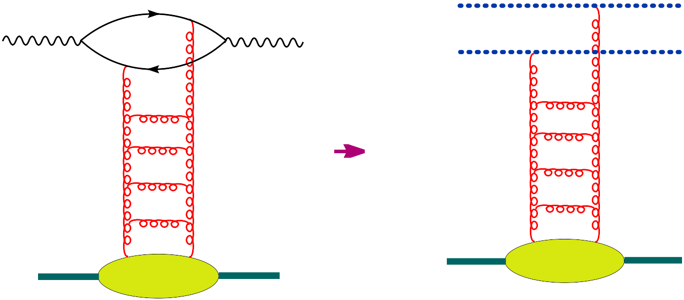

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig \int_{-\infty}^{\infty} du\, n^\mu A_\mu(un + x_\perp)\Big]$$ Wilson line

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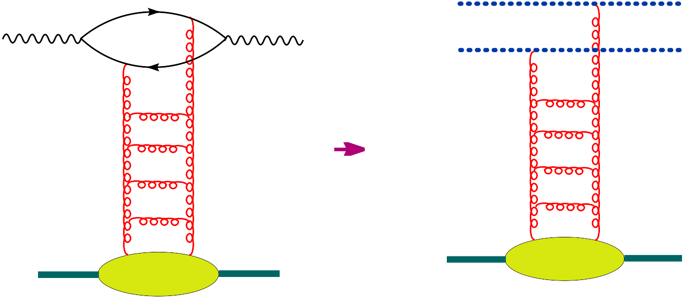

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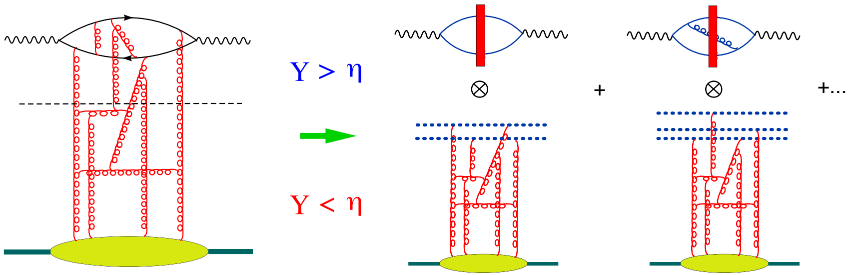
Rapidity factorization



η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")

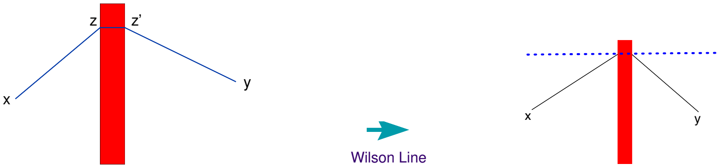
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

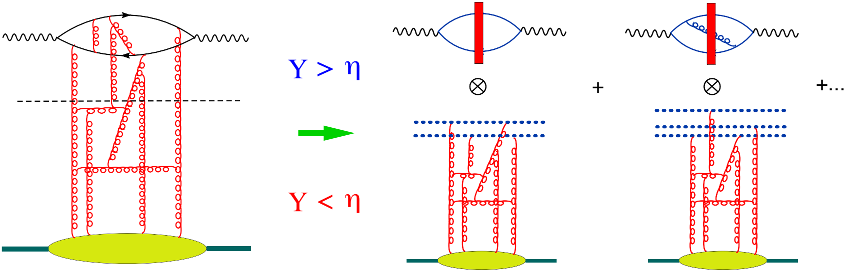
Projectile frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.



[$x \to z$: free propagation]$\times$
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

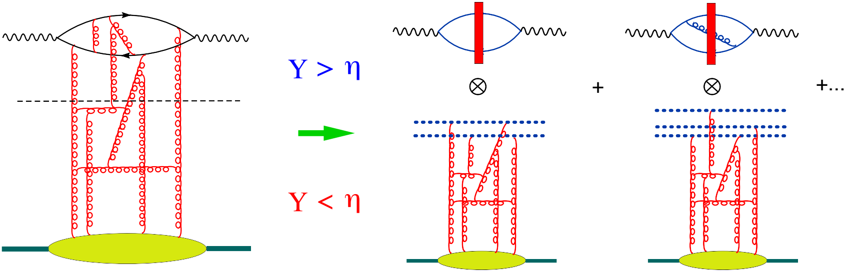
High-energy expansion in color dipoles



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+ \text{ NLO contribution}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

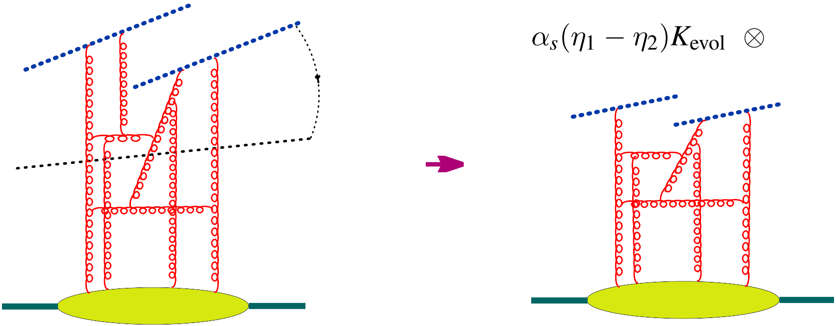
Evolution equation for color dipoles

$$\begin{aligned}\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} &= \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} \\ &- N_c \mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)\end{aligned}$$

(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

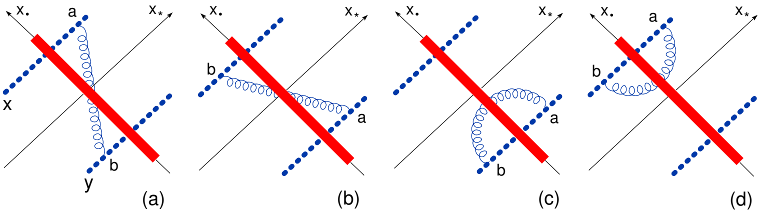
To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \; = \; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$, $\alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To check conformal invariance of the NLO BK equation(in $\mathcal{N}$=4 SYM)
- To determine the argument of the coupling constant of the BK equation(in QCD).
- To get the region of application of the leading order evolution equation.

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; = \; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2$

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$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2z}{\not{z}^4}\frac{(x-y)^2 \;\not{z}^4}{(x-z)^2(z-y)^2}[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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$$[\hat S_0, \hat S_\pm] = \pm \hat S_\pm, \quad \frac{1}{2}[\hat S_+, \hat S_-] = \hat S_0,$$

$$[\hat S_-, \hat U(z,\bar z)] = z^2 \partial_z \hat U(z,\bar z), \quad [\hat S_0, \hat U(z,\bar z)] = z\partial_z \hat U(z,\bar z), \quad [\hat S_+, \hat U(z,\bar z)] = -\partial_z \hat U(z,\bar z)$$

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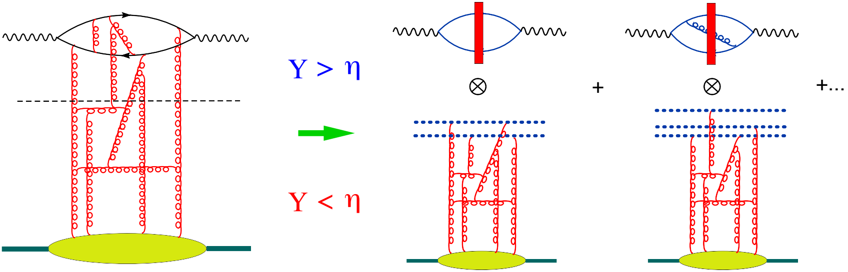
In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



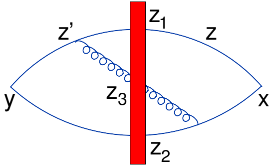
The high-energy operator expansion is $\qquad \mathcal{O} \equiv \mathrm{Tr}\{Z^2\}$

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 \; I^{\mathrm{LO}}(z_1,z_2)\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 \; I^{\mathrm{NLO}}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

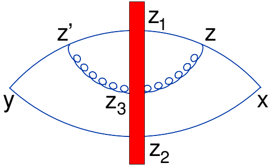
In the leading order - conf. invariant impact factor

$$I_{\mathrm{LO}} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)^2_\perp}{x_+} - \frac{(y-z_i)^2_\perp}{y_+} \qquad CCP, 2007$$

NLO impact factor



(a)



(b)

$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \ = \ -\,I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big]$$

The NLO impact factor is not Möbius invariant $\Leftarrow$ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \ = \ \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\ \frac{\lambda}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2 z_{23}^2}\ +\ O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle + O(\alpha_s^3)$$

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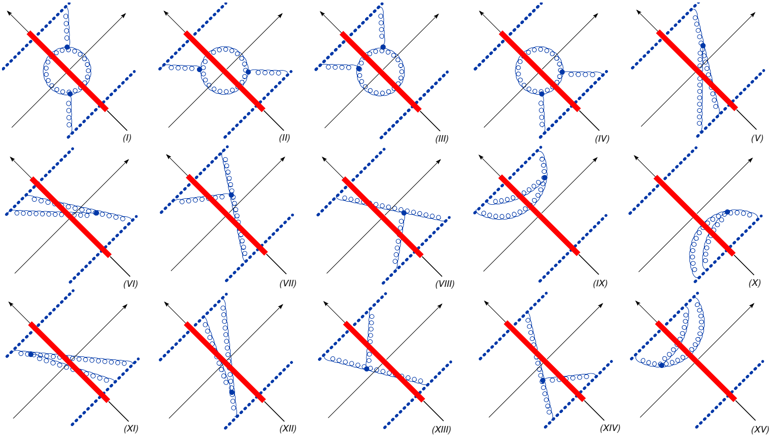
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

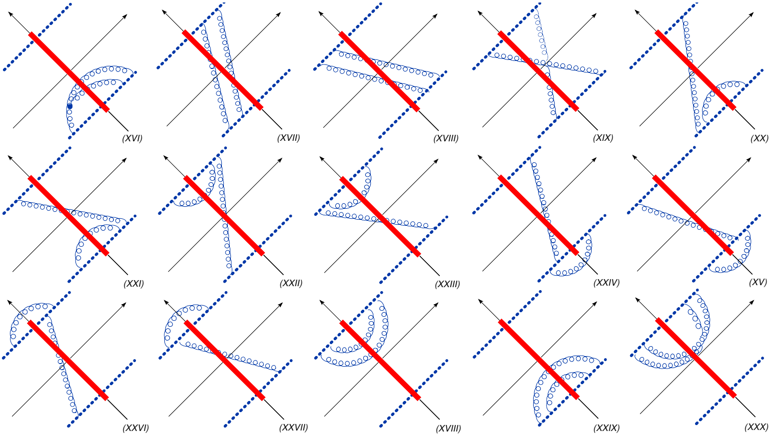
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

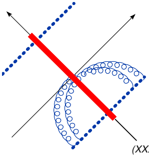
Gluon part of the NLO BK kernel: diagrams



Diagrams for 1→3 dipoles transition



Diagrams for 1→3 dipoles transition



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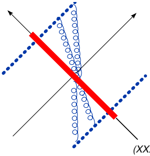

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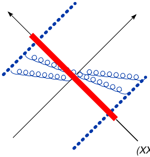

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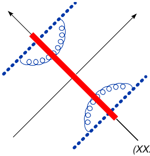
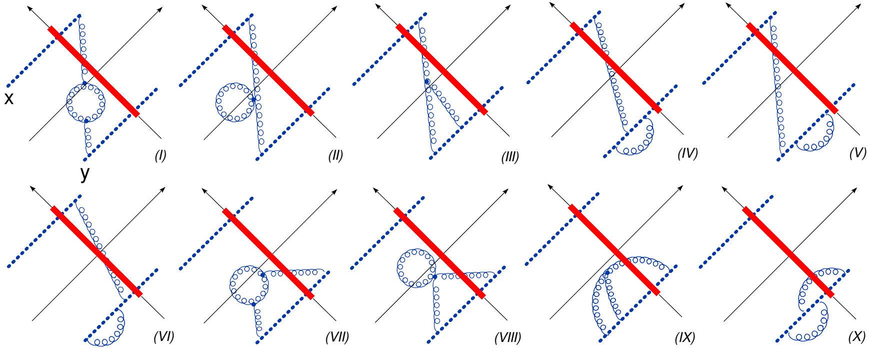
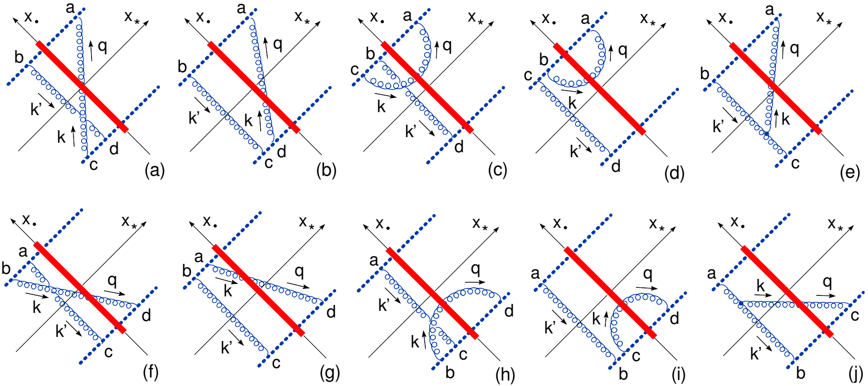


(XXXIV)

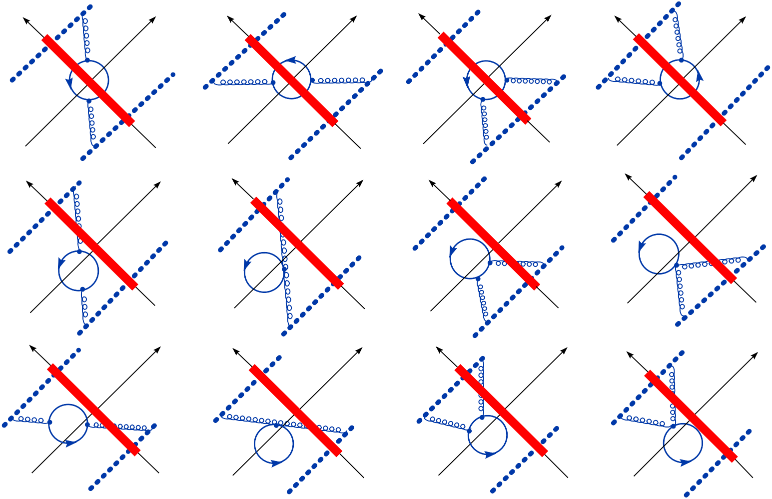
"Running coupling" diagrams



1 → 2 dipole transition diagrams



Gluino and scalar loops



Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times[\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

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Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N}=4$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$
$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$
$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}[(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4})^{bb'}-(z_4\to z_3)]$$

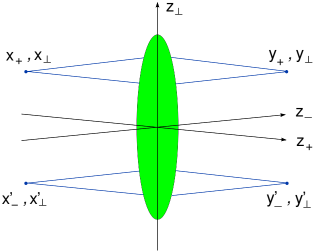
Now Möbius invariant!

Exersize: 4-point correlator in $\mathcal{N}$=4 SYM theory in the Regge limit

Small-x (Regge) limit in the coordinate space

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

Regge limit: $x_+\to\rho x_+,\ x'_+\to\rho x'_+,\ y_-\to\rho' y_-,\ y'_-\to\rho' y-_-\qquad \rho,\rho'\to\infty$



Regge limit symmetry in a conformal theory: 2-dim conformal Möbius group $SL(2,C)$.

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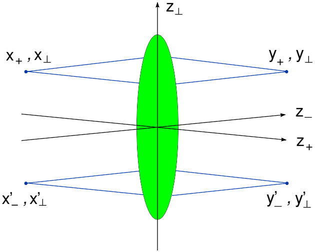

LLA: $\alpha_s \ll 1,\ \alpha_s \ln\rho \sim 1,\ \Rightarrow \sum(\alpha_s \ln\rho)^n \equiv$ BFKL pomeron.
LLA $\Leftrightarrow$ tree diagrams $\Rightarrow$ the BFKL pomeron is Möbius invariant .

NLO LLA: extra α_s: $\sum \alpha_s(\alpha_s \ln\rho)^n \equiv$ NLO BFKL

In conformal theory ($\mathcal{N}=4$ SYM) the NLO BFKL for composite conformal dipole operator is Möbius invariant.

NLO Amplitude in $\mathcal{N}$=4 SYM theory

The pomeron contribution to a 4-point correlation function in $\mathcal{N}=4$ SYM can be represented as $\lambda \equiv g^2 N_c$

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$
$$= \frac{i}{8\pi^2}\int d\nu\, \tilde{f}_+(\nu)\tanh\pi\nu\frac{\sin\nu\rho}{\sinh\rho}F(\nu,\lambda)R^{\frac{1}{2}\omega(\nu,\lambda)}$$

Cornalba(2007)

$\omega(\nu,\lambda)=\frac{\lambda}{\pi}\chi(\nu)+\lambda^2\omega_1(\nu)+...$ is the pomeron intercept,
$\chi(\nu)=2\psi(1)-\psi(\gamma)-\psi(1-\gamma), \quad \gamma\equiv\frac{1}{2}+i\nu$
$\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor.

$F(\nu,\lambda)=F_0(\nu)+\lambda F_1(\nu)+...$ is the "pomeron residue".

R and r are two conformal ratios:

$$R = \frac{(x-x')(y-y')^2}{(x-y)^2(x'-y')^2}, \quad r = R\Big[1-\frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2}+\frac{1}{R}\Big]^2, \quad \cosh\rho=\frac{\sqrt{r}}{2}$$

In the Regge limit $s\to\infty$ the ratio R scales as s while r does not depend on energy.

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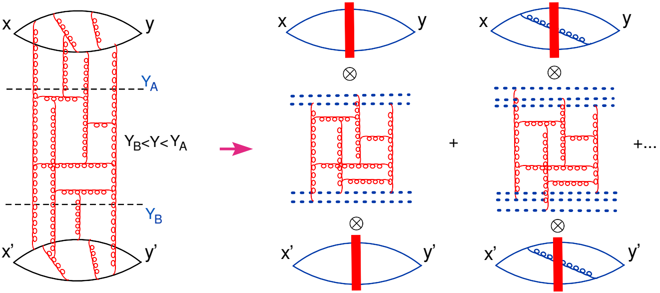
We reproduced $\omega_1(\nu)$ (Lipatov & Kotikov, 2000) and found $F_1(\nu)$

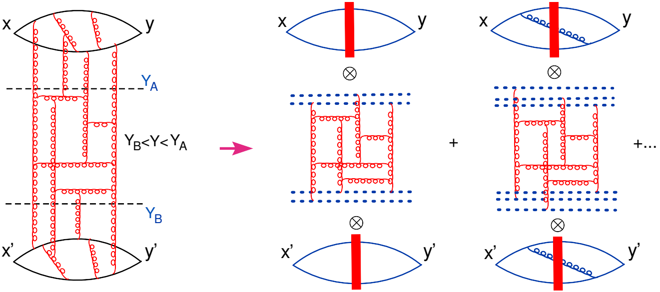
NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy

$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ ($a_0b_0 = R$)



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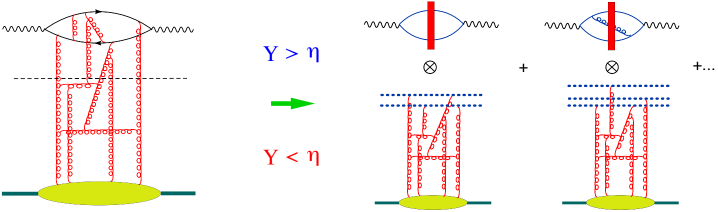
Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_s N_c}{\pi}\left[-\frac{2\pi^2}{\cosh^2\pi\nu}+\frac{\pi^2}{2}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$

In QCD



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\rm LO}_{\mu\nu}(z_1,z_2)\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{1}{\sqrt{s}y^+}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

Composite “conformal” dipole $[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$ - same as in $\mathcal{N}=4$ case.

$$
\begin{aligned}
&(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\}\\
&= \int\frac{d^2z_1d^2z_2}{z_{12}^4}\,\Big\{I^{\mu\nu}_{\rm LO}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\\
&+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)}-2C\Big)I^{\mu\nu}_{\rm LO}+\ I^{\mu\nu}_2\Big]\\
&\qquad\qquad\times\ [\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(I_2)_{\mu\nu}(z_1,z_2,z_3)\ =\ &\frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)}\\
&+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big]\\
&+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]+(\zeta_1\leftrightarrow\zeta_2)\Big\}
\end{aligned}
$$

Photon Impact Factor at NLO G. Chirilli and I.B.

With two-gluon (NLO BFKL) accuracy

$$\frac{1}{N_c}(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \frac{\partial\kappa^\alpha}{\partial x^\mu}\frac{\partial\kappa^\beta}{\partial y^\nu}\int\frac{dz_1dz_2}{z_{12}^4}\,\hat{\mathcal{U}}_{a_0}(z_1,z_2)\big[\mathcal{I}^{\rm LO}_{\alpha\beta}\big(1+\frac{\alpha_s}{\pi}\big)+\mathcal{I}^{\rm NLO}_{\alpha\beta}\big]$$

$$\mathcal{I}^{\alpha\beta}_{\rm LO}(x,y;z_1,z_2) = \mathcal{R}^2\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)-\zeta_1^\alpha\zeta_2^\beta-\zeta_2^\alpha\zeta_1^\beta}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

$$\begin{aligned}
\mathcal{I}^{\alpha\beta}_{\rm NLO}(x,y;z_1,z_2) = \frac{\alpha_sN_c}{4\pi^7}\mathcal{R}^2\Big\{&\frac{\zeta_1^\alpha\zeta_2^\beta+\zeta_1\leftrightarrow\zeta_2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[4{\rm Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+\frac{2\ln\mathcal{R}}{1-\mathcal{R}}+\frac{\ln\mathcal{R}}{\mathcal{R}}\\
&-4\ln\mathcal{R}+\frac{1}{2\mathcal{R}}-2+2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)-4C-\frac{2C}{\mathcal{R}}\Big]\\
&+\Big(\frac{\zeta_1^\alpha\zeta_1^\beta}{(\kappa\cdot\zeta_1)^2}+\zeta_1\leftrightarrow\zeta_2\Big)\Big[\frac{\ln\mathcal{R}}{\mathcal{R}}-\frac{2C}{\mathcal{R}}+2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{1}{2\mathcal{R}}\Big]-\frac{2}{\kappa^2}\Big(g^{\alpha\beta}-2\frac{\kappa^\alpha\kappa^\beta}{\kappa^2}\Big)\\
&+\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]\Big[-2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{\ln\mathcal{R}}{\mathcal{R}}+\ln\mathcal{R}-\frac{3}{2\mathcal{R}}+\frac{5}{2}+2C+\frac{2C}{\mathcal{R}}\Big]\\
&+\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[\frac{2\pi^2}{3}-4{\rm Li}_2(1-\mathcal{R})\\
&-2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}+\frac{1}{2\mathcal{R}^2}-3\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)+6\ln\mathcal{R}-\frac{2}{\mathcal{R}}+2+\frac{3}{2\mathcal{R}^2}\Big]\Big\}
\end{aligned}$$

5 tensor structures (CCP, 2009)

NLO impact factor for DIS

$$I^{\mu\nu}(q,k_\perp) = \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\Big(\frac{k_\perp^2}{Q^2}\Big)^{\frac{1}{2}-i\nu}$$
$$\times\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} + \Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big]$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \qquad P_2^{\mu\nu} = \frac{1}{q^2}\Big(q^\mu - \frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu - \frac{p_2^\nu q^2}{q\cdot p_2}\Big)$$

$$\mathcal{F}_{1(2)}(\nu) = \Phi_{1(2)}(\nu) + \chi_\gamma\Psi(\nu),$$
$$\Psi(\nu) \equiv \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma), \qquad \gamma\equiv\frac{1}{2}+i\nu$$

$$\Phi_1(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2}$$
$$\Phi_2(\nu) = F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma}$$

$$F(\gamma) = \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}$$

NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. Chirilli

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm comp}_a$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{23}}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{23}}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{{z_{23}}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times\,[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{{z_{23}}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-{z_{23}}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{{z_{23}}^2z_{23}^2}\Big]$$

$$\times\,[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b=\frac{11}{3}N_c-\frac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Argument of coupling constant

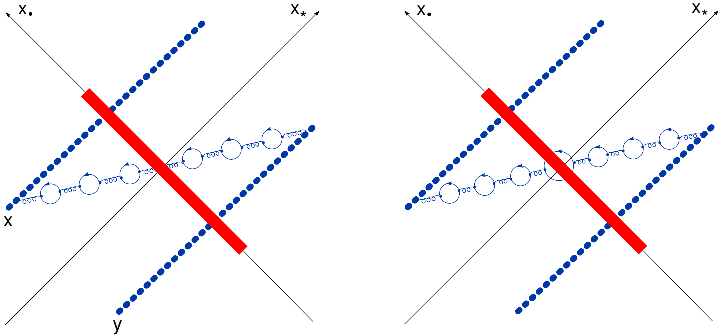
$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$
$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant (rcBK)

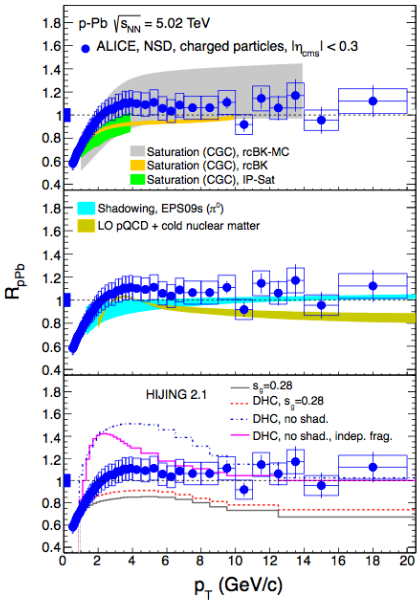
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$
$$\times\Big[\frac{z_{12}^2}{z_{13}^2z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2N^{pPb}_{\rm ch}/d\eta dp_T}{\langle T_{pPb}\rangle d^2\sigma^{\rm pp}_{\rm ch}/d\eta dp_T}$$

N^{pPb} $\equiv$ charged particle yield in p-Pb collisions.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N}=4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel and NLO photon impact factor (in QCD) are calculated.

Two selected applications:

- QCD structure constants in the BFKL limit
- Rapidity evolution of gluon TMDs

Structure constants in the BFKL limit

Consider "forward" leading-twist operators in $\mathcal{N}=4$ SYM

$$\begin{aligned}
\Phi^l_n(x_\perp) &= \int du\ \bar\Phi^a_{AB}\nabla^l_n\phi^{ABa}(un+x_\perp),\\
\Lambda^l_n(x_\perp) &= \int du\ i\bar\lambda^a_A\nabla^{l-1}_n\sigma_n\lambda^a_A(un+x_\perp)\\
F^l(x_\perp) &= \int du\ F^a_{ni}\nabla^{l-2}_n F^{ai}_n(un+x_\perp),
\end{aligned}$$

The renorm-invariant operators are

$$\begin{aligned}
S^l_{1n} &= F^l_n+\frac{1}{4}\Lambda^l_n-\frac{1}{2}\Phi^l_n, \quad S^l_{2n} = F^l_n-\frac{1}{4(l-1)}\Lambda^l_n+\frac{(l+1)}{6(l-1)}\Phi^l_n\\
S^l_{3n} &= F^l_n-\frac{l+2}{2(l-1)}\Lambda^l_n-\frac{(l+1)(l+2)}{2l(l-1)}\Phi^l_n
\end{aligned}$$

and tensor structures of 3-point CFs reduce to one ($x_\perp\cdot n_1=x_\perp\cdot n_2=x_\perp\cdot n_3=0$)

$$\begin{aligned}
&< S^{l_1}_{n_1}(x_{1_\perp})S^{l_2}_{n_2}(x_{2_\perp})S^{k_3}_{n_3}(x_{3_\perp}) >=\\
&= C(g^2,l_i)\frac{(n_1\cdot n_2)^{\frac{l_1+l_2-l_3-1}{2}}(n_1\cdot n_3)^{\frac{l_1+l_3-l_2-1}{2}}(n_2\cdot n_3)^{\frac{l_2+l_3-l_1-1}{2}}}{|x_{12_\perp}|^{(\Delta_1+\Delta_2-\Delta_3-1}|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}
\end{aligned}$$

Structure constants in the BFKL limit

Consider “forward” leading-twist operators in $\mathcal{N} = 4$ SYM

$$\begin{aligned}
\Phi^l_n(x_\perp) &= \int du\ \bar{\Phi}^a_{AB}\nabla^l_n\phi^{ABa}(un + x_\perp),\\
\Lambda^l_n(x_\perp) &= \int du\ i\bar{\lambda}^a_A\nabla^{l-1}_n\sigma_n\lambda^a_A(un + x_\perp)\\
F^l(x_\perp) &= \int du\ F^a_{ni}\nabla^{l-2}_n F^{ai}_n(un + x_\perp),
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S^l_{3n} &= F^l_n - \frac{l+2}{2(l-1)}\Lambda^l_n - \frac{(l+1)(l+2)}{2l(l-1)}\Phi^l_n
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&= C(g^2, l_i)\frac{(n_1 \cdot n_2)^{\frac{l_1+l_2-l_3-1}{2}}(n_1 \cdot n_3)^{\frac{l_1+l_3-l_2-1}{2}}(n_2 \cdot n_3)^{\frac{l_2+l_3-l_1-1}{2}}}{|x_{12_\perp}|^{(\Delta_1+\Delta_2-\Delta_3-1}|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}
\end{aligned}$$

Our aim is to find the structure constants $C(g^2, l_i)$ in the “BFKL limit” $l_i \to 1$

Light-ray operators

Gluon light-ray (LR) operator of twist 2

$$F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp)$$

Forward matrix element - gluon parton density

$$z^\mu z^\nu \langle p|F^a_{\mu\xi}(z)[z,0]^{ab} F^{b\xi}_\nu(0)|p\rangle^\mu \stackrel{z^2=0}{=} 2(pz)^2 \int_0^1 dx_B\ x_B D_g(x_B,\mu) \cos(pz)x_B$$

Evolution equation (in gluodynamics)

$$\mu^2 \frac{d}{d\mu^2} F^a_{-i}(x'_+ + x_\perp)[x'_+, x_+]^{ab} F^{b\ i}_-(x_+ + x_\perp)$$
$$= \int_{x_+}^{x'_+} dz'_+ \int_{x_+}^{z'_+} dz_+\ K(x'_+, x_+; z'_+, z_+; \alpha_s) F^a_{-i}(z'_+ + x_\perp)[z'_+, z_+]^{ab} F^{b\ i}_-(z_+ + x_\perp)$$

"Forward" LR operator

$$F(L_+, x_\perp) = \int dx_+\ F^a_{-i}(L_+ + x_+ + x_\perp)[L_+ + x_+, x_+]^{ab} F^{bi}_-(x_+ + x_\perp)$$

Light-ray operators

Expansion in ("forward") local operators

$$F(L_+,x_\perp) \;=\; \sum_{n=2}^{\infty} \frac{L_+^{n-2}}{(n-2)!}\mathcal{O}_n^g(x_\perp), \qquad \mathcal{O}_n^g \;\equiv\; \int dx_+ F^a_{-i}\nabla_-^{n-2}F_-^{ai}(x_+,x_\perp)$$

Evolution equation for $F(L_+,x_\perp)$

$$\mu\frac{d}{d\mu}F(L_+,x_\perp) \;=\; \int_0^1 du\; K_{gg}(u,\alpha_s)F(uL_+,x_\perp)$$

$$\Rightarrow\quad \gamma_n(\alpha_s) \;=\; -\int_0^1 du\; u^{n-2}K_{gg}(u,\alpha_s) \qquad \mu\frac{d}{d\mu}\mathcal{O}_n^g \;=\; -\gamma_n(\alpha_s)\mathcal{O}_n^g$$

$u^{-1}K_{gg}$ - DGLAP kernel

$$u^{-1}K_{gg}(u) = \frac{2\alpha_s N_c}{\pi}\Big(\bar{u}u + \Big[\frac{1}{\bar{u}u}\Big]_+ - 2 + \frac{11}{12}\delta(\bar{u})\Big) + \text{higher orders in } \alpha_s$$

Light-ray operators

Conformal LR operator ($j = \frac{1}{2} + i\nu$)

$$F^\mu(L_+, x_\perp) \;=\; \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} (L_+)^{-\frac{3}{2}+i\nu} \mathcal{F}^\mu_{\frac{1}{2}+i\nu}(x_\perp)$$

$$F^\mu_j(x_\perp) \;=\; \int_0^\infty dL_+ \, L_+^{1-j} F^\mu(L_+, x_\perp)$$

Evolution equation for “forward” conformal light-ray operators

$$\Rightarrow \; \mu^2 \frac{d}{d\mu^2} F_j(z_\perp) \;=\; \int_0^1 du \, K_{gg}(u, \alpha_s) u^{j-2} F_j(z_\perp)$$

$\Rightarrow \gamma_j(\alpha_s)$ is an analytical continuation of $\gamma_n(\alpha_s)$

Supermultiplet of LR operators

Since LR operators are “analytic continuation” of local operators, we expect ($j_1 = \frac{3}{2} + i\nu_1, j_2 = \frac{3}{2} + i\nu_2$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp}) S^{j_2}_{n_2}(x_{2_\perp})\rangle \;=\; \delta(\nu_1 - \nu_2) f(\alpha_s, j) \frac{(n_1 \cdot n_2)^{j_1} (\mu^2)^{-\gamma(j_1,\alpha_s)}}{|x_{12_\perp}|^{\Delta(\alpha_s, j_1)}}$$

for 2-point CF and similarly ($j_i \equiv 1 + \omega_i$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\; S^{j_2}_{n_2}(x_{2_\perp})\; S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{C(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 - \omega_3)(\omega_1 + \omega_3 - \omega_2)(\omega_2 + \omega_3 - \omega_1)}$$
$$\times \frac{(n_1 \cdot n_2)^{\frac{\omega_1 + \omega_2 - \omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1 + \Delta_2 - \Delta_3 - 1}} \frac{(n_1 \cdot n_3)^{\frac{\omega_1 + \omega_3 - \omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1 + \Delta_3 - \Delta_2 - 1}} \frac{(n_2 \cdot n_3)^{\frac{\omega_2 + \omega_3 - \omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2 + \Delta_3 - \Delta_1 - 1}}$$

for the 3-point CF.

Supermultiplet of LR operators

Since LR operators are "analytic continuation" of local operators, we expect $(j_1 = \frac{3}{2} + i\nu_1, j_2 = \frac{3}{2} + i\nu_2)$

$$\langle S^{j_1}_{n_1}(x_{1_\perp}) S^{j_2}_{n_2}(x_{2_\perp})\rangle = \delta(\nu_1 - \nu_2) f(\alpha_s, j) \frac{(n_1 \cdot n_2)^{j_1} (\mu^2)^{-\gamma(j_1,\alpha_s)}}{|x_{12_\perp}|^{\Delta(\alpha_s, j_1)}}$$

for 2-point CF and similarly $(j_i \equiv 1 + \omega_i)$

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\, S^{j_2}_{n_2}(x_{2_\perp})\, S^{j_2}_{n_3}(x_{3_\perp})\rangle = \frac{C(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 - \omega_3)(\omega_1 + \omega_3 - \omega_2)(\omega_2 + \omega_3 - \omega_1)}$$
$$\times \frac{(n_1 \cdot n_2)^{\frac{\omega_1 + \omega_2 - \omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1 + \Delta_2 - \Delta_3 - 1}} \frac{(n_1 \cdot n_3)^{\frac{\omega_1 + \omega_3 - \omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1 + \Delta_3 - \Delta_2 - 1}} \frac{(n_2 \cdot n_3)^{\frac{\omega_2 + \omega_3 - \omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2 + \Delta_3 - \Delta_1 - 1}}$$

for the 3-point CF.

Our aim is to calculate $f(\alpha_s, j)$ and $C(\alpha_s, j_1, j_2, j_3)$ at $j_i = 1 + \omega_i$ in the "BFKL limit" $g^2 \to 0$, $\omega \to 0$, and $\frac{g^2}{\omega}$ = fixed

Supermultiplet of LR operators

Since LR operators are "analytic continuation" of local operators, we expect $(j_1 = \frac{3}{2} + i\nu_1, j_2 = \frac{3}{2} + i\nu_2)$

$$\langle S^{j_1}_{n_1}(x_{1_\perp}) S^{j_2}_{n_2}(x_{2_\perp})\rangle \; = \; \delta(\nu_1 - \nu_2) f(\alpha_s, j) \frac{(n_1 \cdot n_2)^{j_1} (\mu^2)^{-\gamma(j_1,\alpha_s)}}{|x_{12_\perp}|^{\Delta(\alpha_s, j_1)}}$$

for 2-point CF and similarly ($j_i \equiv 1 + \omega_i$)

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\; S^{j_2}_{n_2}(x_{2_\perp})\; S^{j_2}_{n_3}(x_{3_\perp})\rangle \; = \; \frac{C(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1 + \omega_2 - \omega_3)(\omega_1 + \omega_3 - \omega_2)(\omega_2 + \omega_3 - \omega_1)}$$
$$\times \; \frac{(n_1 \cdot n_2)^{\frac{\omega_1 + \omega_2 - \omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1 + \Delta_2 - \Delta_3 - 1}} \frac{(n_1 \cdot n_3)^{\frac{\omega_1 + \omega_3 - \omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1 + \Delta_3 - \Delta_2 - 1}} \frac{(n_2 \cdot n_3)^{\frac{\omega_2 + \omega_3 - \omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2 + \Delta_3 - \Delta_1 - 1}}$$

for the 3-point CF.

Our aim is to calculate $f(\alpha_s, j)$ and $C(\alpha_s, j_1, j_2, j_3)$ at $j_i = 1 + \omega_i$ in the "BFKL limit" $g^2 \to 0$, $\omega \to 0$, and $\frac{g^2}{\omega}$ = fixed

BK equation for evolution of color dipoles $\Rightarrow$
$C(\alpha_s, 1 + \omega_1, 1 + \omega_2, 1 + \omega_3)$ at $\omega_i \to 0$ and $\omega_1 = \omega_2 + \omega_3$

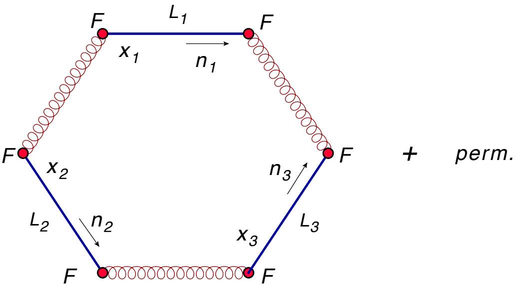
Warm-up exercise: LO

Since LR operators are “analytic continuation” of local operators, we expect

$$\langle S^{j_1}_{n_1}(x_{1_\perp})\, S^{j_2}_{n_2}(x_{2_\perp})\, S^{j_2}_{n_3}(x_{3_\perp})\rangle \;=\; \frac{C(\alpha_s,\omega_1,\omega_2,\omega_3)}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times\;\frac{(n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3-1}}\,\frac{(n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}}\,\frac{(n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}$$

$\Delta \;=\; j+\gamma(j)$ - dimension

Warm-up exercise: LO



Warm-up exercise: LO

$$\langle \mathcal{S}_{n_1}^{1+\omega_1}(x_{1_\perp})\mathcal{S}_{n_2}^{1+\omega_2}(x_{2_\perp})\mathcal{S}_{n_3}^{1+\omega_3}(z_{3_\perp})\rangle =$$
$$= -\frac{N_c^2-1}{32\pi^6 x_{12}^2x_{13}^2x_{23}^2}\Gamma\Big(\frac{\omega_1+\omega_2-\omega_3}{2}\Big)\Gamma\Big(\frac{\omega_2+\omega_3-\omega_1}{2}\Big)\Gamma\Big(\frac{\omega_1+\omega_3-\omega_2}{2}\Big)$$
$$\Big(\frac{2n_1\cdot n_2}{x_{12}^2}\Big)^{\frac{\omega_1+\omega_2-\omega_3}{2}}\Big(\frac{2n_1\cdot n_3}{x_{13}^2}\Big)^{\frac{\omega_1+\omega_3-\omega_2}{2}}\Big(\frac{2n_2\cdot n_3}{x_{23}^2}\Big)^{\frac{\omega_2+\omega_3-\omega_1}{2}}\Phi(\omega_1,\omega_2,\omega_3)$$

$$\Phi(\omega_1,\omega_2,\omega_3)$$
$$= \prod_i \Gamma(1-\omega_i)\Gamma\Big(\frac{\omega_1+\omega_2-\omega_3}{2}+2\Big)\Gamma\Big(\frac{\omega_2+\omega_3-\omega_1}{2}+2\Big)\Gamma\Big(\frac{\omega_1+\omega_3-\omega_2}{2}+2\Big)$$
$$\times \Big\{\Big(e^{i\pi\omega_3}-1\Big)\Big[e^{i\pi(\omega_1-\omega_2)}+e^{i\pi(\omega_2-\omega_1)}-2e^{-i\pi\omega_3}\Big]+\Big(e^{i\pi\omega_1}-1\Big)\Big[e^{i\pi(\omega_2-\omega_3)}+e^{i\pi(\omega_3-\omega_2)}$$
$$-2e^{-i\pi\omega_1}\Big]+\ \Big(e^{i\pi\omega_2}-1\Big)\Big[e^{i\pi(\omega_3-\omega_1)}+e^{i\pi(\omega_1-\omega_3)}-2e^{-i\pi\omega_2}\Big]$$
$$+e^{i\pi(\omega_1+\omega_2-\omega_3)}+e^{i\pi(\omega_2+\omega_3-\omega_1)}+e^{i\pi(\omega_1+\omega_3-\omega_2)}-e^{i\pi(\omega_1+\omega_2+\omega_3)}-2\Big\}$$

At small ω's

$$\Phi(\omega_1,\omega_2,\omega_3)\ \simeq\ -2\pi^2(\omega_1^2+\omega_2^2+\omega_3^2)-\pi^2(\omega_1^2+\omega_2^2+\omega_3^2-2\omega_1\omega_2-2\omega_1\omega_3-2\omega_2\omega_3)$$

In higher orders one should expect

$$\Phi(\omega_1,\omega_2,\omega_3;g^2)\ \simeq\ \Phi(\omega_1,\omega_2,\omega_3)\Big[1+\sum c_n\big(\frac{g^2}{\omega_i}\big)^n\Big]$$

It could be obtained from the CF of three “Wilson frames” with long sides collinear to n_1, n_2, and n_3 and transverse short sides.

Unfortunately, it means analyzing QCD (or N=4 SYM) in the triple Regge limit which is not studied yet.
Triple Regge limit: scattering of 3 particles moving with speed $\sim$ c in x, y, and z directions.

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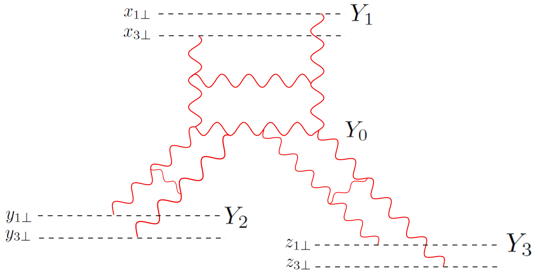
Unfortunately, it means analyzing QCD (or N=4 SYM) in the triple Regge limit which is not studied yet.
Triple Regge limit: scattering of 3 particles moving with speed $\sim$ c in x, y, and z directions.

What we can do in a meantime is to take $n_3\to n_2$ and consider the CF of a Wilson frame in $n_1=n_+$ direction and two Wilson frames in $n_2=n_3=n_-$ directions which can be obtained using the BK evolution.

Using decomposition over Wilson lines we get:

$$\langle S_+^{2+\omega_1}(x_{1\perp},x_{3\perp})S_-^{2+\omega_2}(y_{1\perp},y_{3\perp})S_-^{2+\omega_3}(z_{1\perp},z_{3\perp})\rangle =$$
$$= \mathcal{D}_\perp \int_{-\infty}^{\infty} dx_{1-}\int_{x_{1-}}^{\infty} dx_{3-}x_{31-}^{-2-\omega_1}\int_{-\infty}^{\infty} dy_{1+}\int_{y_{1+}}^{\infty} dy_{3+}y_{31+}^{-2-\omega_2}\int_{-\infty}^{\infty} dz_{1+}\int_{z_{1+}}^{\infty} dz_{3+}z_{31+}^{-2-\omega_3}\times$$
$$\times\langle \mathbf{U}^{\sigma_{1-}}(x_{1\perp},x_{3\perp})\mathbf{V}^{\sigma_{2+}}(y_{1\perp},y_{3\perp})\mathbf{W}^{\sigma_{3+}}(z_{1\perp},z_{3\perp})\rangle,$$

where $\mathcal{D}_\perp = -\frac{N^3}{c(\omega_1)c(\omega_2)c(\omega_3)}(\partial_{x_{1\perp}}\cdot\partial_{x_{3\perp}})(\partial_{y_{1\perp}}\cdot\partial_{y_{3\perp}})(\partial_{z_{1\perp}}\cdot\partial_{z_{3\perp}})$.



- BK equation:

$$\sigma \frac{d}{d\sigma}\mathbf{U}^{\sigma}(z_1, z_2) = \mathcal{K}_{\text{BK}} * \mathbf{U}^{\sigma}(z_1, z_2),$$

where $\mathcal{K}_{\text{BK}}$ in LO approximation:

$$\mathcal{K}_{\text{LO BK}} * \mathbf{U}(z_1, z_2) =$$

$$= \frac{2g^2}{\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\mathbf{U}(z_1, z_3) + \mathbf{U}(z_3, z_2) - \mathbf{U}(z_1, z_2) - \mathbf{U}(z_1, z_3)\mathbf{U}(z_3, z_2)\right].$$

- Schematically calculation of correlation function of 3 dipoles can be wrote as:

$$\int dY_0 (\mathbf{U}^{Y_1} \to \mathbf{U}^{Y_0}) \otimes (\text{BK vertex at } Y_0) \otimes \begin{pmatrix} \langle \mathbf{U}^{Y_0} \mathbf{V}^{Y_2} \rangle \\ \langle \mathbf{U}^{Y_0} \mathbf{W}^{Y_3} \rangle \end{pmatrix}$$

where we introduced rapidity $Y_i = e^{\sigma_i}$

The structure of 3-point correlator (in 2d - $\perp$ space)

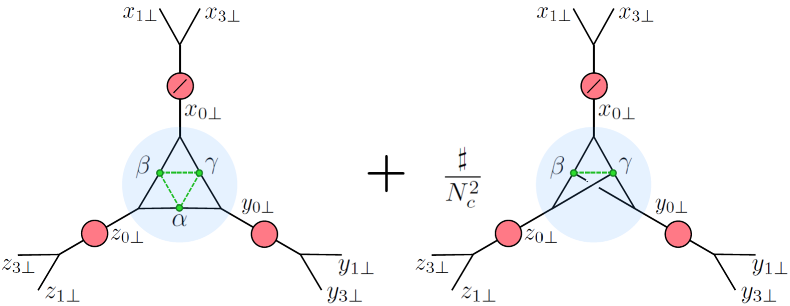


Figure : The structure of 3-point correlator. Red circles correspond to BFKL propagators (the crossed one has extra multiplier $(\frac{1}{4} + \nu_1^2)^2$). The blue blob corresponds to the 3-point functions of 2-dimensional BFKL CFT. The triple veritces correspond to E-functions. The $\alpha\beta\gamma$-triangle in the first, planar, term and $\beta\gamma$-link in the second, nonplanar, term correspond to triple pomeron vertex.

Result:

$$\langle \mathcal{S}^{1+\omega_1}_{n_1}(x_{1\perp},x_{3\perp})\mathcal{S}^{1+\omega_2}_{n_2}(y_{1\perp},y_{3\perp})\mathcal{S}^{1+\omega_3}_{n_2}(z_{1\perp},z_{3\perp})\rangle =$$

$$= -ig^{10}\frac{\delta(\omega_1-\omega_2-\omega_3)}{c(\omega_1)c(\omega_2)c(\omega_3)}H\frac{\Psi(\nu_1^*,\nu_2^*,\nu_3^*)|x_{13}|^{\gamma_1}|y_{13}|^{\gamma_2}|z_{13}|^{\gamma_3}}{|x-y|^{2+\gamma_1+\gamma_2-\gamma_3}|x-z|^{2+\gamma_1+\gamma_3-\gamma_2}|y-z|^{2+\gamma_2+\gamma_3-\gamma_1}}$$

where ν_i^* is a solution of BFKL equation for anomalous dimensions $\omega_i = \aleph(\nu_i^*)$

$$H = \frac{2^{10}(N_c^2-1)^2}{\pi^2 N_c^5}\gamma_1^2(2+\gamma_1)^4(2+\gamma_2)^2(2+\gamma_3)^2\frac{G(\nu_1^*)}{\aleph'(\nu_1^*)}\frac{G(\nu_2^*)}{\aleph'(\nu_2^*)}\frac{G(\nu_3^*)}{\aleph'(\nu_3^*)},$$

$\gamma_i = \gamma(j_i)$ - anomalous dimension ($j_i = 1+\omega_i$) and

$$G(\nu) = \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2}\frac{\pi\Gamma^2(\frac{1}{2}+i\nu)\Gamma(-2i\nu)}{\Gamma^2(\frac{1}{2}-i\nu)\Gamma(1+2i\nu)},$$

$$\Psi(\nu_1,\nu_2,\nu_3) = \Omega(h_1,h_2,h_3) - \frac{2\pi}{N_c^2}\Lambda(h_1,h_2,h_3)\mathrm{Re}(\psi(1)-\psi(h_1)-\psi(h_2)-\psi(h_3)),$$

where $h_i = \frac{1}{2}+i\nu_i = 1+\frac{\gamma_i}{2}$.

Expression for Ω and Λ was obtained by G.Korchemsky in terms of higher hypergeometric and Meijer G-functions.

To identify the function $\Psi(\nu_1^*, \nu_2^*, \nu_3^*)$ with structure constants of CF of three LR operators we need to consider limit $n_2 \to n_3$ in the formula

$$\langle S_{n_1}^{j_1}(x_{1_\perp})\, S_{n_2}^{j_2}(x_{2_\perp})\, S_{n_3}^{j_2}(x_{3_\perp})\rangle \;=\; \frac{C(\alpha_s, \omega_1, \omega_2, \omega_3)}{(\omega_1+\omega_2-\omega_3)(\omega_1+\omega_3-\omega_2)(\omega_2+\omega_3-\omega_1)}$$
$$\times\; \frac{(n_1\cdot n_2)^{\frac{\omega_1+\omega_2-\omega_3}{2}}}{|x_{12_\perp}|^{\Delta_1+\Delta_2-\Delta_3-1}}\frac{(n_1\cdot n_3)^{\frac{\omega_1+\omega_3-\omega_2}{2}}}{|x_{13_\perp}|^{\Delta_1+\Delta_3-\Delta_2-1}}\frac{(n_2\cdot n_3)^{\frac{\omega_2+\omega_3-\omega_1}{2}}}{|x_{23_\perp}|^{\Delta_2+\Delta_3-\Delta_1-1}}$$

The limit $n_2 \to n_3$ is tricky:
in the limit $n_2 \to n_3$ we get a “zero mode” coming from boost invariance at $n_2 = n_3$

$$\frac{1}{\omega_1-\omega_2-\omega_3}\Big(\frac{(n_2,n_3)}{s}\Big)^{\frac{\omega_2+\omega_3-\omega_1}{2}} \;\stackrel{n_2\to n_3}{\to}\; \int d\xi\, e^{-\xi(\omega_1-\omega_2-\omega_3)} \;=\; \delta(\omega_1-\omega_2-\omega_3)$$

Finally for normalized structure constant $c_{\omega_1,\omega_2,\omega_3} = \frac{C_{+--}(\{\Delta_i\},\{1+\omega_i\})}{\sqrt{b_{1+\omega_1} b_{1+\omega_2} b_{1+\omega_3}}}$ we get:

$$C_{\omega_1,\omega_2,\omega_3} = i^{3/2} g^4 \frac{2}{\pi^5} \frac{\sqrt{N_c^2-1}}{N_c^2} \gamma_1^2 (2+\gamma_1)^2 \sqrt{\frac{G(\nu_1^*)}{\aleph'(\nu_1^*)} \frac{G(\nu_2^*)}{\aleph'(\nu_2^*)} \frac{G(\nu_3^*)}{\aleph'(\nu_3^*)}} \Psi(\nu_1^*, \nu_2^*, \nu_3^*),$$

where $\omega_i = \aleph(\nu_i^*)$ and

$$G(\nu) = \frac{\nu^2}{(\frac{1}{4}+\nu^2)^2} \frac{\pi \Gamma^2(\frac{1}{2}+i\nu)\Gamma(-2i\nu)}{\Gamma^2(\frac{1}{2}-i\nu)\Gamma(1+2i\nu)},$$

$$\Psi(\nu_1,\nu_2,\nu_3) = \Omega(h_1,h_2,h_3) - \frac{2\pi}{N_c^2}\Lambda(h_1,h_2,h_3)\mathrm{Re}(\psi(1)-\psi(h_1)-\psi(h_2)-\psi(h_3)),$$

with notation $h_i = \frac{1}{2} + i\nu_i = 1 + \frac{\gamma_i}{2}$, $\omega_i = \aleph(\nu_i)$

Conclusions

- We calculated QCD structure constants in the "BFKL limit" $\omega_i \to 0$ at $\omega_1 = \omega_2 + \omega_3$

Outlook

- Triple Regge limit and effective 2+1 theory?

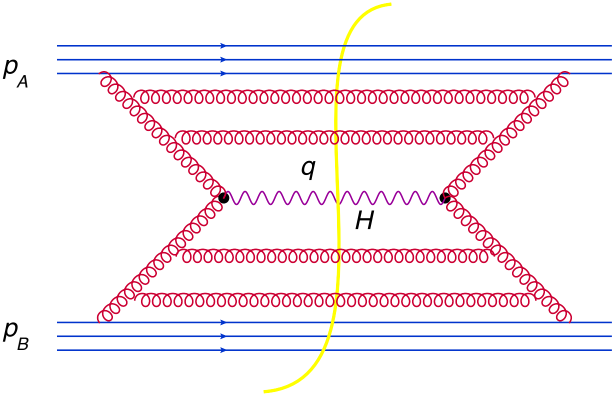
Particle production in pp scattering and evolution of gluon TMDs

Suppose we produce a scalar particle (e.g. Higgs) in a gluon-gluon fusion.

For simplicity, assume the vertex is local:

$$\mathcal{L}_\Phi = g_\Phi \int dz\, \Phi(z) F^2(z), \qquad F^2 \equiv F^a_{\mu\nu} F^{\mu\nu}_a$$

.



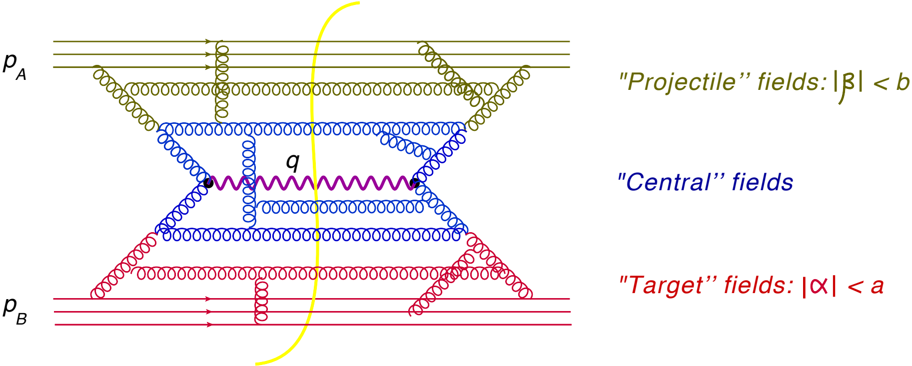
$s >> Q^2 >> Q_\perp^2$

$q^2 = Q^2 = M_H^2$

Rapidity factorization for particle production

Sudakov variables:

$$p \;=\; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A,\;\; p_2 \simeq p_B,\;\; p_1^2 = p_2^2 = 0$$



We integrate over “central” fields in the background of projectile and target fields.

Matrix element between hadron states $\Rightarrow \sum_X = 1$

"Hadronic tensor"

$$\begin{aligned} W(p_A,p_B,q) &\stackrel{\text{def}}{=} \sum_X \int d^4x\, e^{-iqx} \langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle \\ &= \int d^4x\, e^{-iqx} \langle p_A,p_B|F^2(x)F^2(0)|p_A,p_B\rangle \end{aligned}$$

Double functional integral for W

$$\begin{aligned} W(p_A,p_B,q) \;&=\; \sum_X \int d^4x\, e^{-iqx} \langle p_A,p_B|F^2(x)|X\rangle\langle X|F^2(0)|p_A,p_B\rangle \\ = \lim_{\substack{t_f\to\infty \\ t_i\to-\infty}} &\int d^4x\, e^{-iqx} \int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu \int^{\tilde{\psi}(t_f)=\psi(t_f)} D\tilde{\bar{\psi}} D\tilde{\psi} D\bar{\Psi} D\psi\; \Psi^*_{p_A}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i)) \\ \times\; \Psi^*_{p_B}&(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i)) e^{-iS_{\text{QCD}}(\tilde{A},\tilde{\psi})} e^{iS_{\text{QCD}}(A,\psi)} \tilde{F}^2(x) F^2(y) \Psi_{p_A}(\vec{A}(t_i),\psi(t_i)) \Psi_{p_B}(\vec{A}(t_i),\psi(t_i)) \end{aligned}$$

"Left" A,ψ fields correspond to the amplitude $\langle X|F^2(0)|p_A,p_B\rangle$,
"right" fields $\tilde{A},\tilde{\psi}$ correspond to amplitude $\langle p_A,p_B|F^2(x)|X\rangle$
The boundary conditions $\tilde{A}(t_f)=A(t_f)$ and $\tilde{\psi}(t_f)=\psi(t_f)$ reflect the sum over intermediate states X.

TMD factorization with power corrections

In the region $s \gg Q^2 \gg Q_\perp^2$ at the tree level

$$W(p_A,p_B,q) \;=\; \frac{64/s^2}{N_c^2-1}\int d^2x_\perp\; e^{i(q,x)_\perp}\frac{2}{s}\int dx_\bullet dx_*\; e^{-i\alpha_q x_\bullet - i\beta_q x_*}$$

$$\times\;\Big\{\langle p_A|\mathcal{G}_*^{mi}(x_\bullet,x_\perp)\mathcal{G}_*^{mj}(0)|p_A\rangle\langle p_B|\mathcal{F}_{\bullet i}^{n}(x_*,x_\perp)\mathcal{F}_{\bullet j}^{n}(0)|p_B\rangle$$

$$+\;\frac{32}{Q^2}\frac{N_c^2\Delta^{ij,kl}}{(N_c^2-4)(N_c^2-1)}\int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_\bullet\; d^{abc}\langle p_A|\mathcal{G}_{*i}^{a}(x_\bullet,x_\perp)\mathcal{G}_{*j}^{b}(x'_\bullet,x_\perp)\mathcal{G}_{*r}^{c}(0)|p_A\rangle$$

$$\times\int_{-\infty}^{x_\bullet} d\frac{2}{s}x'_*\; d^{mnl}\langle p_B|\mathcal{F}_{\bullet k}^{m}(x_*,x_\perp)\mathcal{F}_{\bullet l}^{n}(x'_*,x_\perp)\mathcal{F}_{\bullet r}^{n}(0)|p_B\rangle\Big\}$$

$$\Delta^{ij,kl} \equiv g^{ij}g^{kl} - g^{ik}g^{jl} - g^{il}g^{jk}$$

$$\mathcal{G}_{*i}^{b}(z_\bullet,z_\perp) \;\equiv\; \big([-\infty_\bullet,z_\bullet]_z^{A_*}\big)^{ab}F_{*i}^{b}(z_\bullet,z_\perp),$$

$$\mathcal{F}_{\bullet i}^{a}(z_*,z_\perp) \;\equiv\; \big([-\infty_*,z_*]_z^{A_\bullet}\big)^{ab}F_{\bullet i}^{b}(z_*,z_\perp)$$

Rapidity evolution: one loop

We study evolution of $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B)\mathcal{F}_j^{a\eta}(y_\perp, x_B)$ with respect to rapidity cutoff η

$$A_\mu^\eta(x) \;=\; \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Matrix element of $\tilde{\mathcal{F}}_i^{a}(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)$ at one-loop accuracy: diagrams in the "external field" of gluons with rapidity $< \eta$.

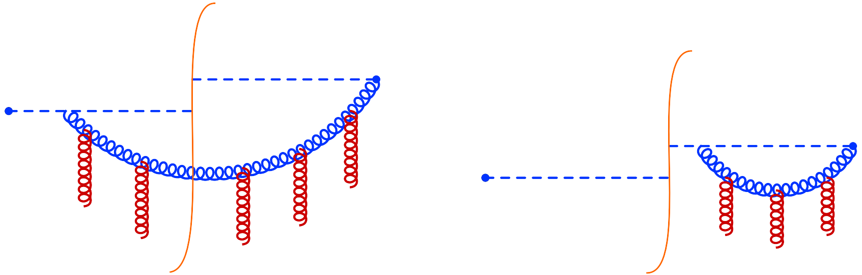
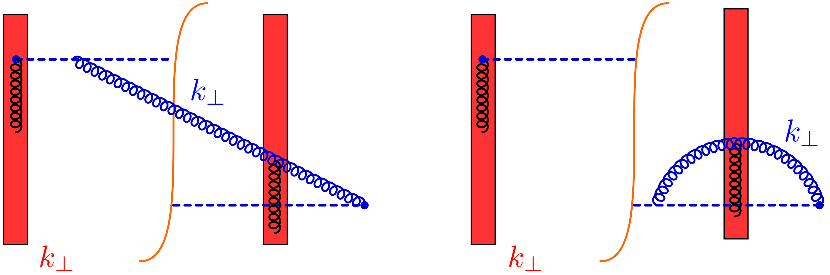


Figure : Typical diagrams for one-loop contributions to the evolution of gluon TMD.

(Fields $\tilde{A}$ to the left of the cut and A to ...)

Shock-wave formalism and transverse momenta

$\alpha \gg \alpha$ and $k_\perp \sim k_\perp$ $\Rightarrow$ shock-wave external field



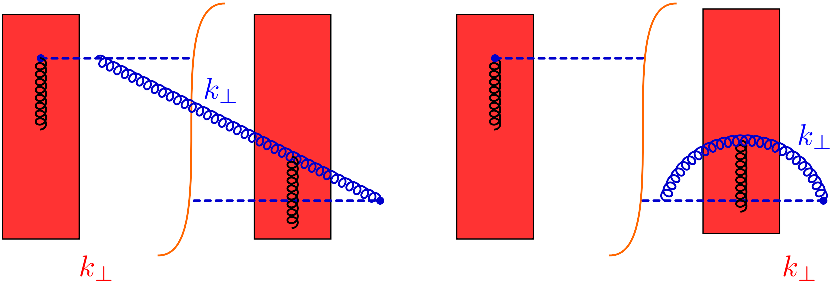
Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow$ $x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow$ $x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \leq k_\perp^2$ $\Rightarrow x_* \gg x_*$
$\Rightarrow$ Diagrams in the shock-wave background at $k_\perp \sim k_\perp$

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp$ $\Rightarrow$ the external field may be wide



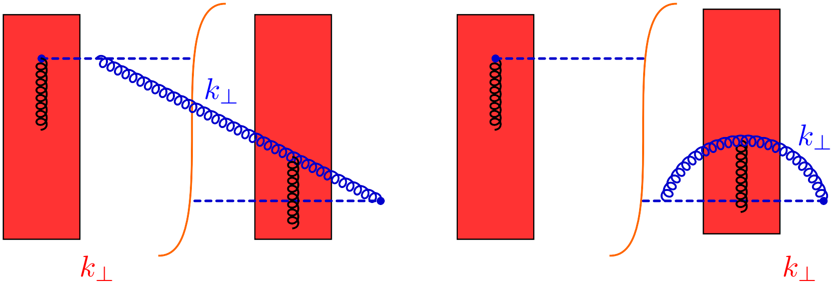
Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta s}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ $\Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid

Problem: different transverse momenta

$\alpha \gg \alpha$ and $k_\perp \gg k_\perp$ $\Rightarrow$ the external field may be wide



Characteristic longitudinal scale of fast fields: $x_* \sim \frac{1}{\beta}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
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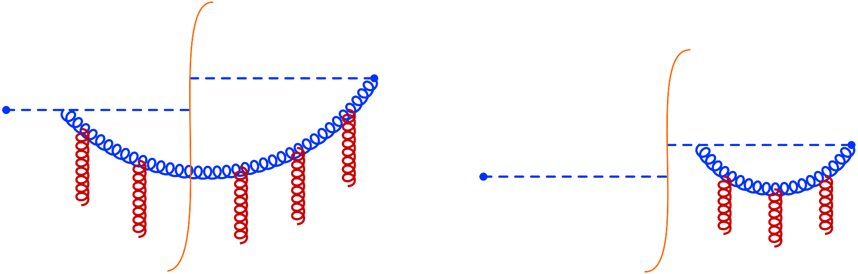
Characteristic longitudinal scale of slow fields: $x_* \sim \frac{1}{\beta s}$, $\beta \sim \frac{k_\perp^2}{\alpha s}$
$\Rightarrow\ x_* \sim \frac{\alpha s}{k_\perp^2}$

If $\alpha \gg \alpha$ and $k_\perp^2 \gg k_\perp^2$ $\Rightarrow x_* \sim x_* \Rightarrow$ shock-wave approximation is invalid

Method of calculation

We calculate one-loop diagrams in the fast-field background



in following way:

if $k_\perp \sim k_\perp \Rightarrow$ propagators in the shock-wave background

if $k_\perp \gg k_\perp \Rightarrow$ light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down "interpolating" formulas correct both at $k_\perp \sim k_\perp$ and $k_\perp \gg k_\perp$

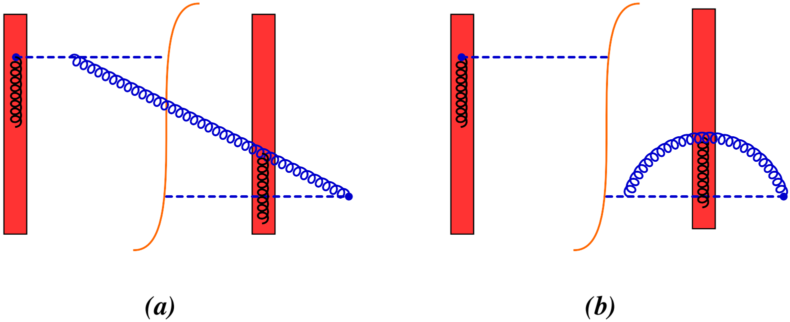


Figure : Typical diagrams for one-loop evolution kernel. The shaded area denotes shock wave of background fast fields.

Shock-wave calculation

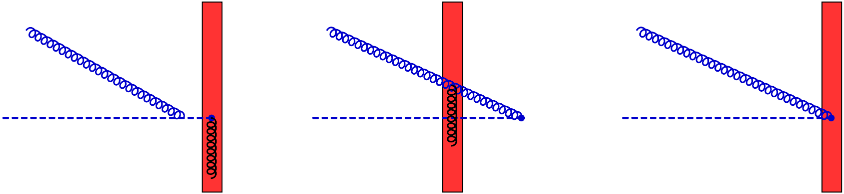
Reminder:

$$\tilde{\mathcal{F}}_i^a(z_\perp, x_B) \equiv \frac{2}{s}\int dz_*\, e^{-ix_B z_*} F^m_{\bullet i}(z_*, z_\perp)[z_*, \infty]_z^{ma}$$

At $x_B \sim 1$ $e^{-ix_B z_*}$ may be important even if shock wave is narrow.
Indeed, $x_* \sim \frac{\alpha s}{k_\perp^2} \ll x_* \sim \frac{\alpha s}{k_\perp^2} \Rightarrow$ shock-wave approximation is OK,
but $x_B\sigma_* \sim x_B\frac{\alpha s}{k_\perp^2} \sim \frac{\alpha s}{k_\perp^2} \geq 1 \Rightarrow$ we need to “look inside” the shock wave.

Technically, we consider small but finite shock wave:
take the external field with the support in the interval $[-\sigma_*, \sigma_*]$ (where $\sigma_* \sim \frac{\alpha s}{k_\perp^2}$), calculate diagrams with points in and out of the shock wave, and check that the σ_*-dependence cancels in the sum of “in” and “out” contributions.

“Lipatov vertex” ($k_\perp$-dependent splitting function)



(a) ***(b)*** ***(c)***

Figure : Lipatov vertex of gluon emission.

Definition

$$L^{ab}_{\mu i}(k, y_\perp, x_B) \;=\; i \lim_{k^2\to 0} k^2 \langle T\{A^a_\mu(k)\mathcal{F}^b_i(y_\perp, x_B)\}\rangle$$

Lipatov vertex at arbitrary momenta

“Interpolating formula” between the shock-wave and light-cone Lipatov vertices

$$L^{ab}_{\mu i}(k,y_\perp,x_B)^{\rm light-like}$$
$$= g(k_\perp|\mathcal{F}^j\Big(x_B+\frac{k_\perp^2}{\alpha s}\Big)\Big\{\frac{\alpha x_B s g_{\mu i}-2k^\perp_\mu k_i}{\alpha x_B s+k_\perp^2}(k_jU+Up_j)\frac{1}{\alpha x_B s+p_\perp^2}U^\dagger$$
$$-2k^\perp_\mu U\frac{g_{ij}}{\alpha x_B s+p_\perp^2}U^\dagger-2g_{\mu j}U\frac{p_i}{\alpha x_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}g_{ij}\Big\}|y_\perp)^{ab}+O(p$$

This formula is actually correct (within our accuracy $\alpha_{\rm fast}\ll\alpha_{\rm slow}$) in the whole range of x_B and transverse momenta

Evoltuion equation for the gluon TMD operator

A. Tarasov and I.B.

$$\frac{d}{d\ln\sigma}\big(\tilde{\mathcal{F}}^a_i(x_\perp,x_B)\mathcal{F}^a_j(y_\perp,x_B)\big)^{\ln\sigma}$$
$$= -\alpha_s\int d\!\!{}^-{}^2k_\perp\ \mathrm{Tr}\{\tilde{L}_i{}^\mu(k,x_\perp,x_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}$$
$$-\ \alpha_s\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(x_\perp,x_B)(y_\perp| -\frac{p^m}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma x_Bs+p_\perp^2}$$
$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|y_\perp)$$
$$+\ (x_\perp|\tilde{U}\frac{1}{\sigma x_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(x_B)\frac{p^m}{p_\perp^2}$$
$$+\ \tilde{\mathcal{F}}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}\ +\ O(\alpha_s^2)$$

This expression is UV and IR convergent.

It describes the rapidity evolution of gluon TMD operator in for any x_B and transverse momenta!

Light-cone limit

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma} = \frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-x_B-\beta)$$
$$\times\Big[\frac{1}{\beta}-\frac{2x_B}{(x_B+\beta)^2}+\frac{x_B^2}{(x_B+\beta)^3}-\frac{x_B^3}{(x_B+\beta)^4}\Big]\langle p|\tilde{\mathcal{F}}_i^n(x_B+\beta,x_\perp)$$
$$\times\mathcal{F}^{ni}(x_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{x_B}{\beta(x_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}$$

In the LLA the cutoff in $\sigma \Leftrightarrow$ cutoff in transverse momenta

$$\langle p|\tilde{\mathcal{F}}_i^n(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{k_\perp^2<\mu^2} = \frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\,\{\text{same}\}$$

$\Rightarrow$ DGLAP equation $\Rightarrow$ $(z'\equiv\frac{x_B}{x_B+\beta})$ DGLAP kernel

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(x_B,0_\perp,\eta) = \frac{\alpha_s}{\pi}N_c\int_{x_B}^1\frac{dz'}{z'}\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{x_B}{z'},0$$

Low-x case: BK evolution of the WW distribution

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$

$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1$

$\Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2) \ =$$

$$-\frac{g^2}{8\pi^3}\mathrm{Tr}\big\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\big\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the BK kernel

This eqn holds true also at small x_B up to $x_B \sim \frac{(x-y)_\perp^{-2}}{s}$ since in the whole range of evolution $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ one can neglect $\sigma x_B s$ in comparison to $p_\perp^2$ in the denominators $(p_\perp^2 + \sigma x_B s) \Leftrightarrow$ effectively $x_B = 0$.

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

One can show that the non-linear terms are power suppressed $\Rightarrow$

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$
$$= 4\alpha_s N_c\int\frac{d^2p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$
$$-\frac{\sigma x_B s}{\sigma x_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow\ \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

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$$-\frac{\sigma x_B s}{\sigma x_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

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$\Rightarrow$ Sudakov double logs

$$\mathcal{D}(x_B,k_\perp,\ln\sigma) \sim \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}\Big(x_B,k_\perp,\ln\frac{k_\perp^2}{s}\Big)$$

Replace

$\infty_* \to -\infty_*$ everywhere

and

$x_B \to -x_B$ in the virtual correction:

$$\frac{d}{d\ln\sigma}\langle p|\big(\mathcal{F}_i^a(x_\perp,x_B)\mathcal{F}_j^a(y_\perp,x_B)\big)^{\ln\sigma}|p\rangle$$

$$= -\alpha_s\int d^2k_\perp\ \langle p|\mathrm{Tr}\{L_i^{\ \mu}(k,x_\perp,x_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,x_B)^{\mathrm{light-like}}\}|p\rangle$$

$$-\ \alpha_s\langle p|\mathrm{Tr}\Big\{\mathcal{F}_i(x_\perp,x_B)(y_\perp|U^\dagger\frac{1}{\sigma x_Bs-p_\perp^2+i\epsilon}U(2\delta_m^k\delta_j^l-g_{jm}g^{kl})(i\partial_l+U_l)\mathcal{F}_k(x_B)$$

$$+\ \mathcal{F}_j(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs-p_\perp^2+i\epsilon)}|y_\perp)$$

$$+\ (x_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_l(x_B)(i\overleftarrow{\partial}_k+U_k)(2\delta_i^k\delta_m^l-g_{im}g^{kl})U^\dagger\frac{1}{\sigma x_Bs-p_\perp^2-i\epsilon}U$$

$$+\ \mathcal{F}_i(x_B)\frac{\sigma x_Bs}{p_\perp^2(\sigma x_Bs-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j\big(y_\perp,x_B\big)\Big\}|p\rangle\ +\ O(\alpha_s^2)$$

Conclusions and outlook

1. Conclusions
   - The evolution equation for gluon TMD at any x_B and transverse momenta.
   - Interpolates between linear DGLAP and Sudakov limits and the non-linear low-x BK regime
2. Outlook
   - Conformal invariance (for N=4 SYM)?
   - Transition between collinear factorization and k_T factorization.

Conclusions and outlook

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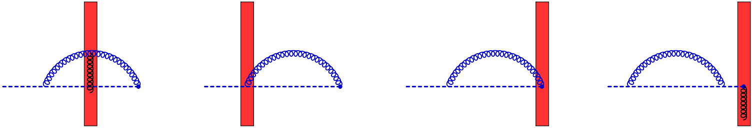
Thank you for attention!

Backup: Virtual corrections and UV cutoff



(a) *(b)* *(c)* *(d)*

Figure : Virtual gluon corrections.

Result of the calculation (in light-like and background-Feynman gauges)

$$\langle \mathcal{F}_i^n(y_\perp, x_B)\rangle^{\text{Fig. 5}} = -ig^2 f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!/\alpha}{\alpha}(y_\perp| - \frac{p^j}{p_\perp^2}\mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l + U_l)$$
$$\times\ (2\delta_j^k\delta_i^l - g_{ij}g^{kl})U\frac{1}{\alpha x_B s + p_\perp^2}U^\dagger + \mathcal{F}_i(x_B)\frac{\alpha x_B s}{p_\perp^2(\alpha x_B s + p_\perp^2)}|y$$

NB: with $\alpha < \sigma$ cutoff there is no UV divergence.

Rapidity *vs* UV cutoff

Regularizing the IR divergence with a small gluon mass m^2 we obtain

$$\int_0^\sigma \frac{d\alpha}{\alpha} \int d^2p_\perp \frac{\alpha x_B s}{(p_\perp^2 + m^2)(\alpha x_B s + p_\perp^2 + m^2)} \simeq \frac{\pi}{2} \ln^2 \frac{\sigma x_B s + m^2}{m^2} \quad (1)$$

Simultaneous regularization of UV and rapidity divergence is a consequence of our specific choice of cutoff in rapidity.

For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff.

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We calculated

$$\int \frac{{d\!\!^-}\alpha\, {d\!\!^-}\beta\, {d\!\!^-}\beta'\, {d\!\!^-}^2 p_\perp}{(\beta - i\epsilon)(\beta' + x_B - i\epsilon)(\alpha\beta s - p_\perp^2 - m^2 + i\epsilon)(\alpha\beta' s - p_\perp^2 - m^2 + i\epsilon)}$$

by taking residues in the integrals over Sudakov variables β and β' and cutting the obtained integral over α from above by the cutofl by $\alpha < \sigma$

Instead, let us take the residue over α:

$$ix_B\int\frac{d\!\!\bar{}^2p_\perp}{m^2+p_\perp^2}\int d\!\!\bar{}\beta d\!\!\bar{}\beta'\frac{\theta(\beta)\theta(-\beta')-\theta(-\beta)\theta(\beta')}{(\beta'+x_B-i\epsilon)(\beta-i\epsilon)(\beta'-\beta)}$$

$$=\int\frac{d\!\!\bar{}^2p_\perp}{m^2+p_\perp^2}\int\frac{d\!\!\bar{}\beta d\!\!\bar{}\beta'}{\beta'+x_B-i\epsilon}\frac{ix_B\theta(\beta)}{(\beta-i\epsilon)(\beta'-\beta+i\epsilon)} = x_B\int\frac{d\!\!\bar{}^2p_\perp}{m^2+p_\perp^2}\int_0^\infty\frac{d\!\!\bar{}}{\beta(\beta\ldots}$$

which is integral (1) with change of variable $\beta=\frac{p_\perp^2}{\alpha s}$.

Rapidity *vs* UV cutoff

Instead, let us take the residue over α:

$$ix_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int d\!\!{}^-\beta d\!\!{}^-\beta'\frac{\theta(\beta)\theta(-\beta')-\theta(-\beta)\theta(\beta')}{(\beta'+x_B-i\epsilon)(\beta-i\epsilon)(\beta'-\beta)}$$

$$=\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int\frac{d\!\!{}^-\beta d\!\!{}^-\beta'}{\beta'+x_B-i\epsilon}\frac{ix_B\theta(\beta)}{(\beta-i\epsilon)(\beta'-\beta+i\epsilon)}\;=\;x_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^\infty\frac{d\!\!{}^-}{\beta(\beta\ \cdots}$$

which is integral (1) with change of variable $\beta=\frac{p_\perp^2}{\alpha s}$.

A conventional way of rewriting this integral in the framework of collinear factorization approach is

$$x_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^\infty\frac{d\beta}{\beta(\beta+x_B)}\;=\;\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^1\frac{dz}{1-z}$$

where $z=\frac{x_B}{x_B+\beta}$ is a fraction of momentum $(x_B+\beta)p_2$ of “incoming gluon” (described by $\mathcal{F}_i$ in our formalism) carried by the emitted “particle” with fraction x_Bp_2.

If we cut the rapidity of the emitted gluon by cutoff in fraction of momentum z, we would still have the UV divergent expression which