JIMWLK evolution - beside and beyond.

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High Energy Scattering is Interesting

The goal is to understand QCD scattering amplitudes (and other observables) in the high energy regime.
Regge: $s \to \infty$, $t/s \to 0$.
Hadronic cross sections grow with energy

$\sigma_{p-p} \sim s^{.08}$

$\frac{d\sigma^{DIS}}{dQ^2} \sim s^{.2-.3}$

Direct calculation of course involves infinite sum of diagrams, since $\alpha_s \ln s/s_0 \sim 1$, so all powers of α_s enhanced by the log of energy have to be re-summed.
The strategy then is similar to Renormalization group: don't do everything at once. Calculate the change in the amplitude under small increase in energy of the collision, and then turn it into evolution equation.

BFKL regime

Within perturbative QCD the growth is described by BFKL equation

$\phi(p_T; x_{Bj})$ - TMD, transverse momentum dependent gluon density

$$\frac{d\phi(p_T)}{dY} = \int_{k_T} K_{BFKL}(p_T, k_T)\phi(k_T)$$

$Y = \ln 1/x_{Bj} \propto \ln s$ - rapidity

BFKL - linear homogeneous equation, with solutions

$$\phi(Y) = e^{\lambda_i Y}\phi_0$$

with the maximal eigenvalue $\lambda = \frac{\alpha_s N}{\pi} \ln 2$

Scattering amplitude $\propto \phi(Y)$

Power growth in energy violates Froissart bound.

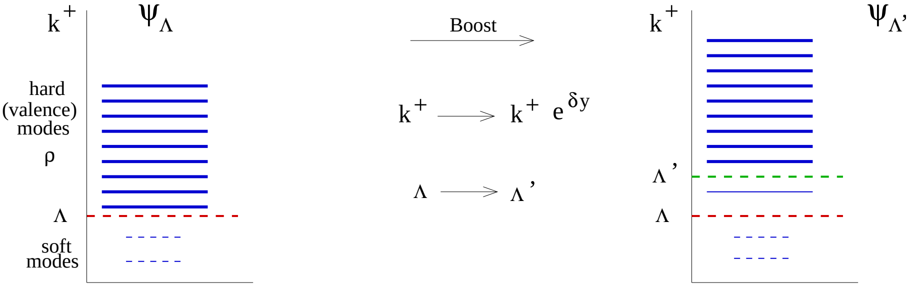
Cross section has to unitarize.

Two distinct effects contribute to the growth of cross section: growth of local gluon density, and growth of transverse size of a hadron.

The maximal BFKL eigenvalue reflects the growth of density.

Growth of transverse size is more subtle - subleading in BFKL, but very stubborn. Cannot be tackled by perturbative methods, and is outside the scope.

Why Does the Gluon Density grow?



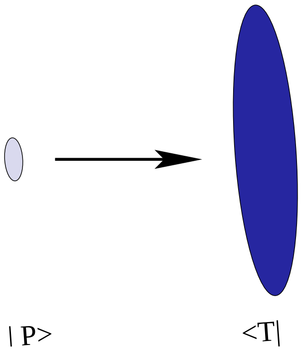
Under Boost Longitudinal Momenta grow.

New Gluons Rise From The "Bottomless Pit" which is the **zero mode**.

Color Field becomes strong because of these extra WEIZSACKER-WILLIAMS gluons.

The “Paradigm”.

Scatter EIKONALLY a “projectile” hadron $|P\rangle$ on a ”target” hadron $|T\rangle$ at high energy



$|P\rangle$ - a distribution of color charge density $\rho^a(x)$.
$|T\rangle$- an ensemble of (possibly strong) color fields $\alpha^a(x)$.

At very high energies evolution is nonlinear.

Rewind: $\phi \propto < \rho^a(x)\rho^a(y) >$
At "low" energies evolution is linear (BFKL): the change in color charge density is proportional to color charge density itself:

$$\delta\rho(x) \propto \rho(x)$$

But the gluon density grows - nonlinear effects become important.

Also **dense objects scatter: cross sections are not proportional to gluon number. Multiple scatterings are important.**

So in principle we have to tackle both nonlinear aspects.

The eikonal S - matrix

Every projectile gluon keeps its transverse position but acquires a color "phase"

$$|x, a\rangle \to S^{ab}(x)|x, b\rangle$$

with

$$S^{ab}(x) = \mathcal{P} \exp\left\{ i \int dx^- \; T^a \, \alpha_T^a(x, x^-) \right\}^{ab} .$$

The forward scattering amplitude of $|P\rangle$:

$$\mathcal{S} \; = \langle \mathrm{IN}|\mathrm{OUT}\rangle = < \langle P|\hat{S}|P\rangle >_T$$

$$= < \int d\rho \; W^P[\rho] \;\; \exp\left\{ i \int \; d^2x \, \rho_P^a(x) \; \alpha_T^a(x) \right\} >_T$$

$W^P[\rho]$ is the probability distribution of the projectile color charge density.

The "Hamiltonian" evolution.

Boost the projectile - more gluons become resolvable by the target ("active").

The S-matrix changes, since $W^P[\rho]$ changes.

The change is due to "materialization" of the soft modes (growth of coherence time of soft fluctuations).

We need to know the "soft gluon" part of the hadronic wave function, to find the change of $S - matrix$.

$$\mathcal{S}_{Y+\Delta Y} = \langle \mathrm{IN}|\mathrm{OUT}\rangle = \langle P_{\rm valence}|\langle P_{\rm soft}|\hat{S}|P_{\rm soft}\rangle|P_{\rm valence}\rangle$$

Here the "Vacuum" of the soft gluons in the presence of "valence" charge density ρ:

$$|P_{\rm soft}\rangle \equiv P_{\rm soft}[a^\dagger_{soft}; \rho(x)]|0_{soft}\rangle$$

The phase space of $|P_{\rm soft}\rangle$ is proportional to ΔY.

Thus we can write

$$\langle P_{\rm soft}|\hat{S}|P_{\rm soft}\rangle = [1-\mathcal{H}[\rho,\delta/\delta\rho]\ \Delta Y + ...]\hat{S}_{valence}$$

Or

$$\mathcal{S}_{Y+\Delta Y} = [1-\mathcal{H}[\rho,\delta/\delta\rho]\ \Delta Y]\mathcal{S}_Y$$

More generally, $\mathcal{H}$ generates a Hamiltonian evolution for any observable which is calculated as average over the probability distribution W:

$$\frac{d}{dY}W^P[\rho] = -\mathcal{H}[\rho,\delta/\delta\rho]\ W^P[\rho]$$

This is the JIMWLK framework - the RG-like evolution equation.

Hamiltonian Assembly Instructions.

1. Calculate the soft gluon wave function at fixed valence color charge density $P_{soft}[a^{a\dagger}(x), \rho^a(x)]$.

2. Eikonally propagate $|P_{soft}\rangle$ through the target fields:

$$|IN\rangle = P_{soft}[a^\dagger(x), \rho(x)] \to |OUT\rangle = P_{soft}[S(x)^{ab} a^{b\dagger}(x), S(x)^{ab}\rho^a(x)]$$

3. Calculate the soft gluon part of the overlap: $\langle IN|OUT\rangle$

4. Expand to first order in ΔY and extract $\mathcal{H}$.

5. If we can do it for arbitrarily dense projectile and arbitrarily dense target, we have solved QCD and we can probably buy a ticket to Stockholm.

The JIWMLK limit.

JIMWLK Hamiltonian: projectile is allowed to be dense $\alpha_s \rho_P(x) \sim 1$, but the target is assumed to be dilute $\alpha_T(x) \sim g$.

Or more conveniently: projectile is assumed to be dilute, $\rho_P(x) \sim 1$, but the target is allowed to be dense $\alpha_T(x) \sim 1/g$.

The eigenfunction P_{soft} is found to leading perturbative order

The eikonal S-matrix on the target field is not expanded in α_s.

The Wave Function.

The soft gluon vacuum is found perturbatively. It is just a cloud of the Weiszacker-Williams gluons hovering around the valence charges:

$$|\Psi\rangle = \left\{ \left[1 - \frac{\delta Y}{2\pi}\int d^2x (b_i^a(x) b_i^a(x))\right] \right.$$

$$\left. +i\int d^2x b_i^a(x) \int_{(1-\delta Y)\Lambda}^{\Lambda} \frac{dk^+}{\pi^{1/2}|k^+|^{1/2}} a_i^{\dagger a}(k^+, x) \right\} |v\rangle$$

$a^\dagger(k^+)$ - the soft gluon creation operator, $b_i(x)$ - “classical field”

$$\partial_i b_i^a + g\epsilon^{abc} b_i^b(x) b_i^c(x) = g\rho^a(x)$$
$$\epsilon_{ij}[\partial_i b_j^a - \partial_j b_i^a + g\epsilon^{abc} b_i^b b_j^c] = 0$$

Perturbatively

$$b_i^a(x) = g\frac{\partial_i}{\partial^2}(x-y)\rho^b(y)$$

JIMWLK Hamiltonian.

We end up with a 2+1 dimensional Euclidean quantum field theory, wit 2 transverse spatial dimensions, and role of time is played by rapidity.

$$H^{JIMWLK} = \frac{\alpha_s}{2\pi^2} \int d^2z\, Q_i^a(z)\, Q_i^a(z)$$

the Hermitian amplitudes $Q_i^a(z)$ are "single inclusive gluon emission amplitude"

$$Q_i^a(z) = \int d^2x \frac{(x-z)_i}{(x-z)^2}[S^{ab}(z) - S^{ab}(x)]J_R^b(x)\,.$$

the generators of color rotation J_R

$$J_R^a(x) = -\mathrm{tr}\left\{S(x)\,T^a \frac{\delta}{\delta S^\dagger(x)}\right\}$$

D. D. D.

How come H depends on S?

$$\mathcal{S} =< \int d\rho\ W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ \alpha^a_T(x)\right\} >_T$$

$$= - \int d\alpha d\rho\, W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ \alpha^a_T(x)\right\} W^T[\alpha]$$

$$\frac{d}{dY}\mathcal{S} - = \int d\alpha d\rho\ \mathcal{H}[\rho,\delta/\delta\rho]W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ \alpha^a_T(x)\right\} W^T[\alpha]$$

$$= - \int d\alpha d\rho\ W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ \alpha^a_T(x)\right\} \mathcal{H}[-i\delta/\delta\alpha, i\alpha]W^T[\alpha]$$

Dense-Dilute Duality: $S = e^{iT^a\alpha^a(x)} \leftrightarrow R \equiv e^{T^a\frac{\delta}{\delta\rho^a(x)}}$

If W^T evolves with $\mathcal{H}[S]$, then W^P evolves with $\mathcal{H}[R]$.

Balitsky hierarchy and JIWMLK evolution.

Like any QFT, Hamiltonian formulation is equivalent to Dyson-Schwinger equations.

E.g. act on a dipole $d(x,y) \equiv \frac{1}{N} tr[S^\dagger(x)S(y)]$

$$\frac{d}{dY} d(x,y) = -H^{JIMWLK} d(x,y)$$

$$= -\frac{\alpha_s N}{\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2\,(y-z)^2}\,[d(x,y) - d(x,z)d(z,y)]$$

Same for other functions of S

Balitsky hierarchy = Dyson-Schwinger equations of the JIMWLK “field theory”.

NEXT TO LEADING ORDER.

NLO JIMWLK $\equiv$ NLO calculation of the Wave Function

M. Lublinsky and Y. Mulian, 2016

But NLO JIMWLK was written in 2013. Mild violation of causality...

A.K., M. Lublinsky and Y. Mulian

S. Caron Huot

A shortcut: Ian Balitsky.

Balitsky-Chirilli calculated NLO evolution of a dipole. Subsequently Grabovsky calculated some elements of the evolution of a "baryon" in $SU(3)$.

It turns out that these results are (almost) enough to write down the full NLO JIMWLK kernel by inspection!

NLO JIMWLK - The General Structure.

For $N = 4$ SUSY (like QCD, but a little bit simpler)

$$
\begin{aligned}
H^{NLO\ JIMWLK} &= \int_{x,y} K_{2,0}(x,y)\,[J_L^a(x)\,J_L^a(y) + J_R^a(x)\,J_R^a(y)] \\
&- 2\int_{x,y,z} K_{2,1}(x,y,z)\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y) \\
&+ \int_{x,y,z,z'} K_{2,2}(x,y;z,z')\left[f^{abc}\,f^{def}\,J_L^a(x)\,S_A^{be}(z)\,S_A^{cf}(z')\,J_R^d(y) - N_c\,J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right] \\
&+ \int_{w,x,y,z,z'} K_{3,2}(w;x,y;z,z')f^{acb}\left[J_L^d(x)\,J_L^e(y)\,S_A^{dc}(z)\,S_A^{eb}(z')\,J_R^a(w)\right. \\
&\quad \left. - J_L^a(w)\,S_A^{cd}(z)\,S_A^{be}(z')\,J_R^d(x)\,J_R^e(y)\right] \\
&+ \int_{w,x,y,z} K_{3,1}(w;x,y;z)\,f^{bde}\left[J_L^d(x)\,J_L^e(y)\,S_A^{ba}(z)\,J_R^a(w) - J_L^a(w)\,S_A^{ab}(z)\,J_R^d(x)\,J_R^e(y)\right] \\
&+ \int_{w,x,y} K_{3,0}(w,x,y)f^{bde}\left[J_L^d(x)\,J_L^e(y)\,J_L^b(w) - J_R^d(x)\,J_R^e(y)\,J_R^b(w)\right].
\end{aligned}
$$

$$K_{2,2}(x,y;z,z') \;=\; \frac{\alpha_s^2}{16\,\pi^4}\left[\frac{(x-y)^2}{X^2Y'^2(z-z')^2}\left(1+\frac{(x-y)^2(z-z')^2}{X^2Y'^2-X'^2Y^2}\right)-\right.$$

$$\left.-\frac{(x-y)^2}{X'^2Y^2(z-z')^2}\left(1+\frac{(x-y)^2(z-z')^2}{X'^2Y^2-X^2Y'^2}\right)\right]\;\ln\frac{X^2Y'^2}{X'^2Y^2}$$

$$K_{2,1}(x,y,z)=\frac{\alpha_s^2N_c}{48\pi}\frac{(x-y)^2}{X^2Y^2}$$

$$K_{2,0}(x,y)=\frac{\alpha^2N_c}{16\pi^3}\int_z\frac{(x-y)^2}{X^2Y^2}\left[\frac{\pi^2}{3}+2\ln\frac{Y^2}{(x-y)^2}\;\ln\frac{X^2}{(x-y)^2}\right]$$

$$K_{3,2}(w;x,y;z,z')=$$

$$=\frac{i}{2}\left[M_{x,y,z}M_{y,z,z'}+M_{x,w,z}M_{y,w,z'}-M_{y,w,z'}M_{x,z',z}-M_{x,w,z}M_{y,z,z'}\right]\ln\frac{W^2}{W'^2}$$

$$K_{3,1}(w;x,y;z)=\int_{z'}\left[K_{3,2}(y;w,x;z,z')-K_{3,2}(x;w,y;z,z')\right]$$

$$K_{3,0}(w,x,y)\;=\;-\,\frac{1}{3}\left[\int_{z,z'}K_{3,2}(w,x,y;z,z')\;+\;\int_z K_{3,1}(w,x,y;z)\right]$$

with

$$X\equiv x-z;\;\; X'\equiv x-z';\text{etc.},\quad\text{and } M(x,y,z)\equiv\frac{\alpha_s}{2\,\pi^2}\,\frac{(x-y)^2}{X^2\,Y^2}$$

Conformal symmetry and what is it good for?

$N = 4$ SUSY is conformally invariant. So is QCD at the tree level.

So is LO JIMWLK.

But NLO JIMWLK is apparently not!

Not totally suprising: NLO JIMWLK was derived with sharp rapidity cutoff, wich is not conformally invariant.

Balitsky, Chirilli - evolution of a dipole at NLO is not invariant, but can define a "conformal dipole" which does satisfy conformally invariant equation. Is it general? How to redefine other operators?

With operatorial NLO JIMWLK we can resolve these questions.

BUT ALSO CS HELPS US TO DETERMINE THE VIRTUAL TERM!

Is NLO JIMWLK really noninvariant?

Effective theory is obtained by integrating some degrees of freedom:

$$\mathcal{L}(\alpha,\beta) \to \mathcal{L}'(\beta)$$

Suppose $\mathcal{L}$ was symmetric under

$$\alpha \to \alpha + g(\alpha,\beta); \qquad \beta \to \beta + f(\alpha,\beta)$$

Even though α mixes with β in the transformation, integrating out α does not destroy the symmetry in $\mathcal{L}'$, but modifies it

$$f(\alpha,\beta) \to f'(\beta) \neq f(\alpha = 0,\beta)$$

Noninvariant cutoff is similar - the "fast" and "slow" degrees of freedom mix under conformal transformation.

Can we find a modified conformal transformation, which is an exact symmetry of H^{NLO}_{JIMWLK}?

Modified inversion symmetry.

Naive inversion transformation ($x_\pm = x_1 \pm i x_2$):

$$\mathcal{I}_0 : S(x_+, x_-) \to S(1/x_-, 1/x_+);\; J_{L,R}(x_+, x_-) \to \frac{1}{x_+ x_-} J_{L,R}(1/x_-, 1/x_+)$$

$$\mathcal{I}_0 H^{NLO}_{JIMWLK} \mathcal{I}_0 = H^{NLO}_{JIMWLK} + O(\alpha_s)$$

But it is easy to check that to order α_s there is invariance under

$$\mathcal{I} = \mathcal{I}_0 \Big[1 - C \Big]$$

$$C = -\frac{1}{2} \int M_{xyz} \ln\left(\frac{z^2}{a^2}\right) [J^a_L(x) J^a_L(y) + J^a_R(x) J^a_R(y) - 2 J^a_L(x) S^{ab}_A(z) J^b_R(y)]$$

IF THE VIRTUAL TERM IS WHAT WE HAVE WRITTEN ABOVE.

CONFORMAL OPERATORS

Can give operatorial definition to conformal Wilson line operator: that transform "naively" under Conformal Transformation:

$$\mathcal{I}\; U(x)\; \mathcal{I} = U(1/x)$$

Solution:

$$U(x) \;=\; S(x) \;+\; \frac{1}{2}\,[\bar{\mathcal{C}}, S(x)]$$

$$\bar{\mathcal{C}} = -\frac{1}{2}\int M_{xyz} \ln \frac{(x-y)^2 a^2}{(x-z)^2(z-y)^2}[J_L^a(x)J_L^a(y)+J_R^a(x)J_R^a(y)-2J_L^a(x)S_A^{ab}(z)J$$

$S(x)$ - Wilson line in the "sharp rapidity cutoff scheme"; $U(x)$ - Wilson line in the "conformal' scheme.

AND NOW FOR SOMETHING COMPLETELY DIFFERENT: UNITARIY OF REGGEON FIELD THEORY.

A.K., E. Levin, M. Lublinsky, 2016

Reggeon field theory $\equiv$ JIMWLK (like) Hamiltonian restricted to act on dipoles.

$$\mathcal{S} = \int d\rho d\alpha_T \delta(\rho) W_P[R] e^{i\int_z g^2 \rho^a(z)\alpha_T^a(z)} \tilde{W}_T[\alpha_T] = \int d\rho \delta(\rho) W_P[R] W_T[S]$$

$$R_x = e^{t^a \frac{\delta}{\delta \rho_x^a}}; \qquad S_x = e^{ig^2 t^a \alpha_x^a}$$

The "POMERONS"

$$P(x,y) = 1 - \frac{1}{N_c} tr[R_x R_y^\dagger]; \qquad \bar{P}(x,y) = 1 - \frac{1}{N_c} tr[S_x S_y^\dagger]$$

"Dipole Model" $\equiv$: Large N_c $\equiv$ "Pomeron Calculus"

$$\mathcal{S} = \int d\bar{P} \delta(\bar{P}) W_P[P] W_T[\bar{P}]$$

In dipole model:

$$W_P = \sum_{n,\{x,\bar{x}\}} F^n(\{x,\bar{x}\}) \prod_{i=1}^{n} [1 - P(x_i, \bar{x}_i)]$$

$$W_T = \sum_{n,\{x,\bar{x}\}} \bar{F}^n(\{x,\bar{x}\}) \prod_{i=1}^{n} [1 - \bar{P}(x_i, \bar{x}_i)]$$

Probability densities:

$$0 < F^n, \bar{F}^n < 1 : \quad \sum_n \int_{\{x,\bar{x}\}} F^n(\{x,\bar{x}\}) = 1; \qquad \sum_n \int_{\{x,\bar{x}\}} \bar{F}^n(\{x,\bar{x}\}) = 1$$

Unitary evolution preserves this property of F^n, $\bar{F}^n$ at all energies!

Evolution of the S-matrix:

$$\mathcal{S} = \int d\bar{P}\delta(\bar{P})\, W_P[P]\, e^{-H_{RFT}[P,\bar{P}]Y}\, W_T[\bar{P}]$$

BK Hamiltonian (JIMWLK)

$$H_{BK} = \frac{\bar{\alpha}_s}{2\pi} \int K(x,y|z)\, P^\dagger(x,y)\, [P(x,z) + P(z,y) - P(x,y) - P(x,z)P(z,$$

Braun Hamiltonian (DDD invariant)

$$H_B = \frac{N_c^2}{2\pi\bar{\alpha}_s} \int \bar{P}(x,y)\nabla_x^2\nabla_y^2\,[K(x,y|z)[P(x,z) + P(z,y) - P(x,y) - P(x,z$$

$$-P(x,y)\nabla_x^2\nabla_y^2\left[K(x,y|z)\bar{P}(x,z)\bar{P}(z,y)\right]$$

Algebra of P and $\bar{P}$? In "dilute limit":

$$P^\dagger(x,y) = \frac{N_c^2}{4\pi^4\bar{\alpha}_s^2}\nabla_x^2\nabla_y^2\bar{P}(x,y)$$

Escape into toy world.

Zero transverse dimensions: all dipole sit at the same transverse position.

m dipoles scatter on n dipoles:

$$\langle m|\bar{n}\rangle = \int d\bar{P}\delta(\bar{P})(1-P)^m(1-\bar{P})^{\bar{n}}$$

This amplitude is evolved in energy according to

$$\langle m|\bar{n}\rangle_Y = \int d\bar{P}\delta(\bar{P})(1-P)^m e^{HY}(1-\bar{P})^{\bar{n}}$$

Act with H to the right - evolve the target wave function; act with H to the left - evolve the projectile wave function.

Question: Does the evolution preserve unitarity of both wave functions?

Trouble in the toy world.

$$H_{BK} = -\frac{1}{\gamma}\left[\bar{P}P - \bar{P}P^2\right]$$

Dilute limit algebra:

$$[\bar{P}, P] = \gamma; \qquad \gamma \sim \alpha_s^2 > 0$$

$$\langle m|e^{\Delta H} \approx (1 - \Delta m)\langle m| + \Delta m\langle m+1|$$

$$e^{\Delta H}|\bar{n}\rangle = (1+\Delta\bar{n})|\bar{n}\rangle - \Delta\bar{n}[1 + \gamma(\bar{n} - 1)]|\bar{n} - 1\rangle + \Delta\gamma\bar{n}(\bar{n} - 1)|\bar{n} - 2\rangle$$

Projectile evolves unitarily, but target evolution generates negative probabilities!

Maybe Braun Hamiltonian?

$$H_B = -\frac{1}{\gamma}\left[\bar{P}P - \bar{P}P^2 - \bar{P}^2 P\right]$$

$$e^{\Delta H_B}|\bar{n}\rangle = (1-\Delta\bar{n})|\bar{n}\rangle + \Delta\bar{n}|\bar{n}+1\rangle - \Delta\gamma\bar{n}(\bar{n}-1)|\bar{n}-1\rangle + \Delta\gamma\bar{n}(\bar{n}-1)|\bar{n}-2\rangle$$

Unitarity is violated by an amount γ, but for both, projectile and target.

New and improved algebra.

Maybe the commutators are to blame? Relax the "dilute limit algebra":

$$(1-P)(1-\bar{P}) = [1-\gamma](1-\bar{P})(1-P)$$

Takes into account multiple scatterings.

$$e^{\Delta H_{BK}}|\bar{n}\rangle = \left[1 - \frac{\Delta}{\gamma}\left[1-(1-\gamma)^{\bar{n}}\right](1-\gamma)^{\bar{n}}\right]|\bar{n}\rangle$$

$$+\frac{\Delta}{\gamma}\left[1-(1-\gamma)^{\bar{n}}\right](1-\gamma)^{\bar{n}}|\bar{n}+1\rangle$$

$$\langle m|e^{\Delta H_{BK}} = \langle m| - \frac{\Delta}{\gamma}[1-(1-\gamma)^m]\langle m+1| + \frac{\Delta}{\gamma}[1-(1-\gamma)^m]\langle m+2|$$

Now the projectile evolution violates unitarity!

Braun Hamiltonian only makes things worse...

The Panacea?

Question: Can we construct a unitary, DDD invariant Toy Hamiltonian, which reduces to BK in the appropriate limit?

The answer is YES!

$$H_{UTM} = -\frac{1}{\gamma}\bar{P}P$$

$$e^{\Delta H_{UTM}}|\bar{n}\rangle = \left[1 - \frac{\Delta}{\gamma}[1-(1-\gamma)^{\bar{n}}]\right]|\bar{n}\rangle + \frac{\Delta}{\gamma}[1-(1-\gamma)^{\bar{n}}]|\bar{n}+1\rangle$$

This is unitary. It also has nice properties close to saturation.

REAL QCD: PROBLEMS ARE THE SAME, SOLUTION IS PENDING...