Isospin conventions for HVP and $\sigma_{\pi N}$ from a phenomenological perspective

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May 30, 2023

Converging on QCD+QED prescriptions

Higgs Centre for Theoretical Physics, Edinburgh

How to define the isospin limit?

- Typical ChPT convention for **isospin limit**

  $\hookrightarrow$ **charged particle masses** M_{π^+}, M_{K^+}, m_p, ...
- Why? Most data available for charged particles
- Examples:
  - $\pi\pi$: $\pi^+\pi^-$ atoms, $\pi N \to \pi\pi N$ data
  - πN: $\pi^- p$, $\pi^- d$ atoms, $\pi N \to \pi N$ data

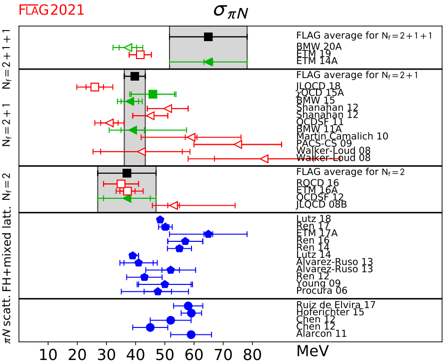
  $\hookrightarrow$ natural to use charged-particle masses to minimize corrections
- Standard example: **$\pi\pi$ scattering lengths** Colangelo, Gasser, Leutwyler 2001

$$a_0^0 = 0.220(5) \qquad a_0^2 = -0.0444(10)$$

- IB corrections important when comparing to $K \to 3\pi$ and $K_{\ell 4}$ data due to neutral-pion thresholds
- a_0^I vanish in the chiral limit

  $\hookrightarrow$ choice of isospin conventions matters

Pion–nucleon σ-term: lattice vs. phenomenology



- **Pion–nucleon σ-term $\sigma_{\pi N}$**

$$\sigma_{\pi N} = \hat{m}\langle N|\bar{u}u + \bar{d}d|N\rangle$$

$$\hat{m} = \frac{m_u + m_d}{2}$$

$\hookrightarrow$ defined in the isospin limit

- Which one?
  - Phenomenology: charged pion mass
  - Lattice QCD: neutral pion mass

Pion–nucleon σ-term: choice of isospin limit

- Does it matter?
- Currently claimed precision:
  - **Pionic atoms**: $\sigma_{\pi N} = 59.0(3.5)\,\text{MeV}$
  - **Low-energy cross sections**: $\sigma_{\pi N} = 58(5)\,\text{MeV}$
  - **Mainz** 2303.08741: $\sigma_{\pi N} = 43.6(3.8)\,\text{MeV}$
  - **RQCD** 2211.03744: $\sigma_{\pi N} = 43.9(4.7)\,\text{MeV}$

  $\hookrightarrow$ 3.0σ (Mainz), 2.6σ (RQCD) tension with pionic atoms
- But: $\sigma_{\pi N} = \mathcal{O}(M_\pi^2)$

  $\hookrightarrow$ expect $2\Delta_\pi / M_\pi \simeq 6\%$ correction

Pion–nucleon σ-term: choice of isospin limit

- Estimate within ChPT 2305.07045:

$$\sigma_{\pi N} = -4c_1 M_\pi^2 - \frac{9g_A^2 M_\pi^3}{64\pi F_\pi^2} + \mathcal{O}(M_\pi^4)$$

$$\bar{\sigma}_{\pi N} = -4c_1 M_{\pi^0}^2 - \frac{9g_A^2 M_{\pi^0}^3}{64\pi F_\pi^2} + \mathcal{O}(M_\pi^4)$$

$$\Delta\sigma_{\pi N} = \sigma_{\pi N} - \bar{\sigma}_{\pi N} = \Big\{3.7(1), 3.2(1), 3.1(1)(4)\Big\}\,\text{MeV} = 3.1(5)\,\text{MeV}$$

$\hookrightarrow$ reduces tension to 2.4σ (Mainz), 2.0σ (RQCD)

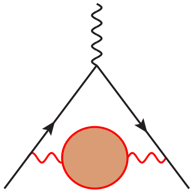
- Not talking about this here, but:
  - Role of excited states 2105.12095, 2203.13862
  - Chiral extrapolation: 2301.06837 finds $\sigma_{\pi N} = 58.7(1.2)$ MeV from the RQCD data

Isospin breaking in HVP

effect	$\pi^0\gamma$	$\eta\gamma$	ρ–ω mixing	FSR	M_{π^0} vs. $M_{\pi^\pm}$	total
size **in units of 10^{-10}**	4.64(4)	0.65(1)	2.71(1.36)	4.22(2.11)	$-4.47(4.47)$	7.8(5.1)

BMWc 2017, Jegerlehner

- Detailed comparison between e^+e^- data and lattice QCD
  $\hookrightarrow$ window quantities, **isospin breaking**
- Can do much better than previous estimates, but still caveats:
  - Cannot cover all channels
  - Scheme dependence
- Dominant effects:
  - **Radiative channels $\pi^0\gamma$, $\eta\gamma$**: data
  - **ρ–ω mixing**: residue in dispersive representation
  - **FSR**: scalar QED + dispersive corrections
  - **M_{π^0} vs. $M_{\pi^\pm}$ for 2π channel**: IAM + Omnès
  - **$\bar{K}K$**: resonance/threshold enhancement



Dispersive representation of 2π contribution

- Decomposition of **pion form factor**

$$F_\pi^V(s) = \underbrace{\Omega_1^1(s)}_{\text{elastic } \pi\pi \text{ scattering}} \times \underbrace{G_\omega(s)}_{\text{isospin-breaking } 3\pi \text{ cut}} \times \underbrace{G_{\text{in}}(s)}_{\text{inelastic effects: } 4\pi, \ldots}$$

- Omnès factor

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

$\hookrightarrow$ can get **pion-mass dependence from IAM** Guo et al. 2009

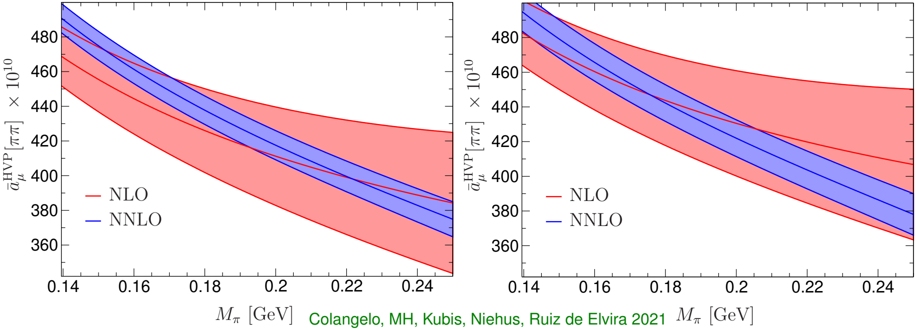
- $G_\omega(s)$ describes ρ–ω mixing in terms of residue $\epsilon_{\rho\omega}$
- $G_{\text{in}}(s)$ parameterized as normal or conformal polynomial

$\hookrightarrow$ free parameters can be matched to $\langle r_\pi^2\rangle$ (and c_π)

- Pion-mass dependence of $\langle r_\pi^2\rangle$ at two loops known Bijnens, Colangelo, Talavera 1998

$\hookrightarrow$ new LEC r_{V1}^r (from resonance saturation or lattice calculation of $\langle r_\pi^2\rangle$)

Predicting the pion-mass dependence from the IAM



Colangelo, MH, Kubis, Niehus, Ruiz de Elvira 2021

- $\bar{a}_\mu^{\rm HVP}[\pi\pi]$ only $I = 1$ correlator (with $\epsilon_{\rho\omega} = 0$)
- **Isospin breaking** due to pion mass difference:

$$a_\mu^{\rm HVP}[\pi\pi]\big|_{M_{\pi^\pm}} - a_\mu^{\rm HVP}[\pi\pi]\big|_{M_{\pi^0}} = -7.67(4)_{\rm ChPT}(3)_{\rm polynomial}(4)_{\langle r_\pi^2\rangle}(21)_{r_{V1}^r}[22]_{\rm total}$$

$\hookrightarrow$ **threshold effect**, almost exclusively in LD window

ρ–ω mixing and FSR

- Can get FSR and $\epsilon_{\rho\omega}$ contributions from dispersive fits to 2π
- Higher-order terms $\mathcal{O}(e^2\epsilon_{\rho\omega})$ small, $\lesssim 0.1$
- Line shape matters Wolfe, Maltman 2009, we use

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\operatorname{Im} g_\omega(s')}{s'(s'-s)}\left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4$$

$$g_\omega(s) = 1 + \epsilon_{\rho\omega}\frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s} \qquad \epsilon_{\rho\omega} \to \operatorname{Re}\epsilon_{\rho\omega} + i\operatorname{Im}\epsilon_{\rho\omega}\frac{\left(1-\frac{M_{\pi^0}^2}{s}\right)^3}{\left(1-\frac{M_{\pi^0}^2}{M_\omega^2}\right)^3}\theta(s-M_{\pi^0}^2)$$

to account for radiative channels $\rho \to \pi^0\gamma, \ldots \to \omega$

- Results:

$$a_\mu^{\text{HVP}}[\pi\pi, \text{FSR}, \text{Born}] = 4.24(2) \qquad a_\mu^{\text{HVP}}[\pi\pi, \rho\text{–}\omega] = 3.68(17)$$

$\hookrightarrow$ to leading order, ρ–ω mixing is purely $\mathcal{O}(\delta) = \mathcal{O}(m_u - m_d)$

Isospin breaking in $\bar{K}K$ channel

- Why $\bar{K}K$?
  - ϕ resonance close to $\bar{K}K$ threshold
  - **Isospin breaking from masses enhanced**
- Threshold region dominated by isoscalar form factor via ϕ Stamen et al. 2022

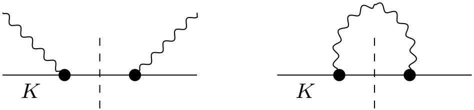
  $\hookrightarrow$ analyzed in terms of ϕ resonance parameters
- Significant isospin breaking in residues

$$c_\phi^{K^+K^-} = 0.977(6) \qquad c_\phi^{\bar{K}^0K^0} = 1.001(6)$$

  $\hookrightarrow$ dominant source of uncertainty
- Need separation of kaon mass into $\mathcal{O}(e^2)$ and $\mathcal{O}(\delta)$

  $\hookrightarrow$ **Cottingham formula**

Kaon mass separation: Cottingham formula



- Cottingham formula for EM self energies:

$$(M_K^2)_{\text{EM}} = \frac{ie^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{T^\mu_\mu}{k^2+i\epsilon}$$

$$\stackrel{\text{elastic}}{\rightarrow} \frac{\alpha}{8\pi}\int_0^\infty ds\,\big[F_K(-s)\big]^2\bigg(4W+\frac{s}{M_K^2}(W-1)\bigg) \qquad W=\sqrt{1+\frac{4M_K^2}{s}}$$

- Fit of dispersive representation to e^+e^- data Stamen et al. 2022

$$\left.\begin{aligned}(M_{K^\pm}^2)_{\text{EM}} &= 2.12(2)(17)\times10^{-3}\,\text{GeV}^2\\ (M_{K^0}^2)_{\text{EM}} &= 7(2)(17)\times10^{-6}\,\text{GeV}^2\\ (\Delta M_\pi^2)_{\text{EM}} &= 1.3(3)\times10^{-3}\,\text{GeV}^2\end{aligned}\right\}\ \epsilon=\frac{(\Delta M_K^2)_{\text{EM}}}{(\Delta M_\pi^2)_{\text{EM}}}-1=0.63(40)$$

gives

$$M_{K^\pm}=\big(494.58-3.05_\delta+2.14_{e^2}\big)\,\text{MeV} \qquad M_{K^0}=\big(494.58+3.03_\delta\big)\,\text{MeV}$$

Isospin breaking in $\bar{K}K$ channel

- Results

$$a_\mu^{\text{HVP}}[K^+K^-, \leq 1.05\,\text{GeV}] = 18.45(20) \qquad a_\mu^{\text{HVP}}[K^0\bar{K}^0, \leq 1.05\,\text{GeV}] = 11.83(15)$$

$$a_\mu^{\text{HVP}}[K^+K^-, \text{FSR}] = 0.75(4)$$

$$a_\mu^{\text{HVP}}[K^+K^-, e^2] = -3.24(17) \qquad a_\mu^{\text{HVP}}[K^0\bar{K}^0, e^2] = -0.02(0)$$

$$a_\mu^{\text{HVP}}[K^+K^-, \delta] = 4.98(26) \qquad a_\mu^{\text{HVP}}[K^0\bar{K}^0, \delta] = -4.62(23)$$

$$a_\mu^{\text{HVP}}[K^+K^-, e^2\delta] = -0.33(1)$$

- Note:
  - K^0 self energy negligible, but indirect $\mathcal{O}(e^2)$ effect from the $K^\pm$ contribution to the ϕ spectral function
  - Remaining differences between "isospin limit" K^+K^- (16.29) and $\bar{K}^0K^0$ (16.47) due to c_ϕ and isovector form factor

- Isospin-breaking effects huge due to **threshold/resonance enhancement**

  $\hookrightarrow$ higher-order terms $\mathcal{O}(e^2\delta)$ in K^+K^- larger than in 2π!

Other channels

- **3π channel** MH, Hoid, Kubis, Schuh, to appear:
  - Estimate for FSR $\simeq 0.5$
  - BaBar 2021 quotes $\rho \to 3\pi$ contribution in VMD fit ($\simeq -0.6$ in a_μ^{HVP} Boito et al. 2022), but within model context
  - Can one really extract a (significant) ρ–ω mixing effect? Surprisingly, yes!
  - Effect is large and cancels large part of 2π analog
  - Why? Naive scaling

$$\left|\frac{a_\mu^{\rho-\omega}[3\pi]}{a_\mu^{\rho-\omega}[2\pi]}\right| \simeq \frac{9\epsilon_{\rho\omega}\, a_\mu[3\pi]}{\epsilon_{\rho\omega}\, a_\mu[2\pi]} \simeq 1$$

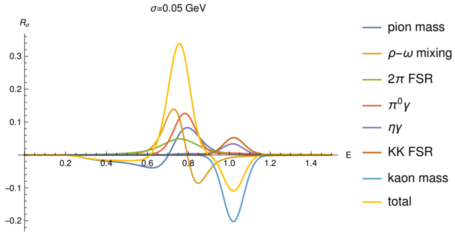
    actually seems to work
- ***R*-ratio from perturbative QCD**
  - QED corrections included in `rhad` Harlander, Steinhauser 2003, but $\mathcal{O}(10^{-3})$ compared to leading-order result, and $10^{-3} a_\mu^{\text{HVP}}[\geq 1.8\,\text{GeV}] \lesssim 0.05$

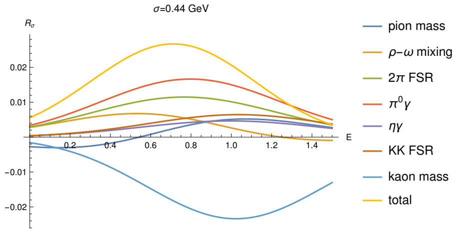
Summing everything up

	SD window		int window		LD window		full HVP	
	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$	$\mathcal{O}(e^2)$	$\mathcal{O}(\delta)$
$\pi^0\gamma$	0.16(0)	–	1.52(2)	–	2.70(4)	–	4.38(6)	–
$\eta\gamma$	0.05(0)	–	0.34(1)	–	0.31(1)	–	0.70(2)	–
ρ–ω mixing	–	0.05(0)	–	0.83(6)	–	2.79(11)	–	3.68(17)
FSR (2π)	0.11(0)	–	1.17(1)	–	3.14(3)	–	4.42(4)	–
M_{π^0} vs. $M_{\pi^\pm}$ (2π)	0.04(1)	–	$-0.09(7)$	–	$-7.62(14)$	–	$-7.67(22)$	–
FSR (K^+K^-)	0.07(0)	–	0.39(2)	–	0.29(2)	–	0.75(4)	–
kaon mass (K^+K^-)	$-0.29(1)$	0.44(2)	$-1.71(9)$	2.63(14)	$-1.24(6)$	1.91(10)	$-3.24(17)$	4.98(26)
kaon mass $(\bar{K}^0K^0)$	0.00(0)	$-0.41(2)$	$-0.01(0)$	$-2.44(12)$	$-0.01(0)$	$-1.78(9)$	$-0.02(0)$	$-4.62(23)$
total	0.14(1)	0.08(3)	1.61(12)	1.02(20)	$-2.44(16)$	2.92(17)	$-0.68(29)$	4.04(39)
BMWc 2020	–	–	$-0.09(6)$	0.52(4)	–	–	$-1.5(6)$	1.9(1.2)
RBC/UKQCD 2018	–	–	0.0(2)	0.1(3)	–	–	$-1.0(6.6)$	10.6(8.0)
JLM 2021	–	–	–	–	–	–	–	3.32(89)

- Reasonable agreement with BMWc 2020, RBC/UKQCD 2018, and James, Lewis, Maltman 2021
- Adding 3π (FSR and ρ–ω mixing) will remove tension in $\mathcal{O}(\delta)$
- **Cancellation of individually sizable corrections!**

Role of isospin breaking: energy dependence





- Alternative to windows: **Gaussian smearing** ETMC 2022

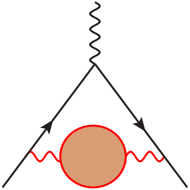
$$R_\sigma(s) = \int_0^\infty ds' \, G_\sigma(\sqrt{s'} - \sqrt{s}) R(s') \qquad G_\sigma(\omega) = \frac{e^{-\omega^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}$$

- Cancellation for a_μ seems to involve a delicate balance with kernel $K(s)$

Conclusions

- Comparison of pion–nucleon σ-term between lattice QCD and phenomenology is getting sensitive to isospin conventions

  $\hookrightarrow$ reduces tension by about 3 MeV
- Phenomenological estimates for **isospin-breaking effects** from $\pi^0\gamma$, $\eta\gamma$, $2\pi(\gamma)$, $\bar{K}K(\gamma)$, $3\pi(\gamma)$
- Decomposition of kaon mass from Cottingham formula in agreement with quark mass scheme
- Cancellations among various $\mathcal{O}(e^2)$ and $\mathcal{O}(\delta)$ effects
- Agreement with BMWc 2020 seems to improve when including 3π channel



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