TWO-PARTON SCATTERING IN THE HIGH-ENERGY LIMIT AND THE THREE-REGGEON CUT

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Iterated integrals and the Regge limit, Edinburgh, 11/04/2017

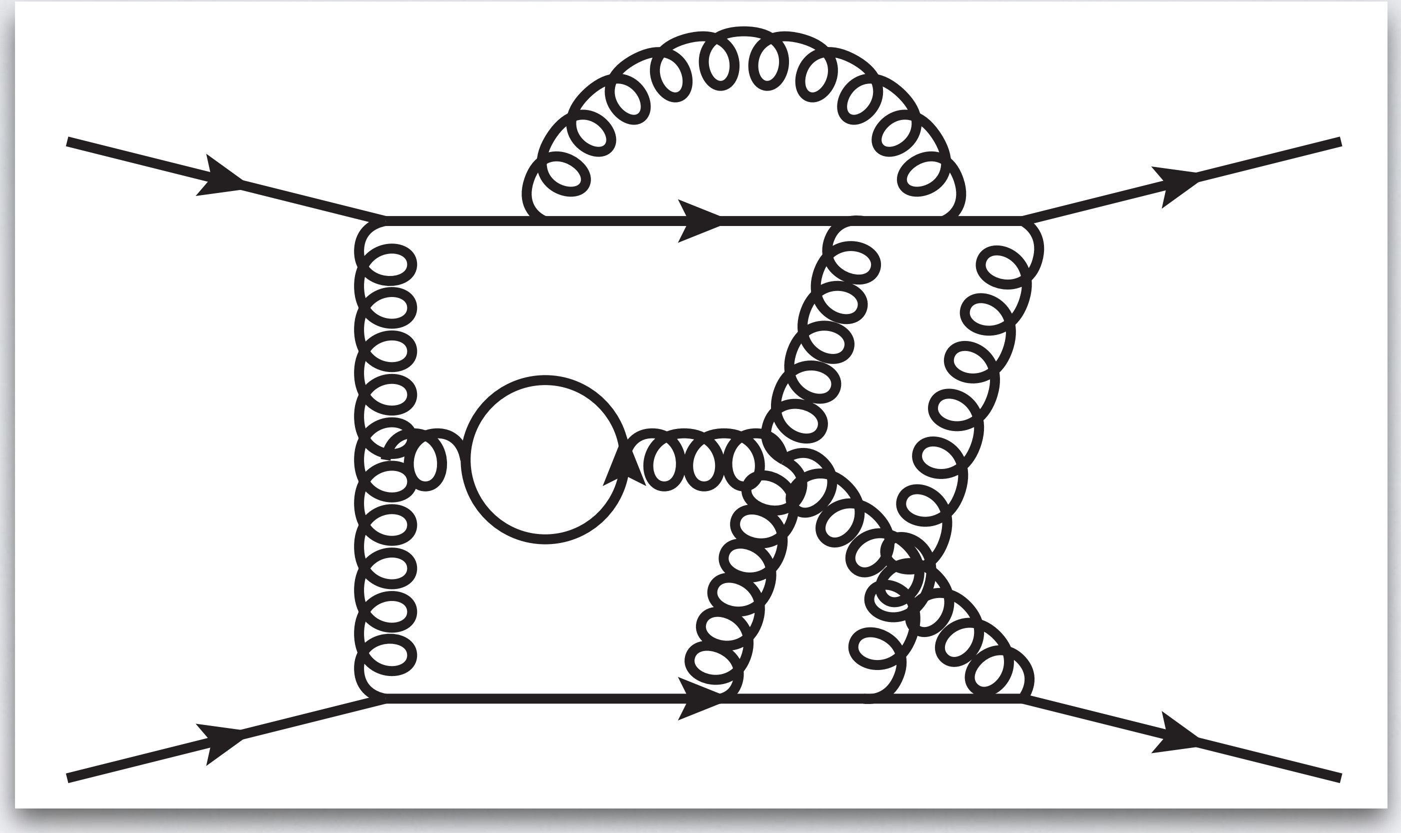
OUTLINE

- **Aspects of 2 → 2 scattering amplitudes in the high-energy limit**
- **The Balitsky-JIMWLK equation and the three loop amplitude**
- **Comparison between Regge and infrared factorization**

In collaboration with Simon Caron-Huot and Einan Gardi,

Based on arXiv:1701.05241

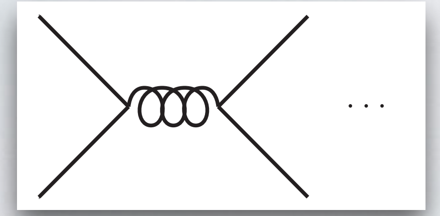
ASPECTS OF 2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

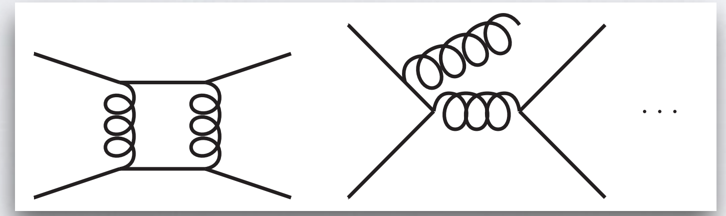


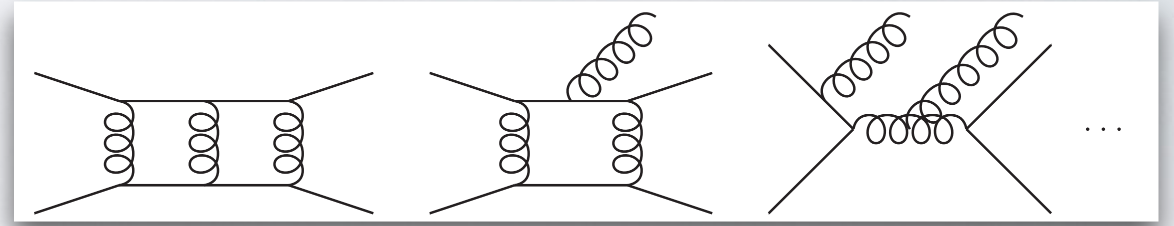
2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Calculation of scattering amplitudes at high order in perturbation theory is one of the main ingredients for the program of precision physics at the LHC

$$\mathcal{M} = 4\pi\alpha_s \left[\mathcal{M}^{(0)} + \frac{\alpha_s}{4\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \mathcal{M}^{(2)} + \ldots\right]$$



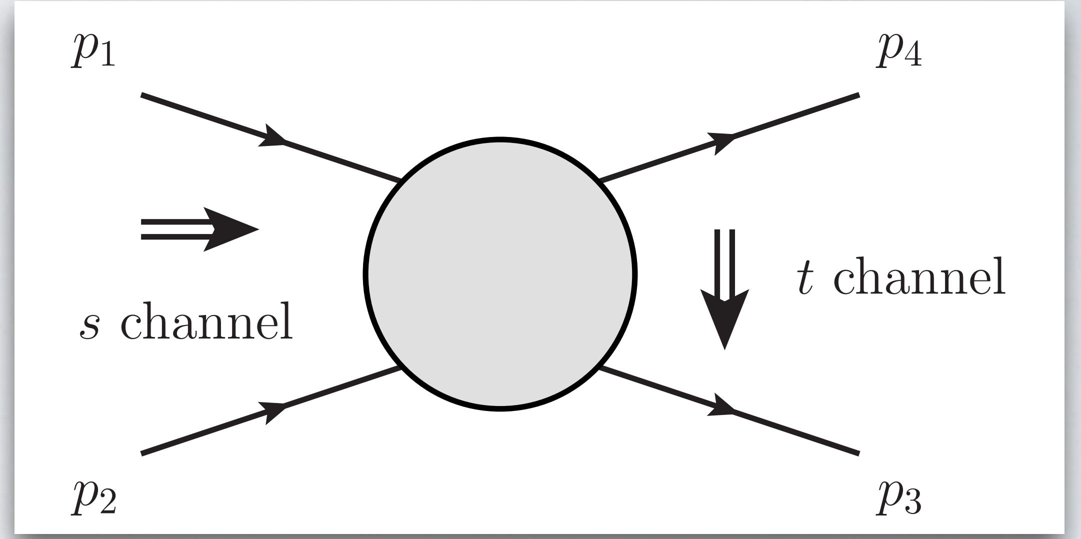




- Amplitudes are complicated functions of the kinematical invariants, their calculation is non-trivial, and it is subject of intense study.
  - Express Feynman integrals in terms of known functions (harmonic polylogarithms, elliptic integrals, etc)
  - Amplitudes contains infrared divergences, which must cancel when summing virtual and real corrections.

2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Information and constraints can be obtained by considering kinematical limits:
  - it reduces the number of invariants;
  - it helps identifying factorisation properties and iterative structures of the amplitude;
  - it may be relevant for phenomenology: because of soft and collinear enhancement, amplitudes in specific kinematic limit develops large logarithms, which may spoil the convergence of the perturbative expansion in that region of the parameter space.



- Consider 2 → 2 scattering amplitudes in the high-energy limit:

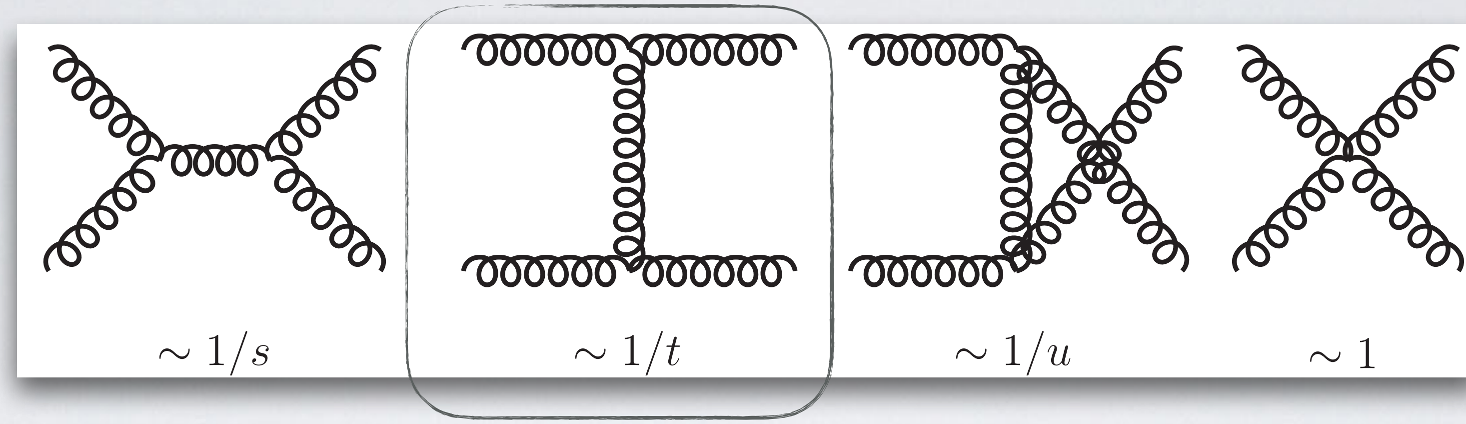
$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0$$

- The amplitude becomes a function of the ratio |s/t|; here we consider the leading power term in this expansion

$$\mathcal{M}(s, t, \mu) = \mathcal{M}_{LP}\left(\frac{s}{-t}, \frac{-t}{\mu^2}\right) + \mathcal{O}\left(\frac{-t}{s}\right)$$

2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Consider, as an example, the gluon-gluon scattering amplitude at tree level:



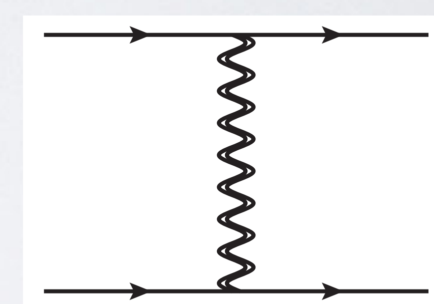
- In the high-energy limit only the second diagram contributes at leading power. The amplitude is simply

$$\mathcal{M}(s,t) = 4\pi\alpha_s \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \mathcal{M}^{(n)}(s,t), \qquad \mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t}\,(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}.$$

- The amplitude at higher orders contains logarithms of the ratio |s/t|. In the sixties the dominant behaviour in the high-energy limit was characterised in terms of Regge poles and cuts. These can now be studied in the context of QCD. One has

Regge, Gribov

$$\mathcal{M}_{ij\to ij}|_{\rm LL} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}\,C_A\,\alpha_g^{(1)}(t)} 4\pi\alpha_s\,\mathcal{M}^{(0)}_{ij\to ij},$$



- where the function $\alpha_g(t)$ is known as the Regge trajectory:

$$\alpha_g(t) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \alpha_g^{(n)}(t), \qquad \alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \overset{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon},$$

- and r_Γ is a ubiquitous 1-loop factor:

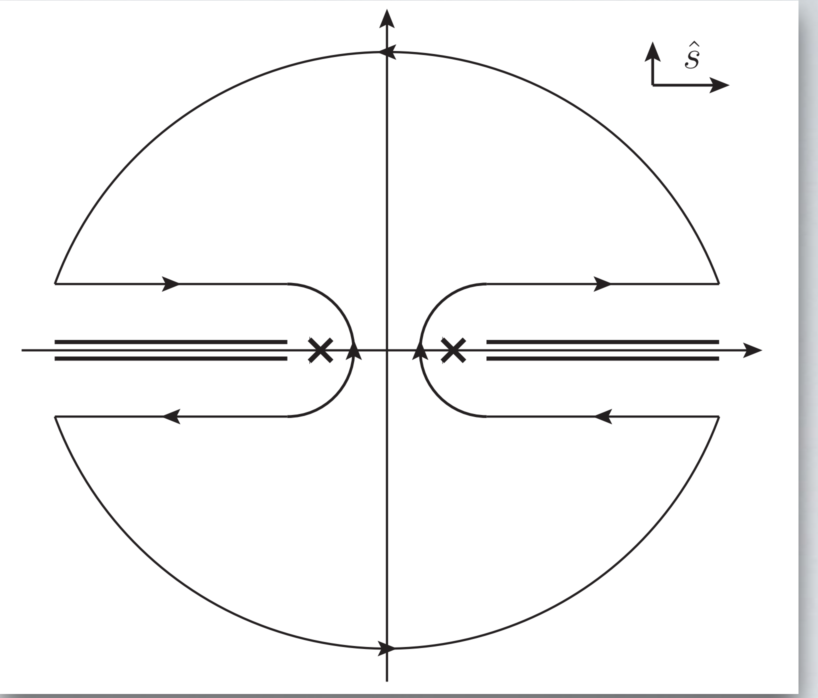
$$r_\Gamma = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1 - \frac{1}{2}\zeta_2\,\epsilon^2 - \frac{7}{3}\zeta_3\,\epsilon^3 + \dots$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: ANALYTIC STRUCTURE

- Beyond LL the structure of factorisation of high-energy logarithms becomes much richer.
- We need to investigate more carefully the analytic structure of the amplitude, which can be summarised via the dispersion relation

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i0} D_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i0} D_u(\hat{u},t)$$

- where D_s and D_u are discontinuities of $M(s, t)$ in the s- and u-channels.



Regge, Gribov, .. see also Collins

- D_s and D_u are real (spectral density of positive energy states propagating in the s- and u-channels). Parametrise them as a sum of power laws by means of a Mellin transformation:

$$a_j^s(t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}} D_s(\hat{s},t)\left(\frac{\hat{s}}{-t}\right)^{-j}, \qquad D_s(s,t) = \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} dj\, a_j^s(t)\left(\frac{s}{-t}\right)^j,$$

- Note that the reality condition of $D_s(s,t)$ implies that the Fourier coefficients admit

$$\left(a_{j^*}^s(t)\right)^* = a_j^s(t),$$

- Substituting the inverse transform into the dispersive representation, swapping the order of integration and integrating over $\hat{s}$ and $\hat{u}$, one obtains a Mellin representation of the amplitude:

$$\mathcal{M}(s,t) = \frac{-1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)}\left(a_j^s(t)\left(\frac{-s-i0}{-t}\right)^j + a_j^u(t)\left(\frac{s+t-i0}{-t}\right)^j\right).$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: ANALYTIC STRUCTURE

- The dispersion relation allows us to infer useful properties concerning the projection of the amplitude onto eigenstates of signature, that is crossing symmetry $s \leftrightarrow u$:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

- $M^{(+)}$ and $M^{(-)}$ are referred respectively to as the even and odd amplitudes. Restricting to the region $s > 0$ and working to leading power as $s \gg |t|$, the formula then evaluates to

$$\mathcal{M}^{(+)}(s,t) = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\tfrac{\pi j}{2}\right) a_j^{(+)}(t)\, e^{jL}\,,$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\tfrac{\pi j}{2}\right) a_j^{(-)}(t)\, e^{jL}\,,$$

- Where $a_j^{(\pm)}(t) = 1/2(a_j^s(t) \pm a_j^u(t))$, and L is the natural signature-even combination of logs:

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

- The reality properties of $a_j^s(t)$, $a_j^u(t)$ implies that $M^{(+)}$ and $M^{(-)}$ are imaginary and real, respectively, when expressed in powers of L (not $\log|s/t|$).
- At leading power in t/s the Mellin variable j is identical to the spin j which enters conventional partial wave expansion.
- One could easily extend the discussion to subleading powers replacing the Mellin transform by the partial wave expansion. For example, $(s/t)^{-j-1}$ and $(s/t)^j$ would be replaced respectively by the associated Legendre function $Q_j(1 + 2\, s/t)$ and Legendre polynomials $P_j(1 + 2\, s/t)$.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: ANALYTIC STRUCTURE

- The simplest conceivable asymptotic behaviour is a pure power law, whose Mellin transform is a simple Regge pole, namely

$$a_j^{(-)}(t) \simeq \frac{1}{j-1-\alpha(t)} \quad \Rightarrow \quad \mathcal{M}^{(-)}(s,t)|_{\text{Regge pole}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\frac{s}{t}\,e^{L\,\alpha(t)} + \ldots .$$

- where the ellipsis indicated subleading contributions. Regge poles give the correct behaviour of the 2 → 2 amplitude at LL in perturbation theory, where $\alpha(t)$ is interpreted as the gluon Regge trajectory, $\alpha(t) = \alpha_g(t) \sim O(\alpha_s(t))$.
- In order to get the precise behavior at higher orders in perturbation theory one needs to take into account the contribution of Regge cuts, which arises from $a_j^{(-)}(t)$ of the form

$$a_j^{(-)}(t) \simeq \frac{1}{(j-1-\alpha(t))^{1+\beta(t)}} \quad \Rightarrow \quad \mathcal{M}^{(-)}(s,t)|_{\text{Regge cut}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\,\frac{s}{t}\,\frac{L^{\beta(t)}\,e^{L\,\alpha(t)}}{\Gamma\left(1+\beta(t)\right)} + \ldots$$

- which has a branch point from $1 + \alpha(t)$ to $-\infty$, or a multiple pole if $\beta(t)$ is a positive integer.
- While Regge poles contribute to LL accuracy, therefore to the odd amplitude, Regge cuts start contributing at the NLL order, to the even amplitude.

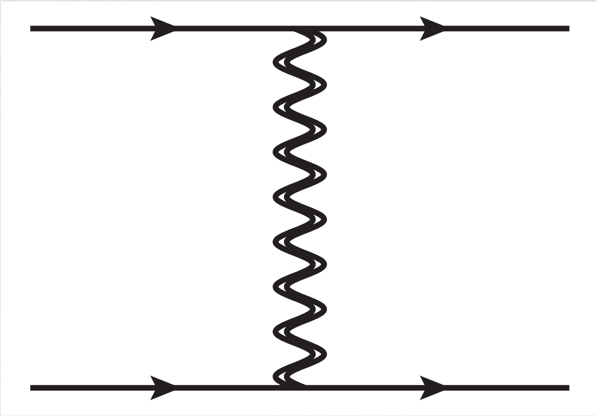
AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY

- Write the amplitude as the sum of odd and even component,

$$\mathcal{M}(s,t) = \mathcal{M}^{(-)}(s,t) + \mathcal{M}^{(+)}(s,t),$$

- with expansion in the strong coupling constant

$$\mathcal{M}^{(\pm)}(s,t) = 4\pi\alpha_s \sum_{l,m} \left(\frac{\alpha_s}{\pi}\right)^l L^m \, \mathcal{M}^{(\pm,l,m)} \, .$$

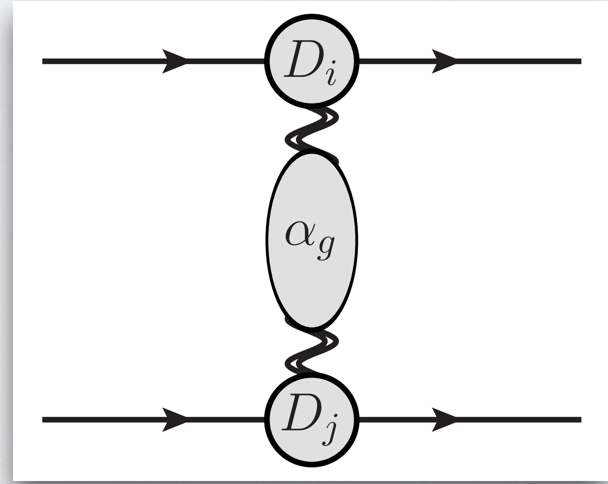


- The (odd) LL contribution to the amplitude is expected to receive corrections starting at NLL: these are expected to be of the form

$$\mathcal{M}^{(-)}_{ij\to ij}|_{\mathrm{NLL}} \sim e^{C_A\, \alpha_g(t)\, L}\, Z_i(t) D_i(t)\; Z_j(t) D_j(t)\; 4\pi\alpha_s\, \mathcal{M}^{(0)}_{ij\to ij},$$

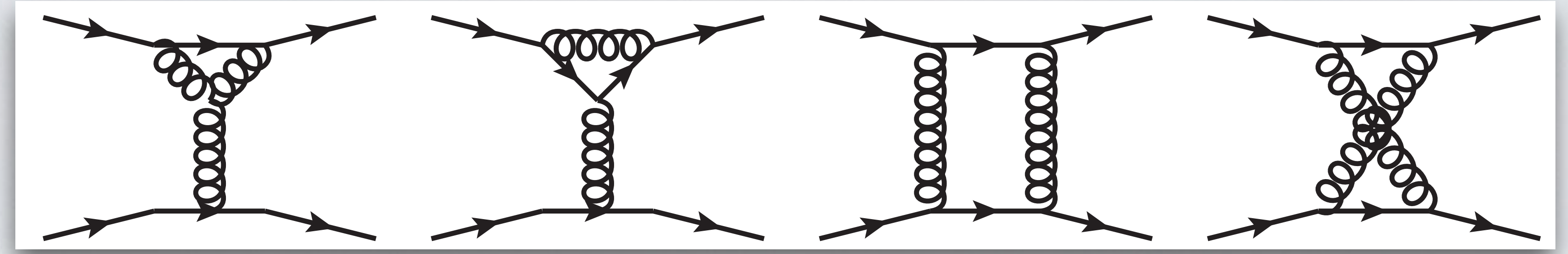
- At NLL, $\alpha_g(t)$ contains the first two terms of its power expansion; $Z_{i,j}(t)$ $D_{i,j}(t)$ are impact factors, representing corrections to the effective parton-parton-Reggeon vertex: their power expansion reads

$$Z_i(t) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n Z_i^{(n)}(t), \qquad D_i(t) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n D_i^{(n)}(t).$$



- The impact factors are written in a factorised form, according to the infrared factorisation theorem: $Z_{i,j}(t)$ collects the infrared divergences of the impact factors.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY



- What about the Regge cut contribution at NLL? It involves the exchange of two Reggeized gluons, and the symmetry properties of this state dictate that it contributes to the even amplitude, i.e. to $M^{(+)}$.
- From the point of view of perturbation theory this contribution arises from diagrams like the two boxes in the picture above. These diagrams introduce new color structures compared to the tree-level color factor.
- Using color-flow space notation, we write the amplitude as a vector in color space:

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\, \mathcal{M}^{[i]}(s,t).$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY

- It is convenient to decompose the amplitude in a color orthonormal basis in the t-channel.
- Consider for instance gluon-gluon amplitude:

$$8 \otimes 8 = 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27 \oplus 0 \qquad \Rightarrow$$

$$\begin{aligned}
c^{[1]} &= \frac{1}{N_c^2-1}\, \delta^{a_4}{}_{a_1}\, \delta^{a_3}{}_{a_2}\,, \\
c^{[8_s]} &= \frac{N_c}{N_c^2-4}\, \frac{1}{\sqrt{N_c^2-1}}\, d^{\,a_1 a_4 b}\, d^{\,a_2 a_3}{}_b\,, \\
c^{[8_a]} &= \frac{1}{N_c}\, \frac{1}{\sqrt{N_c^2-1}}\, f^{\,a_1 a_4 b}\, f^{\,a_2 a_3}{}_b\,, \\
c^{[10+\overline{10}]} &= \sqrt{\frac{2}{(N_c^2-4)(N_c^2-1)}}\left[\frac{1}{2}\left(\delta^{a_1}{}_{a_2}\,\delta^{a_3}{}_{a_4} - \delta^{a_3}{}_{a_1}\,\delta^{a_4}{}_{a_2}\right) - \frac{1}{N_c} f^{\,a_1 a_4 b}\, f^{\,a_2 a_3}{}_b\right], \\
&\dots
\end{aligned}$$

- In this basis the symmetry of the color structure mirrors the signature of the corresponding amplitude coefficients, which can thus be separated into signature odd and even:

$$\text{odd:}\ \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even:}\ \mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]} \qquad (gg\ \text{scattering})\,.$$

- The exchange of one Reggeized gluon contributes only to the antisymmetric octet, so that at LL only this structure is nonzero:

$$\mathcal{M}_{gg\to gg}(s,t)\big|_{LL} = c_{[8_a]}\, \mathcal{M}^{[8_a]}(s,t)\big|_{LL}.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY

- In order to display the Regge-cut contributions in the most transparent way, it proves useful to define a "reduced" amplitude by removing from it the Reggeized gluon and collinear divergences as follows:

$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\mathbf{T}_t^2\,\alpha_g(t)\,L} \, \mathcal{M}_{ij\to ij} \,,$$

- where T_t^2 represents the colour charge of a Reggeized gluon exchanged in the t-channel and $Z_{i,j}$ stand for collinear divergences. The color operator T_t^2, together with other two useful operators T_s^2, T_u^2 are defined as

$$\begin{aligned}
\mathbf{T}_s =& \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{T}_3 - \mathbf{T}_4, \\
\mathbf{T}_u =& \mathbf{T}_1 + \mathbf{T}_3 = -\mathbf{T}_2 - \mathbf{T}_4, \\
\mathbf{T}_t =& \mathbf{T}_1 + \mathbf{T}_4 = -\mathbf{T}_2 - \mathbf{T}_3.
\end{aligned}$$

- These operators are subject to color conservation constraints:

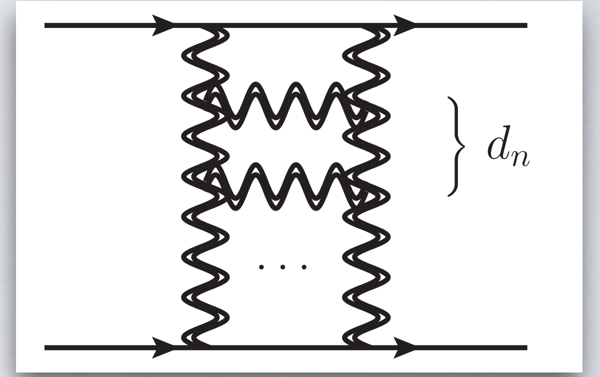
$$(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4)\,\mathcal{M} = 0, \qquad \mathbf{T}_s^2 + \mathbf{T}_u^2 + \mathbf{T}_t^2 = \sum_{i=1}^{4} C_i \equiv C_{\rm tot}.$$

- In terms of the reduced amplitude, the NLL odd contribution reads

$$\hat{\mathcal{M}}^{(-)}_{ij\to ij} = \left[1 + \frac{\alpha_s}{\pi}\left(D_i^{(1)}(t) + D_j^{(1)}(t)\right)\right] 4\pi\alpha_s \, \hat{\mathcal{M}}^{(0)}_{ij\to ij} \,.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY

- The two-Reggeon cut contribution at NLL reads   Caron-Huot, 2013



$$\hat{\mathcal{M}}^{(+)}_{ij\to ij}|_{\rm NLL} = i\pi \sum_{\ell=1}^{\infty} \frac{1}{\ell!}\left(\frac{\alpha_s}{\pi}\right)^{\ell} L^{\ell-1} \times d_\ell \times 4\pi\alpha_s\, \hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

- where the coefficients d_ℓs follows from BFKL evolution equation (more later), and have been calculated up to 4 loops:

$$d_1 = \mathbb{d}_1 \mathbf{T}^2_{s-u}, \qquad \mathbb{d}_1 = r_\Gamma \frac{1}{2\epsilon},$$

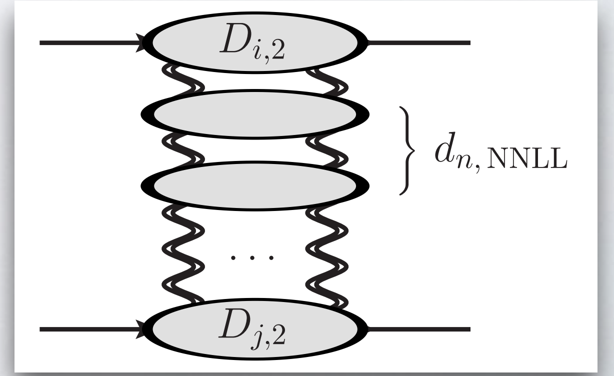
$$d_2 = \mathbb{d}_2 [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}], \qquad \mathbb{d}_2 = (r_\Gamma)^2 \left(-\frac{1}{4\epsilon^2} - \frac{9}{2}\epsilon\zeta_3 - \frac{27}{4}\epsilon^2\zeta_4 + \mathcal{O}(\epsilon^3)\right),$$

$$d_3 = \mathbb{d}_3 [\mathbf{T}^2_t, [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]], \qquad \mathbb{d}_3 = (r_\Gamma)^3 \left(\frac{1}{8\epsilon^3} - \frac{11}{4}\zeta_3 - \frac{33}{8}\epsilon\zeta_4 - \frac{357}{4}\epsilon^2\zeta_5 + \mathcal{O}(\epsilon^3)\right),$$

$$d_4 = \mathbb{d}_{4a} [\mathbf{T}^2_t, [\mathbf{T}^2_t, [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]]] \qquad \mathbb{d}_{4a} = (r_\Gamma)^4 \left(-\frac{1}{16\epsilon^4} - \frac{175}{2}\epsilon\zeta_5 + \mathcal{O}(\epsilon^2)\right),$$

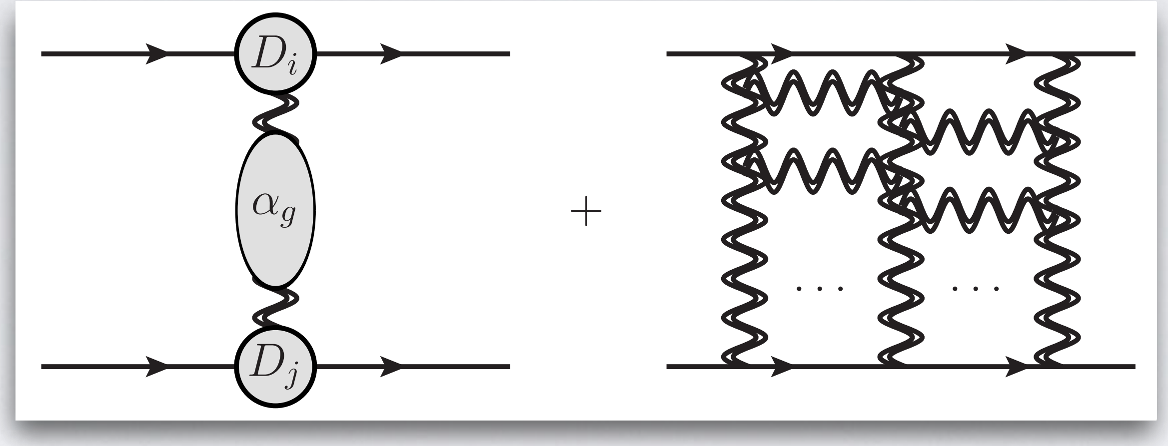
$$+ \mathbb{d}_{4b}\, C_A\, [\mathbf{T}^2_t, [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]], \qquad \mathbb{d}_{4b} = (r_\Gamma)^4 \left(\boxed{-\frac{1}{8\epsilon}\zeta_3} - \frac{3}{16}\zeta_4 - \frac{167}{8}\epsilon\zeta_5 + \mathcal{O}(\epsilon^2)\right).$$

- The color operator $T_{s-u}^2 = 1/2(T_s^2 - T_u^2)$ is odd under $s \leftrightarrow u$ crossing.
- At NNLL, the even amplitude is expected to receive corrections, similarly to what happens to the odd amplitude, when going from LL to NLL. →



AMPLITUDES IN THE HIGH-ENERGY LIMIT: PERTURBATION THEORY

- More interesting are the corrections concerning the odd amplitude at NNLL accuracy.
- In this case one has to take into account for the first time the exchange of three Reggeized gluons. This implies that, starting at NNLL, one has mixing between one- and three-Reggeons exchange:



Del Duca, Glover, 2001; Del Duca, Falcioni, Magnea, LV, 2013

- The mixing between one- and three-Reggeons exchange has significant consequences:
  - It is at the origin of the breaking of the simple power law one has up to NLL accuracy. Such breaking appears for the first time at two loops.
  - It implies that, starting at three loops, there will be a single-logarithmic contribution originating from the three-Reggeon exchange, and from the interference of the one- and three-Reggeon exchange: the interpretation of the Regge trajectory at three loops needs to be clarified.
- Schematically, the whole amplitude at NNLL is composed of

$$\hat{\mathcal{M}}_{ij\to ij}|_{\text{NNLL}} = \hat{\mathcal{M}}^{(-)}_{ij\to ij}|_{\text{1-Reggeon + 3-Reggeon}} + \hat{\mathcal{M}}^{(+)}_{ij\to ij}|_{\text{2-Reggeon}}.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL THEORY ABRIDGED

- The high-energy limit correspond to a configuration of forward scattering:

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -\frac{s}{2}(1 - \cos\theta),$$
$$u = (p_1 - p_3)^2 = (p_2 - p_4)^2 = -\frac{s}{2}(1 + \cos\theta),$$
$$s \gg -t \quad \Rightarrow \theta \to 0.$$

- The high-energy logarithm correspond to the rapidity difference between the target and the projectile:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

- Such kinematical configuration is described conveniently in terms of Wilson lines stretching from $-\infty$ to $+\infty$. The Wilson lines follow the paths of color charges inside the projectile, and are thus null and labelled by transverse coordinates **z**: Korchemskaya, Korchemsky, 1994, 1996

$$U(z_\perp) = \mathcal{P}\exp\left[ig_s\int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+T^a\right].$$

- The idea is to approximate, to leading power, the fast projectile and target by Wilson lines and then compute the scattering amplitude between Wilson lines. Babansky, Balitsky, 2002
- The full transverse structure needs to be retained. As a consequence, due to quantum fluctuations, a projectile necessarily contains multiple color charges at different transverse positions: the number of Wilson lines cannot be held fixed.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL THEORY ABRIDGED

- However, in perturbation theory, the unitary matrices U(z) will be close to identity and so can be usefully parametrised by a field W:

$$U(z) = e^{ig_s\, T^a W^a(z)}\,.$$

- The color-adjoint field W sources a BFKL Reggeised gluon. A generic projectile, created with four-momentum p_1 and absorbed with p_4, can thus be expanded at weak coupling as

$$|\psi_i\rangle \;\equiv\; \frac{Z_i^{-1}}{2p_1^+} a_i(p_4) a_i^\dagger(p_1)|0\rangle \;\sim g_s\, D_{i,1}(t)\,|W\rangle + g_s^2\, D_{i,2}(t)\,|WW\rangle + g_s^3\, D_{i,3}(t)\,|WWW\rangle + \ldots$$

$$\equiv |\psi_{i,1}\rangle + |\psi_{i,2}\rangle + |\psi_{i,3}\rangle + \ldots .$$

- The factors $D_{i,j}$ depend on the transverse coordinates of the W fields, but not on the center of mass energy. They correspond to the impact factors for the exchange of one-, two- and three-Reggeons.
- The energy dependence enters from the fact that the Wilson lines have rapidity divergences which must be regulated, which leads to a rapidity evolution equation (Balitsky-JIMWLK):

$$-\frac{d}{d\eta}\,|\psi_i\rangle = H\,|\psi_i\rangle.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL THEORY ABRIDGED

- A key feature of the Balitsky-JIMWLK equation is that the Hamiltonian is diagonal in the leading approximation:

Caron-Huot, 2013

$$H\begin{pmatrix} W \\ WW \\ WWW \\ \dots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \dots \\ 0 & H_{2\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \dots \end{pmatrix} \sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^2 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \dots \end{pmatrix}$$

- After using the rapidity evolution equation to resum all logarithms of the energy, the amplitude is obtained from the scattering amplitude between equal-rapidity Wilson lines, which depends only on the transverse scale t:

$$\frac{i(Z_i Z_j)^{-1}}{2s}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-HL}|\psi_i\rangle,$$

- Or, in terms of the reduced amplitude,

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} = \langle\psi_j|e^{-\hat{H}L}|\psi_i\rangle, \qquad \hat{H} \equiv H + \mathbf{T}_t^2\,\alpha_g(t).$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL THEORY ABRIDGED

- The inner product is by definition the scattering amplitude of Wilson lines renormalized to equal rapidity.
- For our purposes, it suffices to know that it is Gaussian to leading-order:

$$G_{11'} \equiv \langle W_1 | W_{1'} \rangle = i \, \frac{\delta^{a_1 a_1'}}{p_1^2} \, \delta^{(2-2\epsilon)}(p_1 - p_1') + \mathcal{O}(g_s^2).$$

- Multi-Reggeon correlators are obtained by Wick contractions:

Caron-Huot, 2013

$$\langle W_1 W_2 | W_{1'} W_{2'} \rangle = G_{11'} G_{22'} + G_{12'} G_{21'} + \mathcal{O}(g_s^2),$$
$$\langle W_1 W_2 W_3 | W_{1'} W_{2'} W_{3'} \rangle = G_{11'} G_{22'} G_{33'} + (5 \text{ permutations}) + \mathcal{O}(g_s^2),$$
$$\ldots$$

- There are also off-diagonal elements, which can be defined to have zero overlap (at equal rapidity):

$$\langle W_1 W_2 W_3 | W_4 \rangle = \langle W_4 | W_1 W_2 W_3 \rangle = 0.$$

- Starting from a scheme in which the inner products is ≠0, it is always possible to perform a scheme transformations (e.g. WWW → WWW − g_s^2 G W) such as to reduce to the condition above.
- Choosing the 1-W and 3-W states to be orthogonal, combined with symmetry of the Hamiltonian, (boost invariance):

$$\frac{d}{d\eta} \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = 0 \quad \Leftrightarrow \quad \langle H \mathcal{O}_1 | \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 | H \mathcal{O}_2 \rangle \equiv \langle \mathcal{O}_1 | H | \mathcal{O}_2 \rangle,$$

- implies that in this scheme $H_{k \to k+2} = H_{k+2 \to k}$. This relation is known as projectile-target duality.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL THEORY ABRIDGED

- We can now list the ingredients which build up the amplitude up to three loops. Since the odd and even sectors are orthogonal and closed under the action of $\hat{H}$ (signature symmetry), we have

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij}\right) \equiv \langle\psi_j^{(+)}|e^{-\hat{H}L}|\psi_i^{(+)}\rangle + \langle\psi_j^{(-)}|e^{-\hat{H}L}|\psi_i^{(-)}\rangle.$$

- Using that multi-Reggeon impact factors are coupling-suppressed, $|\psi_{ik}\rangle \sim g_s^k$, and using the suppression by powers of α_s of off-diagonal elements in H, the signature odd amplitude becomes to three loops:

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{tree}}_{ij\to ij} = \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{LO})},$$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = -L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NLO})},$$

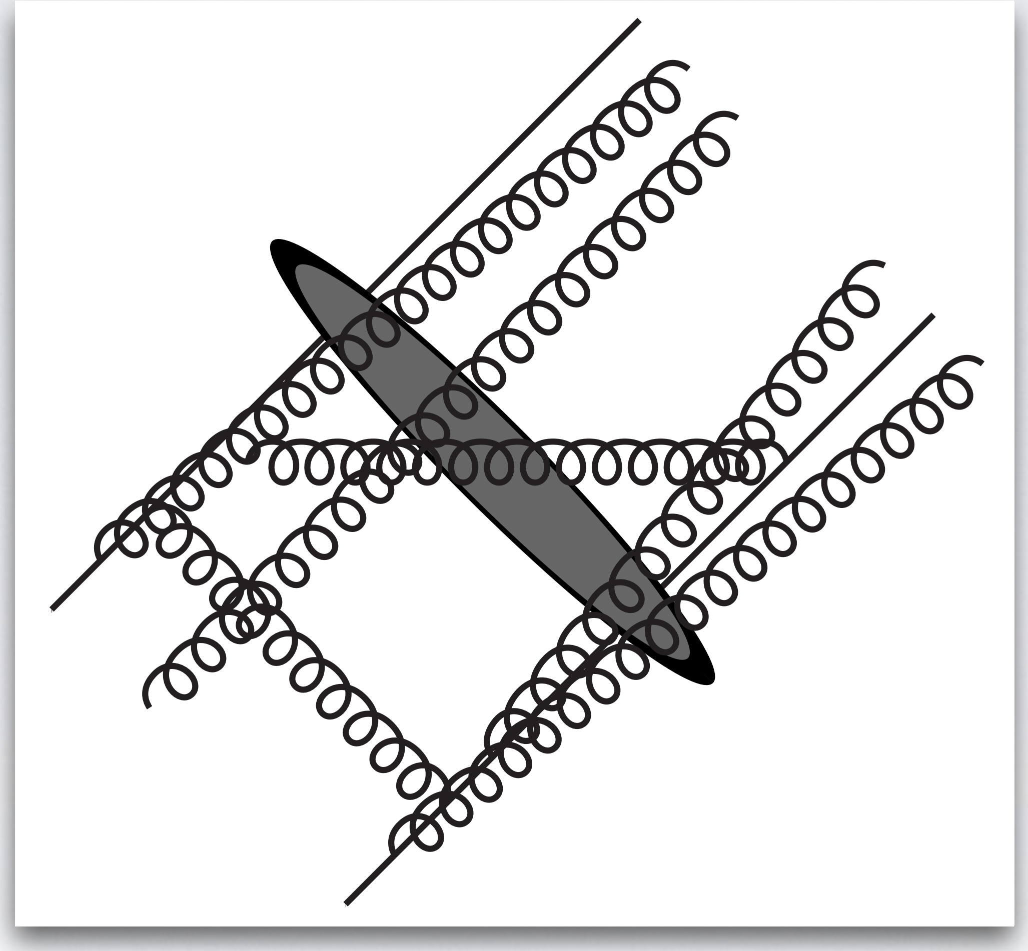
$$\begin{aligned}\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} &= +\frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{LO})} - L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NLO})}\\ &+ \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NNLO})},\end{aligned}$$

$$\begin{aligned}\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{3-loops}}_{ij\to ij} &= -\frac{1}{6}L^3\langle\psi_{j,1}|(\hat{H}_{1\to1})^3|\psi_{i,1}\rangle^{(\text{LO})} + \frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{NLO})}\\ &- L\Big\{\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NNLO})} + \Big[\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle + \langle\psi_{j,3}|\hat{H}_{1\to3}|\psi_{i,1}\rangle\\ &\qquad + \langle\psi_{j,1}|\hat{H}_{3\to1}|\psi_{i,3}\rangle\Big]^{(\text{LO})}\Big\} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{NLO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}.\end{aligned}$$

- Recall that we are considering the reduced amplitude: Given that the 1→1 Hamiltonian is equal minus the Regge trajectory, the 1→1 reduced Hamiltonian is actually zero, and many terms vanish:

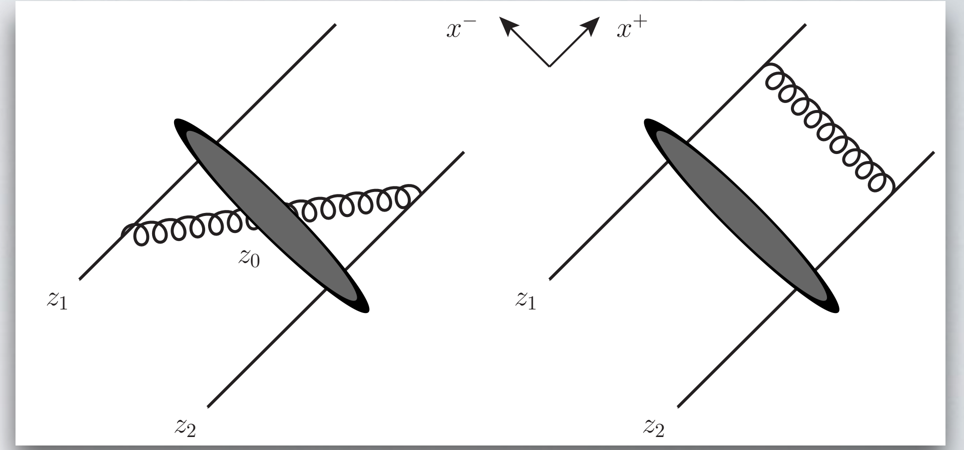
$$H_{1\to1} = -C_A\,\alpha_g(t) \quad\Rightarrow\quad \hat{H}_{1\to1} = 0, \quad \langle\psi_{j,1}|(\hat{H}_{1\to1})^n|\psi_{i,1}\rangle^{(\dots)} = 0.$$

THE BALITSKY-JIMWLK EQUATION AND THE THREE LOOP AMPLITUDE



AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- The Balitsky-JIMWLK equation for an arbitrary number of Wilson lines $U(z_i)$ can be written in the form



$$-\frac{d}{d\eta}\Big[U(z_1)\ldots U(z_n)\Big] = \sum_{i,j=1}^{n} H_{ij}\cdot\Big[U(z_1)\ldots U(z_n)\Big],$$

- with

Caron-Huot, 2013

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int [dz_i][dz_j][dz_0]K_{ij;0}\Big[T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\Big] + \mathcal{O}(\alpha_s^2).$$

- We work now in dimensional regularisation with $2\text{-}2\varepsilon$ dimensions, and $dz = d^{2-2\varepsilon}z$, and $T_{L/R}$'s are generators for left and right color rotations:

Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

$$T^a_{i,L} = [T^a U(z_i)]\,\frac{\delta}{\delta U(z_i)}, \qquad T^a_{i,R}(z) = [U(z_i)T^a]\,\frac{\delta}{\delta U(z_i)}.$$

- In our analysis we need only the leading-order conformal invariant kernel K_{ij}, which has a very simple dimension-independent expression in momentum space:

$$K_{ij;0} \equiv S_\epsilon(\mu^2)\int [\text{đ}q][\text{đ}p]\, e^{iq\cdot(z_i-z_0)}e^{ip\cdot(z_j-z_0)}(-2\pi^2)\frac{(q+p)^2}{q^2p^2} = S_\epsilon(\mu^2)\frac{\Gamma(1-\epsilon)^2}{\pi^{-2\epsilon}}\,\frac{z_{0i}\cdot z_{0j}}{(z_{0i}^2 z_{0j}^2)^{1-\epsilon}},$$

- The corrections to the Balitsky-JIMWLK Hamiltonian are suppressed by α_s in a power-counting where the Wilson lines are generic, $U \sim 1$. This is more general than the perturbative counting where $1 - U \sim g_s W \sim g_s$, implying that the equation resums infinite towers of Reggeon iterations.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- To see this, expand U in powers of W:

$$U = e^{ig_s\, W^a T^a} = 1 + ig_s\, W^a\, T^a - \frac{g_s^2}{2} W^a W^b\, T^a T^b - i\frac{g_s^3}{6} W^a W^b W^c\, T^a T^b T^c + \frac{g_s^4}{24} W^a W^b W^c W^d\, T^a T^b T^c T^d + \mathcal{O}(g_s^5\, W^5).$$

- The expansion of the color generators follows by using the Backer-Campbell-Hausdorff formula. Then, it is possible to expand the leading Hamiltonian H_{ij} in powers of g_s:

$$H = H_{k\to k} + H_{k\to k+2} + \dots$$

- We get

$$H_{k\to k} = \frac{\alpha_s C_A}{2\pi^2} \int [dz_i][dz_0] K_{ii;0}\, (W_i - W_0)^a \frac{\delta}{\delta W_i^a} - \frac{\alpha_s}{2\pi^2} \int [dz_i][dz_j][dz_0] K_{ij;0} (W_i - W_0)^x (W_j - W_0)^y (F^x F^y)^{ab} \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

- The first non-linear correction is new:

$$H_{k\to k+2} = \frac{\alpha_s^2}{3\pi} \int [dz_i][dz_0]\, K_{ii;0}\, (W_i - W_0)^x W_0^y (W_i - W_0)^z\, \mathrm{Tr}\big[F^x F^y F^z F^a\big] \frac{\delta}{\delta W_i^a} + \frac{\alpha_s^2}{6\pi} \int [dz_i][dz_j][dz_0]\, K_{ij;0}\, (F^x F^y F^z F^t)^{ab} \Big[(W_i - W_0)^x W_0^y W_0^z (W_j - W_0)^t - W_i^x (W_i - W_0)^y W_0^z (W_j - W_0)^t - (W_i - W_0)^x W_0^y (W_j - W_0)^z W_j^t \Big] \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- More on the Balitsky-JIMWLK power counting ($U \sim 1$) vs the BFKL power-counting ($W \sim 1$):
- Inserting the expansion of U in terms of W in the leading-order Balitsky-JIMWLK equation, one finds that an $m \to m+k$ transition is proportional to g_s^{2l+k}. Thus for $k \geq 0$, all the leading interactions can be extracted from the leading-order equation.

$$H \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \cdots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \cdots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \cdots \end{pmatrix}.$$

- On the other hand, interactions with $k < 0$ are suppressed by at least $g_s^{2l+|k|}$, which means that they can first appear in the $(|k|+1)$-loop Balitsky-JIMWLK Hamiltonian.
- Thus to obtain the $m \to m-2$ transition by direct calculation of the Hamiltonian would require three- loop non-planar computation.
- For our purposes this is unnecessary, since the symmetry of H predicts the result.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- The actual calculation is easier in momentum space: introduce Fourier transform of the Reggeon-fields W:

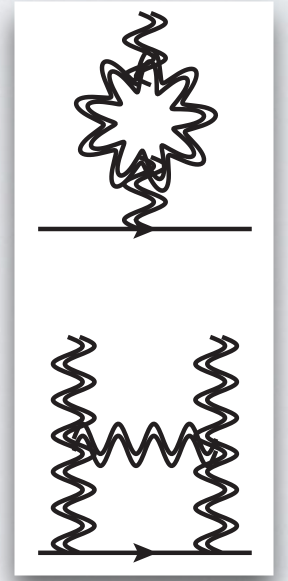
$$W^a(p) = \int [dz]\, e^{-ipz}\, W^a(z), \qquad W^a(z) = \int [\text{đ}p]\, e^{ipz}\, W^a(p).$$

- The k → k transitions then read

$$H_{k\to k} = -\int [dp]\, C_A\, \alpha_g(p)\, W^a(p) \frac{\delta}{\delta W^a(p)}$$
$$+\ \alpha_s \int [\text{đ}q][dp_1][dp_2]\, H_{22}(q;p_1,p_2)\, W^x(p_1{+}q) W^y(p_2{-}q) (F^x F^y)^{ab} \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_2)},$$

- where

$$\alpha_g(p) = \frac{\alpha_s}{\pi}\, \alpha_g^{(1)}(p^2) + \mathcal{O}(\alpha_s^2)$$
$$= -\alpha_s(\mu) S_\epsilon(\mu^2) \int [\text{đ}q] \frac{p^2}{q^2(p-q)^2} + \mathcal{O}(\alpha_s^2) = \frac{\alpha_s(\mu)\, r_\Gamma}{2\pi\epsilon} \left(\frac{\mu^2}{p^2}\right)^{\epsilon} + \mathcal{O}(\alpha_s^2),$$



- is the Regge trajectory, and

$$H_{22}(q;p_1,p_2) = \frac{(p_1+p_2)^2}{p_1^2 p_2^2} - \frac{(p_1+q)^2}{p_1^2 q^2} - \frac{(p_2-q)^2}{q^2 p_2^2}.$$

- represents the BFKL kernel.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- More interesting is the 1 → 3 (3 → 1) transition: one has

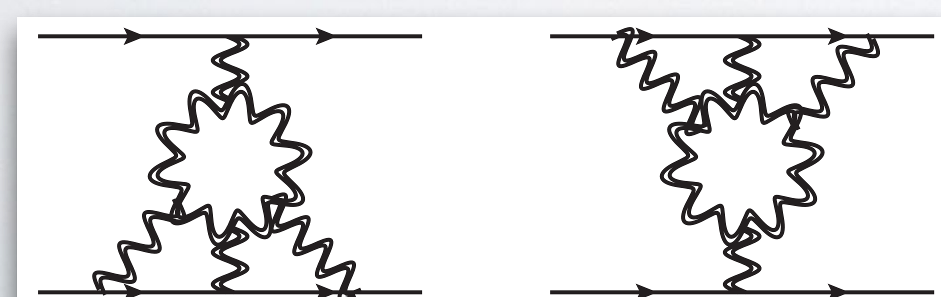
$$H_{1\to 3} = \alpha_s^2 \int [\text{đ}p_1][\text{đ}p_2][dp]\, \mathrm{Tr}[F^a F^b F^c F^d]\, W^b(p_1) W^c(p_2) W^d(p_3)\, H_{13}(p_1,p_2,p_3)\, \frac{\delta}{\delta W^a(p)},$$

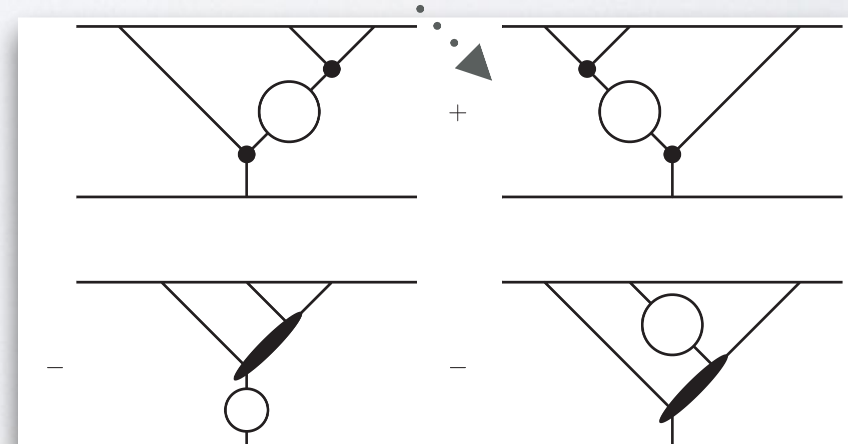
- and by symmetry the 3 → 1 transition by symmetry reads

$$H_{3\to 1} = \alpha_s^2 \int [dp_1][dp_2][dp_3]\, \mathrm{Tr}[F^a F^b F^c F^d]\, W^d(p_1{+}p_2{+}p_3)\, \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_2)} \frac{\delta}{\delta W^c(p_3)} \times (-1) \frac{(p_1{+}p_2{+}p_3)^2}{p_1^2 p_2^2 p_3^2}\, H_{13}(p_1,p_2,p_3),$$

- with kernel

$$H_{13}(p_1,p_2,p_3) = \frac{2\pi}{3} S_\epsilon(\mu^2) \int [\text{đ}q] \left[\frac{(p_1{+}p_2)^2}{q^2(p_1{+}p_2{-}q)^2} + \frac{(p_2{+}p_3)^2}{q^2(p_2{+}p_3-q)^2} - \frac{(p_1{+}p_2{+}p_3)^2}{q^2(p_1{+}p_2{+}p_3{-}q)^2} - \frac{p_2^2}{q^2(p_2{-}q)^2} \right]$$
$$= \frac{r_\Gamma}{3\epsilon} \left[\left(\frac{\mu^2}{(p_1{+}p_2{+}p_3)^2} \right)^\epsilon + \left(\frac{\mu^2}{p_2^2} \right)^\epsilon - \left(\frac{\mu^2}{(p_1{+}p_2)^2} \right)^\epsilon - \left(\frac{\mu^2}{(p_2{+}p_3)^2} \right)^\epsilon \right].$$





AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- Given the Hamiltonian, all one needs to compute the amplitude are the target and projectile impact factors:

$$
\begin{aligned}
|\psi_i\rangle^{(\mathrm{LO})} &= ig\, \mathbf{T}_i^a W^a(p) - \frac{g^2}{2}\mathbf{T}_i^a\mathbf{T}_i^b \int [\bar{d}q]\, W^a(q) W^b(p-q) \\
&\quad - \frac{ig^3}{6}\mathbf{T}_i^a\mathbf{T}_i^b\mathbf{T}_i^c \int [\bar{d}q_1][\bar{d}q_2]\, W^a(q_1) W^b(q_2) W^c(p-q_1-q_2) + \mathcal{O}(\mathrm{N^3LL}), \\
|\psi_i\rangle^{(\mathrm{NLO})} &= \frac{\alpha_s}{\pi}\Big[ig\, \mathbf{T}_i^a W^a(p) D_i^{(1)}(p) - \frac{g^2}{2}\mathbf{T}_i^a\mathbf{T}_i^b \int [\bar{d}q]\, \psi_i^{(1)}(p,q)\, W^a(q) W^b(p-q) + \mathcal{O}(\mathrm{N^3LL}) \Big], \\
|\psi_i\rangle^{(\mathrm{NNLO})} &= \left(\frac{\alpha_s}{\pi}\right)^2 \Big[ig\, \mathbf{T}_i^a W^a(p) D_i^{(2)}(p) + O(\mathrm{N^3LL}) \Big].
\end{aligned}
$$

- The Wilson line is in the representation of particle i, and p in the transferred momentum, $p^2 = -t$.

- At higher orders in the coupling, the color charge of the projectile is no longer concentrated in a single point, which leads to a nontrivial momentum dependence for multi-Reggeon impact factors.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: ONE AND TWO LOOPS

- To get the signature-odd amplitude to two loops we need exchanges of one- and three-Reggeons, the latter first appearing at two loops.

- Let us consider first the single Reggeon exchange: to all orders one has

$$\langle \psi_{j,1} | e^{-\hat{H}_{1\to 1} L} | \psi_{i,1} \rangle = D_i(t)\, D_j(t)\, \frac{i}{2s}\, 4\pi\alpha_s\, \hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

- which up to NNLL gives

$$\langle \psi_{j,1} | \psi_{i,1} \rangle^{(\mathrm{NLO})} = \frac{\alpha_s}{\pi} \left(D_i^{(1)}(t) + D_j^{(1)}(t) \right) \frac{i}{2s}\, 4\pi\alpha_s\, \hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

$$\langle \psi_{j,1} | \psi_{i,1} \rangle^{(\mathrm{NNLO})} = \left(\frac{\alpha_s}{\pi}\right)^2 \left(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)} \right) \frac{i}{2s}\, 4\pi\alpha_s\, \hat{\mathcal{M}}^{(0)}_{ij\to ij}.$$

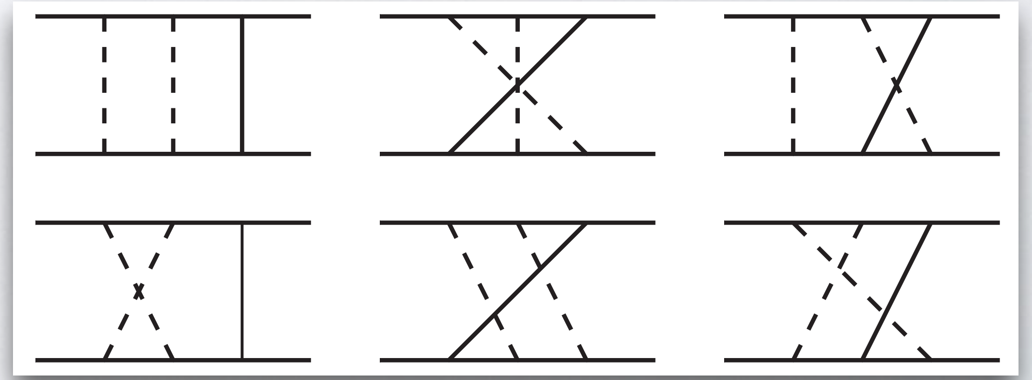
- the $3 \to 3$ transition appears first at two loops: it can be cast into the form

$$\langle \psi_{j,3} | \psi_{i,3} \rangle^{(\mathrm{LO})} = -i\pi^2\, (r_\Gamma)^2\, \mathcal{I}[1]\, \frac{g^2}{t} \left(\frac{\alpha_s}{\pi}\right)^2 C_{33}^{(2)}$$

- where the color structure can be written in terms of color operators acting on the tree-level color structure:

$$C_{33}^{(2)} = \frac{1}{36} \sum_{\sigma \in \mathcal{S}_3} \left(\mathbf{T}_i^{\sigma(a)} \mathbf{T}_i^{\sigma(b)} \mathbf{T}_i^{\sigma(c)} \right)_{a_1 a_4} \left(\mathbf{T}_j^{a} \mathbf{T}_j^{b} \mathbf{T}_j^{c} \right)_{a_2 a_3}$$

$$= \frac{1}{24} \left[(\mathbf{T}^2_{s-u})^2 - \frac{1}{12} (C_A)^2 \right] (T_i^b)_{a_1 a_4} (T_j^b)_{a_2 a_3}.$$



AMPLITUDES IN THE HIGH-ENERGY LIMIT: ONE AND TWO LOOPS

- The momentum structure reads

$$\mathcal{I}[N] \equiv \left(\frac{4\pi S_\epsilon(p^2)}{r_\Gamma}\right)^2 \int [đp_1][đp_2]\, \frac{p^2}{p_1^2 p_2^2 (p-p_1-p_2)^2}\, N$$

- Up to three loops, the momentum structure is determined in terms of simple bubble integrals:

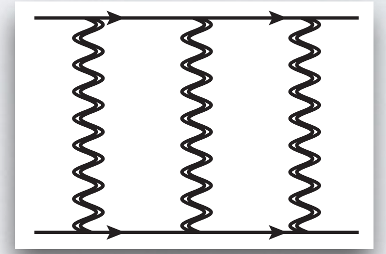
$$\int \frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}} \frac{1}{[k^2]^\alpha [(p-k)^2]^\beta} = \frac{B_{\alpha,\beta}(\epsilon)}{(4\pi)^{1-\epsilon}}\, (p^2)^{1-\epsilon-\alpha-\beta}$$

$$\text{with} \qquad B_{\alpha,\beta}(\epsilon) \equiv \frac{\Gamma(1-\alpha-\epsilon)\Gamma(1-\beta-\epsilon)\Gamma(\alpha+\beta-1+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2-2\epsilon-\alpha-\beta)}\,.$$

- The transition reads

$$\langle\psi_{j,3}|\psi_{i,3}\rangle^{(\mathrm{LO})} = -\frac{\pi^2}{24}\left(\frac{\alpha_s}{\pi}\right)^2 (r_\Gamma)^2\, \mathcal{I}[1]\, \left[(\mathbf{T}_{s-u}^2)^2 - \tfrac{1}{12}(C_A)^2\right] \frac{i}{2s}\, 4\pi\alpha_s\, \hat{\mathcal{M}}^{(0)}_{ij\to ij}.$$

- Up to two loops the amplitude then reads



Three-Reggeon cut

$$\hat{\mathcal{M}}^{(-,1)}_{ij\to ij} = \left(D_i^{(1)} + D_j^{(1)}\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

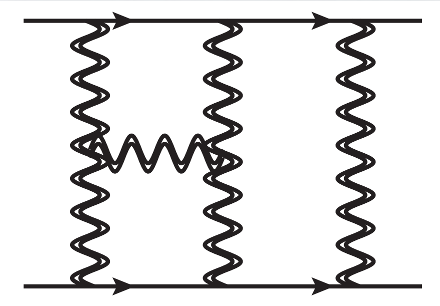
$$\hat{\mathcal{M}}^{(-,2)}_{ij\to ij} = \left[D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)} + \pi^2 R^{(2)}\left((\mathbf{T}_{s-u}^2)^2 - \tfrac{1}{12}(C_A)^2\right)\right]\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

- with

$$R^{(2)} \equiv -\frac{1}{24}(r_\Gamma)^2\, \mathcal{I}[1] = -\frac{(r_\Gamma)^2}{6\epsilon^2}\, \frac{B_{1,1+\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = (r_\Gamma)^2\left(-\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \ldots\right),$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THREE LOOPS

- At three loops we need to take into account for the first time the $H_{3\to3}$, $H_{1\to3}$ and $H_{3\to1}$ evolutions. The action of $H_{3\to3}$ gives



$$\hat{H}_{3\to3}\, W^a(p_1)W^b(p_2)W^c(p_3)\Big|_{\mathcal{S}^3}$$
$$\simeq \frac{\alpha_s r_\Gamma}{2\pi\epsilon}\left[\mathbf{T}_t^2 - 3C_A\left(\frac{p^2}{p_1^2}\right)^\epsilon\right] W^a(p_1)W^b(p_2)W^c(p_3)$$
$$-\,\alpha_s\left(\mathbf{T}_t^2 - 3C_A\right) S_\epsilon \int [\textit{đ}q] H_{22}(q;p_1,p_2)\, W^a(p_1{+}q)W^b(p_2{-}q)W^c(p_3),$$

- which gives rise to the following transition:

$$\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle = \frac{\pi^2}{48}\left(\frac{\alpha_s}{\pi}\right)^3 (r_\Gamma)^3\Big[\mathbf{T}_t^2\left(2\mathcal{I}_b{-}\mathcal{I}_a{-}\mathcal{I}_c\right) + 3C_A\left(\mathcal{I}_c - \mathcal{I}_b\right)\Big]$$
$$\cdot\,\Big[(\mathbf{T}_{s-u}^2)^2 - \tfrac{1}{12}(C_A)^2\Big]\frac{i}{2s}\,4\pi\alpha_s\,\hat{\mathcal{M}}^{(0)}_{ij\to ij}.$$

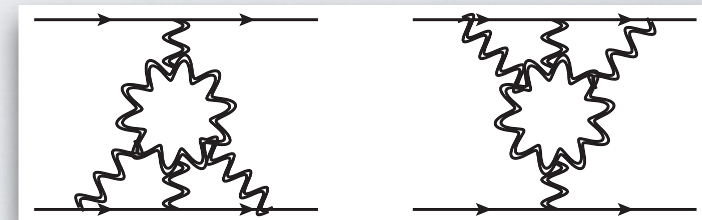
- The transition is completely determined in terms of bubble integrals:

$$\mathcal{I}_a \equiv \mathcal{I}\left[\frac{1}{\epsilon}\right] = \frac{4}{\epsilon^3}\frac{B_{1,1+\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = \frac{3}{\epsilon^3} - 18\zeta_3 - 27\epsilon\zeta_4 + \ldots$$
$$\mathcal{I}_b \equiv \mathcal{I}\left[\frac{1}{\epsilon}\left(\frac{p^2}{p_1^2}\right)^\epsilon\right] = \frac{4}{\epsilon^3}\frac{B_{1+\epsilon,1+\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = \frac{2}{\epsilon^3} - 44\zeta_3 - 66\epsilon\zeta_4 + \ldots$$
$$\mathcal{I}_c \equiv \mathcal{I}\left[\frac{1}{\epsilon}\left(\frac{p^2}{(p_1+p_2)^2}\right)^\epsilon\right] = \frac{4}{\epsilon^3}\frac{B_{1,1+2\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = \frac{8}{3\epsilon^3} - \frac{128}{3}\zeta_3 - 64\epsilon\zeta_4 + \ldots.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: THE BALITSKY-JIMWLK EQUATION

- The 1 → 3, 3 → 1 transitions are determined in terms of the same bubble integrals,

$$\langle\psi_{j,3}|\hat{H}_{1\to 3}|\psi_{i,1}\rangle + \langle\psi_{j,1}|\hat{H}_{3\to 1}|\psi_{i,3}\rangle = \frac{i}{12}\left(\frac{\alpha_s}{\pi}\right)^3 \pi^2 (r_\Gamma)^3 \left[2\mathcal{I}_c - \mathcal{I}_a - \mathcal{I}_b\right] \frac{g^2}{t} C^{(3)}_{13+31}.$$



- and the color structure reads

$$C^{(3)}_{13+31} \equiv \frac{1}{6}\sum_{\sigma\in\mathcal{S}^3} \mathrm{Tr}\left[F^a F^{\sigma(b)} F^{\sigma(c)} F^{\sigma(d)}\right] \left[(T_i^a)_{a_1a_4}(T_j^b T_j^c T_j^d)_{a_2a_3} + (T_i^b T_i^c T_i^d)_{a_1a_4}(T_j^a)_{a_2a_3}\right]$$

$$= \frac{1}{4}\Big(2\mathbf{T}^2_{s-u}[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}] - [\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} - (\mathbf{T}^2_{s-u})^2 C_A - \tfrac{1}{12}(C_A)^3\Big)(T_i^b)_{a_1a_4}(T_j^b)_{a_2a_3}$$

- Collecting the results, we obtain the three loop contribution to the odd amplitude:

$$\hat{\mathcal{M}}^{(-,3,1)}_{ij\to ij} = \pi^2\Big(R_A^{(3)}\,\mathbf{T}^2_{s-u}[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}] + R_B^{(3)}\,[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R_C^{(3)}\,(C_A)^3\Big)\hat{\mathcal{M}}^{(0)}_{ij\to ij}\,,$$

- where the loop functions $R_{A,B,C}$ are

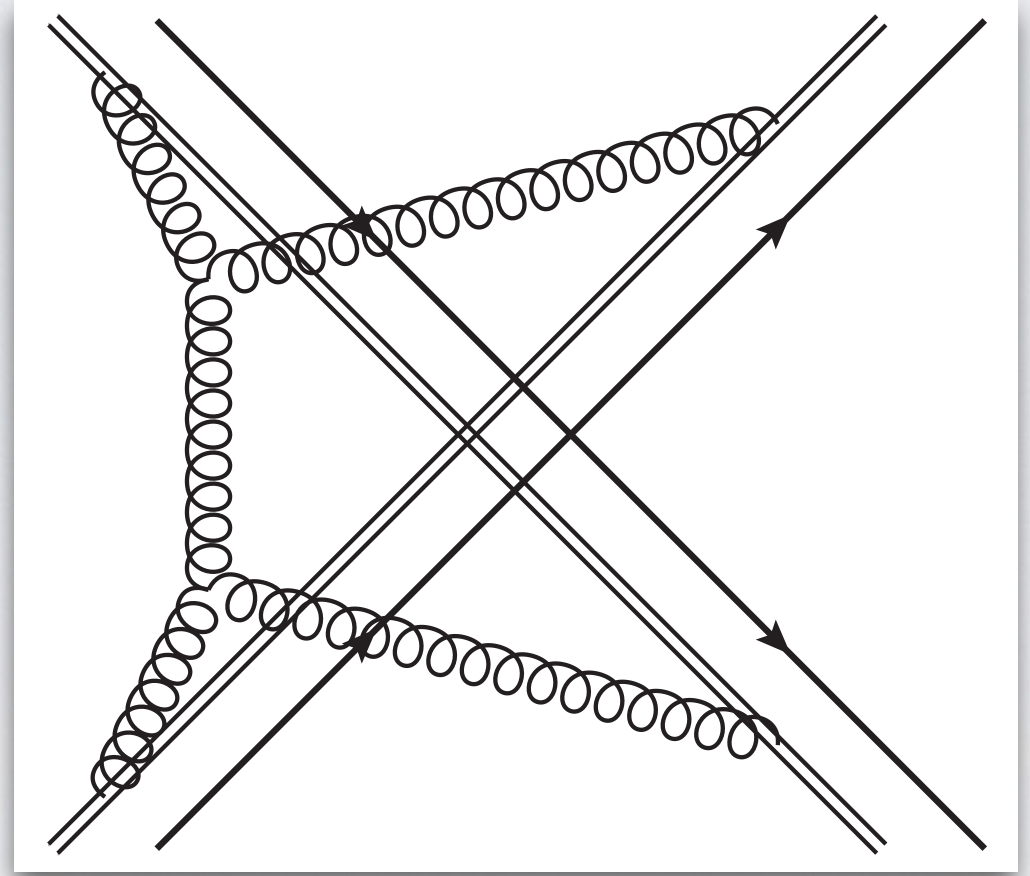
Caron-Huot, Gardi, LV, 2017

$$R_A^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_a - \mathcal{I}_c) = (r_\Gamma)^3\left(\frac{1}{48\epsilon^3} + \frac{37}{24}\zeta_3 + \ldots\right),$$

$$R_B^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_c - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{24\epsilon^3} + \frac{1}{12}\zeta_3 + \ldots\right),$$

$$R_C^{(3)} = \frac{1}{288}(r_\Gamma)^3(2\mathcal{I}_c - \mathcal{I}_a - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{864\epsilon^3} - \frac{35}{432}\zeta_3 + \ldots\right).$$

COMPARISON BETWEEN REGGE AND INFRARED FACTORIZATION



AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- The prediction for the reduced amplitude is based solely on evolution equations of the Regge limit, and has taken no input from the theory of infrared divergences.
- It is therefore a highly nontrivial consistency test that this prediction is consistent with the known exponentiation pattern and the anomalous dimensions governing infrared divergences.
- Conversely, the prediction for the reduced amplitude can also be seen as a constraint on the soft anomalous dimension: the high-energy limit of the latter has a very special structure, which may ultimately help in determining it beyond three loops.
- The infrared divergences of scattering amplitudes are controlled by a renormalization group equation, whose integrated version takes the form

Becher, Neubert, 2009; Gardi, Magnea, 2009

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

- where $\mathbf{Z}$ is given as a path-ordered exponential of the soft-anomalous dimension:

$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- the dependence on the scale is both explicit and via the $4-2\varepsilon$ dimensional coupling. The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012, …

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- Γ^{dip}_n involves only only pairwise interactions amongst the hard partons, and is therefore referred to as the "dipole formula":

$$\Gamma_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = -\frac{\gamma_K(\alpha_s)}{2}\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,.$$

- The term $\Delta_n(\rho_{ijkl})$ involves interactions of up to four partons, and is called the "quadrupole correction":

$$\Delta_n(\{\rho_{ijkl}\}) = \sum_{i=3}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^i \Delta_n^{(i)}(\{\rho_{ijkl}\}).$$

- The three loop correction has been calculated recently, and reads

$$\begin{aligned}\Delta_n^{(3)}(\{\rho_{ijkl}\}) = \frac{1}{4} f^{abe} f^{cde} \sum_{1\le i<j<k<l\le n} \Big[&\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ikjl},\rho_{iljk}) \\ &+ \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj})\Big] \\ &- \frac{C}{4} f^{abe} f^{cde} \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n,\\ j,k\neq i}} \{\mathbf{T}_i^a,\mathbf{T}_i^d\}\mathbf{T}_j^b\mathbf{T}_k^c,\end{aligned}$$

Almelid, Duhr, Gardi, 2015,2016

- where $\mathcal{F}$ is a function of cross ratios: $\rho_{ijkl} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$. Explicitly, one has

$$\mathcal{F}(\rho_{ikjl},\rho_{ilkj}) = F(1-z_{ijkl}) - F(z_{ijkl}), \quad \text{with} \quad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\Big(\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)\Big),$$

- where the $\mathcal{L}$ are Brown's single-valued harmonic polylogarithms, and the constant term reads

$$C = \zeta_5 + 2\zeta_2\zeta_3.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

Del Duca, Duhr, Gardi, Magnea, White, 2011

- In the high-energy limit the dipole formula reduces to

$$\Gamma^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) \xrightarrow{\text{Regge}} \frac{\gamma_K(\alpha_s)}{2}\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2 + \frac{C_{\text{tot}}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$

- and the quadrupole correction reads:

$$\mathbf{\Delta}^{(3)} = i\pi\,[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\frac{1}{4}\left[\zeta_3 L + 11\zeta_4\right] + \frac{1}{4}[\mathbf{T}_{s-u}^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\left[\zeta_5 - 4\zeta_2\zeta_3\right]$$
$$-\frac{\zeta_5+2\zeta_2\zeta_3}{8}\left\{f^{abe}f^{cde}\left[\{\mathbf{T}_t^a,\mathbf{T}_t^d\}\left(\{\mathbf{T}_{s-u}^b,\mathbf{T}_{s-u}^c\}+\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\right)\right.\right.$$
$$\left.\left.+\{\mathbf{T}_{s-u}^a,\mathbf{T}_{s-u}^d\}\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\right] - \frac{5}{8}C_A^2\mathbf{T}_t^2\right\},$$

Only NNLL term

- where

$$\mathbf{T}_{s-u}^a \equiv \frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a - \mathbf{T}_u^a\right), \qquad \mathbf{T}_{s+u}^a \equiv \frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a + \mathbf{T}_u^a\right).$$

Caron-Huot, Gardi, LV, 2017

- Because of the form of Γ^{dip}_n and $\Delta_n(\rho_{ijkl})$ in the High-energy limit, the **Z** factor factorises

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) Z_i\left(t,\mu,\alpha_s(\mu^2)\right) Z_j\left(t,\mu,\alpha_s(\mu^2)\right),$$

- where the relevant bit for us is

$$\tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) = \exp\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\mathbf{\Delta}}^{(3)}\right\}$$

- The factor K involve an integral over the scale:

$$K\left(\alpha_s(\mu^2)\right) = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\alpha_s(\lambda^2)\right) = \frac{1}{2\epsilon}\frac{\alpha_s(\mu^2)}{\pi} + \ldots,$$

- and the quadrupole interaction is contained in the term $Q_{\mathbf{\Delta}}$:

$$Q_{\mathbf{\Delta}}^{(3)} = -\frac{\mathbf{\Delta}^{(3)}}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^3 = \frac{\mathbf{\Delta}^{(3)}}{6\epsilon}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- The scalar factors $Z_{i,j}$ are the same as those we removed from the reduced amplitude in the BFKL context, and at LL accuracy the exponent in $\tilde{\mathbf{Z}}$ is also very similar to the gluon Regge trajectory subtracted in the reduced amplitude. This makes the relation between the "infrared-renormalized" amplitude (hard function) H and reduced matrix element particularly simple:

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \exp^{-1}\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\boldsymbol{\Delta}}^{(3)}\right\}$$
$$\cdot\exp\left\{\alpha_g(t) L\,\mathbf{T}_t^2\right\}\hat{\mathcal{M}}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right).$$

- This equation allows us to pass from directly from the reduced amplitude predicted using BFKL theory, to the hard function.
- In particular, the statement that the left-hand-side H is finite, which is equivalent to the exponentiation of infrared divergences, is a highly nontrivial constraint on our result.
- By using Baker-Campbell-Hausdorff formula one gets

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \left(1 + \frac{K^3(\alpha_s)}{3!}\left(2\pi^2\,L\,[\mathbf{T}_{s-u}^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]] - i\,\pi L^2\,[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\right)\right.$$
$$\left. + i\pi\frac{K^2(\alpha_s)}{2}L\,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] - Q_{\boldsymbol{\Delta}}^{(3)}\right)\cdot\exp\left\{-i\pi\,K(\alpha_s)\,\mathbf{T}_{s-u}^2\right\}$$
$$\cdot\exp\left\{\left(\alpha_g(t) - K(\alpha_s)\right)L\,\mathbf{T}_t^2\right\}\hat{\mathcal{M}}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right).$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- In the following we expand in powers of α_s and L, according to

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = 4\pi\alpha_s \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{\alpha_s}{\pi}\right)^n L^k\,\mathcal{H}^{(n,k)}\left(\frac{-t}{\mu^2}\right).$$

- At LL, it is easy to check that one gets

$$\mathcal{H}^{(n,n)}_{ij\to ij} = \frac{1}{n!}\left(\hat{\alpha}_g^{(1)}\right)^n \left(\mathbf{T}_t^2\right)^n \hat{\mathcal{M}}^{(0)}_{ij\to ij}.$$

- where we introduced the "finite" Regge trajectory

$$\hat{\alpha}_g(t) = \alpha_g(t) - K(\alpha_s), \quad \hat{\alpha}_g(t) = \hat{\alpha}_g^{(n)}\left(\frac{\alpha_s(-t)}{\pi}\right)^n,$$

- and the first two orders read

$$\hat{\alpha}_g^{(1)} = \frac{1}{2\epsilon}(r_\Gamma - 1) = -\frac{1}{4}\zeta_2\,\epsilon - \frac{7}{6}\zeta_3\,\epsilon^2 + \mathcal{O}(\epsilon^3),$$

$$\hat{\alpha}_g^{(2)} = C_A\left(\frac{101}{108} - \frac{\zeta_3}{8}\right) - \frac{7n_f}{54} + O(\epsilon).$$

Korchemskaya, Korchemsky, 1994, 1996

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

The analysis proceed in a straightforward way: order by order in α_s we insert the result from Regge theory, and check consistency with the infrared factorisation formula.

For instance at one loop we have

$$\mathcal{H}^{(1,1)} = \hat{\alpha}_g^{(1)}\, \mathbf{T}_t^2\, \hat{\mathcal{M}}^{(0)},$$
$$\mathcal{H}^{(1,0)} = \hat{\mathcal{M}}^{(1,0)} - i\pi\, K^{(1)}\, \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)}.$$

Explicitly, the real and imaginary part of the NLL term are given by

$$\mathrm{Re}[\mathcal{H}^{(1,0)}] = \hat{\mathcal{M}}^{(-,1,0)},$$
$$i\,\mathrm{Im}[\mathcal{H}^{(1,0)}] = \hat{\mathcal{M}}^{(+,1,0)} - i\pi\, K^{(1)}\, \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)}.$$

i.e., from Regge theory,

$$\mathrm{Re}[\mathcal{H}^{(1,0)}] = \left(D_i^{(1)} + D_j^{(1)}\right) \hat{\mathcal{M}}^{(0)},$$
$$i\,\mathrm{Im}[\mathcal{H}^{(1,0)}] = i\pi\left(\mathbb{d}_1 - K^{(1)}\right) \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)} = i\pi\, \hat{\alpha}_g^{(1)}\, \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)},$$

Some coefficients, like the impact factors, are not predicted explicitly from Regge theory: in that case, we can use these equations in the reverse direction, and get

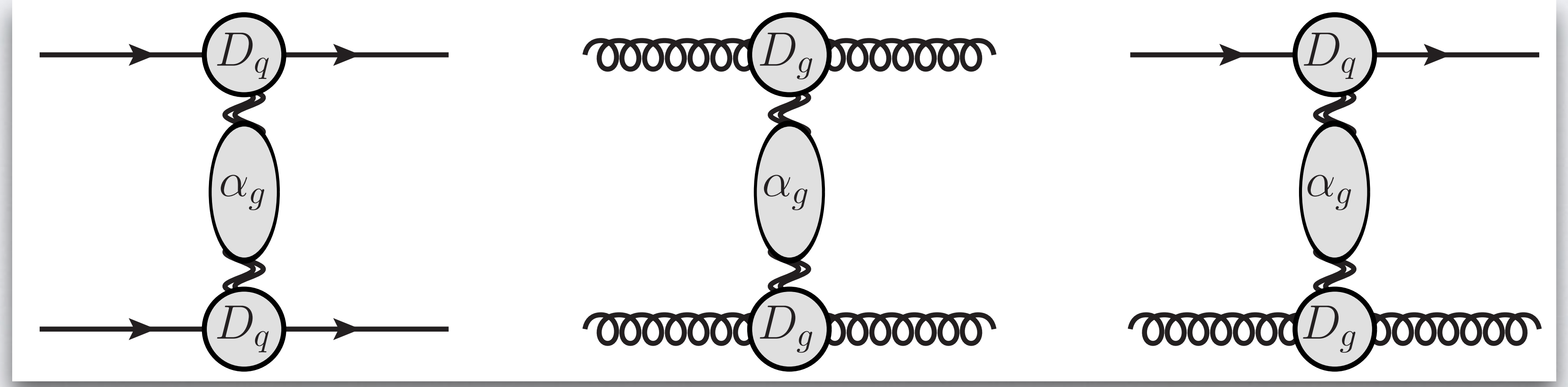
$$D_i^{(1)} = \frac{1}{2}\, \frac{\mathrm{Re}[\mathcal{H}_{ii\to ii}^{(1,0)[8_a]}]}{\mathcal{H}_{ii\to ii}^{(0)[8_a]}}.$$

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- We get:

$$D_g^{(1)} = -N_c\left(\frac{67}{72}-\zeta_2\right)+\frac{5}{36}n_f+\epsilon\left[N_c\left(-\frac{101}{54}+\frac{11}{48}\zeta_2+\frac{17}{12}\zeta_3\right)+n_f\left(\frac{7}{27}-\frac{\zeta_2}{24}\right)\right]$$
$$+\epsilon^2\left[N_c\left(-\frac{607}{162}+\frac{67}{144}\zeta_2+\frac{77}{72}\zeta_3+\frac{41}{32}\zeta_4\right)+n_f\left(\frac{41}{81}-\frac{5}{72}\zeta_2-\frac{7}{36}\zeta_3\right)\right]+\mathcal{O}(\epsilon^3)\,,$$
$$D_q^{(1)} = N_c\left(\frac{13}{72}+\frac{7}{8}\zeta_2\right)+\frac{1}{N_c}\left(1-\frac{1}{8}\zeta_2\right)-\frac{5}{36}n_f+\epsilon\left[N_c\left(\frac{10}{27}-\frac{\zeta_2}{24}+\frac{5}{6}\zeta_3\right)\right.$$
$$\left.+\frac{1}{N_c}\left(2-\frac{3}{16}\zeta_2-\frac{7}{12}\zeta_3\right)+n_f\left(-\frac{7}{27}+\frac{\zeta_2}{24}\right)\right]+\epsilon^2\left[N_c\left(\frac{121}{162}-\frac{13}{144}\zeta_2-\frac{7}{36}\zeta_3+\frac{35}{64}\zeta_4\right)\right.$$
$$\left.+\frac{1}{N_c}\left(4-\frac{\zeta_2}{2}-\frac{7}{8}\zeta_3-\frac{47}{64}\zeta_4\right)+n_f\left(-\frac{41}{81}+\frac{5}{72}\zeta_2+\frac{7}{36}\zeta_3\right)\right]+\mathcal{O}(\epsilon^3)\,.$$

- The result for the impact factor must satisfy a nontrivial constraint:



- Quark and gluon impact factor extracted from quark-quark and gluon-gluon amplitude must give the correct quark-gluon amplitude.

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- Proceeding in a similar way, the infrared factorisation at two loops predicts

$$\mathcal{H}^{(2,2)} = \frac{1}{2}(\hat{\alpha}_g^{(1)})^2(\mathbf{T}_t^2)^2\,\hat{\mathcal{M}}^{(0)},$$

$$\mathcal{H}^{(2,1)} = \hat{\mathcal{M}}^{(2,1)} + \hat{\alpha}_g^{(1)}\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(1,0)} + \hat{\alpha}_g^{(2)}\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(0)} + i\pi K^{(1)}\left[\tfrac{1}{2}K^{(1)}[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] - \hat{\alpha}_g^{(1)}\mathbf{T}_{s-u}^2\mathbf{T}_t^2\right]\hat{\mathcal{M}}^{(0)},$$

$$\mathcal{H}^{(2,0)} = \hat{\mathcal{M}}^{(2,0)} - \frac{\pi^2}{2}(K^{(1)})^2(\mathbf{T}_{s-u}^2)^2\hat{\mathcal{M}}^{(0)} - i\pi\left[K^{(2)}\mathbf{T}_{s-u}^2\hat{\mathcal{M}}^{(0)} + K^{(1)}\mathbf{T}_{s-u}^2\hat{\mathcal{M}}^{(1,0)}\right].$$

- Inserting results from the Regge theory one gets

$$\mathrm{Re}[\mathcal{H}^{(2,1)}] = \left[\hat{\alpha}_g^{(2)} + \hat{\alpha}_g^{(1)}\left(D_i^{(1)} + D_j^{(1)}\right)\right]\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(0)},$$

$$i\,\mathrm{Im}[\mathcal{H}^{(2,1)}] = i\pi\left[\left(\tfrac{1}{2}\mathrm{d}_2 + \tfrac{1}{2}(K^{(1)})^2 + K^{(1)}\hat{\alpha}_g^{(1)}\right)[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] + \left(\hat{\alpha}_g^{(1)}\right)^2\mathbf{T}_t^2\mathbf{T}_{s-u}^2\right]\hat{\mathcal{M}}^{(0)}.$$

- for the NLL coefficient, which is consistent with infrared factorisation. At NNLL we are able to predict the real part:

$$\mathrm{Re}[\mathcal{H}^{(2,0)}] = \left[D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)} - \pi^2 R^{(2)}\tfrac{1}{12}(C_A)^2 + \pi^2\left(R^{(2)} + \tfrac{1}{2}(K^{(1)})^2 + K^{(1)}\hat{\alpha}_g^{(1)}\right)(\mathbf{T}_{s-u}^2)^2\right]\hat{\mathcal{M}}^{(0)}.$$

- Here we see explicitly for the first time the appearance of the contribution from the three-Reggeon cut: because of it, Regge factorisation (interpreted as exponentiation of the Regge pole) is broken starting at two loops. Del Duca, Glover, 2001; Del Duca, Falcioni, Magnea, LV, 2013

AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- Our framework can be used to extract the impact factors at two loops: this is given by taking the projection of the amplitude onto the antisymmetric octet component:

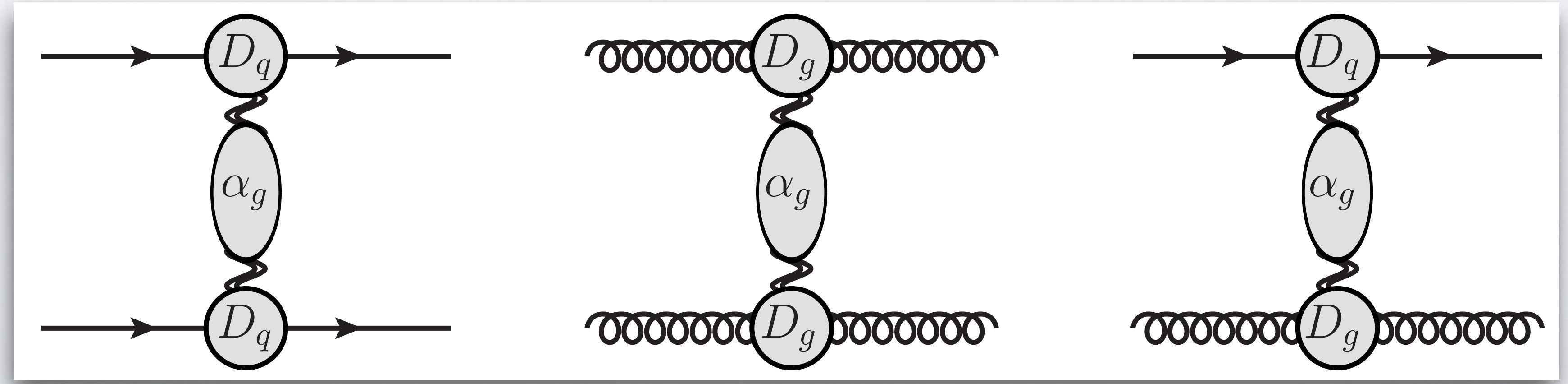
$$2D_g^{(2)} = \frac{\mathcal{H}^{(2,0)[8_a]}_{gg\to gg}}{\mathcal{H}^{(0)[8_a]}_{gg\to gg}} - (D_g^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2+24}{4},$$

$$D_q^{(2)} + D_g^{(2)} = \frac{\mathcal{H}^{(2,0)[8_a]}_{qg\to qg}}{\mathcal{H}^{(0)[8_a]}_{qg\to qg}} - D_q^{(1)} D_g^{(1)} + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2+4}{4},$$

$$2D_q^{(2)} = \frac{\mathrm{Re}[\mathcal{H}^{(2,0)[8_a]}_{qq\to qq}]}{\mathcal{H}^{(0)[8_a]}_{qq\to qq}} - (D_q^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^4 - 4N_c^2 + 12}{4N_c^2}.$$

Caron-Huot, Gardi, LV, 2017

- The effect of the three-Reggeon cut is evident from the color-dependent term in the equations above. Once again, consistency requires the three equations above to be satisfied simultaneously.



AMPLITUDES IN THE HIGH-ENERGY LIMIT: BFKL VS INFRARED FACTORISATION

- At three loops, at LL and NLL, the infrared factorisation formula predicts

$$\mathcal{H}^{(3,3)} = \frac{1}{6}\left(\hat{\alpha}_g^{(1)}\right)^3 \left(\mathbf{T}_t^2\right)^3 \hat{\mathcal{M}}^{(0)},$$

$$\mathcal{H}^{(3,2)} = \hat{\mathcal{M}}^{(3,2)} + \hat{\alpha}_g^{(1)}\,\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(2,1)} + \tfrac{1}{2}(\hat{\alpha}_g^{(1)})^2(\mathbf{T}_t^2)^2\hat{\mathcal{M}}^{(1,0)} + \hat{\alpha}_g^{(1)}\hat{\alpha}_g^{(2)}\,(\mathbf{T}_t^2)^2\,\hat{\mathcal{M}}^{(0)}$$
$$+ i\pi\Big(- \tfrac{1}{2}(\hat{\alpha}_g^{(1)})^2 K^{(1)}\mathbf{T}_{s-u}^2(\mathbf{T}_t^2)^2 + \tfrac{1}{2}\hat{\alpha}_g^{(1)}(K^{(1)})^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_t^2 - \tfrac{1}{6}(K^{(1)})^3[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\Big)\hat{\mathcal{M}}^{(0)}.$$

- which is consistent with Regge exponentiation and (dipole) infrared factorisation. More in details,

$$\mathrm{Re}[\mathcal{H}^{(3,2)}] = \hat{\alpha}_g^{(1)}\left[\hat{\alpha}_g^{(2)} + \tfrac{1}{2}\hat{\alpha}_g^{(1)}\left(D_i^{(1)} + D_j^{(1)}\right)\right](\mathbf{T}_t^2)^2\,\hat{\mathcal{M}}^{(0)} = \mathcal{O}(\epsilon),$$

$$i\,\mathrm{Im}[\mathcal{H}^{(3,2)}] = i\pi\Big[\tfrac{1}{6}\Big(\mathbb{d}_3 - (K^{(1)})^3 - 3K^{(1)}(\hat{\alpha}_g^{(1)})^2 - 3(K^{(1)})^2\hat{\alpha}_g^{(1)}\Big)[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]$$
$$+ \tfrac{1}{2}\hat{\alpha}_g^{(1)}\Big(\mathbb{d}_2 + (K^{(1)})^2 + 2K^{(1)}\hat{\alpha}_g^{(1)}\Big)\mathbf{T}_t^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] + \tfrac{1}{2}(\hat{\alpha}_g^{(1)})^3(\mathbf{T}_t^2)^2\mathbf{T}_{s-u}^2\Big]\hat{\mathcal{M}}^{(0)}$$
$$= i\pi\left(-\frac{11}{24}\zeta_3 + \mathcal{O}(\epsilon)\right)[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]] + \mathcal{O}(\epsilon).$$

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- At NNLL, we see for the first time the effect of the quadrupole correction:

$$\mathcal{H}^{(3,1)} = \hat{\mathcal{M}}^{(3,1)} + \hat{\alpha}_g^{(1)}\mathbf{T}_t^2\hat{\mathcal{M}}^{(2,0)} + \hat{\alpha}_g^{(2)}\mathbf{T}_t^2\hat{\mathcal{M}}^{(1,0)} + \hat{\alpha}_g^{(3)}\mathbf{T}_t^2\hat{\mathcal{M}}^{(0)}$$
$$+ \frac{\pi^2}{6}\Big[-3\hat{\alpha}_g^{(1)}(K^{(1)})^2(\mathbf{T}_{s-u}^2)^2\mathbf{T}_t^2 + (K^{(1)})^3\Big(2\mathbf{T}_{s-u}^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] + [\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_{s-u}^2\Big)\Big]\hat{\mathcal{M}}^{(0)}$$
$$+ i\pi\Big[-K^{(1)}\mathbf{T}_{s-u}^2\hat{\mathcal{M}}^{(2,1)} + \Big(\tfrac{1}{2}(K^{(1)})^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] - K^{(1)}\hat{\alpha}_g^{(1)}\mathbf{T}_{s-u}^2\mathbf{T}_t^2\Big)\hat{\mathcal{M}}^{(1,0)}$$
$$+ \Big(K^{(1)}K^{(2)}[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] - K^{(2)}\hat{\alpha}_g^{(1)}\mathbf{T}_{s-u}^2\mathbf{T}_t^2 - K^{(1)}\hat{\alpha}_g^{(2)}\mathbf{T}_{s-u}^2\mathbf{T}_t^2 - \frac{\zeta_3}{24\epsilon}[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\Big)\hat{\mathcal{M}}^{(0)}\Big].$$

- The effect is in the even sector, therefore we cannot check it explicitly with our computation. However, the calculation of the odd sector within Regge theory gives

$$\mathrm{Re}[\mathcal{H}^{(3,1)}] = \Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)}\Big)\Big]\mathbf{T}_t^2\,\hat{\mathcal{M}}^{(0)}$$
$$+ \pi^2\Big[R_C^{(3)} - \tfrac{1}{12}\hat{\alpha}_g^{(1)}R^{(2)}\Big](\mathbf{T}_t^2)^3\,\hat{\mathcal{M}}^{(0)} + \pi^2\,\hat{\alpha}_g^{(1)}\,\hat{R}^{(2)}\,\mathbf{T}_t^2(\mathbf{T}_{s-u}^2)^2\,\hat{\mathcal{M}}^{(0)}$$
$$+ \pi^2\Big[R_A^{(3)} + \tfrac{1}{6}K^{(1)}\Big(2(K^{(1)})^2 + 3\hat{\alpha}_g^{(1)}K^{(1)} + 3\mathbb{d}_2\Big)\Big]\mathbf{T}_{s-u}^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\,\hat{\mathcal{M}}^{(0)}$$
$$+ \pi^2\Big[R_B^{(3)} - \tfrac{1}{3}K^{(1)}\Big((K^{(1)})^2 + 3\hat{\alpha}_g^{(1)}K^{(1)} + 3(\hat{\alpha}_g^{(1)})^2\Big)\Big][\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_{s-u}^2\,\hat{\mathcal{M}}^{(0)}.$$

- Which is consistent with infrared factorisation. This is a rather non-trivial check, given that the two calculations are done in two completely different ways.

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- The Regge theory we have developed, however, allows us also to get some parts of the finite amplitude. Let's have a more detailed look at the amplitude: we have

$$\mathrm{Re}[\mathcal{H}^{(3,1)}] = \Bigg[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)}\Big) + C_A^2 \frac{\pi^2}{864}\Big(\frac{1}{\epsilon^3} - \frac{15\zeta_2}{4\epsilon} - \frac{175\zeta_3}{2}\Big)\Bigg] C_A\, \hat{\mathcal{M}}^{(0)} + \pi^2 \frac{5\zeta_3}{12} \mathbf{T}_{s-u}^2 [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\, \hat{\mathcal{M}}^{(0)} + \pi^2 \frac{\zeta_3}{12} [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)} + \mathcal{O}(\epsilon).$$

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- Going to an orthonormal basis in the t-channel, in components we have:

$$\mathrm{Re}[\mathcal{H}^{(3,1),[8_a]}] = \Bigg\{ C_A \Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)}\Big)\Big] + C_A^3 \frac{\pi^2}{864}\Big(\frac{1}{\epsilon^3} - \frac{15\zeta_2}{4\epsilon} - \frac{175\zeta_3}{2}\Big) - C_A \pi^2 \frac{2\zeta_3}{3} + \mathcal{O}(\epsilon)\Bigg\} \hat{\mathcal{M}}^{(0),[8_a]},$$

$$\mathrm{Re}[\mathcal{H}^{(3,1),[10+\overline{10}]}] = \sqrt{2} C_A \sqrt{C_A^2 - 4} \Bigg\{ \frac{11\pi^2\zeta_3}{24} + \mathcal{O}(\epsilon)\Bigg\} \hat{\mathcal{M}}^{(0),[8_a]}.$$

- The antisymmetric octet amplitude cannot be predicted entirely, given the unknown Regge trajectory at three loops; The $10+\overline{10}$ component, however, can be predicted exactly, and it agrees with a recent calculation of the gluon-gluon scattering amplitude at three loops in N=4 SYM.

Henn, Mistlberger, 2016

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- Last, consider the relation between the three-loop "gluon Regge trajectory" and the logarithmic terms in the three-loop amplitude.
- Starting from three loops the "gluon Regge trajectory" is scheme-dependent. Here we defined it to be the $1 \to 1$ matrix element of the Hamiltonian $\alpha_g(t) = -H_{1\to 1}/C_A$, in the scheme where states corresponding to a different number of Reggeon are orthogonal.
- This can be related to fixed-order amplitudes by taking the logarithm of the amplitude projected onto the signature-odd adjoint channel:

$$\log \frac{\mathcal{M}^{[8_a]}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}} = L\Big\{ - H_{1\to 1}(t) + \left(\frac{\alpha_s}{\pi}\right)^3 \pi^2 \Big[N_c\Big(- 2R_A^{(3)} + 2R_B^{(3)}\Big) + N_c^3 R_C^{(3)}\Big]\Big\} + \mathcal{O}(L^0, \alpha_s^4),$$

- Thanks to a recent calculation of the gluon-gluon amplitude in N=4 SYM, in this theory one has

$$\log \frac{\mathcal{M}^{[8_a], \mathcal{N}=4}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}}\Bigg|_L = N_c \left[\frac{\alpha_s}{\pi} k_1 + \left(\frac{\alpha_s}{\pi}\right)^2 k_2 + \left(\frac{\alpha_s}{\pi}\right)^3 k_3 + \cdots\right],$$

- where

$$k_1 = \frac{1}{2\epsilon} - \epsilon\frac{\zeta_2}{4} - \epsilon^2 \frac{7}{6}\zeta_3 - \epsilon^3 \frac{47}{32}\zeta_4 + \epsilon^4\left(\frac{7}{12}\zeta_2\zeta_3 - \frac{31}{10}\zeta_5\right) + \mathcal{O}(\epsilon^5),$$

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$$k_2 = N_c\left[-\frac{\zeta_2}{8}\frac{1}{\epsilon} - \frac{\zeta_3}{8} - \epsilon\frac{3}{16}\zeta_4 + \epsilon^2\left(\frac{71}{24}\zeta_2\zeta_3 + \frac{41}{8}\zeta_5\right) + \mathcal{O}(\epsilon^3)\right],$$

$$k_3 = N_c^2\left[\frac{11\zeta_4}{48}\frac{1}{\epsilon} + \frac{5}{24}\zeta_2\zeta_3 + \frac{1}{4}\zeta_5 + \mathcal{O}(\epsilon)\right] + \left[\frac{\zeta_2}{4}\frac{1}{\epsilon^3} - \frac{15\zeta_4}{16}\frac{1}{\epsilon} - \frac{77}{4}\zeta_2\zeta_3 + \mathcal{O}(\epsilon)\right].$$

- Matching these two results we get

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$$-H_{1\to 1}^{\mathcal{N}=4\,\mathrm{SYM}} = N_c \left[\frac{\alpha_s}{\pi}\alpha_g^{(1)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \left(\frac{\alpha_s}{\pi}\right)^2 \alpha_g^{(2)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \left(\frac{\alpha_s}{\pi}\right)^3 \alpha_g^{(3)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \cdots\right],$$

- With

$$\alpha_g^{(1)}|_{\mathcal{N}=4\,\mathrm{SYM}} = k_1, \quad \alpha_g^{(2)}|_{\mathcal{N}=4\,\mathrm{SYM}} = k_2,$$

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- and

$$\alpha_g^{(3)}|_{\mathcal{N}=4\,\mathrm{SYM}} = N_c^2\left[-\frac{\zeta_2}{144}\frac{1}{\epsilon^3} + \frac{49\zeta_4}{192}\frac{1}{\epsilon} + \frac{107}{144}\zeta_2\zeta_3 + \frac{\zeta_5}{4} + \mathcal{O}(\epsilon)\right] + N_c^0\left[0 + \mathcal{O}(\epsilon)\right].$$

- Even though to three loop accuracy the adjoint amplitude may look like a Regge pole, e.g. a pure power-law, it is actually not: starting from two-loops it is really a sum of multiple powers.
- Simply exponentiating the logarithm of the full amplitude at three loops would predict a definitely incorrect four-loop amplitude.
- The correct, predictive, procedure is to exponentiate the action of the BFKL Hamiltonian. With the "trajectory" fixed as above, this procedure does not require any new parameter for the odd amplitude at NNLL to all loop orders.

CONCLUSION

- We have computed the Regge-cut contribution to three loops through NNLL in the signature-odd sector.
- Our formalism is based on using the non-linear Balitsky-JIMWLK rapidity evolution equation to derive an effective Hamiltonian acting on states with a fixed number of Reggeized gluons.
- A new effect occurring first at NNLL is mixing between states with k and k+2 Reggeized gluons due non-diagonal terms in this Hamiltonian.
- Our results are consistent with a recent determination of the infrared structure of scattering amplitudes at three loops, as well as a computation of $2 \to 2$ gluon scattering in $N = 4$ super Yang-Mills theory.
- Combining the latter with our Regge-cut calculation we extract the three-loop Regge trajectory in this theory.
- Our results open the way to predict high-energy logarithms through NNLL at higher-loop orders.