Modern EFT for precision computations

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May 2023

I Method of regions

Toy example

$$\underline{I} = \int d^{4-2\varepsilon}k \; \frac{1}{k^2+m^2} \; \frac{1}{k^2+M^2} \sim \int_0^\infty dk \, \frac{k^{3-2\varepsilon}}{(k^2+m^2)(k^2+M^2)}$$

$$= \frac{1}{m^2-M^2} \int_0^\infty dk \; k^{3-2\varepsilon} \left[\frac{1}{k^2+M^2} - \frac{1}{k^2+m^2} \right]$$

$$= -\frac{1}{2} \frac{\pi}{\sin \pi\varepsilon} \; \frac{1}{m^2-M^2} \left[M^{2-2\varepsilon} - m^{2-2\varepsilon} \right]$$

$$\left\{ \lambda = \frac{m^2}{M^2} \right\} \quad = -\frac{1}{2} \frac{\pi}{\sin\pi\varepsilon} \frac{M^{-2\varepsilon}}{\lambda - 1} \left[1 - \lambda^{1-\varepsilon} \right]$$

$$= -\frac{1}{2} \frac{1}{\varepsilon} \left[\frac{1 - \varepsilon \ln M^2}{\lambda - 1} \right] \left[1 - \lambda + \varepsilon \lambda \ln \lambda \right] + O($$

$$= \frac{1}{2} \frac{1}{\varepsilon} \left[1 - \varepsilon \ln M^2 \right] \left[1 + \varepsilon \frac{\lambda}{1-\lambda} \ln \lambda \right] + O(\varepsilon$$

(*)

$$\underline{I} \sim \int_0^\infty dk \, \frac{k^{3-2\varepsilon}}{(k^2+m^2)(k^2+M^2)} \qquad m \ll \Lambda \ll M \qquad \int_0^\infty dk = \int_0^\Lambda dk + \int_\Lambda^\infty dk$$

$$= \int_0^\Lambda dk \, \frac{k^{3-2\varepsilon} \sum_{n=0}^\infty \left(-\frac{k^2}{M^2} \right)^n}{M^2 (k^2 + m^2)} + \int_\Lambda^\infty dk \, \frac{k^{3-2\varepsilon} \sum_{n=0}^\infty \left(-\frac{m^2}{k^2} \right)^n}{(k^2+M^2) \; k^2}$$

$$k^2 \sim m^2 \qquad\qquad k^2 \sim M^2$$

"soft region"

$$= \int_0^\infty dk \, \frac{k^{3-2\varepsilon} \sum_{n=0}^{\infty} \left(-\frac{k^2}{M^2}\right)^n}{M^2 (k^2+m^2)} - \underbrace{\int_\Lambda^\infty dk \, \frac{k^{3-2\varepsilon}}{M^2 k^2} \sum_{n=0}^{\infty} \left(-\frac{k^2}{M^2}\right)^n \sum_{m=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^m}$$

$$+ \int_0^\infty dk \, \frac{k^{3-2\varepsilon} \sum_{n=0}^{\infty} \left(\frac{-m^2}{k^2}\right)^n}{k^2 (k^2+M^2)} - \underbrace{\int_0^\Lambda dk \, \frac{k^{3-2\varepsilon}}{M^2 k^2} \sum_{n=0}^{\infty} \left(-\frac{k^2}{M^2}\right)^n \sum_{m=0}^{\infty} \left(-\frac{m^2}{k^2}\right)^m}$$

"hard" region

$$\sim \int_0^\infty dk \, k^{\alpha - 2\varepsilon} = 0$$

scaleless contribution

vanish in dim-reg

$$\sim \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$$

$$\alpha = 1 - 2\varepsilon + 2n - 2m$$

homogeneus scaling:

soft region : $k \sim m$

$$\int dk \, \frac{k^{3-2\varepsilon}\left(-\frac{k^2}{M^2}\right)^n}{M^2 (k^2+m^2)} \sim \frac{m \times m^{3-2\varepsilon} \left(\frac{m}{M}\right.}{M^2 m^2}$$

$$\sim \frac{m^{2-2\varepsilon+2n}}{M^{2+2n}}$$

hard region $k \sim M$ $\qquad \frac{m^{2n}}{M^{2n}} M^{-2\varepsilon}$

$n=0$: leading pow

$n=1$ $\left(\frac{m}{M}\right)^2$ power correctio

Hard region

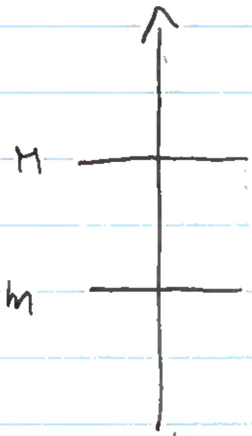
$$\int_0^\infty dk \, \frac{k^{3-2\varepsilon}}{k^2(k^2+M^2)} = \frac{1}{2}\left[\frac{1}{\varepsilon} - \ln M^2\right]$$ original UV pole

$$\int_0^\infty dk \, \frac{k^{3-2\varepsilon}}{k^2(k^2+M^2)}\left(-\frac{m^2}{k^2}\right) = \frac{1}{2}\frac{m^2}{M^2}\left[\frac{1}{\varepsilon} - \ln M^2\right]$$ "new" IR pole

soft region

$$\int_0^\infty dk \, \frac{k^{3-2\varepsilon}}{M^2(k^2+m^2)} = \frac{1}{2}\frac{m^2}{M^2}\left[-\frac{1}{\varepsilon} + \ln m^2\right]$$ "new" UV pole

We reproduced (*) up to higher order terms.



original integral was IR finite

⇝ IR poles

⇝ UV poles

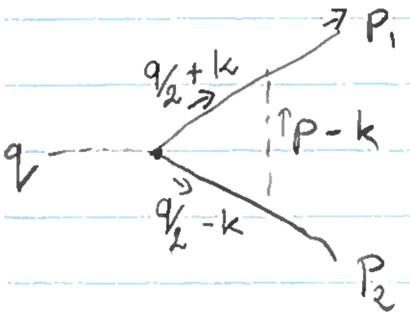
"low energy eft"
UV poles are canceled by short-distance or "hard" contribution.

⇝ EFT logic: separate problem into series of single scale problems

⇝ low energy contribution must capture IR of the full theory.

⇝ 'mistakes' in UV can be fixed by UV counter-terms

Threshold expansion



$$p_1^2 = p_2^2 = m^2$$

$$q = p_1 + p_2$$

$$p = (p_1 - p_2)/2$$

$$y \equiv m^2 - q^2/4 = p^2 \ll q^2$$

We take frame $q = (q_0, \vec{0})$

$$p_1 = (p_0, \vec{p}) \qquad p_2 = (p_0, -\vec{p}) \qquad p = (0, \vec{p})$$

in this frame p_1 and p_2 are non-relativistic

$$q^2 \sim 4m^2 \qquad y = -|\vec{p}|^2 \sim v^2 m$$

Regions:

hard $k \sim m$

potential $k^0 \sim v^2 m$, $\vec{k} \sim v m$

soft $k^0 \sim v m$, $\vec{k} \sim v m$

u-soft $k^0 \sim v^2 m$, $\vec{k} \sim v^2 m$

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k+q/2)^2 - m^2} \frac{1}{(k-q/2)^2 - m^2} \frac{1}{(k-p)^2}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + k\cdot q - y + i\varepsilon} \frac{1}{k^2 - k\cdot q - y + i\varepsilon} \frac{1}{(k-p)^2 + i\varepsilon}$$

Hard region $p \ll k$ & $y \ll k$

$$I^h = \int \frac{d^d k / i\pi^{D/2}}{k^2 (k^2 + qk)(k^2 - qk)} = \left(\frac{4}{q^2}\right)^{1+\varepsilon} \left(-\frac{1}{2}\right) \frac{\Gamma(\varepsilon)}{1+2\varepsilon}$$ * e.x.

Consider quark propagators: q^0 - large
y - small $\sim v^2 m$

$$k^{0\,2} - \vec{k}^2 \pm k_0 q^0 - y$$

if $k^0 \sim |\vec{k}|^2 \sim v^2$ then $-\vec{k}^2 \pm k_0 q_0 - y + i\varepsilon$

$$\Rightarrow k_0 = \pm \frac{\vec{k}^2 + y}{q_0}$$ * e.x.

$$I^p = \frac{2\pi i}{q_0} \int \frac{d^{d-1} k}{i\pi^{d/2}} \frac{1}{(\vec{k}-\vec{p})^2} \frac{1}{-2(\vec{k}^2 + y)} = \frac{y^{-\varepsilon}}{\sqrt{q^2 y}} \frac{\sqrt{\pi}\,\Gamma(\varepsilon + 1/2)}{2\varepsilon}$$

I scaling:

$$v^3 \frac{1}{v^2} \frac{1}{v^2} \sim \frac{1}{v}$$ Coulomb singularity

For soft and u-soft we shift first momentum

$k \to k + p$

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\underset{v^2}{(k+p)^2} - \underset{v}{(k+p)q} - \underset{v^2}{y}} \; \frac{1}{\underset{v^2}{(k+p)^2} - \underset{v}{(k+p)q} - \underset{v^2}{y}} \; \frac{1}{k^2}$$

$p_0 = 0$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k_0 + i\varepsilon} \frac{1}{-k_0 + i\varepsilon} \frac{1}{k_0^2 - \vec{k}^2 + i\varepsilon} \frac{1}{q_0^2}$$

Integral is ill-defined $\to$ poles at $k_0 = 0$ pinch the integration contour

$\to$ quark poles have been accounted for

$\to$ gluon poles only

$k_0 = \pm |\vec{k}|$ $\quad \rightsquigarrow \quad \int \frac{d^{d-1}k}{(2\pi)^d} \frac{1}{|\vec{k}|^3} = 0$

u-soft

$$(k+p)^2 = \underset{v^4}{k^2} + \underset{v^3}{2kp} + \underset{v^2}{p^2}$$

$$(k+p)^2 \pm (k+p)q - y = p^2 + k_0 q_0 - p^2 = \pm k_0 q_0$$

$\to$ the integrand is the same as soft $\to 0$.

NRQED

Potential scaling $E \sim v^2 m \quad \vec{p} \sim vm$

Let's decompose the momentum as

$$p = mv + k \quad , \text{ where } v = (1, \vec{0})$$

$$m^2 = p^2 = m^2 + 2m k_0 + k^2 \quad \text{on-shell condition.}$$

$$\Rightarrow k_0 = \frac{-k_0^2 + \vec{k}^2}{2m} \Leftrightarrow k_0 = \frac{\vec{k}^2}{2m} + O(v^4)$$

QED Lagrandian:

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi$$

We define projection operators $P_\pm = \frac{1 \pm \not{v}}{2}$

Check: $P_\pm^2 = \frac{1 \pm 2\not{v} + \not{v}^2}{4} = \frac{1 + \not{v}}{2}$

$$P_+ P_- = \frac{1 - \not{v}^2}{4} = 0.$$

then define:

$$\Psi = e^{-imv\cdot x}\left[\varphi(x) + \chi(x)\right]$$

with $\varphi(x) = e^{imv\cdot x} P_+ \Psi(x)$

$$\chi(x) = e^{imv\cdot x} P_- \Psi(x)$$

note we write

$\not{D} = \gamma^0 D^0 + \vec{\not{D}}$

$\not{v} D^0 + \not{D}_\perp$

$$\mathcal{L} = \bar{\varphi}\, i D_t\, \varphi - \bar{\chi}\,(i D_t + 2m)\,\chi + \bar{\varphi}\, i \vec{\not{D}}\, \chi + \bar{\chi}\, i \vec{\not{D}}\, \varphi \qquad (1)$$

$\boxed{* \quad \text{ex. 3.}}$
check.

At this point, we can integrate-out χ field

$$0 = \frac{\delta \mathcal{L}}{\delta \bar{\chi}} = -(i D_t + 2m)\chi + i \vec{\not{D}}\, \varphi$$

which can be solved

$$\chi = \frac{1}{i D_t + 2m}\, i \vec{\not{D}}\, \varphi .$$

Inserting into (1) we obtain

$$\mathcal{L} = \bar{\varphi}\, i D_t\, \varphi + \bar{\varphi}\, i \vec{\not{D}}\, \frac{1}{i D_t + 2m}\, i \vec{\not{D}}\, \varphi$$

which is formally equivalent to QED lagrangi
. note it is non-local but $i D_t \ll 2m$
we get a local Lagrangian after expansion

$$\mathcal{L} = \bar{\varphi}\left[i D_t + \sum_{k=0}^{\infty} \frac{1}{2m}\, i \vec{\not{D}} \left(\frac{-i D_t}{2m} \right)^k i \vec{\not{D}} \right] \varphi .$$

the leading term is:

$$\mathcal{L} = \bar{\varphi}\left[i D_t - \frac{\vec{\not{D}}^2}{2m} \right] \varphi \qquad (**)$$

Note that

$$i\not{D} i\not{D} = \gamma^\mu \gamma^\nu iD_\mu iD_\nu = \Big(\tfrac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}} + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\Big) iD_\mu iD_\nu$$

$$= (iD)^2 + \frac{1}{4}[\gamma^\mu,\gamma^\nu]\,[iD_\mu, iD_\nu]$$

$$= (iD)^2 + \frac{1}{4}[\gamma^\mu,\gamma^\nu]\, ieF_{\mu\nu}$$

$$= (iD)^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} \qquad (*)$$

We can define spin operator

$S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$; we can check that $S^i = \frac{1}{4}\varepsilon^{ijk}\sigma^{jk}$

$B^i = -\frac{1}{2}\varepsilon^{ijk}F^{jk}$ magnetic field

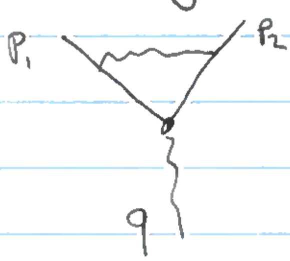
$$\frac{e}{2}\sigma^{ij}F^{ij} = \frac{e}{2}\sigma^{ij}\frac{1}{2}\left(\delta_i^p\delta_j^q - \delta_i^q\delta_j^p\right)F^{pq}$$

$$= \frac{e}{4}\sigma^{ij}\varepsilon_{ijk}\varepsilon^{pqk}F^{pq} = eS^k(-2)B^k$$

So inserting (*) in (**) and using the above, we get

$$\mathcal{L} = \bar{\varphi}\Big[iD_t + \frac{\vec{D}^2}{2m} - \frac{e\vec{S}\vec{B}}{m}\Big)\varphi$$

Matching



$$-ie\bar{\Psi}\left(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)\right)\Psi$$

external spinor $u(p) = N(p)\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi \end{pmatrix}$; $N = \sqrt{\frac{p_0+m}{2p_0}} = 1 - \frac{\vec{p}^2}{8m^2}$

$\bar{\Psi}\gamma^\mu\Psi \to$ take $\mu = 0$ component, and $p_0 = m + O(\vec{p}^2)$

$$\Psi^+\Psi = \chi^+\chi + \chi^+\frac{\vec{\sigma}\,p_2}{2m}\frac{\vec{\sigma}\,p_1}{2m}\chi - \chi^+\chi\left(\frac{\vec{p}_1^{\,2}}{8m^2} + \frac{\vec{p}_2^{\,2}}{8m^2}\right)$$

$$= \chi^+\chi + \chi^+\frac{i\vec{\sigma}\cdot(\vec{p}_1\times\vec{p}_2)}{4m^2}\chi$$

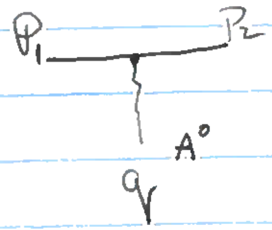
$$- \chi^+\frac{(p_1-p_2)^2}{8m^2}\chi$$

We note that $F_1(q^2) = F_1(0) + F'(0)\,q^2/m^2$

on the EFT side

$$\mathcal{L} = \varphi\left(iD_t + \frac{\vec{D}^2}{2m} + C_F e\frac{\vec{\sigma}\cdot\vec{B}}{2m} + C_D e\frac{\vec{\nabla}\vec{E}}{8m^2}\right.$$
$$\left. + iC_S e\frac{\vec{\sigma}(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})}{8m^2}\right\}\varphi.$$

$$E = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \qquad \phi = A^0$$



$$\bar{\varphi}\, i D_t\, \varphi \rightarrow -ie$$

$$\bar{\varphi}\, c_D e \frac{\vec{\nabla}\vec{E}}{8m^2} \rightarrow \frac{ie c_D |\vec{p}_1 - \vec{p}_2|^2}{8m^2}$$

$$\bar{\varphi}\, i c_S e \frac{\vec{\sigma}(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})}{8m^2}\varphi \rightarrow \frac{e c_S (\vec{p}_2 \times \vec{p}_1)\cdot\vec{\sigma}}{4m^2}$$

so comparing with QED amplitude (F_1 only)

$F_1(0) = 1$ consequence of gauge invariance

$$C_D = F_1(0) + F'(0)\cdot 8 \qquad \text{if we do the same for } F_2 \; + 2F_2(0)$$

$$C_S = F_1(0) \qquad + F_2(0).$$

Complete matching gives

$$C_F = 1 + \frac{\alpha}{2\pi}$$

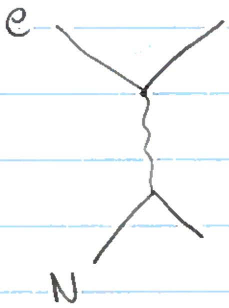
$$C_D = 1 + \frac{\alpha}{\pi}\left(\frac{8}{3}\ln m/\mu\right)$$

* we note this log.

$$C_S = 1 + \frac{\alpha}{\pi}$$

Note that EFT loops are scaleless so we only ~~only~~ include counter-terms $\Leftrightarrow$ drop poles in $\overline{MS}$ $\rightarrow$ we will return to this in SCET

- the NRQED does not have homogeneous power-counting in velocity.
- single diagram contributes to multiple order in $v \sim \alpha$.
- proper EFT with well-defined counting is pNRQED obtained after integrating out soft & potential photon



$$\sim \frac{1}{q^2} \qquad q^0 \sim v^2 \; , \; \vec{q} \sim v$$

we set $m_N \to \infty$

$$\mathcal{L} = \varphi^\dagger(x)\left(i D^0 + \frac{\vec{D}^2}{2m} + \frac{\vec{D}^4}{8m^3} \right) \varphi(x) + N^\dagger i D^0 N - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$+ \int d^3r \, N^\dagger(x) N(x) \, V(\vec{r}) \, \varphi^\dagger(x+\vec{r}) \, \varphi(x+\vec{r})$$

- non-local 4-fermion operator ≡ potential

- We need to perform multipole-expansion for ultra-soft photons.

Potential matching (ignoring spinors)

$$\int d^4x \langle e(p_2)N(p_2') | \int d^3r\, N^+N(x)\, V(\vec{r})\, \varphi^+\varphi(x+\vec{r}) | e(p_1)N(p_1')\rangle$$

$$= \int d^4x \int d^3r\, e^{i(P_2'-P_1')x + i(P_2-P_1)(x+\vec{r})}\, V(\vec{r})$$

$$= (2\pi)^4 \delta^4(P_2+P_2'-P_1-P_1')\, \tilde{V}(\bar{P}_2-\bar{P}_1)$$

where $\tilde{V}(\vec{q}) = \int d^3r\, e^{i\vec{q}\vec{r}}\, V(\vec{r})$

Leading potential

$$= \frac{Ze^2}{q^2} \simeq -\frac{Ze^2}{|\vec{q}|^2} \Rightarrow V(r) = \frac{Z\alpha}{r}$$

Power counting:

$$\langle 0 | \varphi^+(x)\, \varphi(0) | 0\rangle = \int d^4k\, e^{ixk} \frac{1}{k_0 - \vec{k}^2/2m}$$

$$V^2 \times V^3 \qquad\qquad 1/V^2$$

$\Rightarrow \varphi \sim V^{3/2}$

Leading Lagrangian

$$\int d^4x\, \varphi\left(i\partial_t - \frac{\nabla^2}{2m}\right)\varphi \sim \frac{1}{V^2}\,\frac{1}{V^3}\cdot V^{3/2}\cdot V^2\cdot V^{3/2} \sim V^0 \quad \checkmark$$

$$\int d^4x\, d^3r\, N^+N \frac{Z\alpha}{r} \varphi^+\varphi \sim \frac{1}{V^5}\cdot\frac{1}{V^3}\cdot V^3\cdot Z\alpha \times V \times V^3 = Z\alpha \cdot V^{-1}$$

for $\alpha \sim V$ $\Rightarrow$ the potential is a LO term!

$\boxed{ex:}$ derive $1/m^2$ potential. $1/m$ as well
discuss 1-loop corrections

Multipole expansion

u-soft photon. $k \sim v^2 m \Rightarrow x \sim \frac{1}{v^2 m}$

potential fermion $k_0 \sim v^2 m$, $\vec{k} \sim v m$, $x^0 \sim \frac{1}{v^2 m}$; $\vec{x} \sim 1/vm$

$\Rightarrow$ u-soft photons form a long-wavelength background

$$\langle 0 | \varphi_{us}(x) \varphi_p(x) | p_p, p_s \rangle = e^{-i(p_p + p_s)x}$$
$$= \underbrace{e^{-i p_p x - i p_s^0 x^0}}_{1} + \underbrace{-i \vec{p}_s \vec{x}\, e^{-i p_p x - i p_s}}_{v^2 \cdot \frac{1}{v} \sim v}$$

$$i D_t = i\partial_0 - eA^0 = i\partial_0 - e A_0(t, \vec{0}) - e\vec{x}\, \vec{\nabla} A_0(t, 0)$$

v^2 $\quad$ v^4 suppressed $\quad$ v^3

$$i\vec{D} = i\vec{\nabla} + e\vec{A}$$

$$(i\vec{D})^2 = (i\nabla)^2 + i\vec{\nabla}\vec{A} + i\vec{A}\vec{\nabla} + \vec{A}^2 = -\nabla^2 + i(\vec{\nabla}\vec{A}(t,0)) + 2i\vec{A}(t,0)\vec{\nabla} + \vec{A}^2(t,0).$$

$\sim v^4$

so we have

$$\int d^4x\, \varphi^\dagger\left(-i\frac{e}{m}\vec{A}(t,\vec{0})\right)\vec{\nabla}\varphi = \int d^4x\, \varphi^\dagger(-ei)\vec{A}(t,\vec{0})[\vec{x}, \hat{H}]\,\varphi = *$$

Note $[\vec{x}, \hat{H}] = \frac{\vec{\nabla}}{m}$

$i\dot{\varphi} = \hat{H}\varphi$ (equation of motion

$$* = \int d^4x \left[\dot{\varphi}^+(+e)\vec{A}(t,\vec{0})\vec{x}\varphi + \varphi^+(+e)\vec{A}(t,\vec{0})\vec{x}\dot{\varphi}\right.$$

$$= -\int d^4x\, \varphi^+ e \frac{\partial}{\partial t}\vec{A}(t,0)\vec{x}\varphi.$$

So, the leading Lagrangian becomes (up to potential

$$\mathcal{L} = \varphi^+(x)\left[iD^0 + \frac{\vec{D}^2}{2m}\right]\varphi(x) + \text{potential}$$

$$= \varphi^+(x)\left[iD^0(x_0) + \frac{\nabla^2}{2m} - e\vec{x}\left[\vec{\nabla}A_0(t,0) + \frac{\partial}{\partial t}\vec{A}(t,0)\right]\right.$$

$$= \varphi^+(x)\left[iD^0(x_0) + \frac{\nabla^2}{2m} + e\vec{x}\cdot\vec{E}\right]\varphi + \text{potential}$$

we have

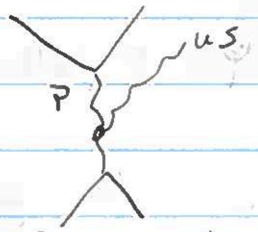
$$iD^0(x_0) = i\partial_0 - eA_0(t,\vec{0})$$

Remarks:

- only D^0 is covariant; gauge symmetry only along "t" direction
- $\vec{x}\cdot\vec{E}$ is manifestly gauge invariant
  → dipole interaction

- Note that we used full e.o.m. i.e. with potential term. → in QCD potential term does not commute with A field

→ the extra term corresponds to the diagram



so the QCD can also be expessed in terms of dipole interaction.

→ Operators proportional to clasical equation of motion do not contribute to on-shell Green's functions

potential ex.

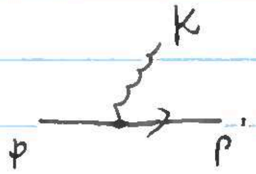
Lomb shift

· Low energy part
· the effect is $v^2\alpha$ suppressed.

(*) 16(a)

$$G(0,\vec{x},E)=\sum_n i\frac{|n\rangle\langle n|}{E-E_n}+\dots$$

∘ Momentum conservation for u-soft interaction



$$\int d^4x \langle p',k|\,\varphi^+(x)\,A^i(t,0)\,\varphi(x)\,|p\rangle$$

$$-\int d^4x\; e^{ip'x}\; e^{ik_0t}\; e^{-ipx}\;\varepsilon^i(k)$$

$$=(2\pi)^4\,\delta^{(3)}(\vec{p}-p')\,\delta(p_0-p_0'-k_0)$$

∵ Only energy is conserved

$$\mathcal{L} = \varphi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m}\right)\varphi + \text{potential}$$

$\rightsquigarrow$ projecting on two-particle space; we recover QM with $H = \frac{p^2}{2m} + V(r)$

- the propagator is

$$\frac{1}{E-H} = \sum_{n,m} |n\rangle\langle n| \frac{1}{E-H} |m\rangle\langle m|$$

$$= \sum_{n,m} |n\rangle \frac{\overbrace{\langle n|m\rangle}^{\delta_{nm}}}{E-E_m} \langle m|$$

$$= \sum_m \frac{|m\rangle\langle m|}{E-E_m}$$

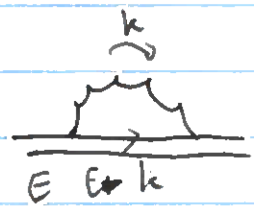
spectral representation

{In Coulomb gauge A^0 does not propagate.}

{So we can use $\varphi^+(-i)\frac{e}{m}\vec{A}(t,0)\vec{\nabla}\varphi$ as interaction to compute $\vec{x}\cdot\vec{E}$ contribution}

* Note x - dependent Feynman rules lead to derivatives of momentum conservation.

we assume $\epsilon \sim E_n$ $(\epsilon \to E_n)$



$$e^2 \frac{i\,|n\rangle\langle n|}{E - E_n}\,\vec{x}\,\vec{E}\,\frac{i\,|m\rangle\langle m|}{E - E_m - k_0}\,\vec{x}\,\vec{E}\,\frac{i\,|n\rangle\langle n|}{E - E_n}$$

for the E field we calculate vacuum

$$\langle 0|\,E_i(t)E_j(t')\,|0\rangle = \langle 0|\left(\nabla_i A_0 + \frac{\partial}{\partial t}\vec{A}_i\right)\left(\nabla_j A_0 + \frac{\partial}{\partial t}A_j\right)|0\rangle$$

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2}\,e^{ik_0(t-t')}\left(k_i k_j - k_0^2\delta_{ij}\right)$$

$$\int \frac{d^dk}{(2\pi)^d}\,e^2\,\frac{|n\rangle\langle n|\,x^i\,|m\rangle\langle m|\,x^j\,|n\rangle\langle n|}{(\epsilon - E_n)(\epsilon - E_m - k_0)(\epsilon - E_n)}\,\frac{1}{k^2}\,k_0^2\left(\frac{k_i k_j}{\vec{k}^2} - \delta_{ij}\right)$$

Recall $\langle n|\,\frac{\vec{p}}{m}\,|m\rangle = \langle n|\,i[\hat{H},x]\,|m\rangle$

$$= i\,(E_n - E_m)\,\langle n|\vec{x}|m\rangle$$

 $= -e^2 \, \frac{|n\rangle\langle n|}{E-E_n} \, \hat{p}^i/m \, |m\rangle\langle m| \, \hat{p}^j/m \, \frac{|n\rangle\langle n|}{E-E_n} \, I_{ij}$

with $I_{ij} = \int \frac{d^D k}{(2\pi)^D} \frac{k_0^2}{(E_n-E_m)^2} \frac{1}{k^2}\left(\frac{k_i k_j}{\vec{k}^2} - \delta_{ij}\right)\frac{1}{E-E_m-}$

We take the residue in k_0 first!

$\boxed{ex}$ show that (in the limit $E \to E_n$)

$$I_{ij} = \frac{(E-E_m)}{6\pi^2}\delta_{ij}\left(\frac{1}{2\varepsilon} + \frac{1}{2}\ln 4\pi + \ln \frac{\mu}{2(E-E_m)} + \frac{5}{6} - \frac{\gamma}{2} - \ln 2\right)$$

(~~Check~~)

We note UV divergence:
the pole part is $\sim \frac{(E-E_m)}{12\pi^2}\frac{1}{\varepsilon}$

$$\sum_m \langle n| \frac{p^i}{m} |m\rangle (E-E_m) \langle m| \frac{p^i}{m} |n\rangle =$$

$$= \langle n| \frac{p^i}{m} (E-\hat{H}) \frac{p^i}{m} |n\rangle$$

$$= \frac{1}{2}\left(\langle n| \underbrace{\frac{p^2}{m^2}(E-\hat{H})} + \frac{p^i}{m}\left[E-\hat{H}, \frac{p^i}{m}\right] + \underbrace{(E-\hat{H})\frac{p^2}{m}} - \left[E-\hat{H}, \frac{p^i}{m}\right]\frac{p^i}{m}\right.$$

In the limit $E \to E_n$ (*) vanis

$$= \langle n| \frac{1}{2}\left[\frac{p^i}{m}, \left[(E-\hat{H}), \frac{p^i}{m}\right]\right] |n\rangle$$

$\hat{p}/m = -i\vec{\nabla}/m$

$$-\frac{1}{2}\left[\frac{p^i}{m}\left[\frac{p^i}{m}, E - \hat{H}\right]\right] = \frac{\Delta}{2m} V = -\frac{Ze^2}{2m^2}\,\delta^{D-1}(\vec{x})$$

still in dim-reg

- this means that we can absorb the divergence into local counter-term in the Lagrangian.

- more importantly; the "low-" and "high-" energy poles cancel

$$c_D = 1 + \frac{\alpha}{\pi}\left(\frac{8}{3}\ln m/\mu\right)$$

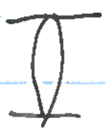
$$\rightarrow \frac{Ze^2}{m^2}\left(-\frac{c_D}{8}\right)\delta^{D-1}(\vec{x}) \Rightarrow -\frac{\alpha}{3\pi}\ln m/\mu \;\frac{Ze^2}{m^2}\delta^{D-1}(\vec{x})$$

Ub-soft: $\frac{e^2}{6\pi^2}\ln \mu/E-\epsilon_m\left(\frac{-Ze^2}{2m^2}\right)\delta^{D-1}(\vec{x})$

$$= -\frac{\alpha}{3\pi}\ln \mu/E-\epsilon_m \;\frac{Ze^2}{m^2}\delta^{D-1}(\vec{x})$$

μ - dependence cancels.

neutrino

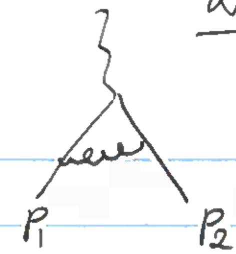


~~ex: calculate total width of n-state.~~

ex: derive Yukawa potential for massive Z boson and discuss $vM \sim m_Z$ and $vM \ll m_Z$ $vM \gg m_Z$ cases
- discuss the potential generated by neutrino → 19a

Summary so far:

- Low energy EFT can be non-local
- homogeneous power-counting is crucial for systematic expansion
- interactions have to be multipole expanded



Back to the method of regions
Sudakov form-factor

$$I = i\pi^{-d/2}\mu^{2\varepsilon}\int d^dk \, \frac{1}{k^2+i\varepsilon} \, \frac{1}{(k+p_1)^2+i\varepsilon} \, \frac{1}{(k+p_2)^2+i\varepsilon}$$

$$p_1^2 \neq 0 \;\&\; p_2^2 \neq 0 \qquad Q^2 = -q^2$$

∘ Introduce light-cone vectors

$$n_+^2 = n_-^2 = 0 \qquad n_+ n_- = 2$$

$$p^\mu = n_+p\,\frac{n_-^\mu}{2} + n_-p\,\frac{n_+^\mu}{2} + p_\perp^\mu$$

$$(p_1+p_2)^2 = \underset{\lambda^2}{p_1^2} + \underset{\lambda^2}{p_2^2} + \underset{1}{n_+p_1\, n_-p_2} + \underset{\lambda\cdot\lambda}{2p_{1\perp}p_{2\perp}} + \underset{\lambda^2\times\lambda^2}{n_-p_1\, n_+p_2}$$

$Q^2 \gg p_i^2 \rightsquigarrow$ expand in the limit

$\lambda = p_1^2/Q^2$ and $n_+p_1 \sim n_-p_2 \sim Q$
$n_-p_1 \sim n_+p_2 \sim \lambda^2 Q$
$p_{\perp_1} = p_{\perp_2} = 0$ frame choice

in general $p_{\perp i} \sim \lambda Q$

· p_1 – collinear
p_2 – anti-collinear

hard region

$k^\mu \sim Q$

$$(k+p_1)^2 = \underset{1}{k^2} + \underset{1}{n_+p_1} \cdot \underset{1}{n_-k} + \underset{\lambda^2}{n_-p_1} \cdot \underset{1}{n_+k} + \underset{\lambda^2}{p_1^2} \qquad ① \quad ⑪$$

$$= k^2 + n_+p_1 \cdot n_-k + O(\lambda^2)$$

$$(k+p_2)^2 = k^2 + n_+k \, n_-p_2 + O(\lambda^2)$$

$$I_{hard} = i\pi^{-D/2} \mu^{2\varepsilon} \int d^dk \, \frac{1}{k^2+i\varepsilon} \, \frac{1}{k^2+n_+p_1 n_-k} \, \frac{1}{k^2+n_+k \cdot n_-p_2}$$

standard integral with massless external momenta

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left[\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln \mu^2/Q^2 + \frac{1}{2}\ln^2 \mu^2/Q^2 - \frac{\pi^2}{6}\right]$$

"IR poles"
- double pole
- $1/\varepsilon \ln \mu^2/Q^2$.

[Note: non-polynomial dependence on n_+p_1 and n_-p_2 : non-locality: expect $f(n_+\partial)$]

collinear region

$k \sim p_1$; $n_+k \sim Q$ $\quad k_\perp \sim \lambda Q \quad n_-k \sim \lambda^2 Q$

$$k \cdot \overset{n_+ \quad \perp \quad n_-}{(1, \lambda, \lambda^2)}\, Q$$

$$(k+p_1)^2 = \underset{\lambda^2}{k^2} + \underset{1}{n_+p_1} \, \underset{\lambda^2}{n_-k} + \underset{\lambda^2}{n_-p_1} \, \underset{1}{n_+k} + \underset{\lambda^2}{p_1^2} \overset{①}{=} (k+p_1)^2$$
no expansi

$$\begin{matrix} & & \lambda^2 & 1 & 1 & \lambda^2 & \lambda^2 & \lambda^2 \end{matrix}$$

$$(k+p_2)^2 = k^2 - n_+k \cdot n_-p_2 + n_-k\, n_+p_2 + p_2^2 = n_+k \cdot n_-p_2$$

collinear + anticolliner = hard

linear propagator → single pole → can be avoided & does not contribute

$$I_C = i\pi^{D/2}\mu^{2\varepsilon}\int d^dk \frac{1}{k^2+i\varepsilon}\frac{1}{(k+p_1)^2+i\varepsilon}\frac{1}{n_+k \cdot n_-p_2+i\varepsilon}$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left[-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\mu^2/p_1^2 - \frac{1}{2}\ln^2\mu^2/p_1^2 + \pi^2/6\right]$$

ex. calculate

anti-colliner : $p_1 \leftrightarrow p_2$

soft region

$$k \sim \lambda^2 Q \quad ; \quad k^2 \sim \lambda^4 Q^2$$

$$(k+p_1)^2 = k^2 + n_+p_1\, n_-k + n_-p_1\, n_+k + p_1^2 = p_1^2 + n_+p_1\, n_-k$$

$$\begin{matrix} \lambda^4 & 1 & \lambda^2 & \lambda^2 \times \lambda^2 & \lambda^2 \end{matrix}$$

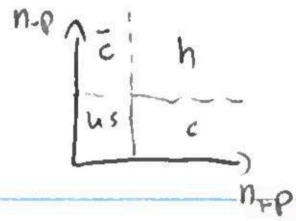
$$(k+p_2)^2 = \qquad\qquad = p_2^2 + n_-p_2\, n_+k$$

$$I_S = i\pi^{D/2}\mu^{2\varepsilon}\int d^dk \frac{1}{k^2+i\varepsilon}\frac{1}{p_1^2+n_+p_1\, n_-k+i\varepsilon}\frac{1}{p_2^2+n_-p_2\, n_+k+i\varepsilon}$$

$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left[1/\varepsilon^2 + \frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{p_1^2p_2^2} + \frac{1}{2}\ln^2\frac{\mu^2Q^2}{p_1^2p_2^2} + \frac{\pi^2}{6}\right]$$

$$I_h + I_c + I_{\bar{c}} + I_{us} = \frac{1}{2}\ln Q^2/p_1^2 \ln Q^2/p_2^2 + \pi^2/3$$

ex. calculate

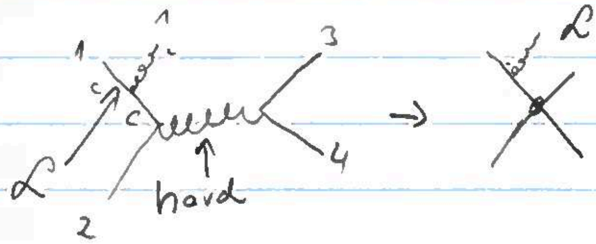


SCET: split fields $\varphi = \sum_i \varphi_i + \varphi_s$

$$\mathcal{L} = \sum_i \mathcal{L}_i(\varphi_{c_i}, \varphi_s) + \mathcal{L}_s(\varphi_s) + J(\varphi_{c_1}, \ldots, \varphi_{c_n}, \varphi_s)$$

$\sum_i$: collinear sectors

J: currents after integrating out hard scale, „N-jet operators"



current

We focus on Lagrangian first

$$P_{+/-} = \frac{\not{n}_\pm \not{n}_\mp}{4}$$

$$\xi = \frac{\not{n}_- \not{n}_+}{4}\psi \qquad \not{n}_- \xi = 0$$

$$\eta = \frac{\not{n}_+ \not{n}_-}{4}\psi$$

$$\langle 0 | T(\xi(x)\bar{\xi}(y)) | 0 \rangle = \frac{\not{n}_- \not{n}_+}{4} \int \frac{d^4p}{(2\pi)^4} \frac{\not{p}}{p^2} e^{-ip(x-y)} \frac{\not{n}_+ \not{n}_-}{4}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{\not{n}_-}{2} \frac{i n_+ p}{p^2} e^{-ip(x-y)}$$

$1 \times \lambda^2 \times \lambda^2 \qquad 1/\lambda^2 \qquad = \lambda^2$

$\Rightarrow \xi \sim \lambda$

for η $n_+ \leftrightarrow n_-$ $\qquad \eta \sim \lambda^2$

soft quark field $\quad \lambda^8 \frac{\lambda^2}{\lambda^4} \Rightarrow q \sim \lambda^3$

$\boxed{ex}$ show that

$n_+ A_c \sim 1 \qquad A_{c\perp} \sim \lambda \qquad n_- A \sim \lambda^2$

$A^\mu_s \sim \lambda^2 \qquad\qquad [* \to 24a]$

Derivation of the Lagrangian

$\mathcal{L} = \bar{\Psi} i \not{D} \Psi \qquad \Psi = \underset{\frac{\not{n}_-\not{n}_+}{4}}{\xi} + \underset{\frac{\not{n}_+\not{n}_-}{4}}{\eta} + q\,, \quad A = A_c + A_s$

$= \bar{\xi} \frac{\not{n}_+}{2} i n_- D \xi +$

$\bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi$

$+ \bar{\eta} \frac{\not{n}_-}{2} i n_+ D \eta \quad + \bar{\xi} g \not{A}_c q + \bar{\eta} g \not{A}_c q$

$+ \bar{q} g \not{A}_c \xi + \bar{q} g \not{A}_c \eta \quad + \bar{q} i \not{D}_s q$

we droped terms with sigle collinear field

example $\bar{q}\, i \not{D} \xi = \bar{q}(\underline{i \not{\partial}} + g \not{A}_c + \underline{g A_s}) \xi$

cannot fulfill momentum conservation

we can use e.o.m *

$\frac{\not{n}_-}{2} i n_+ D \eta = - i \not{D}_\perp \xi - g \not{A}_c q$

derivatives of the fields

- soft field vary over distances $1/\lambda^2 \Rightarrow \partial\phi_s \sim \lambda^2 \phi_s$

collinear $\quad n_+ x \sim \frac{1}{n_- p_c} \sim 1/\lambda^2 \Rightarrow n_- \partial \phi_c \sim \lambda^2 \phi_c$

$$n_- x \sim \frac{1}{n_+ p_c} \sim \lambda^0 \Rightarrow n_+ \partial \phi_c \sim \phi_c$$

$$x_\perp \sim 1/p_{\perp c} \sim 1/\lambda \Rightarrow \partial_\perp \phi_c \sim \lambda \phi_c$$

multiply by $\frac{\not{n}_+}{2}$

[* gaussian path integral over η can be done exactly]

$\frac{\not{n}_+ \not{n}_-}{4}\eta = \eta$ and then

$$\eta = -\frac{1}{in_+D}\frac{\not{n}_+}{2}\left(i\not{D}_\perp \xi + g\not{A}_c q\right)$$

we can now insert this back to the Lagrangian. [* →25.

$$\mathcal{L} = \bar{\xi}\frac{\not{n}_+}{2}\left(in_-D + i\not{D}_\perp \frac{1}{in_+D}\not{D}_\perp\right)\xi + \bar{q}_s \not{D}_s q$$

$$+ \bar{\xi} g\not{A}_c q + \bar{q} g\not{A}_c \xi_c - \bar{\xi} i\not{D}_\perp \frac{1}{in_+D}\frac{\not{n}_+}{2} g\not{A}_c q$$

$$- \bar{q} g\not{A}_c \frac{1}{in_+D}\frac{\not{n}_+}{2} i\not{D}_\perp \xi + \bar{q} g\not{A}_c \frac{\not{n}_+}{2}\frac{1}{in_+D}\frac{\not{n}_-}{2} in_+D \frac{1}{in_+D}\frac{\not{n}_+}{2} g\not{A}_c$$

- the above expression is exact: QCD Lagrangian in a frame where c-ll particles are boosted
- terms with soft quarks require extra care ⇒ q. ins.
- Let us focus on LP

$\boxed{\text{soft quarks are power suppressed}}$

$$\mathcal{L}_{cc} = \bar{\xi}\frac{\not{n}_+}{2}\left(in_-D + i\not{D}_\perp \frac{1}{in_+D}\not{D}_\perp\right)\xi$$

$$= \bar{\xi}\frac{\not{n}_+}{2}\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}\not{D}_{\perp c}\right)\xi +$$

$$\bar{\xi}\frac{\not{n}_+}{2}\left(\underbrace{i\not{D}_{\perp c}\frac{1}{in_+D_c} g\not{A}_s}_{\mathcal{O}(1)} + \underbrace{g\not{A}_s \frac{1}{in_+D_c} i\not{D}_{\perp c}}_{\mathcal{O}(\lambda^2)}\right)\xi + \mathcal{O}(\lambda^2) \quad (*)$$

[!] LP Lagrangian

note

$$\frac{1}{n_+D} = \frac{1}{n_+D_c} + \frac{1}{n_+D_c} g n_+A_s \frac{1}{n_+D_c}.$$

Non locality explicate

Define $[in_+ D W] = 0$ $\Big\{$ or $W^+ in_+ D W = in_+ \partial$

$$W(x) = P \exp[ig \int_{-\infty}^{0} ds \, n_+ A(x+sn_+)]$$

note $W W^+ = 1$

$A = A_c + A_s$

$$\frac{1}{in_+ D} = W \frac{1}{in_+ \partial} W^+ \quad ;$$

and we can write

$$\left[\frac{1}{in_+\partial + i\varepsilon} f(x) = -i \int_{-\infty}^{0} ds \, f(x+sn_+) \right]$$

Multipole expansion

$$\langle \phi_c(x)\,\phi_s(x)\rangle \sim e^{ip_c x + ip_s x} = e^{ip_c x + i n_- p_s n_+ x \cdot \frac{1}{2}} \times \left[1 + i x_\perp p_{s\perp} + \dots\right]$$

$$x_- \equiv n_+ x \,\frac{n_-^\mu}{2}$$

$$\phi_c(x)\,\phi_s(x) = \phi_c(x)\left[\phi_s(x_-) + x_\perp \partial^\perp \phi_s(x_-) + \dots\right]$$

analog of multipole expansion in pNRQED

Remember x has (inverse) collinear scaling

$$x_\perp \partial^\perp \sim 1/\lambda \cdot \lambda^2 = \lambda$$

→ derivative acts only on soft field so it has soft scaling

$$\rightarrow \int d^4x\; \phi_c(x)\left[\partial_\perp \phi_s(x_-)\right] \neq -\int d^4x \left[\partial_\perp \phi_c(x)\right]\phi_s(x_-)$$

$$\bar\xi \frac{\not n_+}{2}\, i n_- D\, \xi = \bar\xi \frac{\not n_+}{2}\, i n_- \not\partial\big|_{s\to x_-}\, \xi + g\bar\xi \frac{\not n_+}{2}\, x_\perp^\mu \left[\partial^\perp_\mu n_- A(x_-)\right]\xi$$

(**)

the gauge covariant form of $\mathcal{L}^{(1)}$

Let us note that

$$i\left[x_\perp^\mu , i\not{D}_{\perp c}\frac{1}{n_+D_c}i\not{D}_{\perp c}\right] = \gamma_\perp^\mu \frac{1}{in_+D_c} i\not{D}_{c\perp} + i\not{D}_\perp \frac{1}{in_+D}\gamma_\perp^\mu$$

{ remember $[x, H] \approx p$

So we can write (*)

$$g\not{A}_{\perp s}\frac{1}{in_+D}i\not{D}_{c\perp} + gi\not{D}_\perp \frac{1}{in_+D}\not{A}_\perp^s = i\left[(g x_\perp A_s^\perp), i\not{D}_{\perp c}\frac{1}{in_+D_c}i\not{D}_{\perp c}\right]$$

we note that

$$in_-D\,\xi_c + i\not{D}_\perp \frac{1}{in_+D} i\not{D}_\perp \xi = 0 \quad \text{is the LP}$$

eom

$$i\left[g x_\perp A_s^\perp , i n_- D\right] = -g x_\perp [n_-\partial A_s^\perp] \quad \text{in the } \underline{\text{QED case}}$$

note that

$$x_\perp^\mu n_-^\nu g F_{\mu\nu}^s = g\left(x_\perp^\mu \partial_\mu^\perp n_- A_s - x_\perp^\mu n_-\partial A_\mu^\perp\right)$$

so (*) + (**) we can get $\mathcal{L}^{(1)}$:

$$\mathcal{L}^{(1)} = \bar{\xi}\frac{\not{n}_+}{2}\left(x_\perp^\mu n_-^\nu g F_{\mu\nu}^s\right)\xi \qquad \text{Analog of } \vec{x}\,\vec{E} = x^\mu v^\nu F_{\mu\nu}$$

Comment on QCD

[ex] Show that e.o.m. application is equivalent to field redefinition

$$\xi \to \left(1 + g\, x_\perp A_{..} + \dots\right) \xi.$$

Show that the new field transforms

$$\xi(x) \to U_s(x_-)\, \xi(x) + O(\lambda^2)$$

· If the Lagrangian contains

$$\varphi_c^+(x)\, \varphi_s(x) \to \varphi_c^+(x)\, U_s^+(x)\, U_s(x)\, \varphi_s(x)$$

but if $\varphi_s(x_-)$ then $U_s^+(x)\, U_s(x_-)$ is not invariant.

to fix it, perform field redefinition and require that:

$$\hat{\varphi}_c \to U_s(x_-)\, \hat{\varphi}_c \qquad \varphi_s(x) \to U_s(x)\, \varphi_s(x)$$

$$A_c \to U_s(x) A_c\, U_s^+(x_-)$$

$$A_s \to U A U^+ + \frac{i}{g} U[\partial, U^+]$$

We can perform field redefinition:

$$\varphi = R\,\hat{\varphi}$$

where $R = P\exp\left[ig\int_{x_-}^{x} dy_\mu\, A_s^\mu(y)\right]$

note that

$$\int_{x_-}^{x} dy_\mu\, A_s^\mu(x) = \int_0^1 ds\,(x-x_-)_\mu\, A^\mu(x_- + s(x-x_-))$$

$$= x_\perp^\mu\, A_\mu(x_-) + \ldots \quad O(\lambda^2)$$

[ex]: Show that

a) $\Phi(x,y) = P\exp\left[-ig\int_x^y dz^\mu A_\mu(z)\right]$

transforms as $U(x)\,\Phi(x,y)\,U^+(y)$

b) $W_c(x) = P\exp\left[-ig\int_{-\infty}^{0} dt\, n_+ A_c(x+tn_+)\right]$

$$W^+(x)\, i\, n_+ D_c\, W(x) = i\, n_+ \partial$$

comments:

- all soft fields are taken at x_-
- no translation invariance $\Rightarrow$ no momentum conservation

Decoupling transformation

We can introduce soft wilson line

$$Y_\pm = P\exp\left[ig\int_{-\infty}^{0} ds\, n_\pm A_s(x+sn_\pm)\right]$$

$$\xi_c(x) \to Y_-(x_-)\,\xi_c^{(0)}$$

$$A_c^\mu(x) \to Y_-(x_-)\,A_c^{\mu(0)}(x)\,Y_-^\dagger(x_-)$$

then the $\mathcal{L}$? lagrangian becomes

$$\mathcal{L}_{LP} = \bar{\xi}\frac{\not{n}_+}{2}\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}\not{D}_{\perp c}\right)\xi$$

$$\to \bar{\xi}^0\frac{\not{n}_+}{2}\left(Y_-^\dagger(x_-)\,in_-D\,Y_-(x_-) + i\not{D}^0_{\perp c}\frac{1}{in_+D_c^0}i\not{D}^0_{\perp c}\right)\xi^0$$

where $iD^0_{\perp,c} = i\partial_\perp + gA^0_{\perp c}$ etc.

We can also show that

$$Y_-^\dagger(x)\,in_-D\,Y_-(x_-) = in_-D^0 = in_-\partial + g\,n_-A_c^0.$$

i.e. the collinear and soft fields do not interact at LP.

$$\mathcal{L}_{LP}^{soft} = \mathcal{L}_{LP}^{(c)} + \mathcal{L}_{LP}^{soft}$$

→ this is crucial property for deriving factorization

Arbitrary state can be written as a product of soft and collinear states, since states are defined with respect to LP Hamiltonian

$$|X\rangle = |X_c\rangle|X_s\rangle \qquad \text{e.g. } J = \phi_c(x)\,\phi_s(x)$$

eq.
$$\langle X|\,J\,|0\rangle = \langle X_c|\phi_c(x)|0\rangle_c \;\langle X_s|\phi_s(x)|0\rangle_s$$

↑ currents will pick up soft wilson lines as we will see

Gauge invariance

- we discussed soft gauge invariance
- each collinear sector has its own gauge symm
- gauge symmetry cannot mix powers of λ.

$$\xi_c \to \mathcal{U}_c\,\xi_c$$

$$A_s \to A_s$$

$$A_{\perp c} \to \mathcal{U}_c A_{\perp c}\mathcal{U}_c^\dagger + \tfrac{i}{g}\mathcal{U}_c[\partial_\perp \mathcal{U}_c^\dagger]$$
$$n_- A_c \to \mathcal{U}_c n A_c\,\mathcal{U}_c^\dagger + \tfrac{i}{g}\mathcal{U}_c[n_- D_s(x_-)\mathcal{U}_c^\dagger]$$

- $W_c \to \mathcal{U}_c W_c$

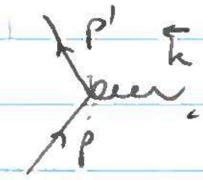
we define collinear gauge invariant building block

$$W^\dagger \xi_c = \chi \qquad \mathcal{A}^\mu = W_c^\dagger D_c^\mu W_c$$

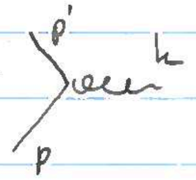
→ 31a

Feynman Rules

$$\frac{i\, n_+k}{k^2+i\varepsilon}\,\frac{\not n_-}{2}$$

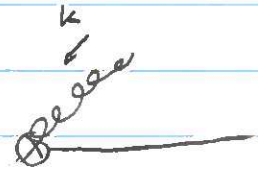


$$ig\, t^a\left(n_-^\mu + \frac{\not p_\perp'}{n_+p'}\gamma_\perp^\mu + \gamma_\perp^\mu\frac{\not p_\perp}{n_+p} - \frac{\not p'_\perp}{n_+p'}\, n_+^\mu\, \frac{\not p_\perp}{n_+p}\right)\frac{\not n_+}{2}$$



$$ig\, t^a\, \frac{\not n_+}{2}\, n_-^\mu \qquad \mathcal{O}(\lambda^0)$$

$$ig\, t^a\, \frac{\not n_+}{2}\, x_\perp^\rho\, n_-^\nu\left(k_\rho\, g_{\nu\mu} - k_\nu\, g_{\rho\mu}\right) \qquad \mathcal{O}(\lambda)$$

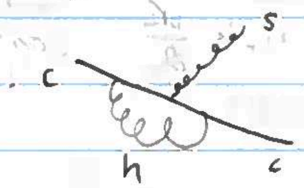


$$\frac{g_s\, t^a\, n_+^\mu}{n_+k}$$

ex
derive

Matching

· Lagrangian is not renormalized

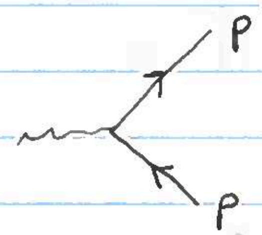


– scaleless loops

$\Rightarrow$ tree level Lagrangian is exact!

We need Sources

example: note our Sudakov problem (think $e^+e^- \to$ jets)



$$J^\mu(x) = \bar\psi \gamma^\mu \psi(x)$$

set $x=0$

$\to$ before decoupling
$\to$ after we would pick Wilson lin.

$$J^\mu_{SCET} = \int dt\, dt'\, C_V(t,t')\, \bar\chi_c(tn_+)\, \gamma^\mu_\perp\, \chi_{\bar c}(t'n_-)$$

* note why we need gauge inv. building block $\to$ [32a]

remember non-locality

The current needs to be renormalized; it is usefull to introduce F.T.

$$\chi(P) = \int dt\, e^{itP} \chi(t n_+)$$

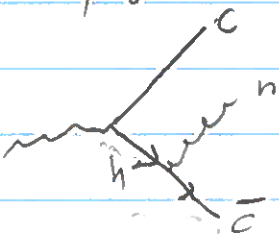
note that matrix element for initial stat

$$\langle 0 | \chi(P) | p\rangle = u_c(p)\, 2\pi\, \delta(P - n_+ p)$$

since $\langle 0 | \chi(tn_+) | p\rangle = e^{-itn_+p} \langle 0 | \chi(0) | p\rangle$

Note that Wilson lines are actually representing infinet set of diagrams

In QCD, we consider



$$\sim \bar{u}_{\bar c}\, \not\varepsilon\, \frac{\not p_{\bar c} + \not k}{(p_{\bar c}+k)^2}\, \gamma^\mu u_c$$

$\not p_{\bar c} \to n_- p_c \frac{\not n_+}{2}$

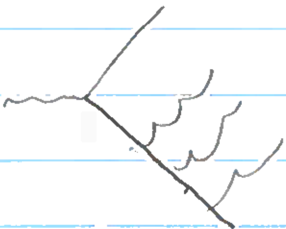
$\not k \to n_+ k \frac{\not n_-}{2}$ ↗ does not contribute

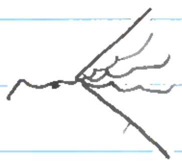
$(p_{\bar c}+k)^2 \to n_+ k \cdot n_- p_{\bar c}$

$$\not\varepsilon \to n_+\varepsilon \cdot \frac{\not n_-}{2}$$

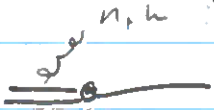
$$\sim \bar{u}_{\bar c}\, \underbrace{n_+\varepsilon\, \frac{\not n_-}{2}}_{1}\, \frac{\not n_+}{2}\, \frac{\cancel{n_- p_c}}{\cancel{n_- p_c}\, n_+ k}\, \gamma^\mu u_c$$

$$\sim \frac{n_+\varepsilon}{n_+ k}\, \bar{u}_{\bar c}\, \gamma^\mu u_c$$



$\to$  $\to$ Wilson line

- Unsuppresed components are controlled to all orders by Wilson lines.
- the argument can be generalized to arbitrary operators; also including heavy quarks



$$J^\mu_{SCET} = \int \frac{dP}{2\pi}\frac{dP'}{2\pi}\, \tilde{C}_V(P,P')\, \bar{\chi}_c(p)\,\gamma^\mu_\perp\, \chi_{\bar{c}}(p')$$

$$\langle 0 | J^\mu_{SCET} | P_1, P_2 \rangle = \int \frac{dP}{2\pi}\frac{dP'}{2\pi}\, C_V(P,P')\, \delta(P - n_+ p)\,\delta(P' - n_- p)$$

$$\times\; \bar{v}_c(p_1)\,\gamma^\mu_\perp\, u(p_2)$$

$$= C_V(n_+ p_1, n_- p_2)\, \bar{v}_c(p_1)\,\gamma^\mu_\perp\, u(p_2)$$

$\boxed{ex}$ discuss current with 2 collinear building blocks in the collinear direction: hint: introduce momentum frac

On the QCD side $\langle 0 | J^\mu | p_1 p_2 \rangle = F(Q^2)\, \bar{v}(p_1)\gamma^\mu v$
with $Q = (p_1 + p_2)^2 \simeq n_+ p_1\, n_- p_2$

hence $C_V(n_+ p_1\ n_- p_2) = F(Q^2)\big|_{LP}$

Operator renormalization in renormalized perturbation theory

$$\langle O_A(\{\phi_{ren}, g_{ren}\})\rangle_{ren} = \sum_B Z_{AB} \prod_{\phi \in B} Z_\phi^{1/2} \prod_{g \in B} Z_g \;\times$$

fields   parameters

$$\langle O_{B,bare}(\{\phi_{ren}\}, \{g_{ren}\})\rangle$$

where $Z_{AB} = \delta_{AB} + \delta Z_{AB}$

At one loop

$$\text{finite} = \langle O_{A\text{-bare}} \rangle_{1\text{-loop}} + \sum_{B} \Big[\delta Z^{(1)}_{AB} + \delta_{AB} \Big(\frac{1}{2} \sum_{\phi \in A} \delta Z_\phi + \sum_{g \in A} \delta Z_g \Big) \Big] \langle O_{B,\text{bare}} \rangle_{\text{tree}}$$

$$\langle 0 | \chi_c(0) | p \rangle_{\text{tree}} = u_c(p)$$

$$\langle 0 | \chi_c(0) | p \rangle_{1\text{-loop}} = \frac{\alpha_s C_F}{4\pi} \left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \ln \frac{\mu^2}{p^2} + \frac{2}{\varepsilon} \right) u_c(p) + \mathcal{O}(\varepsilon^0)$$

[ex] verify the above (c.f. method of region ex)

Adding wave-function renormalizat we get  $= -\frac{\alpha}{4\pi} C_F \frac{1}{\varepsilon}$

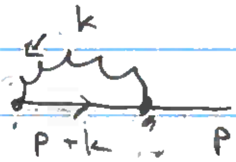
$$\langle 0 | \chi_c(0) | p \rangle_{1\text{-loop}} = J(p^2) \langle 0 | \chi_c(0) | p \rangle_{\text{tree}}$$

with $J(p^2) = 1 + \frac{\alpha_s C_F}{4\pi} \left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \ln \frac{\mu^2}{-p^2} + \frac{3}{2\varepsilon} \right) + \mathcal{O}(\varepsilon$

We can add soft

$$\langle 0 | \bar{\chi}_c \gamma_\perp^\mu \chi_c | p_1, p_2 \rangle^{\text{soft}}_{1\text{-loop}} = \frac{\alpha_s C_F}{4\pi} \left[-\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon} \ln \frac{(-p_1^2)(-p_2^2)}{(-Q^2)\mu^2} \right] +$$

[ex] verify the above (c.f. method of region ex)



$$\int \frac{d^dk}{(2\pi)^d}\, \bar{u}_c(p) \frac{\not n_+}{2}\, i g t^a n_-^\mu \frac{\not n_-}{2} \frac{i n_+ p + n_+ k}{(p+k)^2 + i\varepsilon} \frac{g t^a n_+^\mu}{n_+ k}$$

$$\times \frac{-i}{k^2} \qquad n_+ n$$

$$= i\int \frac{d^dk}{(2\pi)^d}\, \bar{u}_c(p)\, C_F g^2 \cdot 2 \frac{n_+ p + n_+ k}{(p+k)^2 + i\varepsilon} \frac{1}{n_+ k} \frac{1}{k^2}$$

$$\int \frac{d^dk}{(2\pi)^d}\, \bar{u}_c(p) \frac{\not n_+}{2} g t^a n_-^\mu \frac{\not n_-}{2} \frac{i n_+ p_1}{p_1^2 + n_+ p_1 n_- k} \gamma_\perp^\mu \frac{i n_- p_2}{p_2^2 + n_- p_2 n_+ k}$$

$$\frac{\not n_+}{2} \cdot \frac{\not n_-}{2} n_+^\mu g t^a u_{\bar c}(p_2) \qquad \frac{-i}{k^2}$$

$n_+ n$

$$2 C_F g^2\, \bar{u}_c \gamma_\perp^\mu u_{\bar c}\; i\int \frac{d^dk}{(2\pi)^d} \frac{n_+ p_1}{p_1^2 + n_+ p_1 n_- k} \frac{n_- p_2}{p_2^2 + n_- p_2 n_- k} \frac{1}{k^2}$$

ex could we compute the soft diagram after decoupling?

total contribution : coll - a - coll + soft

$$\langle 0|\bar{\chi}_c \gamma_\perp^\mu \chi_c |p_1 p_2\rangle_{1\text{-loop}} = \frac{\alpha_s C_F}{4\pi}\left[\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{\mu^2}{Q^2} + \frac{3}{\varepsilon}\right]$$

$$\times \langle 0|\bar{\chi}_c \gamma_\perp^\mu \chi_c |p_1 p_2\rangle_{\text{tree}} \qquad (*)$$

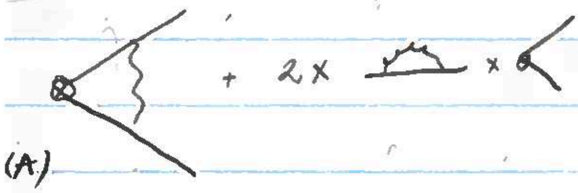
$$\Rightarrow Z_{AB}^{(1)} = \delta_{AB}\ \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln \mu^2/Q^2 - \frac{3}{\varepsilon}\right]$$

no mixing at LP

<u>Matching</u> → why do we need to consider renormalization

. IR poles of QCD must be reproduced by SCET
'true for any EFT'

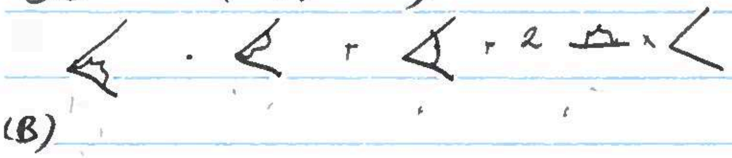
QCD $(p_i^2 = 0)$ [simpler kinematics, since dependence on low-energy parameters must cancel] these are IR poles



$$= \frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln \mu^2/Q^2 - 3/\varepsilon\right.$$

$$\left. - \ln^2 \mu^2/Q^2 - 3\ln \mu^2/Q^2 - 8 + \pi^2/6\right]$$

SCET $(p_i^2 = 0)$



→ scaleless integrals

$$\sim \frac{1}{\varepsilon} - \frac{1}{\varepsilon}$$

UV IR

SCET counter-term (*)

$$\frac{\alpha_s C_F}{4\pi}\left[-\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln \mu^2/Q^2 - \frac{3}{\varepsilon}\right]$$

(C)

UV poles

→ (C) + (B) → UV poles turn into IR poles

→ SCET reproduces QCD, IR

note that B is the same as scaleless integral that we subtracted in the method of regions

$$\begin{aligned} QCD &= SCET \\ A &= B + C \quad \Rightarrow \quad C_V^{(1)} = \frac{\alpha_s}{4\pi} C_F\left[-\ln^2\frac{\mu^2}{Q^2} - 3\ln\mu^2/Q^2 - 8 + \frac{\pi^2}{6}\right. \end{aligned}$$

Resummation

We defined $O_{Ren} = Z\, O_{bare}$

the anomalous dimension is $\underset{\text{matrix}}{\Gamma} = Z \frac{d}{d\ln\mu} \overset{\text{matrix}}{Z^{-1}} = \left(-\frac{d}{d\ln\mu} Z\right) Z^{-1}$

which implies

$$0 = \frac{d}{d\ln\mu} O_{bare} = \frac{d}{d\ln} Z^{-1} O_{ren} = \left(\frac{d}{d\ln\mu} Z^{-1}\right) O_{ren} + Z^{-1}\frac{d}{d\ln} O_{ren}$$

$$\Rightarrow \frac{d}{d\ln\mu} O_{ren} = -Z\left(\frac{d}{d\ln\mu} Z^{-1}\right) O_{ren} = -\Gamma\, O_{ren}$$

$$\frac{d}{d\ln\mu} C_i^r O_i^r = 0 \Rightarrow \left(\frac{d}{d\ln\mu} C^i\right) O_i - \Gamma C_i O_{ren} = 0$$

$$\frac{d}{d\ln\mu} C_i^{ren} = \Gamma_{ij} C_j^{ren}$$

For the vector current:

$$\Gamma_v = -Z^{-1} \frac{d}{d\ln\mu} Z = (-1) \frac{d}{d\ln\mu}\left[\frac{\alpha_s}{4\pi} C_F\left[-\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln\mu^2/Q^2 - \frac{3}{\varepsilon}\right]\right]$$

$$\underset{1+\mathcal{O}(\alpha_s)}{\uparrow}$$

$$\frac{d\alpha_s}{d\ln\mu} = -2\varepsilon\alpha_s + \underbrace{\beta(\alpha)}_{\mathcal{O}(\alpha)}$$

$$= (-1)\, C_F\left(\frac{-2\varepsilon\alpha_s}{4\pi}\left[-\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln\mu^2/Q^2 - 3/\varepsilon\right] + \frac{\alpha}{4\pi}\left(-\frac{4}{\varepsilon}\right)\right)$$

$$= \frac{C_F\alpha_s}{4\pi}\left[-4\ln\mu^2/Q^2 - 6\right]$$

Γ_{cusp} : cusp anomalous dimension: • typical for problem with double log corrections

typical hard function : $d\sigma \sim \langle J^\mu\rangle\langle J^\nu\rangle$
$\sim |C_v|^2 \langle J\, J\rangle_{SCET}$

$$H = |C_v(Q^2)|^2$$

$$\frac{dH}{d\ln\mu} = 2\,\mathrm{Re}\,\gamma H = -\frac{\alpha}{2\pi} C_F\left[4\ln\mu/Q + 6\right] H$$

$\boxed{ex}$ Solve RGE
discuss initial condition

we could resum complete also

Thrust

$e^+e^- \to$ hadrons

$$\tau := 1 - T \quad , \quad T = \max_n \sum_i |\vec{p}_i \cdot \vec{n}| \Big/ \sum_i |\vec{p}_i|$$

$$\sigma = \frac{1}{2Q^2} \sum_X \int dPS_X \, (2\pi)^d \delta^{(d)}(q - P_X) L_{\mu\nu}(q) \times$$
$$\langle 0 | J^{\mu +}(0) | X \rangle \langle X | J^\nu(0) | 0 \rangle$$

with leptonic tensor $L_{\mu\nu}(q) = \frac{-8\pi^2\alpha^2}{3Q^4}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{Q^2} \right)$

we can drop $q_\mu q_\nu$ part using gauge-invariance

$$\frac{d\sigma}{d\tau} = \frac{-8\pi^2\alpha^2}{3Q^4} \cdot \frac{1}{2Q^2} \int dPS_X (2\pi)^d \delta^{(d)}(q - P_X) \, \delta(\tau - \tau(X)) \times$$
$$\langle 0 | J^{\mu +}(0) | X \rangle \langle X | J^\nu(0) | 0 \rangle \qquad (*)$$

we can write $(2\pi)^d \delta^d(q - P_X) = \int d^d x \, e^{i(q - P_X)x}$
and translate one of the currents:

$$\frac{d\sigma}{d\tau} = - \frac{8\pi^2\alpha^2}{3Q^4} \frac{1}{2Q^2} \sum_X \int dPS_X \int d^d x \, e^{iqx} \times$$
$$\langle 0 | J^{\mu +}(x) \, \delta(\tau - \tau(\hat{X})) | X \rangle \langle X | J_\mu(0) | 0 \rangle$$

$$= - \frac{8\pi^2\alpha^2}{3Q^4} \frac{1}{2Q^2} \int d^d x \, e^{iqx} \langle 0 | J^{\mu +}(0) \, \delta(\tau - \tau(\hat{X})) J_\mu(x) | 0 \rangle$$

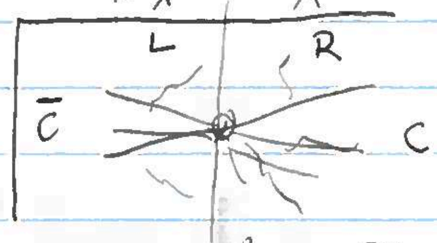
In the limit $\tau \to 0$, the final state particles can either form jets or be soft

$$q = P_{X_c} + P_{X_{\bar c}} + P_{X_s} \qquad n_+ P_{X_c} n_- P_{X_s} + O(\lambda^4) \quad n_+ P_{X_c} n$$

$$Q^2 = P_{X_s}^2 + P_{X_c}^2 + P_{X_{\bar c}}^2 + 2 P_{X_c} P_{X_{\bar c}} + 2 P_{X_s} P_{X_c} + 2 P_{X_s} P_{X_{\bar c}} + O$$

$\sim \lambda^4 \qquad \lambda^2 \qquad \lambda^2$

$\hookrightarrow n_+ P_{X_c} n_- P_{X_{\bar c}} + O(\lambda^4)$



we go to a frame where $P_{X_c}^\perp = 0$ (actually $P_{c\perp} + O(\lambda^2) = 0$ is automatic)

we count $\tau \sim \lambda^2 \Rightarrow \tau = \dfrac{M_R^2 + M_L^2}{Q^2} + O(\tau^2)$

two hemisphere distribution

$M_R = P_{X_c}^2 + n_+ P_{X_s} n_-$

$M_L : P_{X_{\bar c}}^2 + n_+ P_{\bar c}\, n_-$

we use matching:

$$J^\mu = C_V(-Q^2)\, \bar\chi_c\, \gamma_\perp^\mu\, \chi_{\bar c}$$

$$= C_V(-Q^2)\, \bar\chi_c^0\, Y_-^\dagger\, \gamma_\perp^\mu\, Y_+\, \chi_{\bar c}^0$$

we will drop ^{0} from collinear fields

Inserting SCET current into the matrix element (*) (ignoring normalization) and $|X\rangle = |X_c\rangle |X_{\bar c}\rangle |X_s\rangle$

$$d\sigma \sim \sum_X \int dPS_X (2\pi)^d \delta^{(d)}(q - P_{X_c} - P_{X_{\bar c}} - P_{X_s}) \, |C_V(-Q^2)|^2$$

$$\langle 0 | \bar\chi_{\bar c_\alpha} Y_+^\dagger \gamma^\mu_{\perp\alpha\beta} Y_- \chi_{c_\beta} | X \rangle \langle X | \bar\chi_{c_\delta} Y_-^\dagger \gamma_{\perp\mu\,\delta\gamma} Y_+ \chi_{\bar c_\gamma} | 0 \rangle$$

(add indices later)

fields are at $x = 0$.

$$\sigma \sim \sum_X dPS_X (2\pi)^d \delta^{(d)}(q - p_{X_c} - p_{X_{\bar{c}}} - p_{X_s}) \; |C_V(-Q^2)|^2$$

$$\times \; \gamma^\mu_{\alpha\beta} \; \gamma_{\mu\,\delta\gamma}$$

$$\times \; \langle 0 | \, \chi_{c_\beta}(0) \, | X_c \rangle \langle X_c | \, \bar{\chi}_{c_\delta}(0) | 0 \rangle \qquad *$$

$$\langle 0 | \, \bar{\chi}_{\bar{c}\alpha}(0) \, | X_{\bar{c}} \rangle \langle X_{\bar{c}} | \, \chi_{\bar{c}\delta}(0) \, | 0 \rangle$$

$$\langle 0 | \, Y^\dagger_+(0) Y_-(0) \, | X_s \rangle \; \langle X_s | \, Y^\dagger_-(0) Y_+(0) \, | 0 \rangle$$

now we can insert $1 = \int d^d p_c \; \delta^{(d)}(p_c - p_{X_c})$

$\to$ replace everywhere $p_{X_c} \to p_c$

$\to$ rewrite δ as $\int \frac{d^d x}{(2\pi)^d} \, e^{i(p_c - p_{X_c})x}$

and translate collinear fields

$$* \;\to\; \int \frac{d^d x}{(2\pi)^d} \, e^{i p_c x} \, \langle 0 | \, \chi_{c_\beta}(x) \, | X_c \rangle \langle X_c | \, \bar{\chi}_{c_\delta}(0) \, | 0 \rangle$$

we can now define collinear function:

cut propagator!

$$\frac{1}{\frac{1}{X_c}} \frac{1}{2\pi} \int dPS_{X_c} \langle 0 | \, \chi_{c\beta}(x) | X_c \rangle \langle X_c | \, \bar{\chi}_{c\delta}(0) | 0 \rangle \equiv$$

$$= \left(\frac{\not{n}}{2} \right)_{\beta\delta} \int \frac{d^d p}{(2\pi)^d} \, \theta(p^0) \, n_+ p \; J(p^2) \, e^{-ixp}$$

now we can insert it into the matrix element and do integral over $d^d x \to \delta^{(d)}(p_c - p)$

we can do integral over $p \to J(p_c^2)$

We can now insert measurement functions for hemisphere distribution

$$: \delta(M_R^2 - p_c^2 - n_+ p_c \, n_- P_{X_s}^R)$$
$$\delta(M_L^2 - p_{\bar c}^2 - n_- p_{\bar c} \, n_+ P_{X_s}^L)$$

and insert two more unit operators to remove P_{X_s}

$$1 = \int dl_+ dl_- \, \delta(l_+ - n_+ P_{X_s}^L) \, \delta(l_- - n_- P_{X_s}^R)$$

multipole expansion

$$\frac{d\sigma}{dM_R^2 dM_L^2} \sim \sum_{X_s} dPS_{X_s} \, \delta^{(d)}(q - p_c - p_{\bar c} - P_{X_s}) \, |C_V(-q^2)|^2$$
$$\times \mathrm{Tr}\left[\frac{\not{n}_-}{2} \gamma_\perp^\mu \frac{\not{n}_+}{2} \gamma^\perp_\mu\right]$$

we absorb X_c and $X_{\bar c}$ into collinear function

$$\int d^d p_c \int d^d p_{\bar c} \; n_+ p_c \, n_- p_{\bar c} \, J(p_c^2) \, J(p_{\bar c}^2) \, \theta(p_c^0) \theta(p_{\bar c}^0)$$

$$\int dl_+ dl_- \, \delta(M_R^2 - p_c^2 - n_+ p_c \, l_-) \, \delta(M_L^2 - p_{\bar c}^2 - n_- p_{\bar c} \, l_+)$$

$$\delta(l_- - n_- P_{X_s}^R) \, \delta(l_+ - n_+ P_{X_s}^L) \langle 0 | Y_+^\dagger(0) Y_-(0) | X_s \rangle \langle X_s | Y_-^\dagger Y_+ | 0 \rangle$$

Now we can define hemisphere soft function

color factors are ignored.

$$S(l_-, l_+) = \sum_{X_s} dPS_{X_s} \, \delta(l_- - n_- P_{X_s}^R) \, \delta(l_+ - n_+ P_{X_s}^L)$$
$$\langle 0 | Y_+^\dagger(0) Y_-(0) | X_s \rangle \langle X_s | Y_-^\dagger(0) Y_+(0) | 0 \rangle$$

and multipole expand $\delta^{(d)}(\ )$

$$\delta^{(d)}(q - p_c - p_{\bar{c}} - p_{X_s}) = 2\delta(n_+ q - n_- p_c)\,\delta(n_- q - n_-
\delta^{(d-2)}(p_{c\perp} - p_{\bar{c}\perp})$$

$\to$ replace $n_+ p_c$ and $n_- p_{\bar{c}}$ by Q.

use measurement to replace p_c^2 by $M_R^2 - Q\, l_+$
and $p_{\bar{c}}^2$ by $M_L^2 - Q\, l_-$
and integrate over $n_- p_c$, $n_+ p_{\bar{c}}$.

Finally integrate over $d^{d-2} p_{c_\perp}\, d p_{\bar{c}\perp}^{d-2}$.

$$\frac{d\sigma}{dM_R^2\, dM_L^2} = \sigma_0 \int dl_+ dl_- \, |C_V(-Q^2)|^2$$

$$J_c(M_R^2 - Q\, l_-)\; J_c(M_L^2 - Q\, l_+)\; S(l_-, l_+)$$

Note that jet function were defined w/o factors $n_+ p_c\; n_- p_{\bar{c}}$ which combine with trace

$$n_+ p_c\; n_- p_{\bar{c}}\; T\left[\frac{\not{n}_-}{2}\gamma^\mu_\perp \frac{\not{n}_+}{2}\gamma^\perp_\mu\right] =$$

$$Tr\left[\not{p}_c\, \gamma^\mu_\perp\, \not{p}_{\bar{c}}\, \gamma^\perp_\mu\right]$$

which is just LO matrix element squared.

$\to \sigma_0$

$\to$ 42a

thrust

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \int dM_L^2\, dM_R^2\, \delta\left(\tau - \frac{M_L^2+M_R^2}{Q^2}\right)\frac{1}{\sigma_0}\frac{d\sigma}{dM_R^2\, dM_L^2}$$

$$= \int dl_+\, dl_-\, dM_L^2\, dM_R^2\, |C_V(-Q^2)|^2\, J(M_L^2 - Ql_+)\, J(M_R^2 - Ql_-) \times S(l_-, l_+)$$

- shift $M_{L,R}^2 \to M_{L,R}^2 + Ql_{+/-}$
- define $S_T(k) = \int dl_+\, dl_-\, \delta(k - l_- - l_+)\, S(l_+, l_-)$

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \int dk\, dM_L^2\, dM_R^2\, \delta\left(\tau - \frac{M_L^2+M_R^2}{Q^2} - \frac{k}{Q}\right) J(M_L^2)\, J(M_R^2)\, S_T(k)$$

Jet function

$$\sum_{X_c} \frac{1}{2\pi} dPS_X \langle 0| \chi_{c\beta}(x) |X_c\rangle \langle X_c| \bar{\chi}_{c\sigma}(0)|0\rangle =$$

$$\left(\frac{\not{n}_-}{2}\right)_{\beta\sigma} \int \frac{d^d p}{(2\pi)^d} \theta(p^0)\, n_+ p\, J(p^2)\, e^{-ixp}$$

At LO we get

$$\sum_s \frac{1}{2\pi} \int \frac{d^{d-1}p}{(2\pi)^{d-1} 2p^0} \underbrace{u_\beta(p) \bar{u}_\sigma(p)}_{n_+ p \left(\frac{\not{n}_-}{2}\right)_{\beta\sigma}} e^{-ixp}$$

$$= \int \frac{d^d p}{(2\pi)^d} \theta(p^0) \delta(p^2)$$

so comparing both sides $J(p^2) = \delta(p^2)$

We can also write it as cut quark propagator.

[Ex. · discuss / calculate NLO contribution
· discuss renormalization

$$J(p^2) = \frac{1}{\pi} \frac{1}{Q} \operatorname{Im}\left[i \int d^d x\, e^{ipx} \langle 0| T\left[\bar{\chi}_c(x) \frac{\not{n}_+}{2} \chi_c(0)\right]|0\rangle\right]$$

Soft function

$$S(l_+,l_-) = \sum_{X_s} \int dPS_{X_s}\, \delta(l_+ - n_+ p^L_{X_s})\, \delta(l_- - n_- p^R_X)$$

$$\langle 0| Y_+^\dagger(0) Y_-(0) |X_s\rangle \langle X_s| Y_-^\dagger(0) Y_+(0) |0\rangle$$

$$= \sum_{X_s} \int dPS_{X_s}\, \delta\Big(l_- - \sum_{i\in X_s} \theta(n_+k - n_-k)\, n_-k_i\Big)$$

$$\delta\Big(l_+ - \sum_{i\in X_s} \theta(n_-k - n_+k)\, n_+k_i\Big)$$

$$\langle 0| Y_+^\dagger(0) Y_-(0) |X_s\rangle \langle X_s| Y_-^\dagger(0) Y_+(0) |0\rangle$$

At LO $\quad Y_\pm = 1 \qquad |X_s\rangle \to |0\rangle$

$$S(l_+ l_-) = \delta(l_+)\,\delta(l_-)$$

c, : calculate NLO soft function

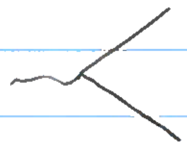
Beyond LP

3 sources of power corrections

- Phase space → kinematic corrections
  not related directly to SCET

- Currents

$$J^\mu = \bar{\Psi}\gamma^\mu\Psi$$



LP

$$\rightarrow \bar{\chi}_c \gamma_\perp^\mu \chi_{\bar{c}} \equiv J^{A0,A0}$$

$\mathcal{O}(\lambda)$ suppressed

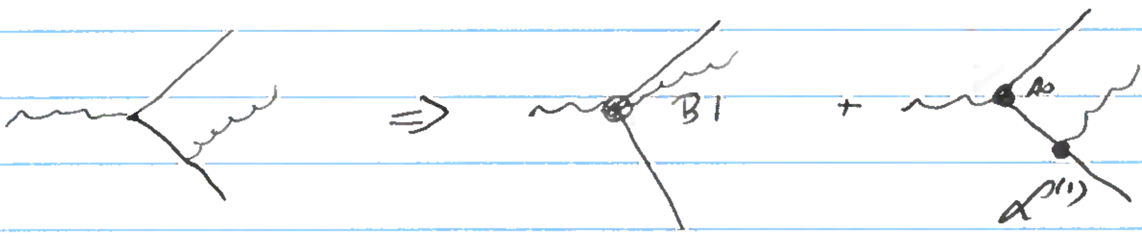
$$+ \bar{\chi}_c n_\pm^\mu \not{\partial}_\perp \chi_{\bar{c}} \equiv J^{A0,A1}$$

NLP

$$+ \bar{\chi}_c n_\pm^\mu \not{A}_c \chi_{\bar{c}} \equiv J^{A0,B1}$$

- T-ordered products

$$T\left(J^{A0,A0}, \mathcal{L}^{(1)}\right) \quad \ldots$$



NLP factorization

B-type currents ⇒ simple generalization of standard LP

example

$$\left| \int_0^1 du\, C^{B1}(u)\, J^{B1A0}(u) \right|^2 \rightarrow$$ 

see prev. exercise

new convolution, previous hard funct. multiplied the rest

$$\int du\, du'\, C^{B1}(u)\, C^{B1*}(u')\, J(u,u',p_R^2)\, J(p_L^2)$$

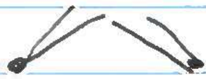
$$J(p^2) = \sum_{X_c} \frac{1}{2\pi} dPS_X \langle 0 | \chi_c(x) | X_c \rangle \langle X_c | \bar{\chi}_c(0) | 0 \rangle$$

↓

$$J(u,u',p^2) = \sum_{X_c} \frac{1}{2\pi} dPS_X \langle 0 | \chi_c(x)\, \bar{\chi}_c(x+tn_+) | X \rangle \langle X_c | \bar{\chi}_c(0)\, \chi_c(t'n_+) | 0 \rangle$$

$$\int dt\, dt'\, e^{itQu - it'Qu'}$$

$$n_+ p = Q$$



- soft function the same as LP but with a different color structure.

A - type needs extra step $\mathcal{L}^{(1)}_{\xi q} = \bar{q}_s(x_-) \not{A}_c \chi_c + h.c.$

$$J^T = T\left(J^{A0,A0}, \mathcal{L}^{(1)}_{\xi q}\right) = i \int d^4x \, \bar{\chi}_c Y_+^{\dagger} \gamma_\perp^\mu Y_- \chi^{(0)}_{\bar{c}} \, \bar{q}_s(x_-) \not{A}_{c\perp}(x) \chi_c(x)$$

$$\langle X | J^T | 0 \rangle = i \gamma^\mu_{\perp\alpha\beta} \langle X_{\bar{c}} | \chi_{\bar{c}\beta}(0) | 0 \rangle \times$$

$$\int d^4x \langle X_s | \bar{q}_{s\gamma}(x_-) Y_+^{\dagger(0)} Y_-(0) | 0 \rangle \times$$

$$\langle X_c | \bar{\chi}_{c\alpha}(0) \, {}_\gamma\not{A}_c(x) \chi_c(x) | 0 \rangle$$

since the soft field depend only on x_- we can split the integral

$$\int \frac{d\omega}{2\pi} S_\gamma(\omega, X_s) e^{-i x_- \omega} = \langle X_s | \bar{q}_{s\gamma}(x_-) Y_+^{\dagger(0)} Y_-(0) | 0 \rangle$$

$$\langle X | J^T | 0 \rangle = i \gamma^\mu_{\perp\alpha\beta} \langle X_{\bar{c}} | \chi_{\bar{c}\beta}(0) | 0 \rangle \int \frac{d\omega}{2\pi} S_\gamma(\omega, X_s)$$

$$\int d^4x \langle X_c | \bar{\chi}_{c\alpha}(0)_\gamma \not{A}(x) \chi_c(x) | 0 \rangle$$

squaring matrix element and repeating all the steps of LP factorization, we get: only collinear

$$\frac{d\sigma^{A0}}{d\tau} \sim |C^{A0}(-Q)|^2 \int dk \, dM_L^2 \, dM_R^2 \, \delta\left(\tau - \frac{M_L^2 + M_R^2}{Q^2} - \frac{k}{Q}\right) J^{A0}(M_L^2)$$

$$\int d\omega \, d\omega' \, J^{NLP}_{coll}(M_R^2, \omega, \omega') \, S(\omega, \omega, k)$$

where

$$S \sim \int dx_-\, dx_-' \; e^{+ix_- \omega - ix_-' \omega'} \, \delta(k - n_+ p_s^L - n_- p_s^R) \, \langle 0 | \, q_s(x_-') \, Y_+^\dagger(0) Y_-(0) \, | X_s \rangle \langle X_s | \, \bar{q}_s(x_-) Y_+(0) Y_-(0)$$

ex: calculate leading order J^{LP}

$$J^{NLP} \approx \int d^4x \, d^4x' \; e^{-ix_-\omega + ix_-'\omega'} \; e^{-ipz}$$

$$\langle 0 | \, \chi(z) \, \bar{\chi}_c \, \mathcal{A}(z+x) \, | X_c \rangle \langle X_c | \, \bar{\chi}_c \, \mathcal{A} \, \chi_c \, | 0 \rangle$$

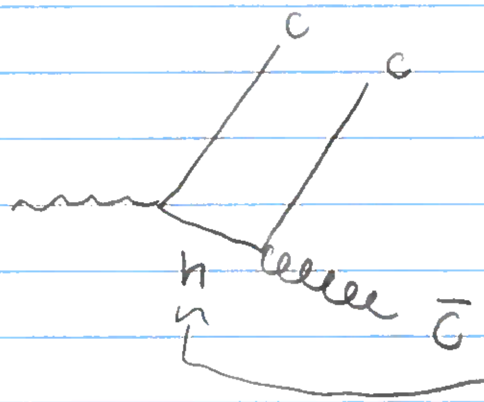
Endpoint divergence

General structure is:

$$\frac{d\sigma}{d\tau} \sim \int_0^1 dx_1 \, dx_2 \; H(x_1, x_2) \otimes J_c(x_1, x_2) \otimes J_{\bar{c}}^{LP} \otimes S^{LP}$$

$$+ \int_0^\infty d\omega_1 \, d\omega_2 \; H^{LP} \otimes J_{\bar{c}}(\omega_1, \omega_2) \otimes J_{\bar{c}}^{LP} \otimes S(\omega_1, \omega_2)$$

new integrals over x_i, ω_i often diverge

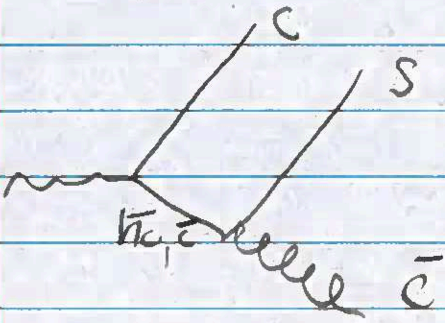


← momentum fraction x_1

$x_1 \, p_c \cdot p_{\bar{c}}$

this propagator →

→ goes on-shell when momentum fraction $x_1 \to 0$.



→ in $x_1 \to 0$ limit there is overlap with the soft quark emission.

→ the overlap is a scaleless integral that has to be subtracted!