Infinite volume QED in finite volume lattice QCD calculations

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QED in Weak Decays

Higgs Centre for Theoretical Physics

Higgs Centre Seminar Room

School of Physics and Astronomy

The University of Edinburgh

Outline

- **Introduction to the finite volume effects in lattice QCD + QED**
- QED correction to hadron masses & the infinite volume reconstruction method

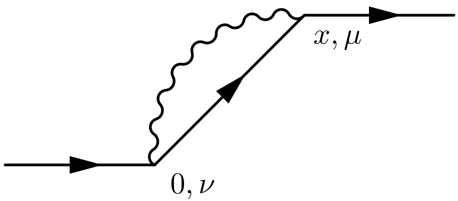
  Feng and Jin [Phys.Rev.D 100 (2019) 9, 094509]

  Christ, Feng, Jin and Sachrajda [Phys.Rev.D 103 (2021) 1, 014507]
- Lattice calculation of the pion mass splitting

  Feng, Jin, and Riberdy [Phys.Rev.Lett. 128 (2022) 5, 052003]
- QED correction to meson leptonic decay rates

  Christ, Feng, Jin, Sachrajda, and Wang [arXiv:2304.08026]
- Summary and outlook

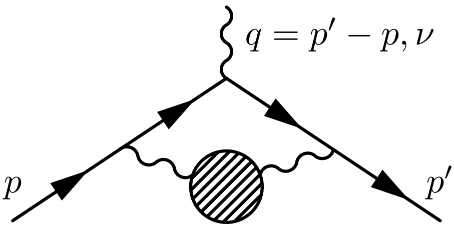
- No massless particles in QCD → Finite volume effects for many observables are **exponentially suppressed** by the spatial lattice size L.
  - Mass of a stable particle M. Lüscher, Commun.Math.Phys. 104, 177-206 (1986)
- QED include massless photon → Use treatments similar to QCD for QED leads to **power-law suppressed** finite volume effects.
  - Mass of a stable particle in QED_L M. Hayakawa and S. Uno, Prog.Theor.Phys. (2008).



$$\Delta M(L) = \Delta M(\infty) - \frac{q^2}{4\pi}\frac{\kappa}{2L}\left(1 + \frac{2}{mL}\right) + \mathcal{O}\left(\frac{1}{L^3}\right) \tag{1}$$

where $\kappa = 2.8372997\cdots$. S. Borsanyi et al., Science 347, 1452 (2015).

- Start the derivation in the **infinite volume** (and in the continuum).
- Treat the QED part of the diagram analytically (and perturbatively).
- The **hadronic part** needs **finite volume** lattice QCD. Finite volume errors introduced.
  - Hadronic vacuum polarization (HVP) contribution to muon $g-2$:



$$a_\mu^{\text{HVP LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t) C(t) \quad (2)$$

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j^{em}(\vec{x}, t) J_j^{em}(0) \rangle_{\text{QCD}} \quad (3)$$

T. Blum (2003) D. Bernecker, H. Meyer (2011)

- Start the derivation in the **infinite volume** (and in the continuum).
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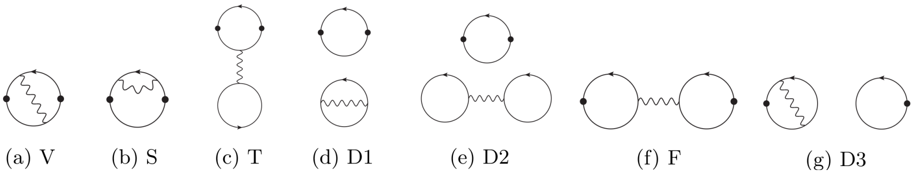
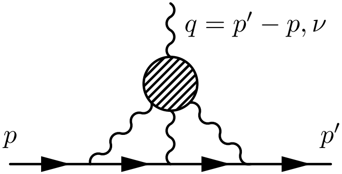
- The **hadronic part** needs **finite volume** lattice QCD. Finite volume errors introduced.
  - QED corrections to the hadronic vacuum polarization (HVP):

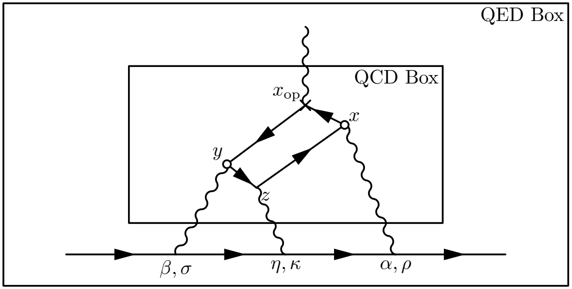


$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \qquad (4)$$

T. Blum et al (2018)

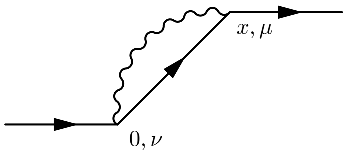
- Start the derivation in the **infinite volume** (and in the continuum).
- Treat the QED part of the diagram analytically (and perturbatively).
- The **hadronic part** needs **finite volume** lattice QCD. Finite volume errors introduced.
  - Hadronic light-by-light (HLbL) contribution to muon $g-2$:





N. Asmussen et al (2016) T. Blum et al (2017)

- Start the derivation in the **infinite volume** (and in the continuum).
- Treat the QED part of the diagram analytically (and perturbatively).
- The **hadronic part** needs **finite volume** lattice QCD. Finite volume errors introduced.
- Does **NOT** work for calculating the QED correction to the mass of a stable hadron.



$$\Delta M = \mathcal{I} = \frac{1}{2}\int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x), \qquad (5)$$

$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|T J_\mu(x) J_\nu(0)|N\rangle, \quad S^\gamma_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \qquad (6)$$

- The hadronic function does not always fall exponentially in the long distance region. When $t \gg |\vec{x}|$:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1) \qquad (7)$$

- Truncate the integral: $\int d^4x \to \int_{-L/2}^{L/2} d^4x$ & Approx the $\mathcal{H}(x)$: $\mathcal{H}(x) \to \mathcal{H}^L(x)$
$\to$ Power-law suppressed finite volume errors.

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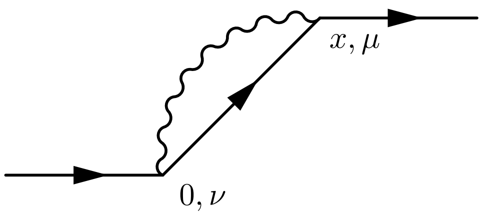
$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|T J_\mu(x) J_\nu(0)|N\rangle$$

$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2}$$



- Evaluate the QED part, the photon propagator, in infinite volume.
- The hadronic function does not always fall exponentially in the long distance region
  $\rightarrow$ Separate the integral into two parts ($t_s \lesssim L$):

$$\Delta M = \mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)}$$

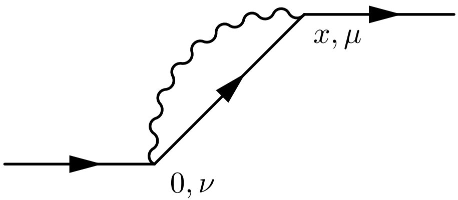
$$\mathcal{I}^{(s)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int d^3x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \quad (8)$$

$$\mathcal{I}^{(l)} = \int_{t_s}^{\infty} dt \int d^3x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \quad (9)$$

- For the short distance part, $\mathcal{I}^{(s)}$ can be directly calculated on a finite volume lattice:

$$\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int_{-L/2}^{L/2} d^3x\, \mathcal{H}^{L}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

- For the **long distance part**, $\mathcal{I}^{(l)}$, a different treatment is required.



- For the long distance part, we can evaluate $\mathcal{H}_{\mu,\nu}(x)$ **indirectly** in the **infinite volume**.

$$\mathcal{I}^{(l)} = \int_{t_s}^{\infty} dt \int d^3x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \qquad (10)$$

- Note that when t is large ($t > t_s$), the intermediate states between the two currents are dominated by the single particle states (possibly with small momentum). Therefore:

$$\mathcal{H}_{\mu,\nu}(x) \approx \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2E_{\vec{p}}} \frac{1}{2M} \langle N|J_\mu(0)|N(\vec{p})\rangle \langle N(\vec{p})|J_\nu(0)|N\rangle \right] e^{i\vec{p}\cdot\vec{x}-(E_{\vec{p}}-M)t} \qquad (11)$$

  - We only need to calculate the form factors: $\langle N(\vec{p})|J_\nu(0)|N\rangle$!
  - Values for all $\vec{p}$ are needed. Inversely Fourier transform the above relation **at** t_s!

$$\int d^3x\, \mathcal{H}_{\mu,\nu}(t_s,\vec{x}) e^{-i\vec{p}\cdot\vec{x}+(E_{\vec{p}}-M)t_s} \approx \frac{1}{2E_{\vec{p}}} \frac{1}{2M} \langle N|J_\mu(0)|N(\vec{p})\rangle \langle N(\vec{p})|J_\nu(0)|N\rangle \qquad (12)$$

- The final expression for QED correction to hadron mass is split into two parts:

$$\Delta M = \mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)}$$

- For the short distance part: $\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int_{L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$ (13)

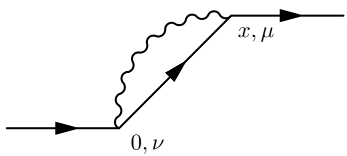
- For the long distance part: $\mathcal{I}^{(l)} \approx \mathcal{I}^{(l,L)} = \int_{-L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(t_s,\vec{x}) L_{\mu,\nu}(t_s,\vec{x})$

- For Feynman gauge:

$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \qquad L_{\mu,\nu}(t_s,\vec{x}) = \frac{\delta_{\mu,\nu}}{2\pi^2}\int_0^{\infty} dp \frac{\sin(p|\vec{x}|)}{2(p+E_p-M)|\vec{x}|} e^{-pt_s}$$

- Only use $\mathcal{H}^L_{\mu,\nu}(t,\vec{x})$ within $-t_s \le t \le t_s$.

- Choose $t_s = L/2$, **finite volume errors and the ignored excited states contribution to $\mathcal{I}^{(l)}$ are both exponentially suppressed by the spatial lattice size L.**



- **NOT** a general finite volume QED scheme.
- Derivation is in the **infinite volume**.
- QED interactions are treated perturbatively in infinite volume.
- Exploit the property of some Euclidean space-time hadronic matrix elements at long distance in infinite volume. e.g.

$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|T J_\mu(x) J_\nu(0)|N\rangle \tag{14}$$

$$\approx \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_{\vec{p}}}\frac{1}{2M}\langle N|J_\mu(0)|N(\vec{p})\rangle\langle N(\vec{p})|J_\nu(0)|N\rangle\right] e^{i\vec{p}\cdot\vec{x}-(E_{\vec{p}}-M)t} \tag{15}$$

  The **infinite volume** hadronic matrix elements can therefore be **reconstructed** by **finite volume** hadronic matrix elements with exponentially suppressed finite volume errors.

- Much more sophisticated treatment is needed for a muti-hadron system.
  Christ, Feng, Karpie, Nguyen [PoS LATTICE2021 (2022) 312]

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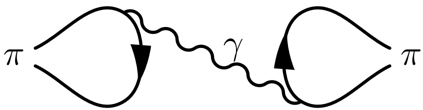
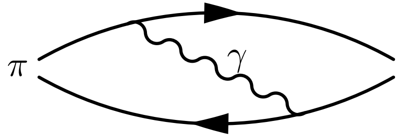


Disconnected diagram



Connected diagram

- Coulomb gauge fixed wall sources are used to interpolate the pion interpolating operators.
- Fixed time separation between the vector current operator and the closest pion interpolating operators: $t_{\text{sep}} \approx 1.5\text{fm}$.

$$\mathcal{H}^L_{\mu,\nu}(t,\vec{x}) = L^3 \frac{\langle \pi(t+t_{\text{sep}}) J_\mu(t,\vec{x}) J_\nu(0) \pi^\dagger(-t_{\text{sep}}) \rangle_L}{\langle \pi(t+t_{\text{sep}}) \pi^\dagger(-t_{\text{sep}}) \rangle_L^{[*]}} \tag{16}$$

- Diagrams are similar to the $\pi^- \to \pi^+ ee$ neutrinoless double beta $(0\nu2\beta)$ decay.
  D. Murphy and W. Detmold (2018), Tuo, Feng, and Jin (2019)
- At $\mathcal{O}(\alpha_{\text{QED}}, (m_u - m_d)/\Lambda_{\text{QCD}})$, all UV divergence are canceled. The two diagrams are the only diagrams contributing to $m_{\pi^\pm} - m_{\pi^0}$. RM123 (2013)
- In particular, the pion mass splitting at leading order does not depend on $m_u - m_d$.

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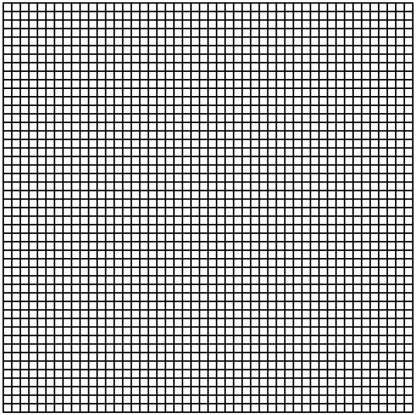
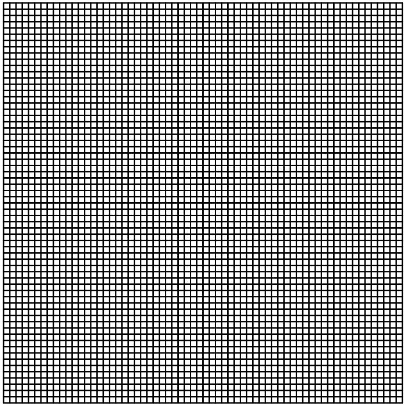
[*]: Need to correct the around the world effects.

Lattice QCD Ensembles from RBC/UKQCD

48I



64I



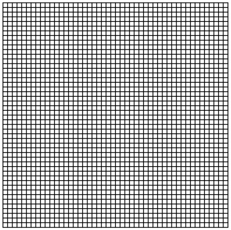
- Domain wall fermion action (preserves Chiral symmetry, no $\mathcal{O}(a)$ lattice artifacts).
- Iwasaki gauge action.
- $M_\pi = 135$ MeV *, $L = 5.5$ fm box, $1/a_{48\text{I}} = 1.73$ GeV, $1/a_{64\text{I}} = 2.359$ GeV.

*: Valence pion mass. Slightly different from the 139 MeV unitary pion mass used in the ensemble generation.

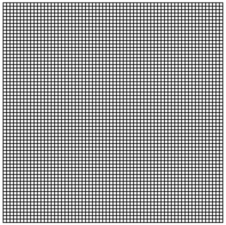
RBC/UKQCD, PRD [arXiv:1411.7017]

Lattice QCD Ensembles from RBC/UKQCD

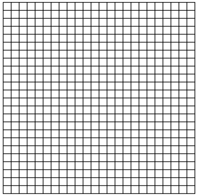
48I



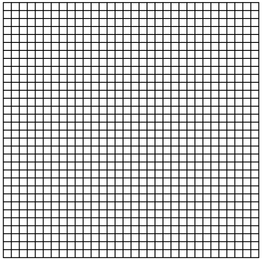
64I



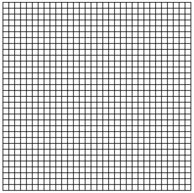
24D



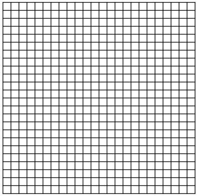
32D



32Dfine

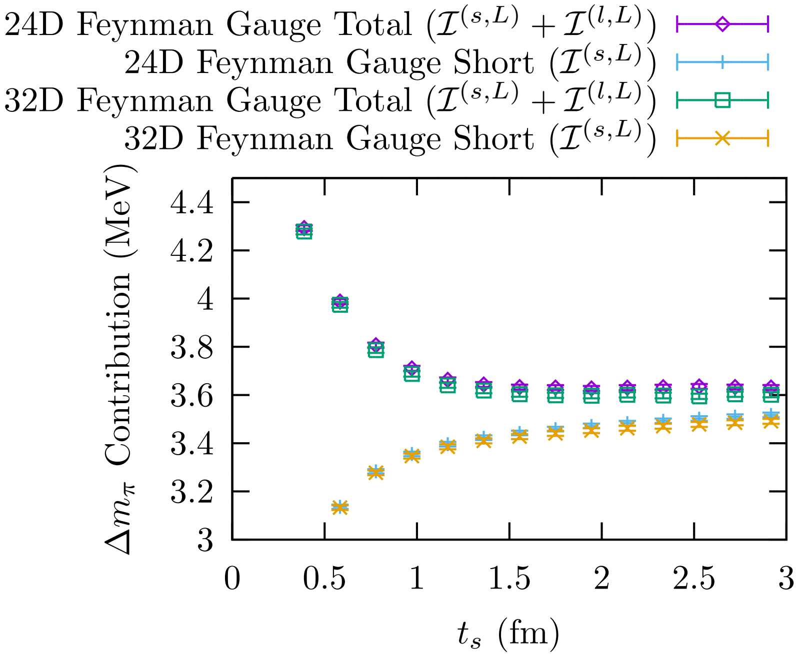


24DH



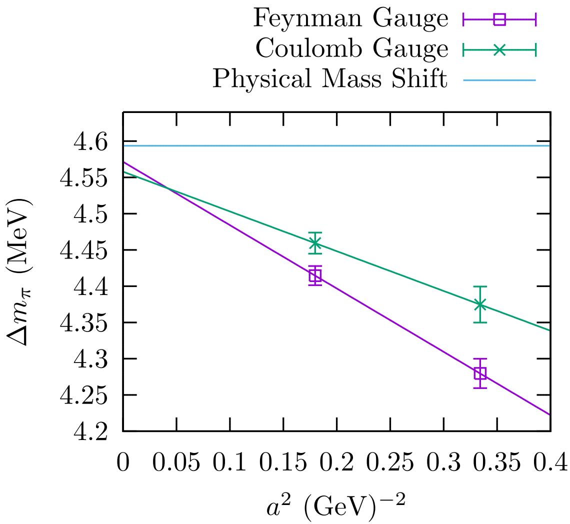
- For 24D, 32D, 32Dfine, $M_\pi \approx 140$ MeV
- For 24DH, $M_\pi \approx 340$ MeV

RBC/UKQCD, PRD [arXiv:1411.7017]



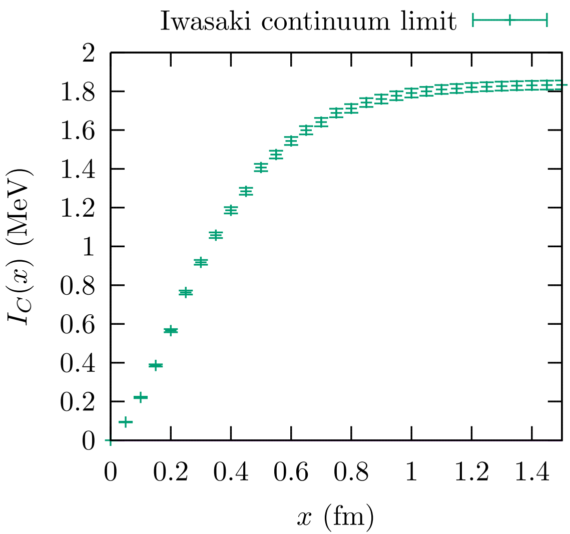
- The difference between 32D and 24D is $-0.035(16)$MeV. This is consistent with a scalar QED calculation, which yields -0.022MeV.

Continuum extrapolation of Δm_π

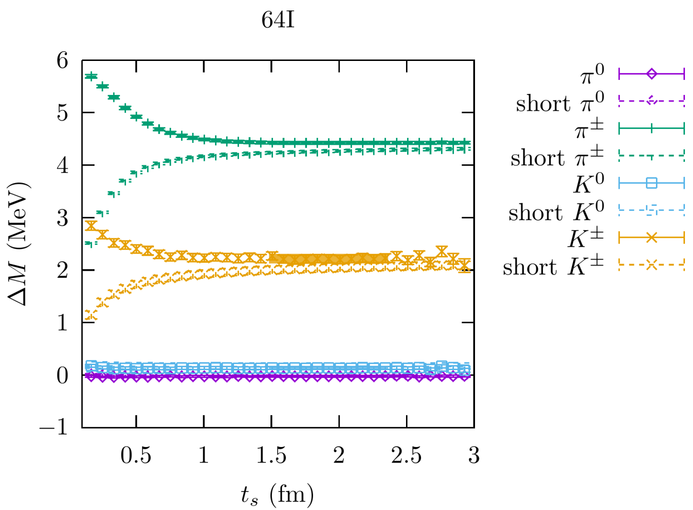


	Disc (MeV)	Conn (MeV)	Total (MeV)
Feyn	0.051(9)(22)	4.483(40)(28)	4.534(42)(43)
Coul	0.052(2)(13)	4.508(46)(42)	4.560(46)(41)
Coul-t	0.018(1)(4)	1.840(22)(39)	1.858(22)(41)

Finite volume corrections (the differences between the 32D and 24D ensembles) are included in table.



- The Coulomb potential contribution to the pion mass difference. The curve is the partial sum respect to the spatial separation of the two equal-time current operators.
- This plot provide some interesting pion shape information.



- QED correction to masses of π^0, $\pi^\pm$, K^0, $K^\pm$.
- Results from the 64I ensemble.
- Plot by Joshua Swaim (current UCONN graduate student).

- Calculation performed by reusing propagators generated for the lattice HLbL calculation at MIRA.

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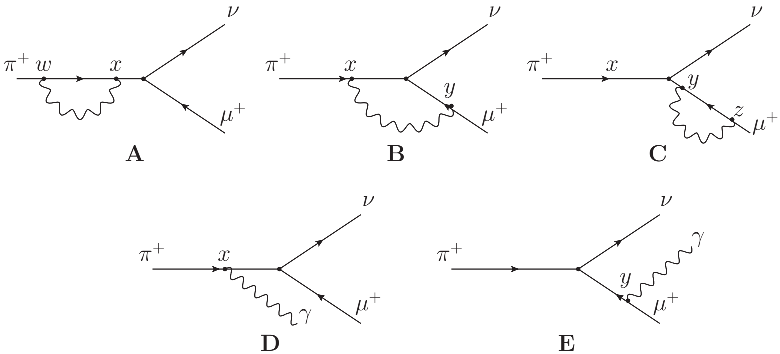


- Diagram A:

$$H^{(2)}_{\mu,\rho,\sigma}(t_1, t_2, \vec{x}) \quad = \quad \int d^3\vec{w}\, \langle 0|T\{J^W_\mu(0) J^{\mathrm{EM}}_\rho(t_1, \vec{w}+\vec{x}) J^{\mathrm{EM}}_\sigma(t_2, \vec{w})\}|\pi(\vec{0})\rangle \qquad (17)$$

- Diagram B and D:

$$H^{(1)}_{\mu,\rho}(x) = H^{(1)}_{\mu,\rho}(x_t, \vec{x}) \quad = \quad \langle 0|T\{J^W_\mu(0) J^{\mathrm{EM}}_\rho(x)\}|\pi(\vec{0})\rangle \qquad (18)$$

- Diagram C and E ($f_\pi \approx 130$ MeV):

$$H^{(0)}_\mu = H^{(0)}_t \delta_{\mu,t} \quad = \quad \langle 0|J^W_\mu(0)|\pi(\vec{0})\rangle = -i m_\pi f_\pi \delta_{\mu,t} \qquad (19)$$

$$H^{(1)}_{\mu,\rho}(x_t,\vec{x}) \quad = \quad \langle 0|T\{J^W_\mu(0)J^{\mathrm{EM}}_\rho(x)\}|\pi(\vec{0})\rangle \tag{20}$$

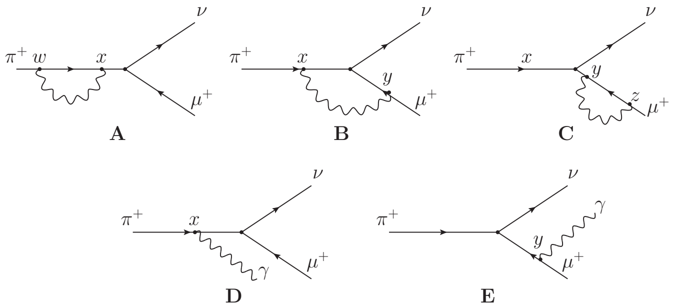
- Goal is to obtain the infinite volume hadron matrix elements with even for large $|x|$.
- Short distance region $x_t \geq -t_s$. Can be directly approximated in finite volume.

$$H^{(1)}_{\mu,\rho}(x_t,\vec{x}) \quad \approx \quad H^{(1,L)}_{\mu,\rho}(x_t,\vec{x}) \tag{21}$$

- Long distance region $x_t \leq -t_s$. Can be approximated by the single pion intermediate states contribution.

$$\begin{aligned}
H^{(1)}_{\mu,\rho}(x_t,\vec{x}) \quad &\approx \quad \int \frac{d^3\vec{p}}{(2\pi)^3}\langle 0|J^W_\mu(0)|\pi(\vec{p})\rangle\frac{e^{-(E_{\pi,\vec{p}}-M_\pi)|x_t|-i\vec{p}\cdot\vec{x}}}{2E_{\pi,\vec{p}}}\langle\pi(\vec{p})|J^{\mathrm{EM}}_\rho(0)|\pi(\vec{0})\rangle && (22)\\
&= \quad \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-(E_{\pi,\vec{p}}-M_\pi)(|x_t|-t_s)+i\vec{p}\cdot\vec{x}}\int d^3\vec{x}' H^{(1)}(-t_s,\vec{x}')e^{-i\vec{p}\cdot\vec{x}'} && (23)\\
&\approx \quad \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-(E_{\pi,\vec{p}}-M_\pi)(|x_t|-t_s)+i\vec{p}\cdot\vec{x}}\int_{-L/2}^{L/2} d^3\vec{x}' H^{(1,L)}(-t_s,\vec{x}')e^{-i\vec{p}\cdot\vec{x}'} && (24)
\end{aligned}$$

- As long as $t_s \lesssim L$, the above two approximations only have exponentially suppressed effects.



- Derivation in infinite volume will encounter logarithmic infrared divergence.
- Fortunately, the divergence cancel analytically between diagrams.
- Use “T” to represent the tree level diagram. We will have IR divergence cancellation between:
  - “TA” and “DD”; “TB” and “DE”
    Christ, Feng, Jin, Sachrajda, and Wang [arXiv:2304.08026]
  - “TC” and “EE” (Pure QED)
    Carrasco, Lubicz, Martinelli, Sachrajda, Tantalo, Tarantino, and Testa [Phys.Rev.D 91 (2015) 7, 074506]

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- **Summary and outlook**

- We invent the infinite volume reconstruction (IVR) method, eliminates all power-law suppressed finite volume errors in QED self-energy calculations.

  Feng and Jin [Phys.Rev.D 100 (2019) 9, 094509]

  Christ, Feng, Jin and Sachrajda [Phys.Rev.D 103 (2021) 1, 014507]

- We have used this method to calculate the pion mass splitting $m_{\pi^\pm} - m_{\pi^0}$.

  Feng, Jin, and Riberdy [Phys.Rev.Lett. 128 (2022) 5, 052003]

Reference	$m_{\pi^\pm} - m_{\pi^0}$(MeV)
RM123 2013	$5.33(48)_{\text{stat}}(59)_{\text{sys}}$
R. Horsley et al. 2015	$4.60(20)_{\text{stat}}$
RM123 2017	$4.21(23)_{\text{stat}}(13)_{\text{sys}}$
This work	$4.534(42)_{\text{stat}}(43)_{\text{sys}}$
RM123 2022	$4.622(64)_{\text{stat}}(70)_{\text{sys}}$

The experimental value is 4.5936(5)MeV.

- The IVR method and the 4-point hadronic function have more applications:
  - Two-photon Exchange Contribution to the muonic-hydrogen Lamb Shift from Lattice QCD. Fu, Feng, Jin and Lu [Phys.Rev.Lett. 128 (2022) 17, 172002]
  - $\pi^- \to \pi^+ e^- e^-$ neutrinoless double beta ($0\nu 2\beta$) decay.

$$g_\nu^{\pi\pi}(\mu)\Big|_{\mu=m_\rho} = -10.89(28)_{\text{stat}}(33)_L(66)_a$$

    Tuo, Feng, and Jin [Phys.Rev.D 100 (2019) 9, 094511]
  - Electroweak box diagrams in $\pi^- \to \pi^0 e^- \bar{\nu}_e$.
    Feng, Gorchtein, Jin, Ma, and Seng [Phys.Rev.Lett. 124 (2020) 19, 192002]
    Ma, Feng, Gorchtein, Jin, and Seng [Phys.Rev.D 103 (2021) 114503]
  - $K \to \ell \nu_\ell \ell'^+ \ell'^-$ Tuo, Feng, Jin and Wang [Phys.Rev.D 105 (2022) 5, 054518]
- QED correction to the meson leptonic decay formulation.
  Christ, Feng, Jin, Sachrajda, and Wang [arXiv:2304.08026]
  There are mature lattice QCD calculations using QED_L.
  RM123 [Phys.Rev.D 100 (2019) 3, 034514], RBC-UKQCD [JHEP 02 (2023) 242]

Thank You!

$$L_{\mu}^{(0)} = \bar{u}(\vec{p}_\nu)\gamma_\mu(1-\gamma_5)v(\vec{p}_\ell) \tag{25}$$

$$L_{\mu,\rho}^{(1)}(ix_{\mathrm{M}}^t,\vec{x}) = \bar{u}(\vec{p}_\nu)\gamma_\mu(1-\gamma_5)S_\ell(0;ix_{\mathrm{M}}^t,\vec{x})\gamma_\rho v(\vec{p}_\ell)e^{-i\vec{p}_\ell\cdot\vec{x}}e^{iE_\ell x_{\mathrm{M}}^t} \tag{26}$$

$$= -i\int\frac{d\vec{p}_{\mathrm{M}}^t}{(2\pi)}\int\frac{d^3\vec{p}}{(2\pi)^3}\tilde{L}_{\mu,\rho}^{(1)}(ip_{\mathrm{M}}^t,\vec{p})e^{i\vec{p}\cdot\vec{x}}e^{-ip_{\mathrm{M}}^t x_{\mathrm{M}}^t} \tag{27}$$

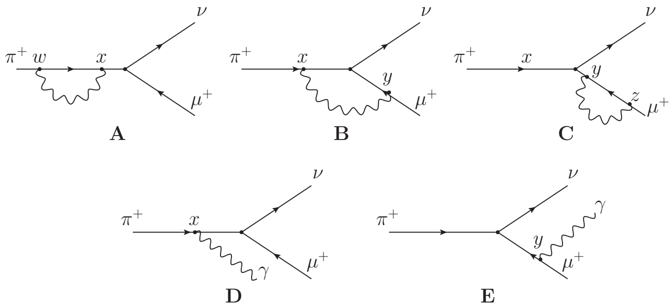
$$\tilde{L}_{\mu,\rho}^{(1)}(ip_{\mathrm{M}}^t,\vec{p}) = \bar{u}(\vec{p}_\nu)\gamma_\mu(1-\gamma_5)\tilde{S}_\ell(-ip_{\mathrm{M}}^t - iE_\ell, -\vec{p}-\vec{p}_\ell)\gamma_\rho v(\vec{p}_\ell) \tag{28}$$

where

$$S_\ell(x;y) = \int\frac{d^4p}{(2\pi)^4}\tilde{S}_\ell(p)e^{ip\cdot(x-y)} \qquad \tilde{S}_\ell(p_t,\vec{p}) = \frac{-i\gamma_\mu p_\mu + m}{p^2+m^2} \tag{29}$$

For small $\vec{k}$, we have:

$$\tilde{L}_{\mu,\rho}^{(1)}(i|\vec{k}|,\vec{k}) \approx -\tilde{L}_{\mu,\rho}^{(1)}(-i|\vec{k}|,-\vec{k}) \tag{30}$$

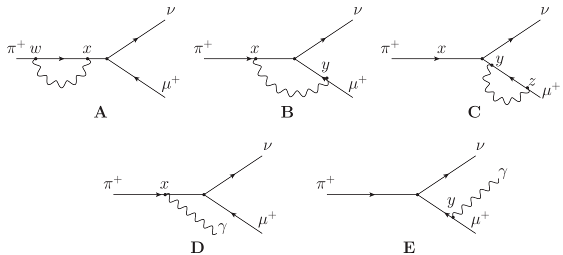


$$i\mathcal{M}_{\mathsf{T}} = -i\frac{G_F}{\sqrt{2}} V_{ud}^{*} H_{\mu}^{(0)} L_{\mu}^{(0)} \tag{31}$$

$$i\mathcal{M}_{B} = -i\frac{G_F}{\sqrt{2}} V_{ud}^{*} (-(-ie)^2) \int d^4x \int d^4y \, H_{\mu,\rho}^{(1)}(x) L_{\mu,\rho'}^{(1)}(y) S_{\rho,\rho'}^{\gamma}(x;y) \tag{32}$$

$$i\mathcal{M}_{D} = -i\frac{G_F}{\sqrt{2}} V_{ud}^{*} (ie) \int d^4x \, H_{\mu,\rho}^{(1)}(x) e^{-i\vec{p}_\gamma \cdot \vec{x}} e^{|\vec{p}_\gamma| x_t} L_{\mu}^{(0)} \epsilon_{\lambda,\rho}^{*}(\vec{p}_\gamma) \tag{33}$$

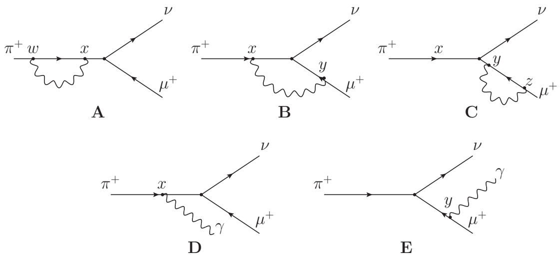
$$i\mathcal{M}_{E} = -i\frac{G_F}{\sqrt{2}} V_{ud}^{*} (-ie) H_{\mu}^{(0)} \tilde{L}_{\mu,\rho}^{(1)}(i|\vec{p}_\gamma|, \vec{p}_\gamma) \epsilon_{\lambda,\rho}^{*}(\vec{p}_\gamma) \tag{34}$$



Focusing on the long distance part of diagram “B” and “D” (source of the divergence)

$$i\mathcal{M}_B^L = -i\frac{G_F}{\sqrt{2}}V_{ud}^*(-(-ie)^2)\int_{-\infty}^{-t_s} dx_t \int d^3\vec{x} \int d^4y\, H_{\mu,\rho}^{(1)}(x_t,\vec{x})L_{\mu,\rho'}^{(1)}(y)S_{\rho,\rho'}^{\gamma}(y;x) \tag{35}$$

$$i\mathcal{M}_D^L = -i\frac{G_F}{\sqrt{2}}V_{ud}^*\,(ie)\int_{-\infty}^{-t_s} dx_t \int d^3\vec{x}\, H_{\mu,\rho}^{(1)}(x_t,\vec{x})e^{-i\vec{p}_\gamma\cdot\vec{x}}e^{-|\vec{p}_\gamma||x_t|}L_\mu^{(0)}\epsilon_{\lambda,\rho}^*(\vec{p}_\gamma) \tag{36}$$



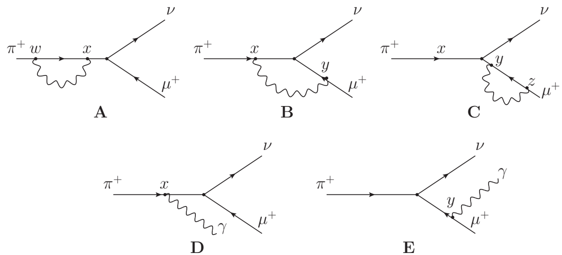
Use Feynman gauge for photon propagator and ignores the region $x_t > y_t$ [*]:

$$i\mathcal{M}_B^L \approx -i\frac{G_F}{\sqrt{2}}V_{ud}^*(-(-ie)^2) \times \int_{-\infty}^{-t_s} dx_t \int d^3\vec{x} \int d^4y\, H_{\mu,\rho}^{(1)}(x_t,\vec{x}) L_{\mu,\rho}^{(1)}(y) \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{y}-\vec{x})-|\vec{k}|(y_t-x_t)}}{2|\vec{k}|} \quad (37)$$

$$= -i\frac{G_F}{\sqrt{2}}V_{ud}^*(-(-ie)^2) \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2|\vec{k}|} \times \int_{-\infty}^{-t_s} dx_t \int d^3\vec{x}\, H_{\mu,\rho}^{(1)}(x_t,\vec{x}) e^{-i\vec{k}\cdot\vec{x}} e^{-|\vec{k}||x_t|} \tilde{L}_{\mu,\rho}^{(1)}(-i|\vec{k}|,-\vec{k}) \quad (38)$$

[*]: Since $x_t \leq -t_s$, the contribution of the region $x_t > y_t$ is small and does not contribute to the IR divergence.

Infrared divergence cancellation “TB” and “DE”



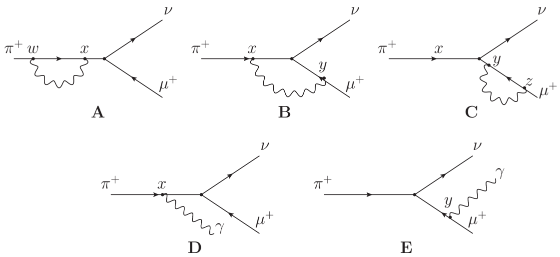
Use IVR for $H^{(1)}_{\mu,\rho}(x_t,\vec{x})$ $(x_t \leq -t_s)$

$$\int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(x_t,\vec{x})e^{-i\vec{p}\cdot\vec{x}} \quad = \quad \int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(-t_s,\vec{x})e^{-i\vec{p}\cdot\vec{x}}e^{-(E_{\pi,\vec{p}}-M_\pi)(|x_t|-t_s)} \tag{39}$$

We obtain:

$$\begin{aligned} i\mathcal{M}^L_B \quad &\approx \quad -i\frac{G_F}{\sqrt{2}}V^*_{ud}(-(-ie)^2)\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2|\vec{k}|} \\ &\qquad \times\frac{e^{-|\vec{k}|t_s}}{E_{\pi,\vec{k}}+|\vec{k}|-M_\pi}\int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(-t_s,\vec{x})e^{-i\vec{k}\cdot\vec{x}}\tilde{L}^{(1)}_{\mu,\rho}(-i|\vec{k}|,-\vec{k}) \end{aligned} \tag{40}$$

$$i\mathcal{M}^L_D = -i\frac{G_F}{\sqrt{2}}V^*_{ud}\,(ie)\frac{e^{-|\vec{p}_\gamma|t_s}}{E_{\pi,\vec{p}_\gamma}+|\vec{p}_\gamma|-M_\pi}\int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(-t_s,\vec{x})e^{-i\vec{p}_\gamma\cdot\vec{x}}L^{(0)}_\mu\epsilon^*_{\lambda,\rho}(\vec{p}_\gamma) \tag{41}$$

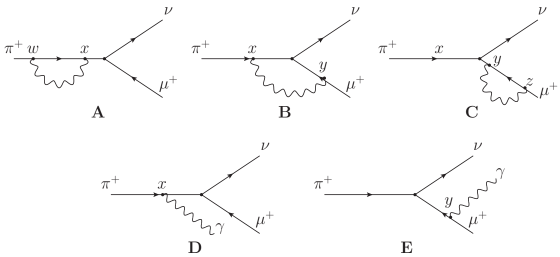


Use $e^{-i\vec{k}\cdot\vec{x}} = 1 + (e^{-i\vec{k}\cdot\vec{x}} - 1)$. The second term vanishes when $\vec{k} \to 0$ and its contribution is IR finite. Pick the IR divergence piece:

$$\int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(-t_s, \vec{x}) e^{-i\vec{k}\cdot\vec{x}} \to \int d^3\vec{x}\, H^{(1)}_{\mu,\rho}(-t_s, \vec{x}) = H^{(0)}_{\mu}\delta_{\rho,t} \tag{42}$$

$$i\mathcal{M}^{L,\text{div}}_{B} \approx -i\frac{G_F}{\sqrt{2}}V^*_{ud}(-(-ie)^2)\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2|\vec{k}|}\frac{e^{-|\vec{k}|t_s}}{E_{\pi,\vec{k}} + |\vec{k}| - M_\pi} H^{(0)}_{\mu}\tilde{L}^{(1)}_{\mu,t}(-i|\vec{k}|, -\vec{k}) \tag{43}$$

$$i\mathcal{M}^{L,\text{div}}_{D} = -i\frac{G_F}{\sqrt{2}}V^*_{ud}\,(ie)\frac{e^{-|\vec{p}_\gamma|t_s}}{E_{\pi,\vec{p}_\gamma} + |\vec{p}_\gamma| - M_\pi} H^{(0)}_{\mu} L^{(0)}_{\mu}\epsilon^*_{\lambda,t}(\vec{p}_\gamma) \tag{44}$$

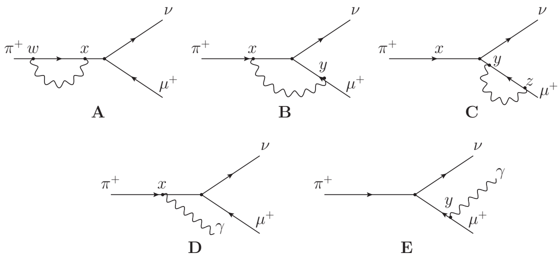


Combining diagram "T" and "B", we obtain

$$\Gamma_{TB}^{L,\text{div}} = \frac{1}{2M_{\pi,\text{phys}}}\int d\Phi_2(E_\pi, \vec{p}_\pi; \vec{p}_l, \vec{p}_\nu)\, 2\text{Re}\left[\mathcal{M}_T^\dagger \mathcal{M}_B^{L,\text{div}}\right] \tag{45}$$

$$\begin{aligned} &\approx \frac{1}{2M_{\pi,\text{phys}}}\int d\Phi_2(E_\pi, \vec{p}_\pi; \vec{p}_l, \vec{p}_\nu) \\ &\times\left|\frac{G_F}{\sqrt{2}}V_{ud}^*\right|^2(-(-ie)^2)H_\nu^{(0)\dagger}H_\mu^{(0)}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2|\vec{k}|} \\ &\times\frac{e^{-|\vec{k}|t_s}}{E_{\pi,\vec{k}}+|\vec{k}|-M_\pi}2\text{Re}\left[L_\nu^{(0)\dagger}\tilde{L}_{\mu,\rho}^{(1)}(-i|\vec{k}|,-\vec{k})\right] \end{aligned} \tag{46}$$

Infrared divergence cancellation “TB” and “DE”



$$\sum_{\lambda} \epsilon_{\lambda,\rho}(\vec{k})\epsilon_{\lambda,\rho'}{}^{*}(\vec{k}) \to \delta_{\rho,\rho'} - 2\delta_{\rho,t}\delta_{\rho',t} \tag{47}$$

Combining diagram “D” and “E” and use the above replacement, we obtain

$$\Gamma^{L,\text{div}}_{DE} = \frac{1}{2M_{\pi,\text{phys}}} \int d\Phi_3(E_\pi, \vec{p}_\pi; \vec{p}_l, \vec{p}_\nu, \vec{p}_\gamma)\, 2\text{Re}\left[\mathcal{M}_E^{\dagger}\mathcal{M}_D^{L,\text{div}}\right] \tag{48}$$

$$\begin{aligned} &\to \frac{1}{2M_{\pi,\text{phys}}} \int \frac{d^3\vec{p}_\gamma}{(2\pi)^3}\frac{1}{2|\vec{p}_\gamma|} \int d\Phi_2(E_\pi - |\vec{p}_\gamma|, \vec{p}_\pi - \vec{p}_\gamma; \vec{p}_l, \vec{p}_\nu) \\ &\quad \times \left|\frac{G_F}{\sqrt{2}}V^*_{ud}\right|^2 (-(ie)^2) H^{(0)\dagger}_{\mu} H^{(0)}_{\nu} \\ &\quad \times \frac{e^{-|\vec{p}_\gamma| t_s}}{E_{\pi,\vec{p}_\gamma} + |\vec{p}_\gamma| - M_\pi} 2\text{Re}\left[\tilde{L}^{(1)}_{\mu,t}(i|\vec{p}_\gamma|, \vec{p}_\gamma)^{\dagger} L^{(0)}_{\nu}\right] \end{aligned} \tag{49}$$

Finally, we verified that $\Gamma^{L,\text{div}}_{TB} + \Gamma^{L,\text{div}}_{DE}$ is IR finite.

Finally, we verify that $\Gamma^{L,\text{div}}_{TB} + \Gamma^{L,\text{div}}_{DE}$ is IR finite.

$$\begin{aligned}\Gamma^{L,\text{div}}_{TB} &\approx \frac{1}{2M_{\pi,\text{phys}}}\int d\Phi_2(E_\pi,\vec{p}_\pi;\vec{p}_l,\vec{p}_\nu) \\ &\quad\times\left|\frac{G_F}{\sqrt{2}}V^*_{ud}\right|^2(-(-ie)^2)H^{(0)\dagger}_\nu H^{(0)}_\mu\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2|\vec{k}|} \\ &\quad\times\frac{e^{-|\vec{k}|t_s}}{E_{\pi,\vec{k}}+|\vec{k}|-M_\pi}2\text{Re}\left[L^{(0)\dagger}_\nu\tilde{L}^{(1)}_{\mu,\rho}(-i|\vec{k}|,-\vec{k})\right] \end{aligned} \tag{50}$$

$$\begin{aligned}\Gamma^{L,\text{div}}_{DE} &= \frac{1}{2M_{\pi,\text{phys}}}\int\frac{d^3\vec{p}_\gamma}{(2\pi)^3}\frac{1}{2|\vec{p}_\gamma|}\int d\Phi_2(E_\pi-|\vec{p}_\gamma|,\vec{p}_\pi-\vec{p}_\gamma;\vec{p}_l,\vec{p}_\nu) \\ &\quad\times\left|\frac{G_F}{\sqrt{2}}V^*_{ud}\right|^2(-(ie)^2)H^{(0)\dagger}_\mu H^{(0)}_\nu \\ &\quad\times\frac{e^{-|\vec{p}_\gamma|t_s}}{E_{\pi,\vec{p}_\gamma}+|\vec{p}_\gamma|-M_\pi}2\text{Re}\left[\tilde{L}^{(1)}_{\mu,t}(i|\vec{p}_\gamma|,\vec{p}_\gamma)^\dagger L^{(0)}_\nu\right] \end{aligned} \tag{51}$$