High-Energy evolution to three loops

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1701.05241 w/ Gardi&Vernazza

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Motivation

- Forward scattering is interesting in many contexts
- phenomenology:
  -DIS at small x
  -saturation, …
  -large rapidity jets ('Mueller-Navelet')
  - …
- theory:
  -partonic amplitudes in (multi)-Regge limit): unique insight into scattering at high loops
  -generally interesting limit (pomeron→graviton in AdS CFT,…)

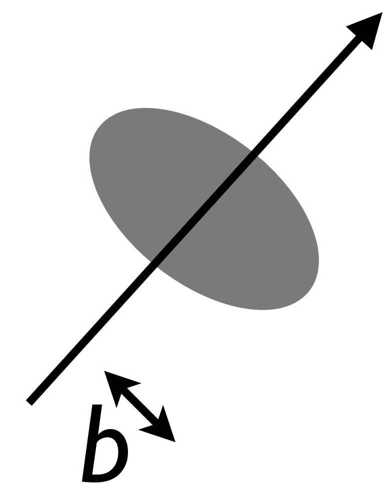
this talk

Outline

- Wilson line approach to forward scattering:
  -Eikonal approximation
  -The Reggeized gluon
  -Expected all-order structure
  -Quantitative tests w/ the 2->2 amplitude
- Systematic improvements
  - A remarkable equivalence: ‘non-global logs’
  - 3-loop evolution

The eikonal approximation

- Fast particles like to go straight



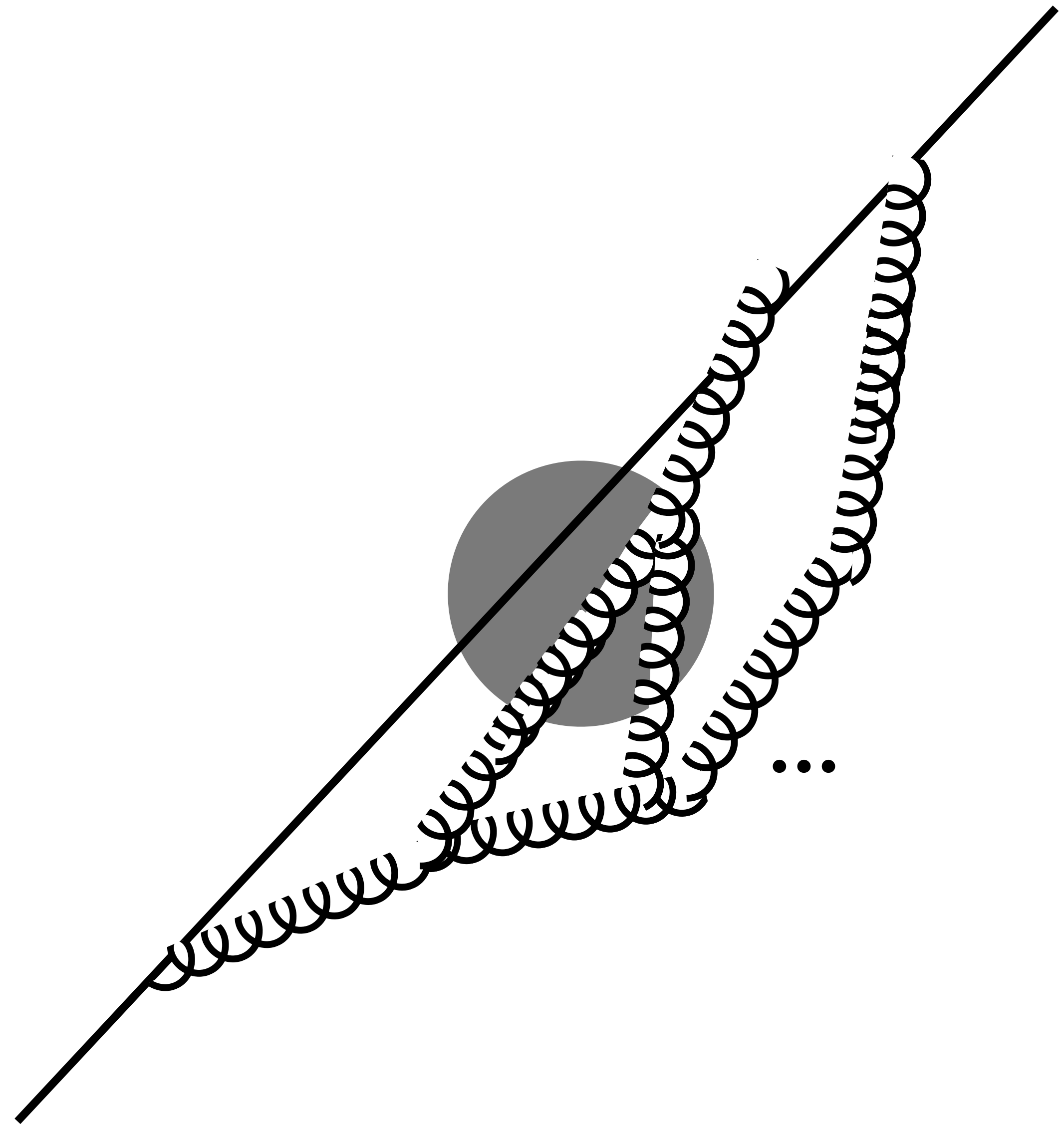
- In gauge theories, natural to dress with Wilson lines:

$$\mathcal{M}(p_i) \approx \int d^2b e^{iq\cdot b} \langle U(b) \rangle_{\text{target}},$$

$$U(b) \equiv \mathcal{P} e^{i \int_{-\infty}^{\infty} T^a A^a_\mu (b+vt) v^\mu dt}$$

- Familiar enough for heavy quarks. Seems natural for fast particles… Q: When is that valid?

A slightly less naive picture of an ultrarelativistic particle:



Q: Which trajectory should one dress with a Wilson line?

- This question was analyzed by many people

- The only possible correct answer is « all »

  (all partons whose rapidity is between that of the projectile and target)

- Increases with energy *(⇒growing σ_{tot})*

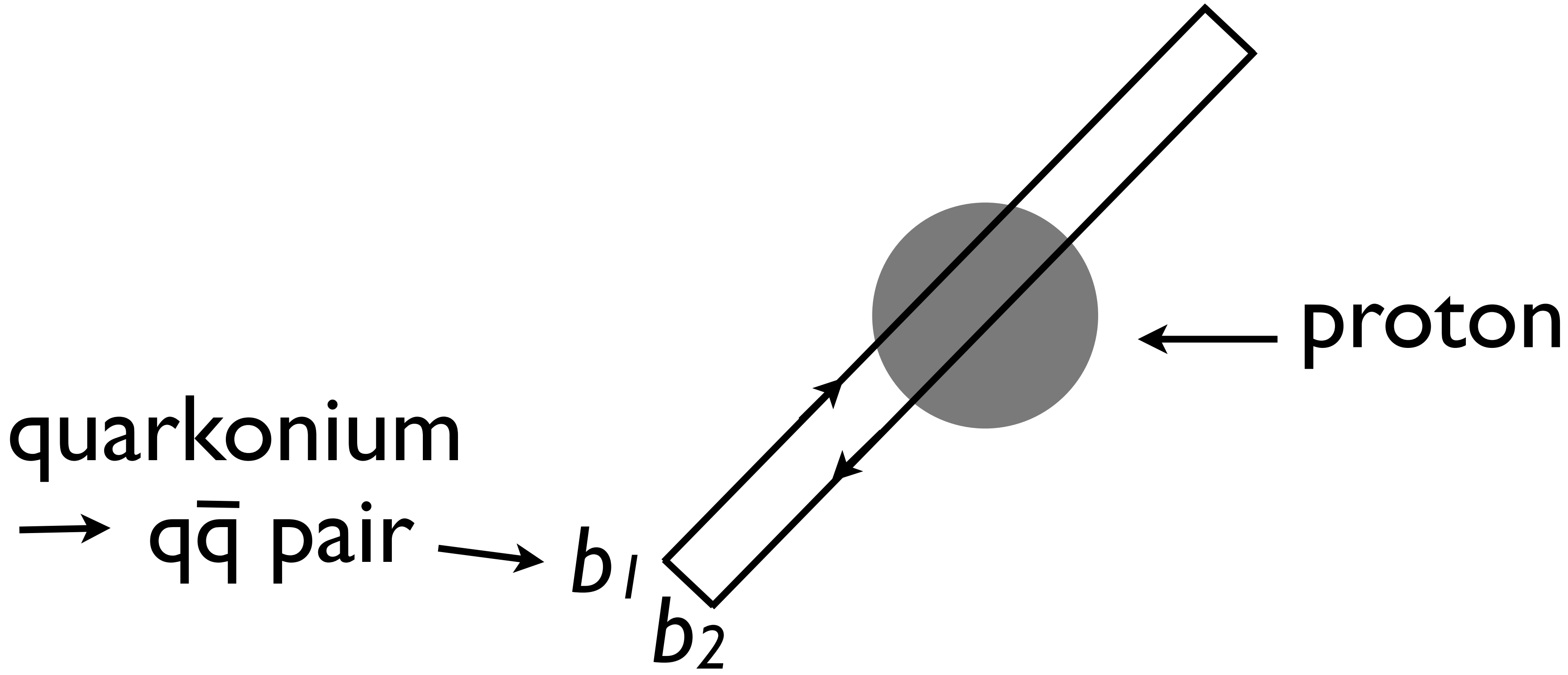
- Looks complicated!

- Successful theory finally developed in the '90'

[Balitsky '95, Mueller, Kovchegov, JIMWLK*]

The eikonal approximation

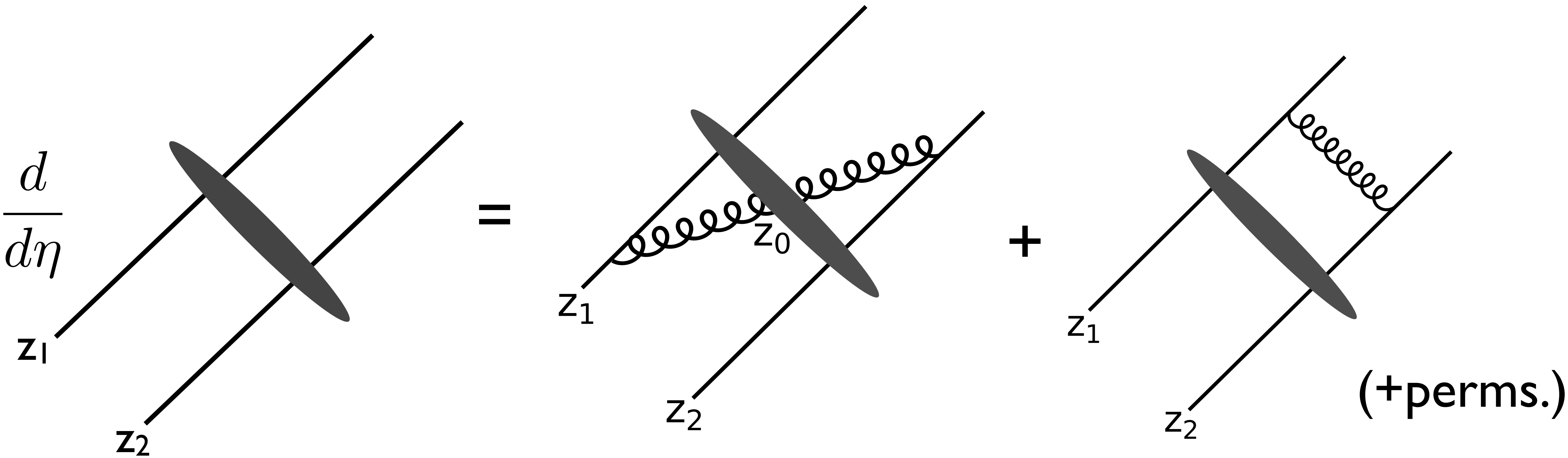
(a more well-defined gauge theory example:)



$$\mathcal{M} \approx \int d^2b_1 d^2b_2 \rho(b_1 - b_2) e^{iq\cdot\frac{b_1+b_2}{2}} \mathrm{Tr}[U(b_1)U^\dagger(b_2)]$$

What are the corrections to this?

The Balitsky-JIMWLK equation



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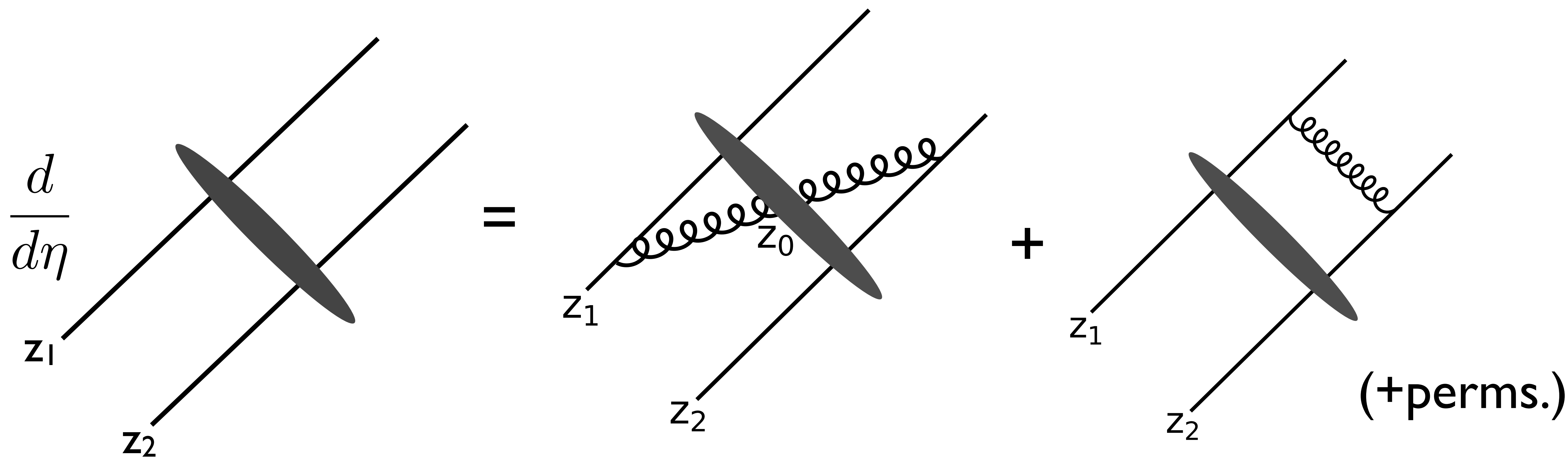
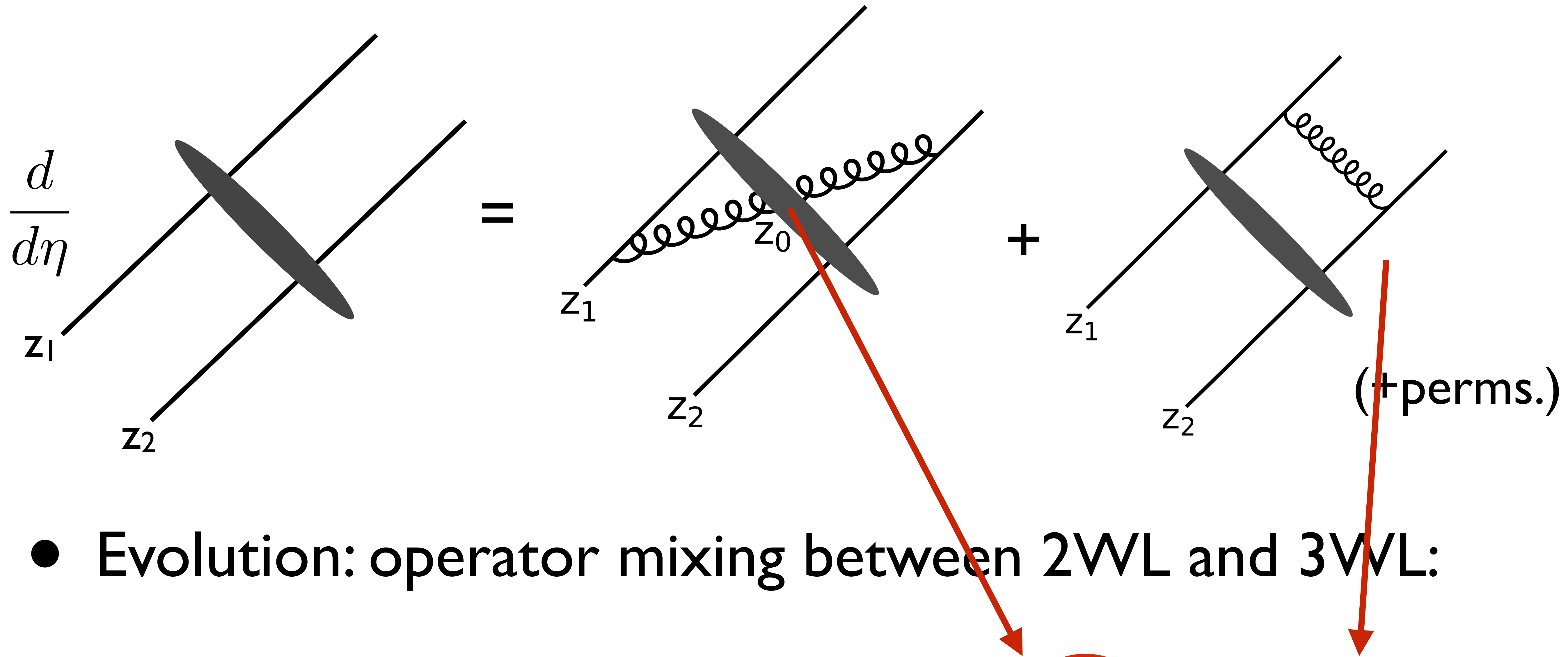


- The ‘shock’ represents Lorentz-contracted target
- The 45° lines represent fast projectile partons
- Each parton that crosses the shock gets a Wilson line

The Balitsky-JIMWLK equation

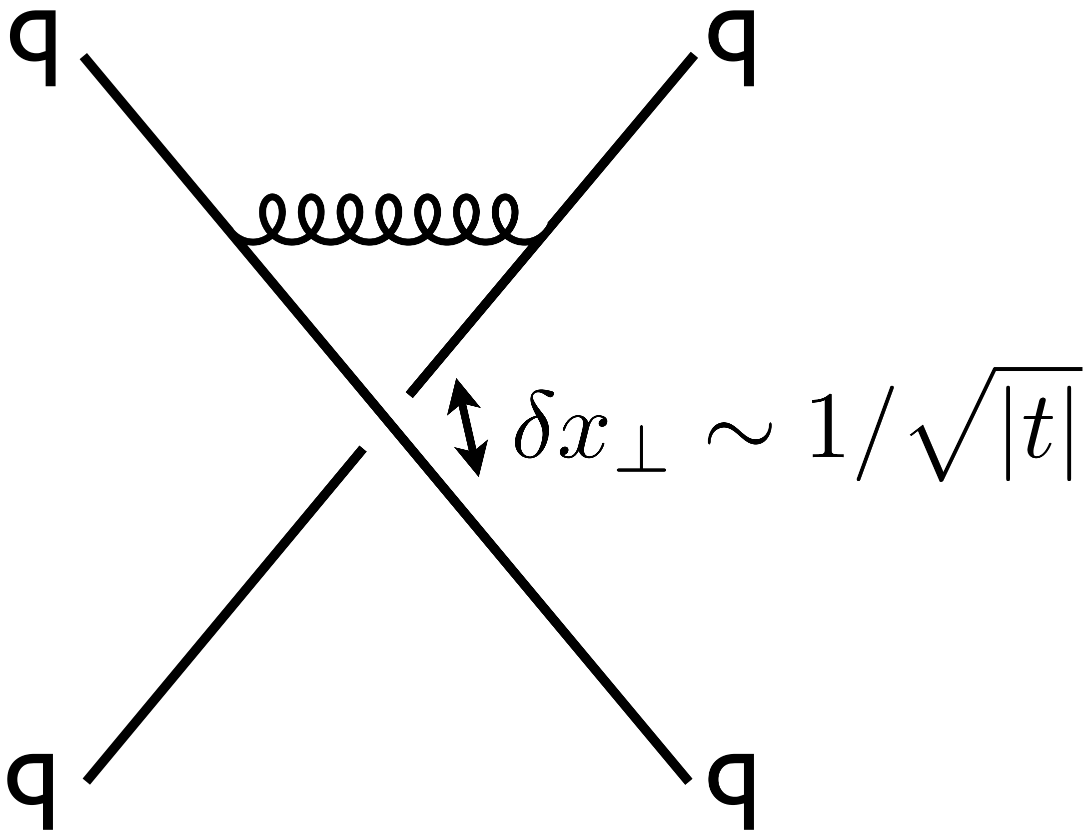


- Evolution: operator mixing between 2WL and 3WL:

$$\frac{d}{d\eta}\mathrm{Tr}[U_{\mathrm{f}}^{\dagger}(z_1)U_{\mathrm{f}}(z_2)] = \frac{\alpha_s}{\pi^2}\int \frac{d^2z_0\, z_{12}^2}{z_{01}^2 z_{02}^2}\left(\mathrm{Tr}[U_{\mathrm{f}}^{\dagger}(z_1)T^aU_{\mathrm{f}}(z_2)T^b]U_{\mathrm{ad}}^{ab}(z_0) - C_F\mathrm{Tr}[U_{\mathrm{f}}^{\dagger}(z_1)U_{\mathrm{f}}(z_2)]\right)$$

Application: 2to2 scattering, I

i) Start from naive eikonal approximation:
scatter fundamental Wilson lines $(s \gg |t|)$



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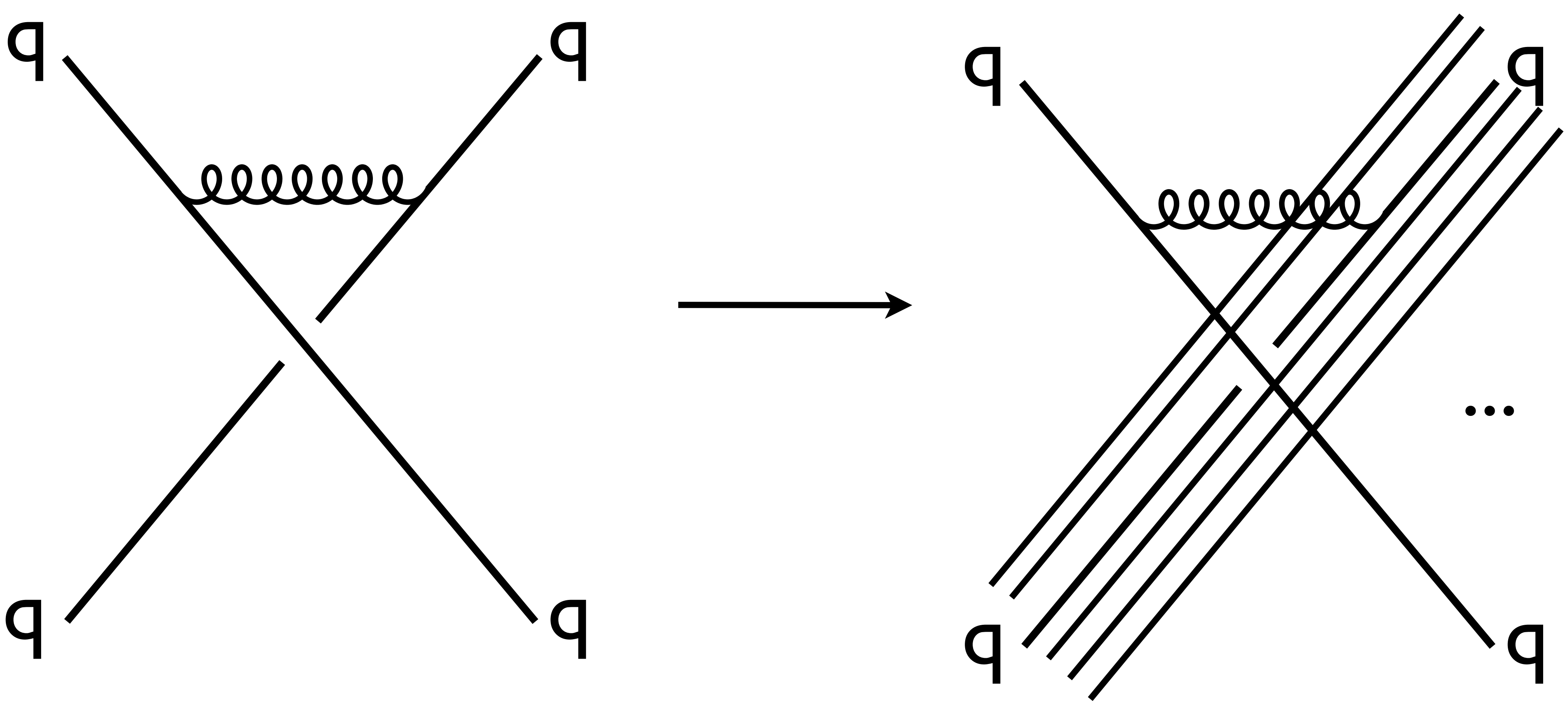

ii) Use B-JIMWLK evolution to resum $\alpha_s \log(s)$ effects

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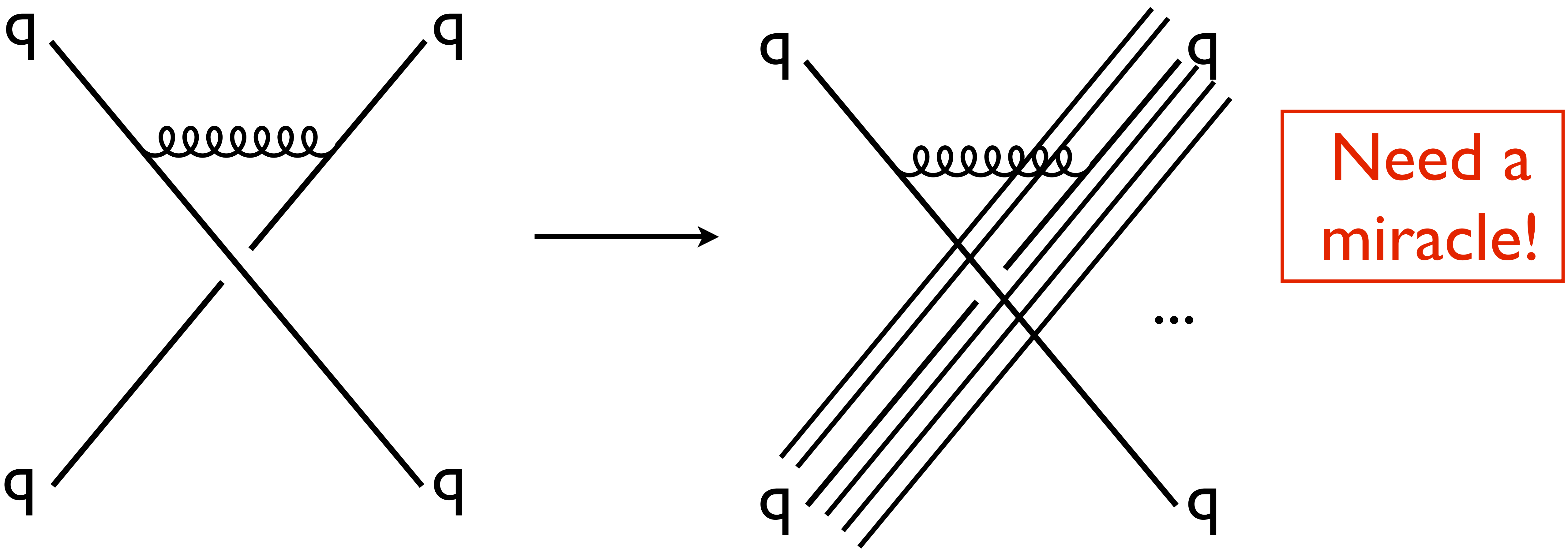
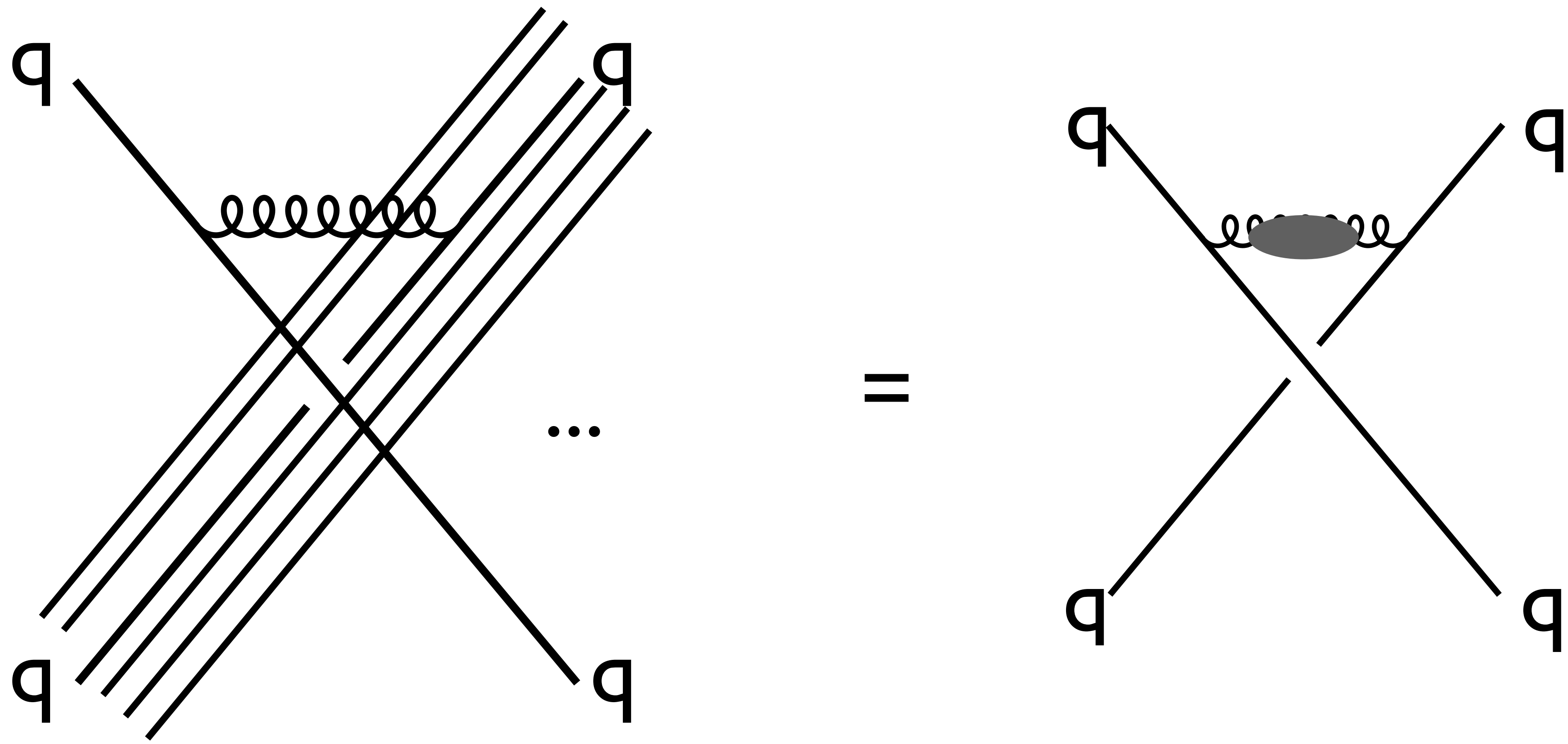


ii) Use B-JIMWLK evolution to resum $\alpha_s \log(s)$ effects

[Fadin, Kuraev& Lipatov 1977,
Balitsky& Lipatov 1978]

Gluon Reggeization



at leading-log: $\mathcal{M} \approx \frac{s}{t}\left(\frac{|s|}{-t}\right)^{\alpha_g(t/\mu_{\rm IR}^2)}$

How does this happen?????

- At LL, most Wilson lines are trivial (U=1)

- Define *Reggeized gluon* operator by taking log:

$$U(x_\perp) \equiv e^{igT^a W^a(x_\perp)}$$

similar to Goldstone's parametrization of pion field

- Gives a gauge-invariant operator*:

[cf Kovner et al, '97]

$$W^a = \int_{-\infty}^{\infty} A_+^a(x^+) dx^+ - g\frac{1}{2} f^{abc} \int_{-\infty}^{\infty} dx_1^+ dx_2^+ A_+^b(x_2) A^+ c(x_1) \theta(x_2^+ - x_1^+) + \ldots$$

- Result is independent of representation used.

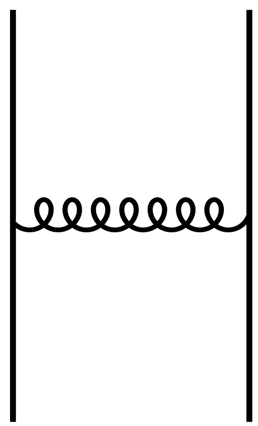
*under gauge transformations which vanish at infinity

- RG equation for W linearizes!

$$\frac{d}{d\eta}W^a(z_1) = \frac{\alpha_s C_A}{2\pi^2}\int \frac{d^2 z_0}{z_{01}^2}\left(W^a(z_0) - W^a(z_1)\right) + \mathcal{O}(g^4 W^3)$$

- Diagonal in momentum space:

$$\frac{d}{d\eta}W^a(p) = \alpha_g(p) W^a(p) + O(g^4 W^3)$$



$$\propto s^{\alpha_g(t)}$$

- Eigenvalue is (LL) 'gluon Regge trajectory'

$$\alpha_g(p) = \frac{\alpha_s C_A}{2\pi^2}\int \frac{d^{2-2\epsilon} z}{(z^2)^{1-2\epsilon}}(e^{p\cdot z} - 1) \sim \frac{\alpha_s C_A}{2\pi\epsilon}\left(\frac{p^2}{\mu^2}\right)^{-\epsilon}$$

- More reggeons: start from B-JIMWLK

$$\frac{-d}{d\eta} = \frac{\alpha_s}{2\pi^2}\int d^2z_i d^2z_j \frac{d^2z_0 \;\; z_{0i}\cdot z_{0j}}{z_{0i}^2 z_{0j}^2}\Big(T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\big(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\big)\Big)$$

- Plug in Goldstone's parametrization

$$U \rightarrow e^{igW}$$

- T's = color rotation generators: CBH formula:

$$igT^a_{j,L} = \frac{\delta}{\delta W^a_j} + \frac{g}{2} f^{abx} W^x_j \frac{\delta}{\delta W^b_j} + \frac{g^2}{12} f^{aex} f^{eby} W^x_j W^y_j \frac{\delta}{\delta W^b_j} - \frac{g^4}{720} WWWW \frac{\delta}{\delta W} + \ldots$$

$$igT^a_{j,R} = \frac{\delta}{\delta W^a_j} - \frac{g}{2} f^{abx} W^x_j \frac{\delta}{\delta W^b_j} + \frac{g^2}{12} f^{aex} f^{eby} W^x_j W^y_j \frac{\delta}{\delta W^b_j} - \frac{g^4}{720} WWWW \frac{\delta}{\delta W} + \ldots$$

[SCH '13]

- Result of expanding in g:

$$H_{k\to k} = -\int [dp]\, C_A\, \alpha_g(p)\, W^a(p) \frac{\delta}{\delta W^a(p)} \tag{3.13}$$
$$+ \alpha_s \int [\bar{d}q][dp_1][dp_2]\, H_{22}(q;p_1,p_2)\, W^x(p_1+q) W^y(p_2-q) (F^x F^y)^{ab} \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_2)},$$

- Act on polynomial $W(p_1)\ldots W(p_n)$ = n-Reggeon state
- First term = gluon Regge trajectories $\sum_i \alpha_g(p_i)$
- Second term = sum over pairwise interactions

$$H_{22}(q;p_1,p_2) = \frac{(p_1+p_2)^2}{p_1^2 p_2^2} - \frac{(p_1+q)^2}{p_1^2 q^2} - \frac{(p_2-q)^2}{q^2 p_2^2}.$$

=BFKL/BJKP

LO B-JIMWLK gives:

$$e^{igW^aT^a} \sim 1 + igWT + \ldots$$

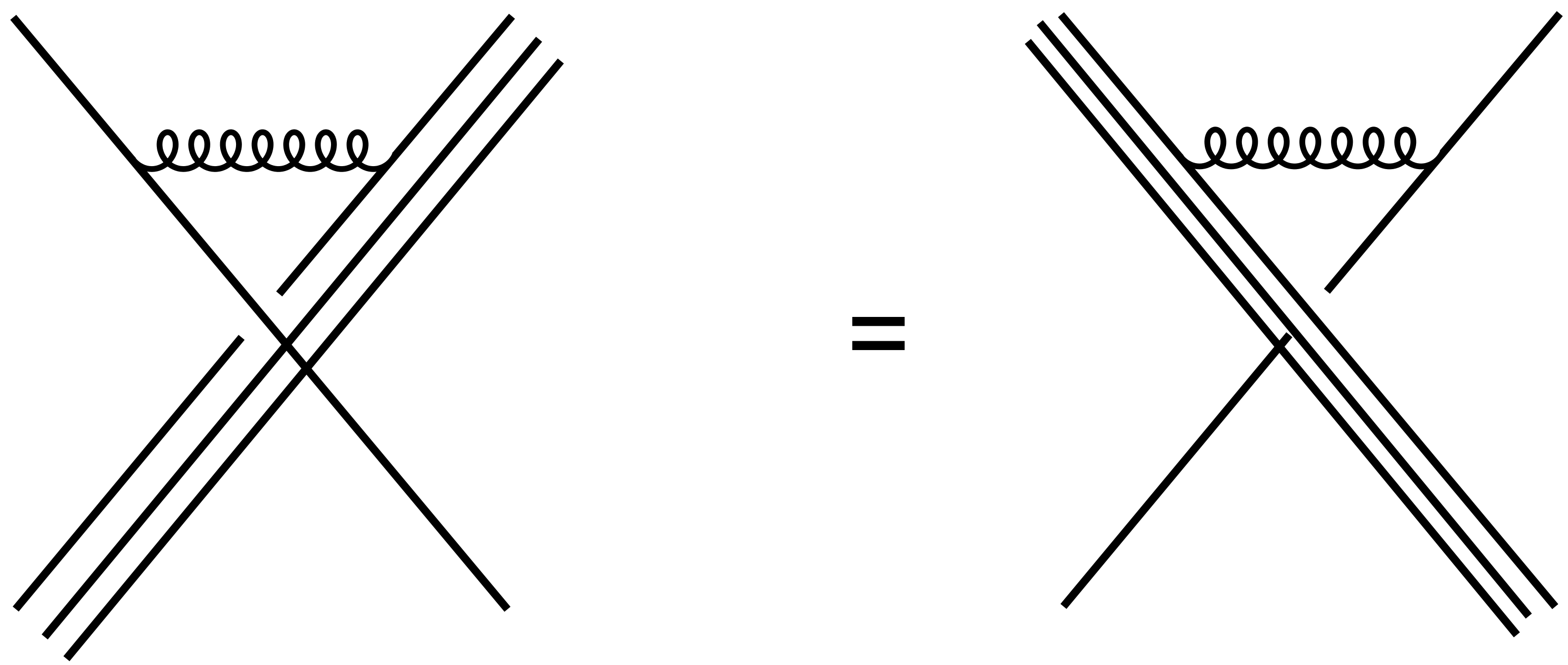
$$\frac{d}{d\eta}\begin{pmatrix} (W)^1 \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ \ldots \end{pmatrix} = \begin{pmatrix} g^2 & 0 & g^4 & 0 & g^6 & \ldots \\ 0 & g^2 & 0 & g^4 & & \\ 0 & 0 & g^2 & 0 & \ldots & \\ 0 & 0 & 0 & g^2 & & \\ & & \ldots & & & \end{pmatrix} \cdot \begin{pmatrix} (W)^1 \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ \ldots \end{pmatrix}$$

Leading BFKL and 'BKP' kernels

off the diagonal: n→n+k Reggeon transitions

all determined by one simple function: $\dfrac{z_{0i} \cdot z_{0j}}{z_{0i}^2 z_{0j}^2}$

Projectile-Target duality



boosting projectile = boosting target

(cf Kovner's talk)

Basically: $H = H^T$

*really: $\langle 0|(\bar{W}_1 \cdots \bar{W}_n)\, H (W_1 \cdots W_m)|0\rangle = \langle 0|H(\bar{W}_1 \cdots \bar{W}_n)\,(W_1 \cdots W_m)|0\rangle$

same as $H=H^T$ in schemes where correlators are diagonal $\sim\delta_{m,n}$:

General structure

$$e^{igW^aT^a} \sim 1 + igWT + \ldots$$

n→n+k transitions: from LO B-JIMWLK

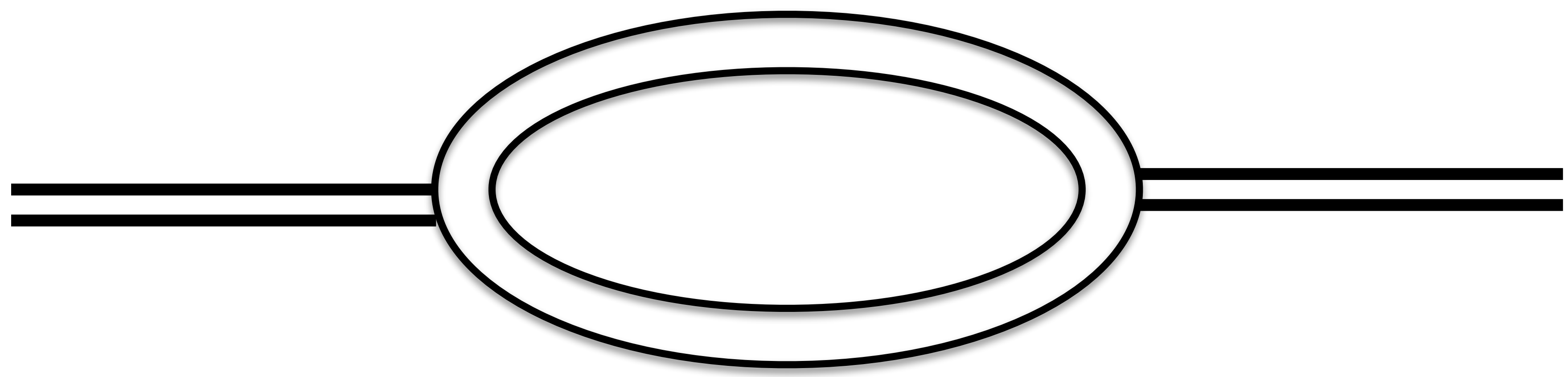
$$\frac{d}{d\eta}\begin{pmatrix}(W)^1\\(W)^2\\(W)^3\\(W)^4\\\cdots\end{pmatrix} = \begin{pmatrix} g^2 & 0 & g^4 & 0 & g^6 & \cdots \\ 0 & g^2 & 0 & g^4 & & \\ g^4 & 0 & g^2 & 0 & \cdots & \\ 0 & g^4 & 0 & g^2 & & \\ & & \cdots & & & \end{pmatrix} \cdot \begin{pmatrix}(W)^1\\(W)^2\\(W)^3\\(W)^4\\\cdots\end{pmatrix}$$

terms in NNLO B-JIMWLK predicted by symmetry $H=H^T$

Leading BFKL and BKP kernels

- Due to Pomeron growth, off-diagonal can't be ignored
- Complete 'Reggeon field theory' remains elusive

Pomeron loop power counting



$H_{2\to4}$ x $H_{4\to2}$

- BFKL (W~1)

  NLL x NLL = NNLL

- B-JIMWLK (log U_{proj}~gW~1)

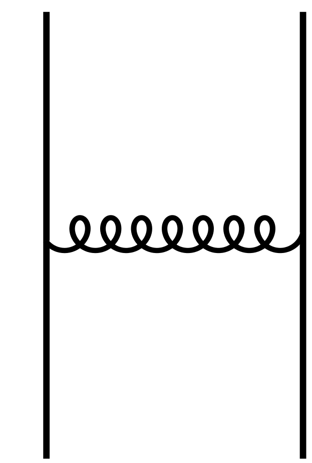
  LL x NNLL = NNLL

- The observable physics is of course the same

How do we know that this is right?

One way: compare w/fixed-order

- Parton amplitude beyond leading-log:

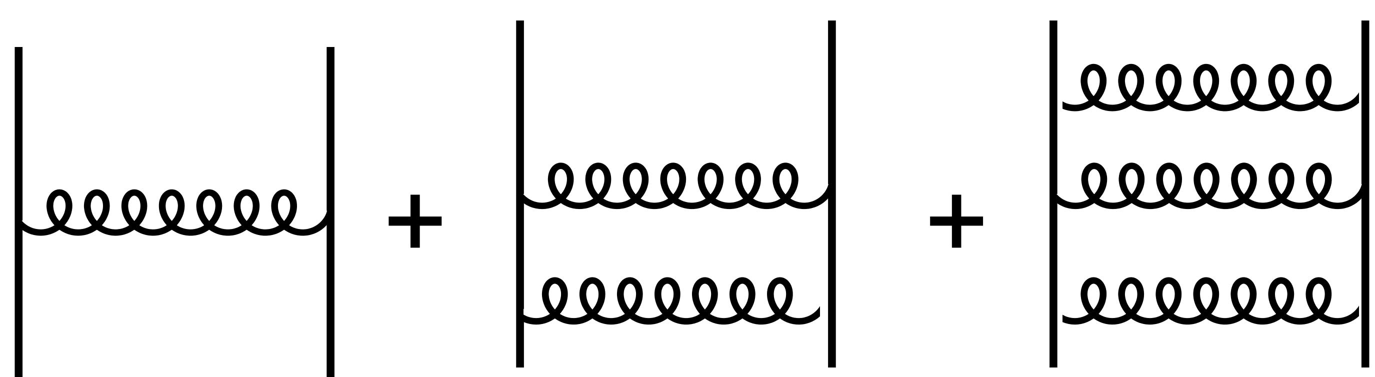


$$a(p_3)a^\dagger(p_2) \sim \ |\psi_i\rangle^{(\mathrm{LO})} = \int [dz] e^{ip\cdot z} U_i(z),$$

- Expand in reggeized gluons: $U \to e^{igT^a W^a}$

$$|\psi_i\rangle^{(\mathrm{LO})} = ig\, \mathbf{T}_i^a W^a(p) - \frac{g^2}{2} \mathbf{T}_i^a \mathbf{T}_i^b \int [\not{d}q]\, W^a(q) W^b(p-q)$$
$$- \frac{ig^3}{6} \mathbf{T}_i^a \mathbf{T}_i^b \mathbf{T}_i^c \int [\not{d}q_1][\not{d}q_2]\, W^a(q_1) W^b(q_2) W^c(p-q_1-q_2) + \mathcal{O}(\mathrm{N^3LL}),$$

$\Rightarrow$



+ ...
(+crossed)

Form of the NLL 4-point amplitude:

$$\mathcal{M}_4^{\rm NLL} = (1 + \alpha_s C^{(1)L} + \alpha_s C^{(1)R})\langle W, W\rangle_{\rm NLL} + \langle WW, WW\rangle_{\rm LL}$$

signature ('CPT') odd

signature even

The first term is a pure power-law, while the second term is a (pure imaginary) 'Regge cut'

- The NLL Regge cut can be computed using just the leading order ('naive') eikonal approximation, +leading order BFKL kernel

$$\mathcal{M}^{aa'bb'}_{ij\to ij}\Big|^{\text{even}}_{\text{NLL}} = i\tilde{\alpha}_{\text{s}} \left(\frac{|s|}{-t}\right)^{\alpha_g(t)\frac{\mathbf{T}_t^2}{C_A}} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(\frac{\tilde{\alpha}_{\text{s}}}{\pi} \log\frac{|s|}{-t}\right)^{\ell-1} d_\ell \, \mathcal{M}^{\text{tree}}_{ij\to ij} \,.$$

$$d_\ell = \frac{\pi p^2 \ell}{c'_\Gamma} \int \frac{\bar{\mu}^{2\epsilon} d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}} \langle \hat{H}^{\ell-1} W_p(k)\rangle \times \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2} .$$ (SCH'13)

powers of BFKL Kernel

- This result was known since (B)FKL '77; actually doing the integrals proves interesting

$$d_1 = \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2} \times \frac{1}{2\epsilon}$$

$$d_2 = [\mathbf{T}_t^2, \mathbf{T}_s^2] \times \left[-\frac{1}{4\epsilon^2} - \frac{9}{2}\epsilon\zeta_3 - \frac{27}{4}\epsilon^2\zeta_4 - \frac{63}{2}\epsilon^3\zeta_5 + \ldots \right]$$

$$d_3 = [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_s^2]] \times \left[\frac{1}{8\epsilon^3} - \frac{11}{4}\zeta_3 - \frac{33}{8}\epsilon\zeta_4 - \frac{357}{4}\epsilon^2\zeta_5 + \ldots \right].$$

- First two lines match 1,2-loop fixed-order calculations ✔
- Leading poles reproduce the (correct) exponentiation of one-loop IR divergences ✔
- No subleading poles at 3-loops!?!?! (cf Fadin's talk)

$$d_4 = [\mathbf{T}_t^2, [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_s^2]]] \times \left[-\frac{1}{16\epsilon^4} - \frac{175}{2}\zeta_5\epsilon + \ldots \right]$$
$$+ C_A[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_s^2]] \times \left[-\frac{1}{8\epsilon}\zeta_3 - \frac{3}{16}\zeta_4 - \frac{167}{8}\zeta_5\epsilon + \ldots \right]$$

Implication for IR divergences:

- resummation of IR divergences:

$$\mathcal{M} = \exp\left(\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\frac{P_i}{\lambda},\alpha(\lambda^2),\epsilon\right)\right)\mathcal{H}\left(\frac{P_i}{\mu},\frac{\mu_f}{\mu},\alpha_s(\mu^2),\epsilon\right)$$

- Dipole conjecture must break in Regge limit at 4-loops: (broken at 3-loops, but only away from Regge limit) ✔

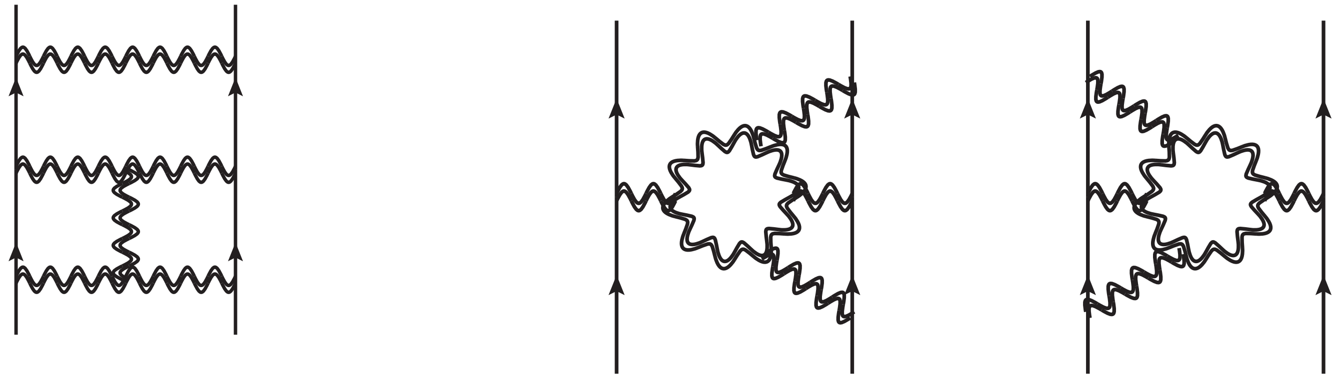
[Gardi&Magnea; Neubert&Becher '09]

[Almelid,Duhr&Gardi '15]

- Regge limit of Γ can now be predicted to all loops

(see Joscha's talk!)

- NNLL: 3-loop amplitude w/ reggeized gluons



- Many color structures can be predicted using only LO evolution

$$\hat{\mathcal{M}}^{(-,3,1)}_{ij\to ij} = \pi^2 \Big(R_A^{(3)} \mathbf{T}^2_{s-u}[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}] + R_B^{(3)} [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R_C^{(3)} (C_A)^3 \Big) \hat{\mathcal{M}}^{(0)}_{ij\to ij} \,,$$

(see Leonardo's talk!)

- Breakdown of Regge pole factorization @ 2-loops ✔ [Del Duca&Glover '01]
- Poles match 3-loop evolution. ✔ [Almelid,Duhr&Gardi '15]
- Full agreement with 3-loop non-planar N=4 ✔ [Henn&Mistlberger '1
- First direct test of projectile/target duality $H=H^T$

Summary so far

Input: Regge scattering factorizes on Wilson lines U

Output: gluon Reggeization

=Regge form $\sim s^{\alpha(t)}$ at LL and NLL;

+Regge cuts beyond (predicted)

(Note: asking if W is Reggeized gluon to all orders
=like asking if π in U= $\exp(i\pi^a T^a/f_\pi)$ is 'pion' to all orders.
It's a just a valid parametrization of the relevant DOF)

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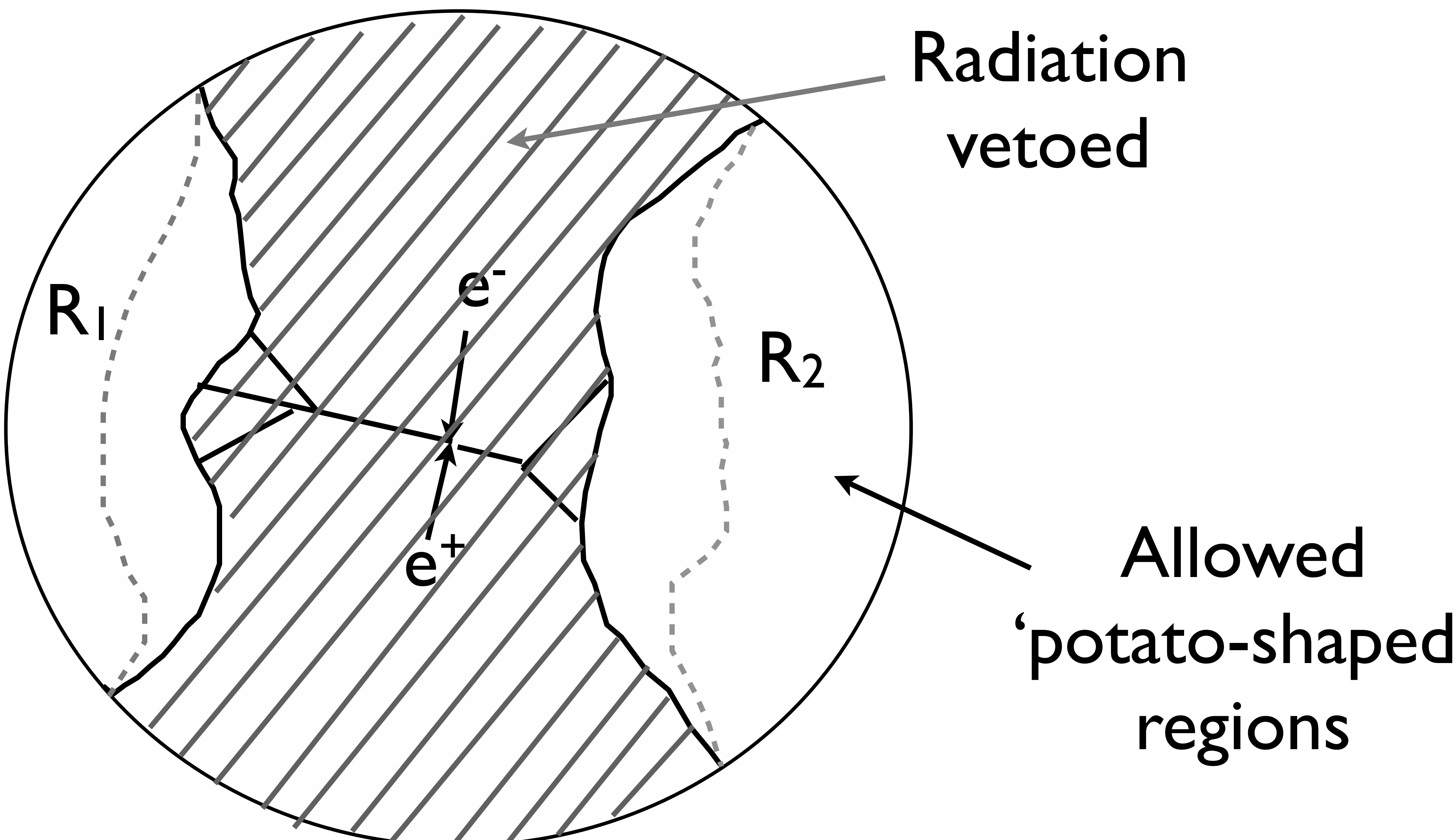
  - 3-loop evolution

Non-global logs

Q: Cross-section for $e^+e^- \to X$, with 'X' energy smaller than E_0 outside some region R



- Suppressed by soft* radiation: large logarithms

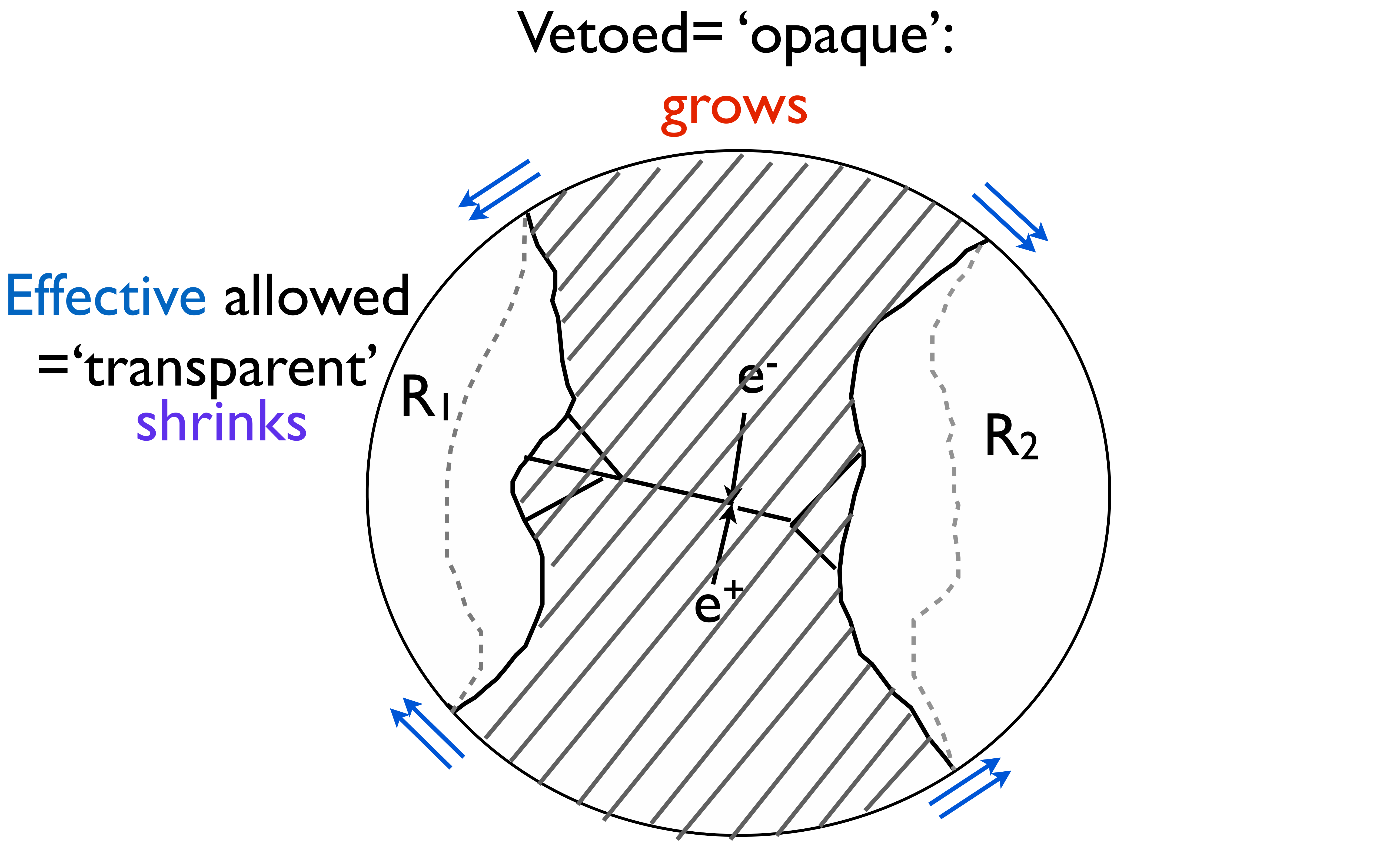
[Salam&Dasupta ‘01
Banfi, Salam& Dasgupta ’03]

- angles not ‘globally integrated’

- Difficult: need to keep track of all radiation in allowed region! [color&angle]

*’soft’=GeV<E<<TeV

As in forward scattering: transparent& opaque regions.
Opaque regions grows with energy



- Quantitative equivalence:

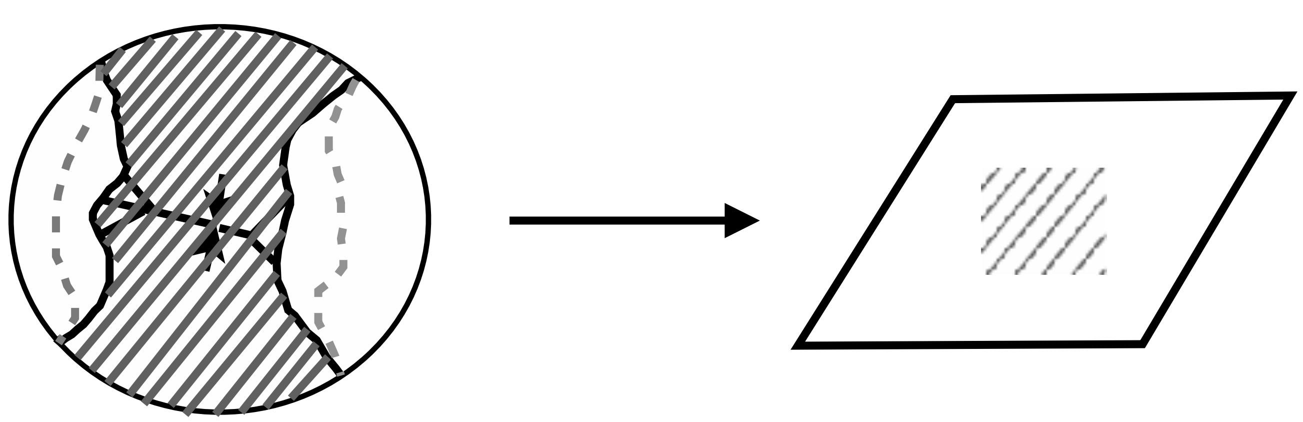
BK: $$\frac{d}{d\eta}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2z_0}{\pi}\frac{z_{12}^2}{z_{10}z_{02}}(U_{10}U_{02}-U_{12})$$ [Regge limit]

BMS: $$E\frac{d}{dE}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}(U_{10}U_{02}-U_{12})$$ [non-global logs]

- Conformal (stereographic) transformation:

$$\alpha_{ij} \equiv \frac{1-\cos\theta_{ij}}{2} \to z_{ij}^2 \equiv (z_i - z_j)^2, \qquad \frac{d\Omega}{4\pi} \to \frac{d^2z}{\pi}$$

[Weigert '03; Hatta '08-…]



Checks at LO and NLO

- Universal amplitude for soft gluon:

$$\lim_{p_0\to 0} M_{n+1} = \sum_i \frac{\epsilon \cdot p_i}{p_0 \cdot p_i} g T_i^a \times M_n$$

[Weinberg]

- Start with two parents and square:

$$|M_3|^2 \simeq \frac{s_{12}}{s_{10}s_{02}} |M_2|^2$$

- Energy logs from phase space integrals:

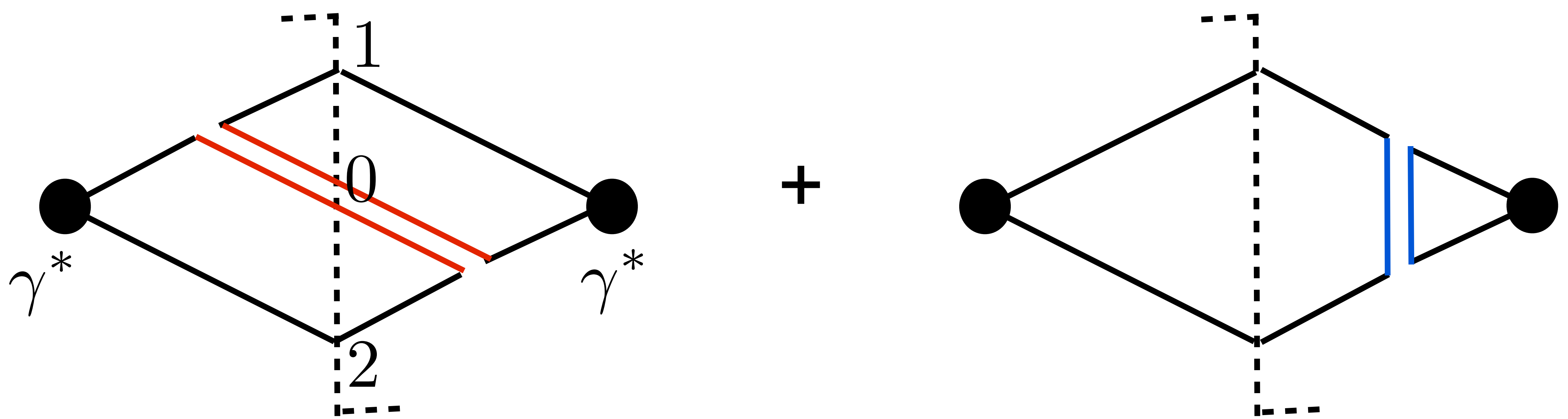
$$\sim \log(Q/E_0)$$

$$\int d\text{Lips}(p_0)|M_3|^2 \to |M_2|^2 \int_{E_0}^{Q} \frac{dp_0}{p_0} \int \frac{d\Omega}{4\pi} \frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}$$

- Similar to textbook computation of IR divergences, except angular integral 'not global'!

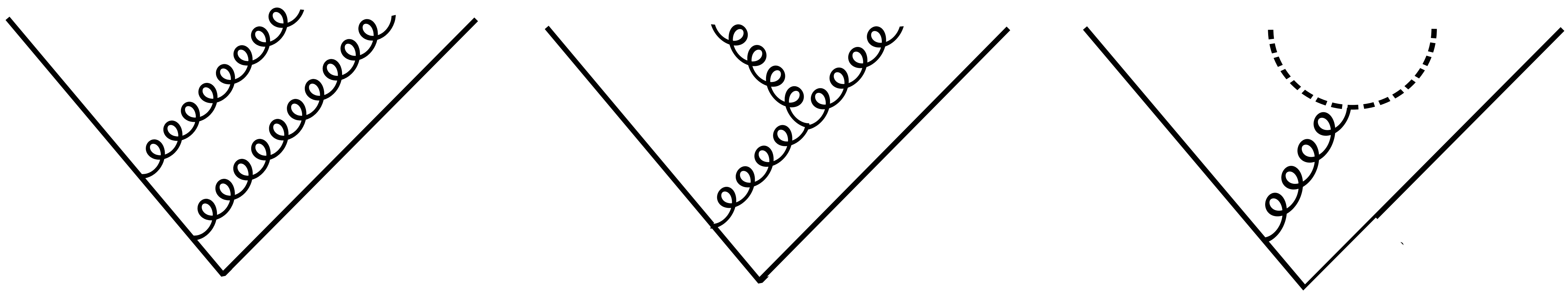
$$E\frac{d}{dE}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}\left(U_{10}U_{02} - U_{12}\right)$$ ✔

- Real& virtual related by KLN [cancel for U=1]



NLO:

[SCH, '15]



Square of tree-level soft current relatively simple:

$$|\mathcal{S}|^2 = \frac{s_{12}}{s_{10}s_{00'}s_{0'2}}\left[1 + \frac{s_{12}s_{00'} + s_{10}s_{0'2} - s_{10'}s_{20}}{2(s_{10}+s_{10'})(s_{02}+s_{0'2})}\right]$$

$$+ (n_F - 4)\frac{s_{12}}{s_{00'}(s_{10}+s_{10'})(s_{20}+s_{20'})}$$

$$+ (\tfrac{1}{2}n_s - n_F + 1)\frac{(s_{10}s_{20'} - s_{10'}s_{20'})^2}{s_{00'}^2(s_{10}+s_{10'})^2(s_{20}+s_{20'})^2}$$

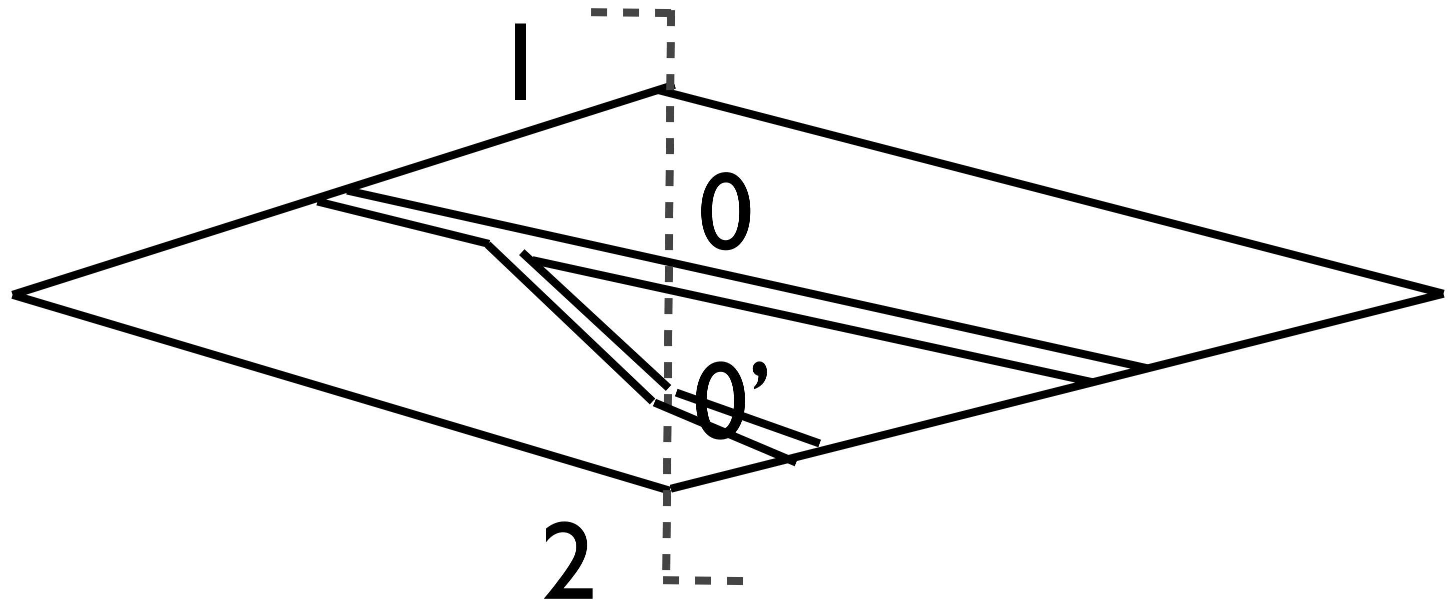
N=4SYM

general gauge thy

- Crucial: two soft gluons not independent

$$|\mathcal{S}|^2 = \frac{s_{12}}{s_{10}s_{00'}s_{0'2}}\left[1 + \frac{s_{12}s_{00'} + s_{10}s_{0'2} - s_{10'}s_{20}}{2(s_{10}+s_{10'})(s_{02}+s_{0'2})}\right]$$

- Amplitude depends on ratio of soft gluon energies
- NLO is basically the integral over that ratio

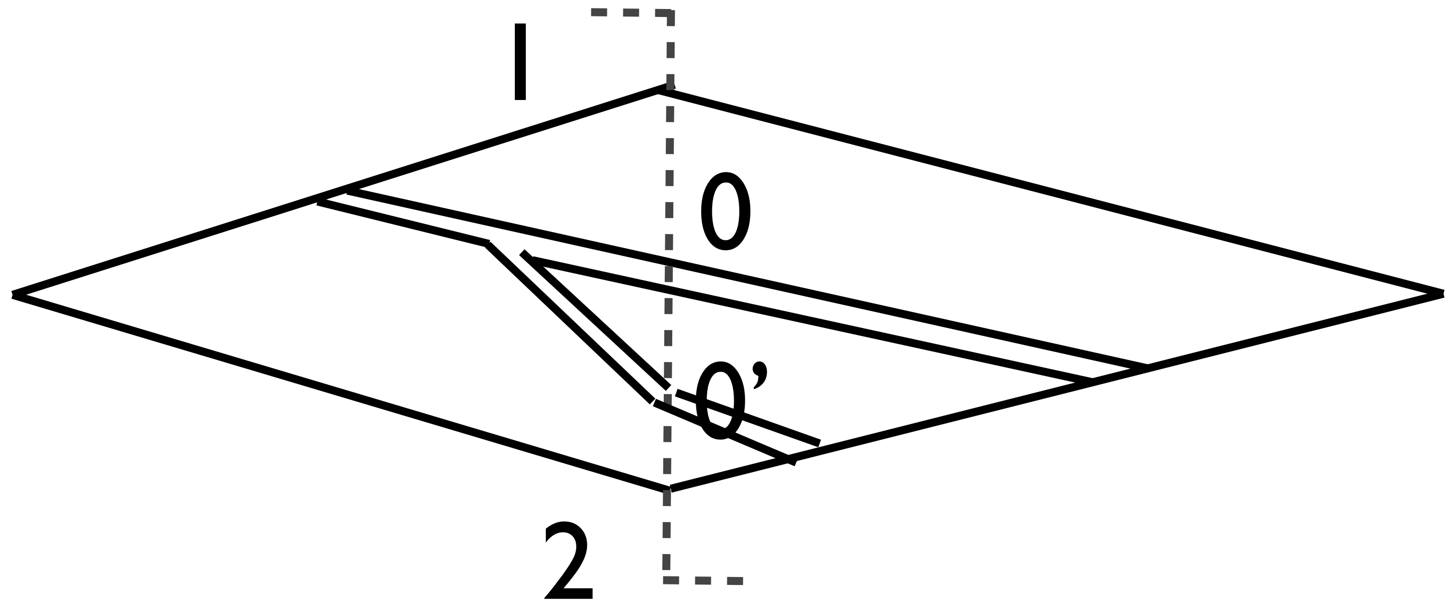


Pull out angular integrals:

$$E\frac{d}{dE}U_{12} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]} U_{10}U_{00'}U_{0'2}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty \tau d\tau \left[\left| \mathcal{S}(\tau\beta_0, \beta_{0'}) \right|^2 \qquad\qquad \right]$$



Pull out angular integrals:

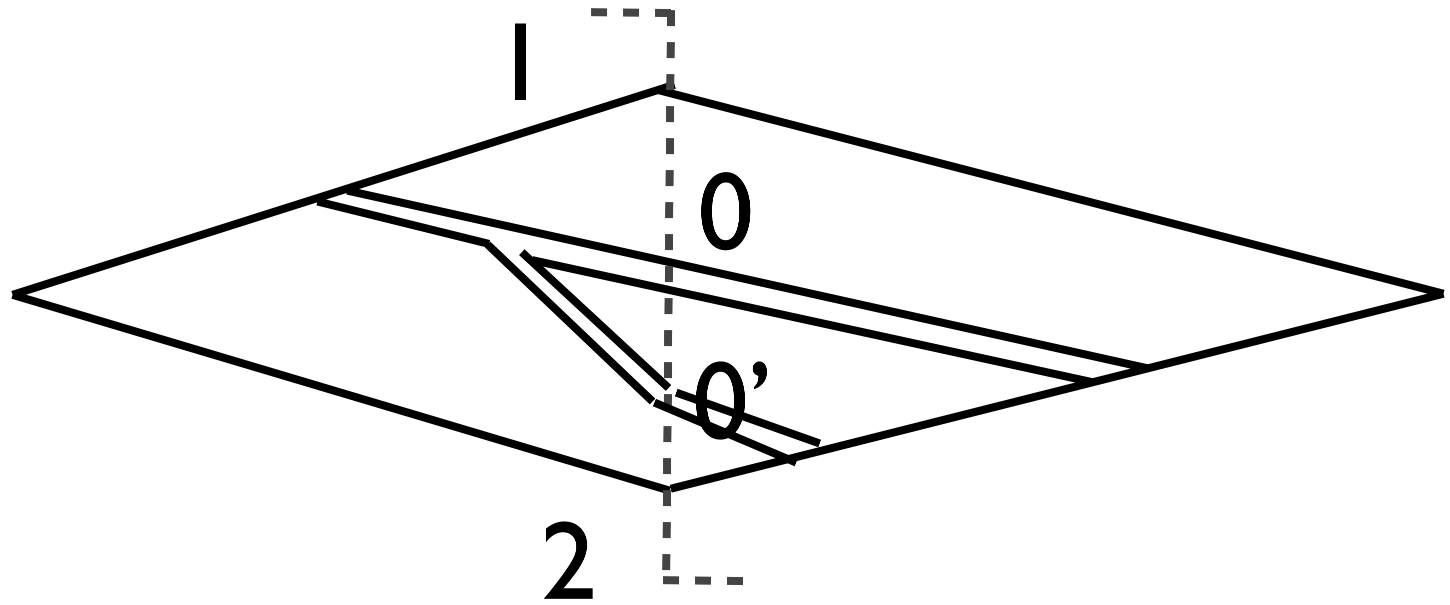
$$E\frac{d}{dE}U_{12} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]} U_{10} U_{00'} U_{0'2}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty \tau d\tau \left[\left| \mathcal{S}(\tau\beta_0, \beta_{0'}) \right|^2 - \big|_{\tau\to 0}\theta(\tau<1) - \big|_{\tau\to\infty}\theta(\tau>1) \right]$$

Subtract iterations of LO

⇒no subdivergences!



Pull out angular integrals:

$$E\frac{d}{dE}U_{12} \supset \int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K_{[1\,00'\,2]} U_{10} U_{00'} U_{0'2}$$

Integrate over relative energies:

$$K_{[1\,00'\,2]} = \int_0^\infty \tau d\tau \left[\begin{array}{l} \left|\mathcal{S}(\tau\beta_0, \beta_{0'})\right|^2 \\ -\Big|_{\tau\to 0}\theta(Q^2_{[1\tau 00']} < Q^2_{[10'2]}) \\ -\Big|_{\tau\to\infty}\theta(Q^2_{[00'2]} < Q^2_{1\tau 02]}) \end{array} \right]$$

Best: use Lorentz-invariant energy scales $Q^2_{[i0j]} \equiv \frac{s_{i0}s_{0j}}{s_{ij}}$

- Full (planar) NLO evolution: (non-global logs&Regge)

$$K^{(2)}U_{12} = \int_{\beta_0,\beta_{0'}} \frac{\alpha_{12}}{\alpha_{10}\alpha_{00'}\alpha_{0'2}} K^{(2)}_{[1\,00'\,2]} \left(U_{10}U_{02}+U_{10'}U_{0'2}-2U_{10}U_{00'}U_{0'2}\right) + \gamma_K^{(2)} K^{(1)} U_{12}$$

$$K^{(2)}_{[1\,00'\,2]} = 2\log\frac{\alpha_{12}\alpha_{00'}}{\alpha_{10'}\alpha_{02}} + \left(1+\frac{\alpha_{12}\alpha_{00'}}{\alpha_{10}\alpha_{0'2}-\alpha_{10'}\alpha_{02}}\right)\log\frac{\alpha_{10}\alpha_{0'2}}{\alpha_{10'}\alpha_{02}}$$

- Precisely Balitsky&Chirilli's (N=4) result!!!

[Balistky&Chirilli '07,'08]

- Eigenvalues match 'Pomeron trajectory'

[Fadin&Lipatov(&Kotikov) '98;
Ciafaloni&Gamici '98]

Note: use Lorentz-invariant energy scales, not ‘energy’!

$$Q^2_{[i0j]} \equiv \frac{s_{i0}s_{0j}}{s_{ij}}$$

This ensures Lorentz-invariance of BMS equation
= conformal invariance of BK equation

Similar to using : $k^+\sqrt{\frac{x_{10}^2 x_{02}^2}{x_{12}^2}}$ instead of k^+ in LO BK,

makes NLO automatically conformal invariant.

- Full non-planar result also available (N=4& QCD)

$$K^{(2)} = \int_{i,j,k}\int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K^{(2)\ell}_{ijk;00'} i f^{abc}\left(L^a_{i;0}L^b_{j;0'}R^c_k - R^a_{i;0}R^b_{j;0'}L^c_k\right)$$
$$+\int_{i,j}\int \frac{d^2\Omega_0}{4\pi}\frac{d^2\Omega_{0'}}{4\pi} K^{(2)N=4,\ell}_{ij;00'}\left(f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'} - \frac{C_A}{2}(U_0^{aa'}+U_{0'}^{aa'})\right)(L^a_i R^{a'}_j + R^{a'}_i L^a_j)$$
$$+\int_{i,j}\int \frac{d^2\Omega_0}{4\pi}\frac{\alpha_{ij}}{\alpha_{0i}\alpha_{0j}}\gamma_K^{(2)}\left(R^a_{i;0}L^a_j + L^a_{i;0}R^a_j\right) + K^{(2)N\neq 4}. \qquad (3.32)$$

(SCH ’15)

Equivalent to NLO B-JIMWLK result

[Kovner,Mulian&Lublinski ’14,
Balitsky&Chirilli ’14]

(cf’s Kovner’s talk)

Wait. QCD is not conformal!

- One can compute QCD non-global logs in the same way:

$$
\begin{aligned}
K^{(2)N\neq 4} = & \int_{i,j}\int \frac{d\Omega_0}{4\pi}\frac{d\Omega_{0'}}{4\pi}\frac{1}{\alpha_{00'}}\left[\frac{\alpha_{ij}\log\frac{\alpha_{0i}\alpha_{0'j}}{\alpha_{0'i}\alpha_{0j}}}{\alpha_{0i}\alpha_{0'j}-\alpha_{0'i}\alpha_{0j}}\right](L_i^a R_j^{a'}+R_i^{a'}L_j^a) \\
& \times\left\{2n_F \mathrm{Tr}_R\big[T^a U_0 T^{a'} U_{0'}^\dagger\big] - 4f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'} - (n_F T_R - 2C_A)(U_0^{aa'}+U_{0'}^{aa'})\right\} \\
& + \int_{i,j}\int \frac{d\Omega_0}{4\pi}\frac{d\Omega_{0'}}{4\pi}\frac{1}{2\alpha_{00'}^2}\left[\frac{\alpha_{0i}\alpha_{0'j}+\alpha_{0'i}\alpha_{0j}}{\alpha_{0i}\alpha_{0'j}-\alpha_{0'i}\alpha_{0j}}\log\frac{\alpha_{0i}\alpha_{0'j}}{\alpha_{0'i}\alpha_{0j}}-2\right](L_i^a R_j^{a'}+R_i^{a'}L_j^a) \\
& \times\left\{\begin{array}{c} 2(n_S-2n_F)\mathrm{Tr}_R\big[T^a U_0 T^{a'} U_{0'}^\dagger\big] + 2f^{abc}f^{a'b'c'}U_0^{bb'}U_{0'}^{cc'} \\ -((n_S-2n_F)T_R + C_A)(U_0^{aa'}+U_{0'}^{aa'}) \end{array}\right\} \\
& + \int_{i,j} 2\pi i b_0 \log(\alpha_{ij})\big(L_i^a L_j^a - R_i^a R_j^a\big).
\end{aligned} \tag{3.34}
$$

- They don't quite agree:

$$K_{Regge} - K_{Soft} = \beta^{(1)} \times \int (\ldots) \left(\frac{z_{ij}^2}{z_{0i}^2 z_{0j}^2} \log(\mu^2 z_{ij}^2) + \frac{z_{0j}^2 - z_{0i}^2}{z_{0i}^2 z_{0j}^2} \log \frac{z_{0i}^2}{z_{0j}^2} \right)$$

- Difference computable from matter loop…

Rapidity vs Soft divergences

- Work in d=4-2ε dimensions:

K_{Soft} does not depend on ε

$K_{Regge}(\epsilon)$ does

- In the conformal dimension, they are equal!

$$K_{Regge}(2\epsilon = -\beta(\alpha_s)) = K_{soft}$$

- Given the ε-dependence at lower loops, they are equivalent to each other!!!

[Vladimirov '16]

Upshot

- Full equivalence: non-global logs/(pert.)Regge limit
- Advantages:
  -Directly in coordinate space

  $$(\text{recall that } x_\perp \Leftrightarrow \theta \text{ stereographically})$$

  -Can use well-studied on-shell building blocks:
  soft currents, KLN theorem, …

- Works in QCD: difference $\propto \beta$ computable

From Ian Balitsky's talk:

NLO evolution of composite "conformal" dipoles in QCD

I. B. and G. Chirilli

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a \;=\;& \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a \\
&\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
&+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{23}{}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{23}{}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{23}{}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big] \\
&\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)] \\
&+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}{}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}{}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}{}^2z_{23}^2}\Big] \\
&\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}
\end{aligned}
$$

$$b=\frac{11}{3}N_c-\frac{2}{3}n_f$$

=O(eps)term in LO BK

=matter loops NGLs

=N=4 NGLs

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

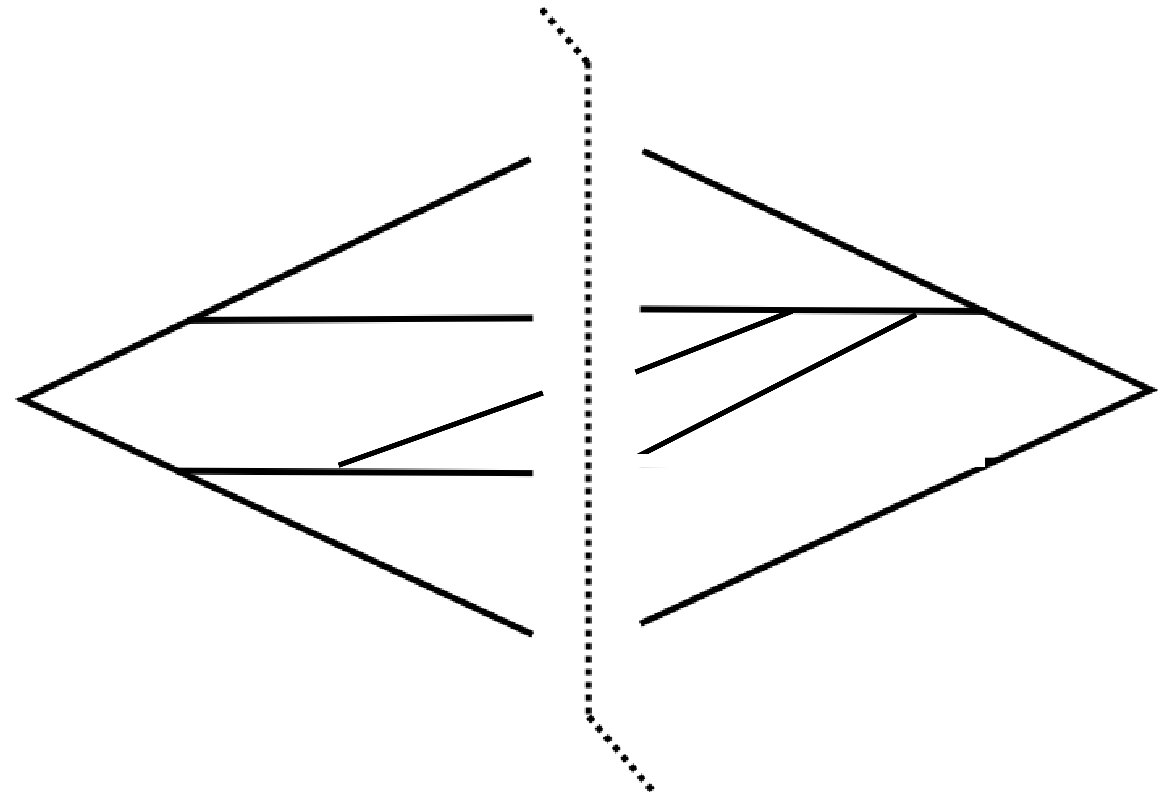
NNLO

[Herranen+SCH, '16]

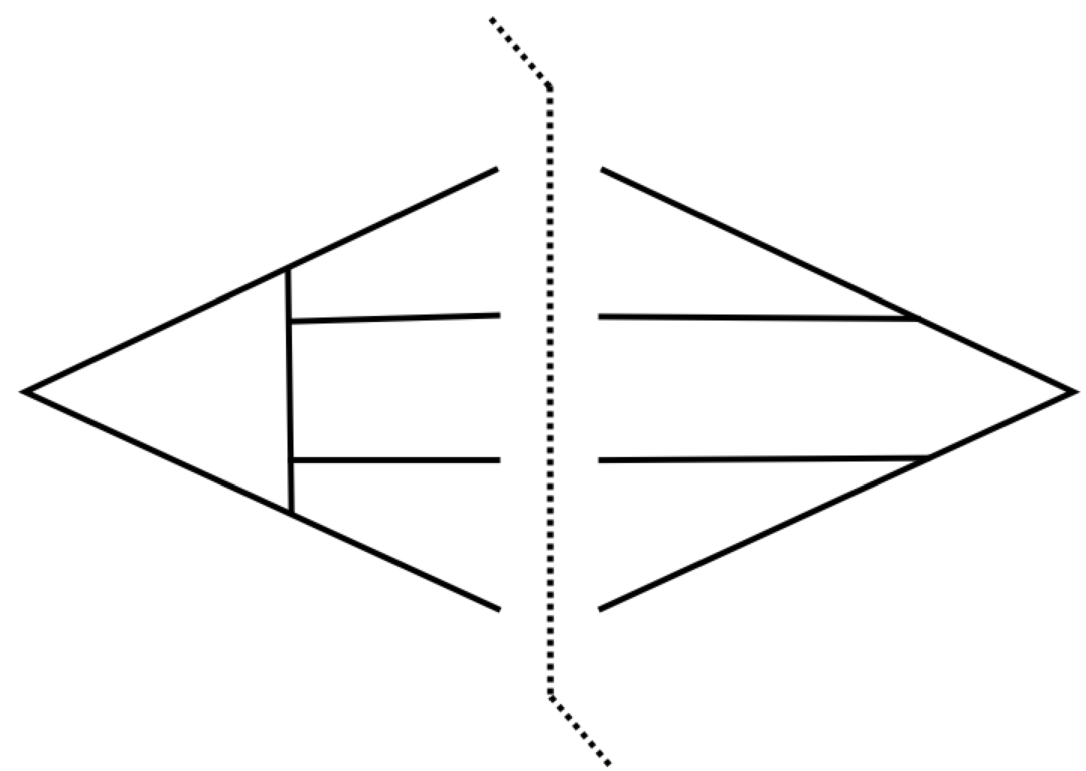
- Triple soft current at tree-level
  ⇒ extract from known 4-particle integrand ✓
- Double soft current at one-loop
  ⇒ extract from known one-loop 6-point ✓
- Single soft current at two-loops
  ⇒ not needed: contribution really just $\gamma_K^{(3)}$ ✓
- Fully virtual IR divergences at three-loops
  ⇒ not needed: KLN fixes it from rest ✓

- Sample graphs we computed/borrowed:

triple soft emission (squared) at tree-level



double soft emission: tree/one-loop interference



- Recursive subtraction of subdivergences:

$$F^{\rm sub}_{[1\,0\,2]} \equiv F_{[1\,0\,2]} = 1, \tag{4.20a}$$

$$F^{\rm sub}_{[1\,00'\,2]} \equiv F_{[1\,00'\,2]} - [1\,0\,0'][1\,0'\,2] - [0\,0'\,2][1\,0\,2], \tag{4.20b}$$

$$\begin{aligned} F^{\rm sub}_{[1\,00'0''\,2]} \equiv\; & F_{[1\,00'0''\,2]} - [1\,0\,0'][1\,0'0''\,2] - [0\,0'\,0''][1\,00''\,2] - [0'\,0''\,2][1\,00'\,2] \\ & -[1\,00'\,0''][1\,0''\,2] - [0\,0'0''\,2][1\,0\,2] \\ & -[1\,0\,0'][1\,0'\,0''][1\,0''\,2] - [0'\,0''\,2][0\,0'\,2][1\,0\,2] - [0\,0'\,0''][1\,0\,0''][1\,0''\,2] \\ & -[0\,0'\,0''][0\,0''\,2][1\,0\,2] - [1\,0\,0'][0'\,0''\,2][1\,0'\,2] - [0'\,0''\,2][1\,0\,0'][1\,0'\,2]. \end{aligned} \tag{4.20c}$$

energy step functions

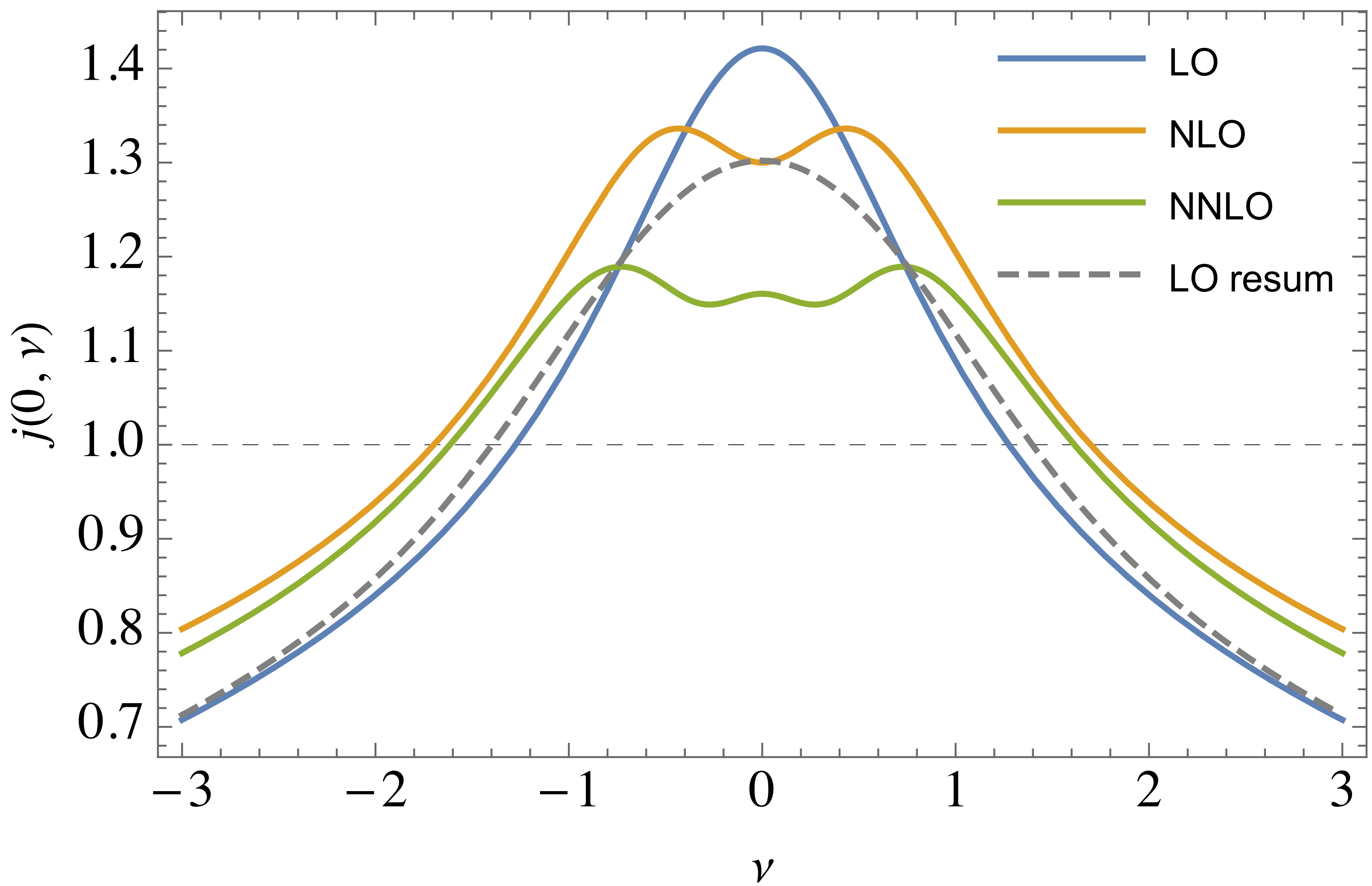
- Cleanly removes iterations of lower-loop evolution
- Compute only finite absolutely convergent integrals ✓

result:

$$K^{(3)}_{[1\,00'\,2]} = \left(1-\frac{u}{1-v}\right)\log v\left[\log u\log\frac{v}{u}-\frac{1}{3}\log^2 v-4\zeta_2\right]+2(1+v-u)\left(\zeta_2\log\frac{u}{v}-2\zeta_3\right)$$
$$+\left(\frac{2u}{1-v}+v-u-1\right)\left[4\mathrm{Li}_3\left(1-\frac{1}{v}\right)+2\mathrm{Li}_2\left(1-\frac{1}{v}\right)\log\frac{v}{u}\right]-\frac{5}{6}\log^3 u$$
$$+4\big(\mathrm{Li}_3(x)+\mathrm{Li}_3(\bar{x})-2\zeta_3\big)-2\big(\mathrm{Li}_2(x)+\mathrm{Li}_2(\bar{x})+2\zeta_2\big)\log u\,. \qquad (4.35)$$

$$K^{(3)}_{[1\,00'0''\,2]} = \left(1-\frac{u_3}{1-v_1v_2}\right)\left[\begin{matrix}2\mathrm{Li}_2\left(1-\frac{1}{v_1v_2}\right)-2\mathrm{Li}_2\left(1-\frac{1}{v_1}\right)-2\mathrm{Li}_2\left(1-\frac{1}{v_2}\right)\\ +\log v_1\log v_2+\log(v_1v_2)\big(\log(u_1u_2)-\frac{3}{2}\log u_3\big)\end{matrix}\right]$$
$$+(u_1u_2-u_1v_2-u_2v_1+v_1+v_2-u_1-u_2+u_3)\left[\mathrm{Li}_2\left(1-\frac{1}{v_1v_2}\right)-\zeta_2\right]$$
$$+3\log u_1\log u_2-\tfrac{3}{2}\log^2 u_3+(1+P)(f+f_1), \qquad (4.23)$$

Attached in computer-friendly format to arXiv submission.



- Pomeron trajectory = linearized eigenvalue

$$U_{ij} = 1 - \frac{1}{N_c}\mathcal{U}_{ij}$$

$$\frac{d}{d\eta}\mathcal{U}_{m,\nu} = \left[j(m,\nu) - 1\right]\mathcal{U}_{m,\nu}$$

for eigenfunction: $\mathcal{U}_{m,\nu} = |z_i - z_j|^{i\nu} e^{im\arg(z_i - z_j)} \quad (\Delta = 2 + i\nu)$

[see Brower,Polchinski,Strassler&Tan]

Tests

- Collinear limit $\nu \to \pm i$ controlled by small-x limit of DGLAP

[Jaroscewicz ’83; Ball, Falgari, Forte,Marzani… 07]

$$\omega^{(3)} \to +g^6\left(\frac{1024}{\gamma^5} - \frac{512}{\gamma^3}\zeta_2 + \frac{576}{\gamma^2}\zeta_3 - \frac{464}{\gamma}\zeta_4 + 840\zeta_5 + 64\zeta_2\zeta_3 + \gamma\left(-40{\zeta_3}^2 - 373\zeta_6\right) + \gamma^2\left(-8\zeta_2\zeta_5 - 86\zeta_3\zeta_4 + \frac{1001}{4}\zeta_7\right)\right). \tag{21}$$

✔

[Velizhanin ’15]

- Analytic expression for m=0 conjectured using Integrability of planar N=4

$$\frac{F^{(3)}_{0,\nu}}{32} = -S_5 + 2S_{-4,1} - S_{-3,2} + 2S_{-2,3} - S_{2,-3} - 2S_{3,-2} + 4S_{-3,1,1} + 4S_{1,-3,1} + 2S_{1,-2,2} + 2S_{1,2,-2} + 2S_{2,1,-2} - 8S_{1,-2,1,1} + \zeta_2\left(S_1S_2 - 3S_{-3} + 2S_{-2,1} - 4S_{1,-2}\right) - \tfrac{49}{2}\zeta_4 S_1 + 7\zeta_3\left(2S_{1,-1} + 2(S_1 - S_{-1})\log 2 - S_{-2} - \log^2 2\right) + (8\zeta_{-3,1} - 17\zeta_4)\left(S_{-1} - S_1 + \log 2\right) - \tfrac{1}{2}\zeta_3 S_2 + 4\zeta_5 - 6\zeta_2\zeta_3 + 8\zeta_{-3,1,1}\,. \tag{C.3}$$

✔

[Gromov,Levkovich-Maslyuk&Sizov, ’15]

Conclusions

- Modern approach to high-energy scattering via Wilson lines: new theoretical control @NNLL
- Evolution now known in planar N=4 SYM:
  -eigenvalue for m=1,2,3,...
  -nonlinear interactions '3-Pomeron vertex'
- Possible extension to (planar?) QCD
- Study convergence& resummations?

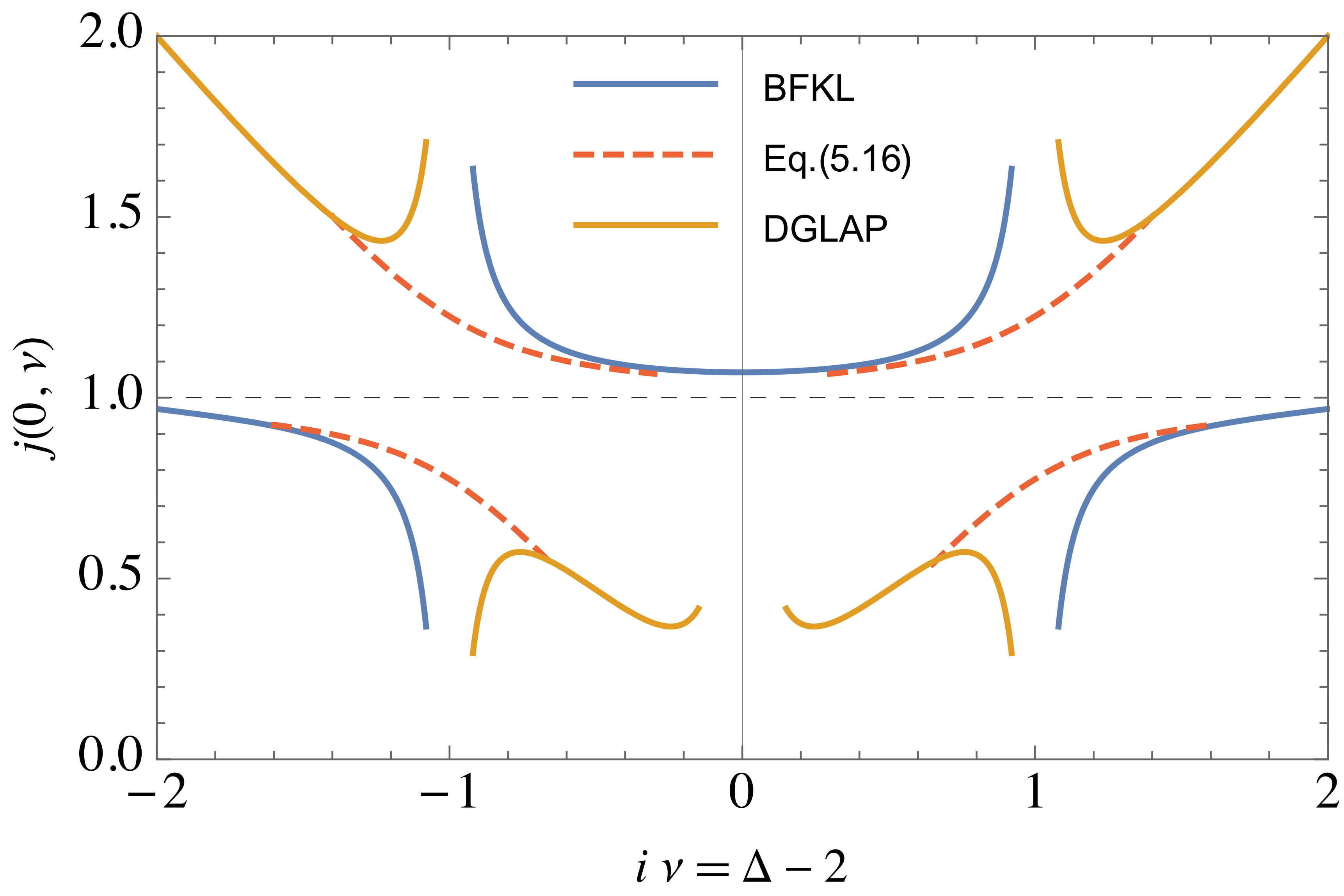
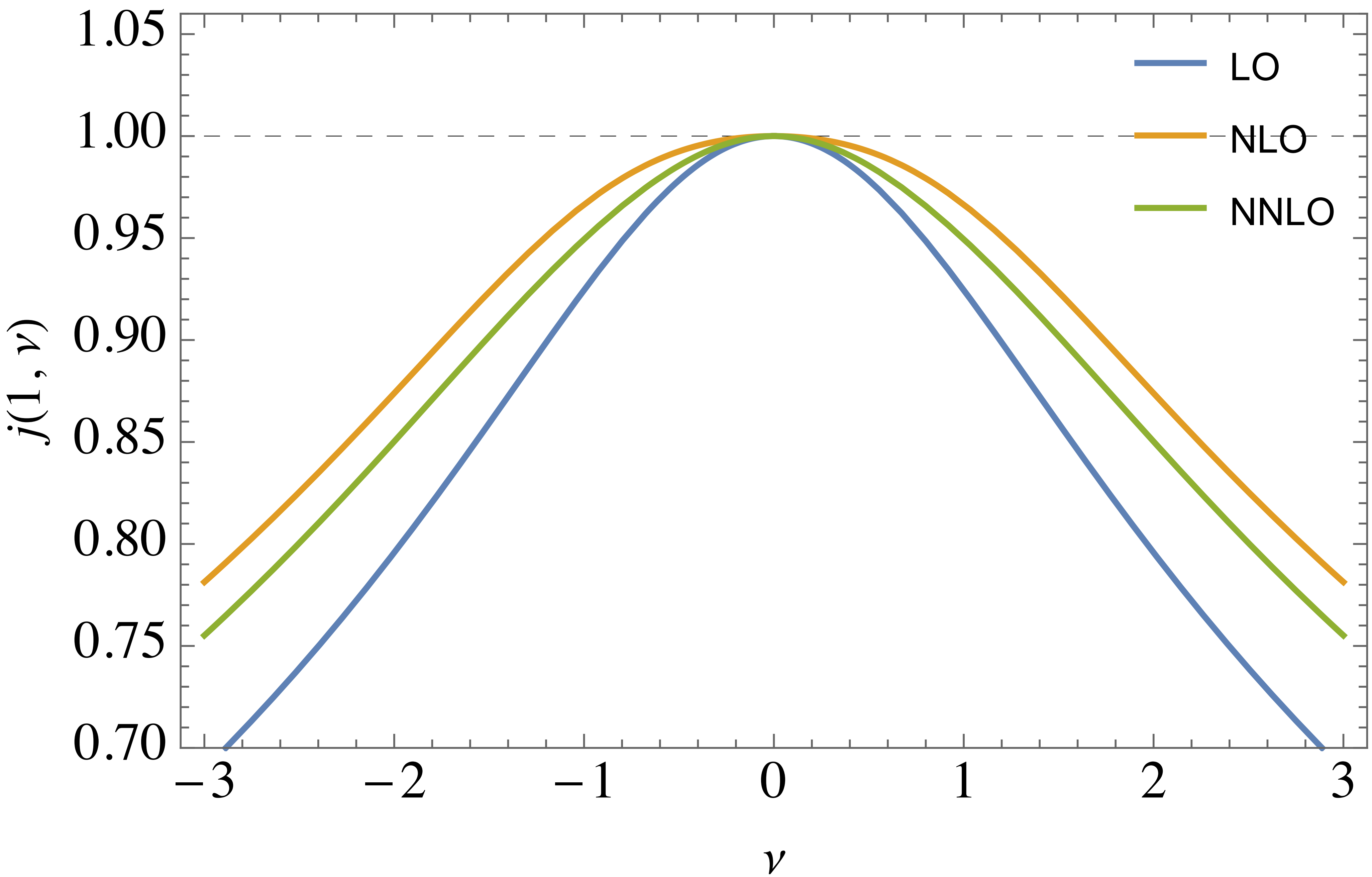


Figure 6. Level repulsion between the Pomeron and DGLAP trajectories for $m = 0$ as a function of scaling dimension, illustrating the $\nu = \pm i$ singularities. (LO expressions plotted with $\lambda = g^2_{\text{YM}} N_c = 1$.)

$$j \approx 1 + \frac{\Delta - 3 \pm \sqrt{(\Delta - 3)^2 + 32 g^2}}{2}, \qquad \Delta = 2 + i\nu. \tag{5.16}$$

m=1 (leading Odderon trajectory)



note: Odderon intercept=1 to all orders in λ. Agrees with strong coupling!

[Tan et al '14]

On the Odderon intercept

- m=1,ν=0 is a very special wavefunction:

$$\mathcal{U}_{12} = 1 - \frac{1}{N_c}(z_1 - z_2)$$

- Strings of dipoles in planar limit telescope:

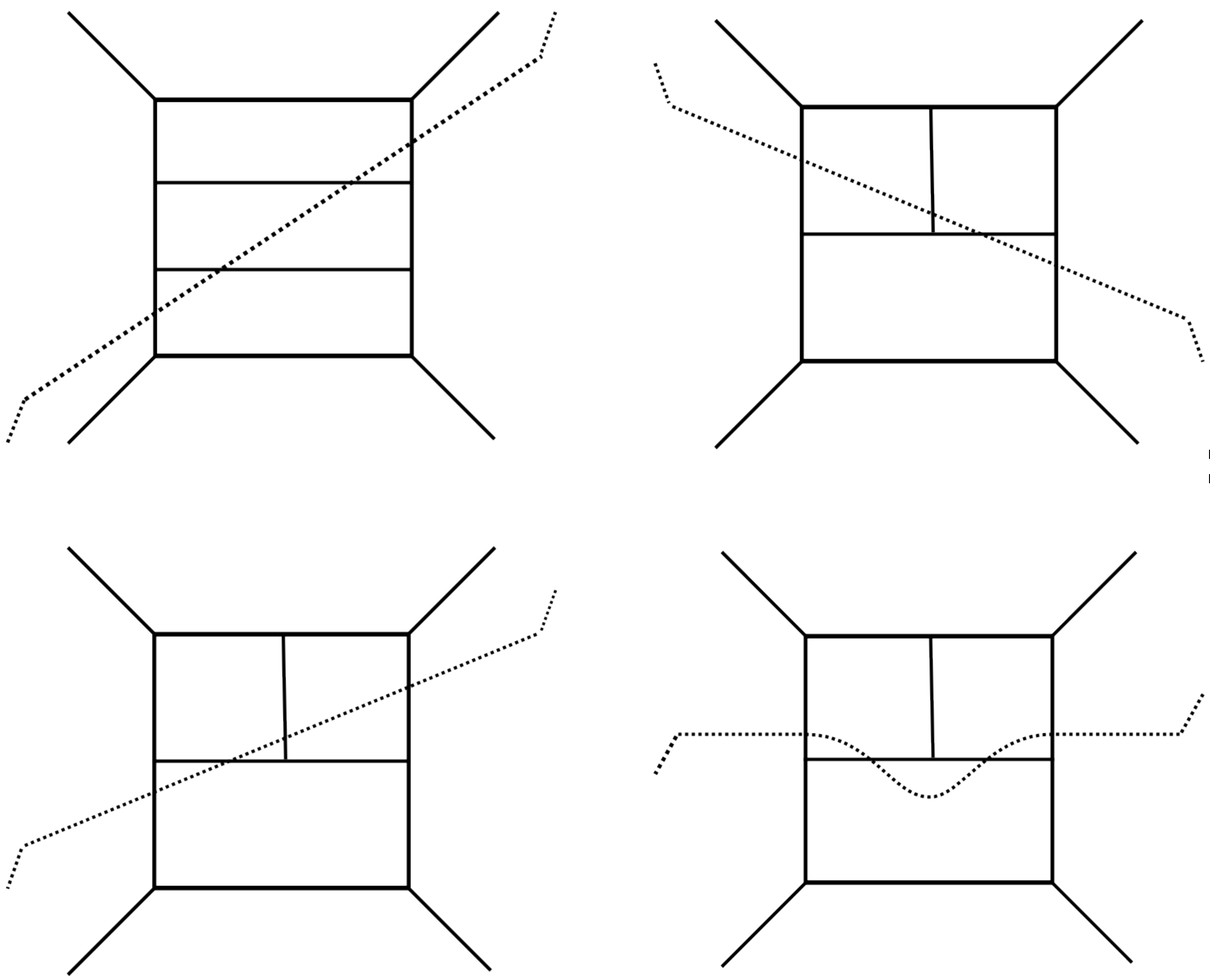
$$\mathcal{U}_{10}\mathcal{U}_{02} = 1 - \frac{1}{N_c}((z_1 - \cancel{z_0}) + (\cancel{z_0} - z_2)) + O(1/N_c^2)$$
$$= 1 - \frac{1}{N_c}(z_1 - z_2) = \mathcal{U}_{12}$$

$$\mathcal{U}_{10}\mathcal{U}_{00'2}\mathcal{U}_{0'2} = \mathcal{U}_{12}$$

...

- Cancel in evolution. Thm: Odderon intercept vanishes to all order in λ in planar limit

- Soft current squared from four-particle planar integrand:



$$= 1 + \frac{s_{12}s_{00'} + s_{10}s_{0'2} - s_{10'}s_{20}}{2(s_{10}+s_{10'})(s_{02}+s_{0'2})}$$

- Known 8-loop integrand $\leftrightarrow$ maximally nonlinear term in 7-loops evolution! [Bourjaily, Heslop&Tran '15]

Linearized kernel

- Start with full evolution, expand w/ $U_{ij} \approx 1 - \frac{1}{N_c}\mathcal{U}_{ij}$

$$K^{(L)}\mathcal{U}_{12} = \left(\gamma_K^{(L)} K^{(1)} + 2C^{(L)}\right)\mathcal{U}_{12} + \int \frac{d^2z_0 d^2z_{0'}}{\pi^2} \frac{(-2)\alpha_{12}\,\mathcal{U}_{00'}}{\alpha_{10}\alpha_{00'}\alpha_{0'2}} K^{(L)\mathrm{lin}}_{[1\,00'\,2]}$$

- The functions K^(L) are relatively simple:

$$\text{(only 5 letters: } \{x,\, \bar{x},\, 1{-}x,\, 1{-}\bar{x},\, x{+}\bar{x}{-}x\bar{x}\})$$

- On translation-invariant states, do one integral:

$$K^{(L)}\mathcal{U}(x) = \left(\gamma_K^{(L)} K^{(1)} + 2C^{(L)}\right)\mathcal{U}(x) - \int \frac{d^2y}{|y|^2}\, H^{(L)}(y)\,\mathcal{U}(xy)\,,$$

- Trajectory = Fourrier-Mellin transform of H

3-loop special functions:

In nonlinear evolution:

$$= 2\mathrm{Re}\left\{\left[1+\frac{\alpha_{0'0''}\langle 0\,2\rangle[2\,1]}{\alpha_{0''2}\langle 0\,0'\rangle[0'1]-\alpha_{0'2}\langle 0\,0''\rangle[0''1]}\right]\left[\mathrm{Li}_2\left(1-\frac{\alpha_{10'}\alpha_{0''2}}{\alpha_{10''}\alpha_{0'2}}\right)-\mathrm{Li}_2\left(1-\frac{\alpha_{00''}\alpha_{0'2}}{\alpha_{00'}\alpha_{0''2}}\right)+\right.\right.$$
$$\left.\left.+\mathrm{Li}_2\left(-\frac{[1\,0][0'\,0'']}{[1\,0''][0\,0']}\right)-\mathrm{Li}_2\left(-\frac{\langle 1\,0\rangle\langle 0'\,0''\rangle}{\langle 1\,0''\rangle\langle 0\,0'\rangle}\right)+\log\frac{\alpha_{10}\alpha_{0'0''}}{\alpha_{10''}\alpha_{00'}}\log\frac{\alpha_{0''2}\langle 0\,0'\rangle[0'1]}{\alpha_{0'2}\langle 0\,0''\rangle|[0''1]}\right]\right\}. \quad (4.22)$$

In linear limit: $(5 \text{ letters: } \{x,\, \bar{x},\, 1{-}x,\, 1{-}\bar{x},\, x{+}\bar{x}{-}x\bar{x}\})$

$$O_1 = 2\big(\mathrm{Li}_3(x)+\mathrm{Li}_3(\bar{x})-2\zeta_3\big)-\log u\big(\mathrm{Li}_2(x)+\mathrm{Li}_2(\bar{x})\big), \quad \text{(B.10a)}$$

$$O_2 = 2\big(\mathrm{Li}_3(1{-}x)+\mathrm{Li}_3(1{-}\bar{x})-2\zeta_3\big)-\log v\big(\mathrm{Li}_2(1{-}x)+\mathrm{Li}_2(1{-}\bar{x})\big), \quad \text{(B.10b)}$$

$$O_3 = \left\{\mathrm{Li}_3\left(\frac{\bar{x}}{x(\bar{x}{-}1)}\right)+\mathrm{Li}_3\left(\frac{x(\bar{x}{-}1)}{\bar{x}}\right)+\frac{1}{2}\left[\mathrm{Li}_2\left(\frac{\bar{x}}{x(\bar{x}{-}1)}\right)-\mathrm{Li}_2\left(\frac{x(\bar{x}{-}1)}{\bar{x}}\right)\right]\log(1{-}x)(1{-}\bar{x})\right.$$
$$-4\mathrm{Li}_3(x)-2\mathrm{Li}_3(1{-}x)+\log(x\bar{x})\mathrm{Li}_2(x)+\frac{1}{6}\log^3(1{-}x)-\frac{1}{2}\log^2(1{-}x)\big(\log(x)-\log(\bar{x})\big)$$
$$\left.-\frac{1}{4}\log^2(1{-}x)\log(1{-}x)(1{-}\bar{x})+\zeta_2\log(1{-}x)\right\}-(x\leftrightarrow\bar{x}). \quad \text{(B.10c)}$$