Amplitudes Conference

Edinburgh 2017

# Loop-level BCJ and KLT relations

Oliver Schlotterer (AEI Potsdam)

based on 1612.00417 with S. He

and 1706.00640 with S. He, Y. Zhang

13.07.2017

# Different formulations of double copy

color ordered amplitudes

manifest gauge-/diffeo'invariance

cubic diagrams $\in \Gamma_{g\text{-loop}}$

manifest locality

KLT relations '86

$$M^{\text{tree}}_{\text{grav}} = A^{\text{tree}}_{\text{YM}} \otimes \tilde{A}^{\text{tree}}_{\text{YM}}$$

$\xrightarrow[\text{kin. Jacobis}]{\text{establish}}$

BCJ '08: YM-kinematics $n_i$

$$M^{\text{tree}}_{\text{grav}} = \sum_{i \in \Gamma_{\text{tree}}} \frac{n_i \, \tilde{n}_i}{\prod_{\text{edge } \alpha_i} k^2_{\alpha_i}}$$

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BCJ' 10: natural $g$-loop version

$$M_{\text{grav}}^{(g)} = \int \mathrm{d}^{g \cdot d} \ell \sum_{i \in \Gamma_g} \frac{n_i(\ell) \, \tilde{n}_i(\ell)}{\prod_{\alpha_i} k_{\alpha_i}^2(\ell)}$$

need Jacobi-satisfying $n_i(\ell)$

or **[see Radu's talk & 1701.02519]**

Universal to any number $d$ of spacetime dimensions & #(SUSY)!

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establish $\longrightarrow$ kin. Jacobis

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**this talk** **[He, OS 1612.00417]**

$$M_{\text{grav}}^{(1)} = \int \frac{\mathrm{d}^d \ell}{\ell^2} \, a_{\text{YM}}^{(1)}(\ell) \otimes \tilde{a}_{\text{YM}}^{(1)}(\ell)$$

**any** representation of $a_{\text{YM}}^{(1)}$

**but** ?? higher-loop KLT ??

rearrange $\longleftarrow$ propagators

? ?

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# KLT relations in gravity $\leftrightarrow$ BCJ relations in YM

Consistency condition of KLT formula $M_{\text{grav}}^{\text{tree}} = A_{\text{YM}}^{\text{tree}} \otimes \tilde{A}_{\text{YM}}^{\text{tree}}$:

BCJ amplitude relations among color-ordered $A_{\text{YM}}^{\text{tree}}(\rho(1,2,\ldots,n))$

$\Longrightarrow$ $(n-3)!$ linearly independent choices of $\rho \in S_n$

**[Piotr Tourkine's talk; Bern, Carrasco, Johansson 2008]**

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Loop-level KLT $M^{(1)}_{\text{grav}} = \int \frac{\mathrm{d}^d \ell}{\ell^2}\, a^{(1)}_{\text{YM}}(\ell) \otimes \tilde{a}^{(1)}_{\text{YM}}(\ell)$ requires ...

... loop-lv BCJ relations ... **[Boels, Isermann; Du, Luo; Primo, Torres-Bobadilla; Piotr's talk; Vanhove, Tourkine; Hohenegger, Stieberger]**

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... loop-lv BCJ relations ... **[Boels, Isermann; Du, Luo; Primo, Torres-Bobadilla; Piotr's talk; Vanhove, Tourkine; Hohenegger, Stieberger]**

... among "partial integrands" $a_{\rm YM}^{(1)}(\rho(\ldots))$ **[He, OS 1612.00417]**

$\Longrightarrow$ at most $(n-1)!$ linearly independent choices of $\rho$

(will see extra degeneracy that depends on supersymmetry)

# Einstein–Yang–Mills (EYM): coupling gauge theory to gravity

∃ numerous tree-level amplitude relations with the flavour of

$$A^{\text{tree}}_{\text{EYM}}\left(\begin{smallmatrix}\text{gauge}\\ \text{gravity}\end{smallmatrix}\right) = A^{\text{tree}}_{\text{YM}}\begin{pmatrix}\text{gauge}\\ \text{only!}\end{pmatrix} \otimes (\text{polarizations for extra gravitons})$$

**[Stieberger, Taylor; Nandan, Plefka, OS, Wen; de la Cruz, Kniss, Weinzierl; OS; Du, Teng, Wu; Du, Feng, Fu, Huang; Chiodaroli, Günaydin, Johansson, Roiban; Feng, Teng]**

Consequence of double-copy formulation EYM = YM ⊗ (YM ⊕ scalar)

**[Chiodaroli, Günaydin, Johansson, Roiban 1408.0764, 1703.00421]**

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Consequence of double-copy formulation EYM = YM ⊗ (YM ⊕ scalar)

**[Chiodaroli, Günaydin, Johansson, Roiban 1408.0764, 1703.00421]**

This talk: one-loop amplitude relations for gauge multiplets in the loop

$$A^{(1)}_{\text{EYM}} = \int \frac{\mathrm{d}^d \ell}{\ell^2} \left[a^{(1)}_{\text{YM}}(\ell) \otimes (\text{extra polarizations})\right] ,$$

also see recent all-loop relations for 1 external graviton

**[Chiodaroli, Günaydin, Johansson, Roiban 1703.00421]**

# Derivations: CHY formalism / ambitwistor strings

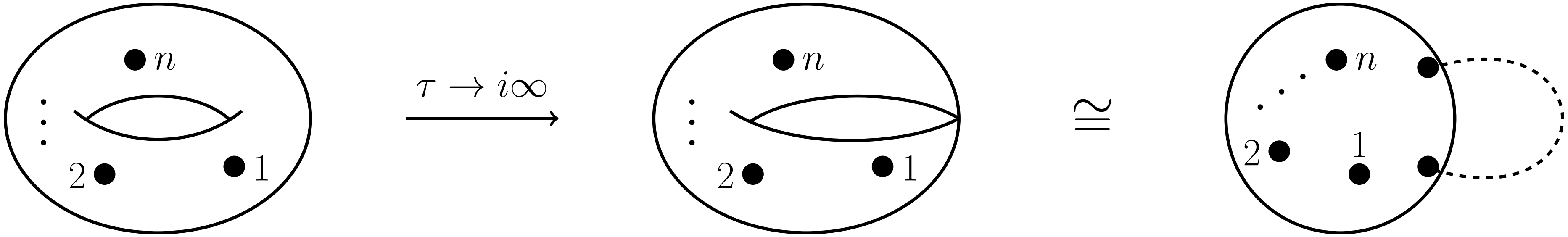
$\longrightarrow$ integrands share double-copy structure with superstrings

& new perspectives on the field-theory limit $\alpha' \to 0$

**[Cachazo, He, Yuan 1306.6575, 1307.2199, 1309.0885]**

**[Mason, Skinner 1311.2564; Adamo, Casali, Skinner 1312.3828]**

@ 1-loop: torus worldsheet $\longrightarrow$ nodal Riemann sphere




**[Lionel Mason's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]**

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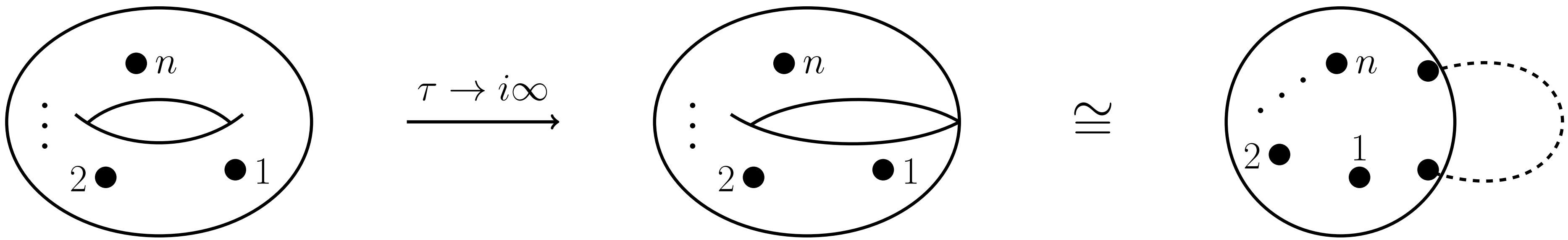


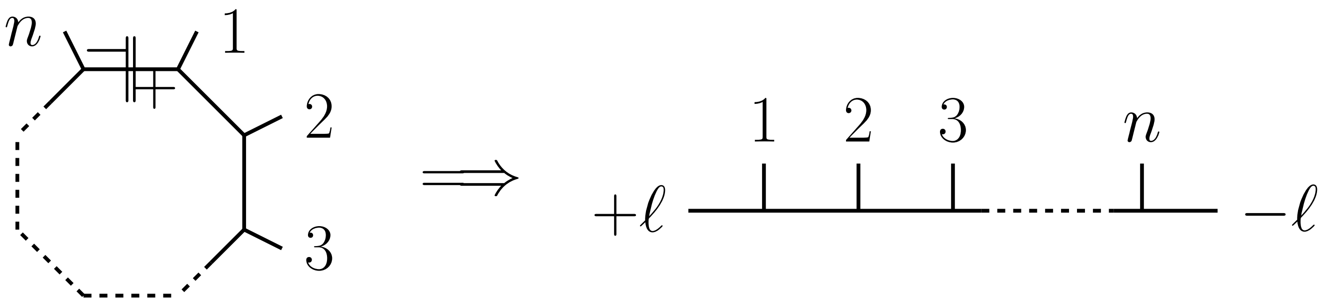

**[Lionel Mason's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]**

- natural framework to define partial integrand $a^{(1)}_{\rm YM}(\ldots)$
- one-loop BCJ relations from scattering equations
- one-loop KLT relations from double-copy structure of the integrand

**[He, OS, Zhang 1706.00640]**

# Outline

I. Partial integrands and one-loop BCJ relations



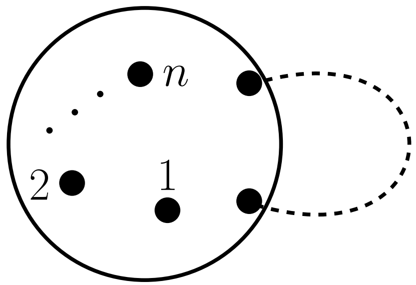

$$+\ \mathrm{cyclic}(1,2,\ldots,n)$$

II. One-loop KLT formula and EYM relations

$$\sum_{\rho,\tau\in S_{n-1}} a(+,\rho(1,2,\ldots,n-1),n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau(1,2,\ldots,n-1),-,n)$$

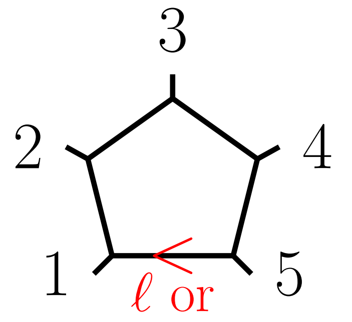
III. CHY derivation of one-loop amplitude relations

$$a(\tau(1,2,\ldots,n,+,-)) \;=\; \int \mathrm{d}\mu_{n+2}^{\mathrm{tree}}$$



# I. 1 New representations of Feynman integrals

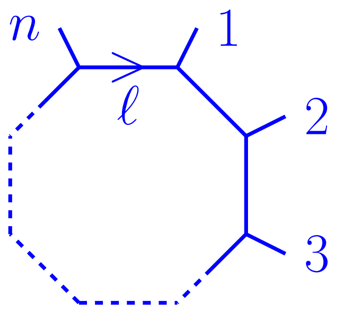
**Ambiguity:** which edge carries the loop momentum?




→ particularly confusing for $\ell$-dependent numerator $n_i(\ell)$

**Resolution:** Canonicalize $\ell$-dependence via

partial fraction (PF) $\Longrightarrow$ democratic sum over positions of $\ell$




$$= \int \frac{d^d\ell}{\ell^2\,(\ell+k_1)^2\,(\ell+k_{12})^2\,\ldots\,(\ell+k_{12\ldots n-1})^2}$$

$$= \sum_{i=0}^{n-1} \int \frac{d^d\ell}{(\ell+k_{12\ldots i})^2} \prod_{\substack{j=0\\ j\neq i}}^{n-1} \frac{1}{(\ell+k_{12\ldots j})^2 - (\ell+k_{12\ldots i})^2}$$

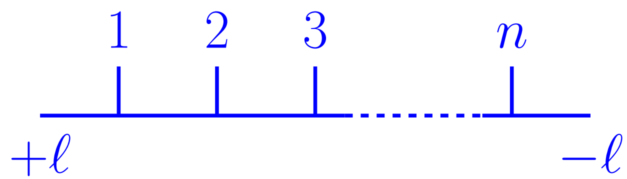
linear in $\ell$

PF

$$= \int \frac{d^d\ell}{\ell^2} \sum_{i=0}^{n-1} \prod_{j=0}^{i-1} \frac{1}{s_{j+1,j+2,\ldots,i,-\ell}} \prod_{j=i+1}^{n-1} \frac{1}{s_{i+1,i+2,\ldots,j,+\ell}}$$

shift $\ell$

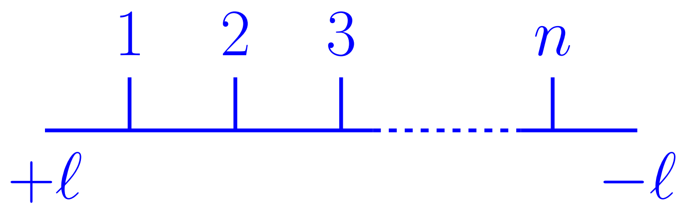
$$= \int \frac{d^d\ell}{\ell^2}$$




$+ \text{cyclic}(1,2,\ldots,n)$

[Lionel's talk & GMMT 1507.00321; Q-cuts: Emil's talk & BBBCDF 1509.02169]

## I. 2 Partial integrands




Each term in partial-fractionized $n$-gon

has a notion of cyclic ordering in $n$+2 legs $(1, 2, \ldots, n, -\ell, +\ell)$.

Decompose single-trace subamplitudes into $n$ "partial integrands"

$$A^{(1)}(1, 2, \ldots, n) \;=\; \int \frac{\mathrm{d}^d \ell}{\ell^2} \sum_{i=1}^{n} \underbrace{a(1, 2, \ldots, i, -, +, i{+}1, \ldots, n)}$$

- all propagators linear in $\ell$
- only those diagrams compatible with cycle $(1, 2, \ldots, i, -, +, i{+}1, \ldots, n)$

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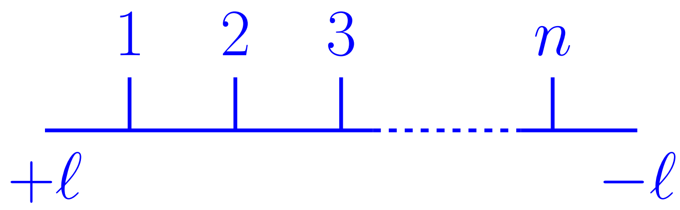


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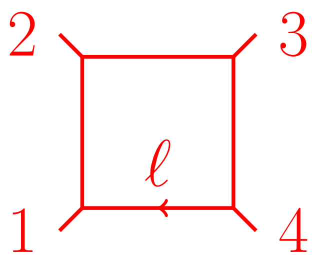
- all propagators linear in $\ell$
- only those diagrams compatible with cycle $(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$

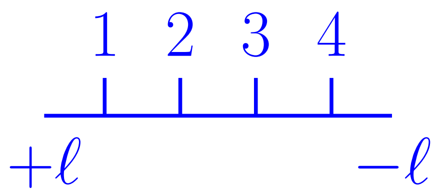
e.g. 4pt @ max. SUSY:

$$A^{(1)}_{\rm max}(1,2,3,4) = n_{\rm box} \times$$




cyc. orbit of

$$\Longrightarrow \ a_{\rm max}(1,2,3,4,-,+) = \frac{n_{\rm box}}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} \longleftrightarrow$$



Decompose single-trace subamplitudes into $n$ "partial integrands"

$$A^{(1)}(1,2,\ldots,n) \;=\; \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=1}^{n} a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$$

Properties of partial integrand $a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$

- all propagators made linear in $\ell$ via partial fraction
- only those diagrams compatible with cycle $(1,2,\ldots,i,-,+,i{+}1,\ldots,n)$
- decomposition is universal $\forall$ # spacetime dim's $d$ and supercharges
- can be viewed as forward limit of color-ordered $(n{+}2)$-point tree

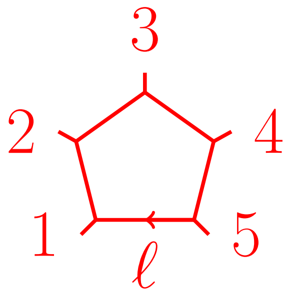
  **[Cachazo, He, Yuan 1512.05001]**

- each partial integrand is separately **gauge invariant**

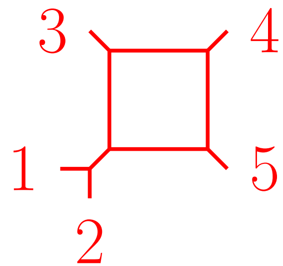
# I. 3 Five-point example @ maximal SUSY

By the no-triangle property, one pentagon & five boxes per single-trace

$$A^{(1)}_{\text{max}}(1,2,3,4,5) \; = $$



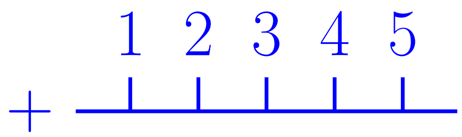

$$+ \Big\{$$




$$+ \,\text{cyc}(1,2,3,4,5) \Big\}$$

Partial integrands only have 4 out of 5 boxes (no massive corner with $k_{51}$):

$$a_{\text{max}}(1,2,3,4,5,-,+) \; = $$

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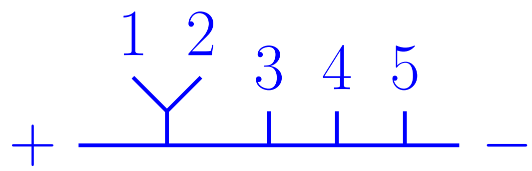


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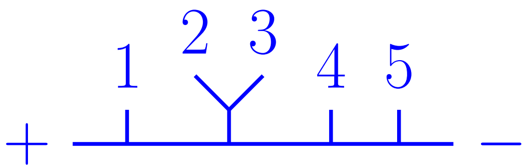


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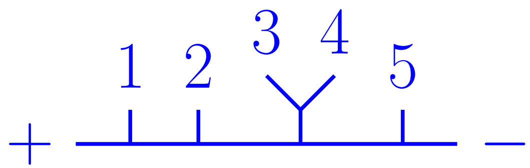


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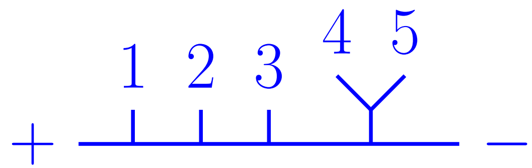



Individual numerators are gauge dependent,

e.g. they change under linearized gauge trf. $e_i \to k_i$.

**cf. 5pt BCJ numerators in pure-spinor superspace [Mafra, OS 1410.0668]**

## Explicit five-point numerators for external gluons via generalized $t_8$-tensor

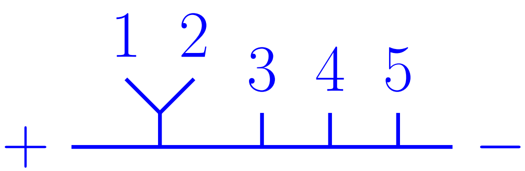
$$t_8(A,B,C,D) \;=\; \mathrm{tr}(f_A f_B f_C f_D) - \frac{1}{4}\,\mathrm{tr}(f_A f_B)\,\mathrm{tr}(f_C f_D) + \mathrm{cyc}(B,C,D)$$

with linearized field-strength $f_1^{mn} = k_1^m e_1^n - k_1^n e_1^m$ & two-particle current

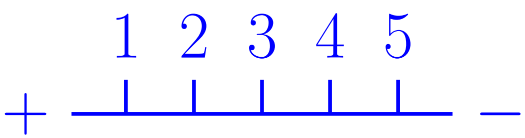
$$e_{12}^m \;=\; e_2^m\,(k_2\cdot e_1) + \tfrac{1}{2}\,k_1^m\,(e_1\cdot e_2) \;-\; (1\leftrightarrow 2)$$

$$f_{12}^{mn} \;=\; k_{12}^m\,e_{12}^n \;-\; s_{12}\,e_1^m\,e_2^n \;-\; (m\leftrightarrow n)$$

Gauge trf. $t_8(12,3,4,5)\big|_{e_1\to k_1} = s_{12}\,t_8(2,3,4,5)$ cancels between diag's




$$\longleftrightarrow \quad \frac{t_8(21,3,4,5)}{s_{12}\;\; s_{12,\ell}\, s_{123,\ell}\, s_{1234,\ell}}$$




$$\longleftrightarrow \quad \frac{\ell_m\Big[e_1^m\,t_8(2,3,4,5) + (1\leftrightarrow 2,3,4,5)\Big]}{s_{1,\ell}\,s_{12,\ell}\,s_{123,\ell}\,s_{1234,\ell}}$$

$$-\frac{1}{2}\,\frac{\Big[t_8(12,3,4,5) + (12\leftrightarrow 13,14,\ldots,45)\Big]}{s_{1,\ell}\,s_{12,\ell}\,s_{123,\ell}\,s_{1234,\ell}}$$

# I. 4 Amplitude relations of partial integrands

From intuition partial integrands $\leftrightarrow$ forward limit of $(n+2)$-point trees

$$a(1,2,\ldots,n,-,+) \;=\; \sum_{\text{states } \phi^*,\phi} A^{\text{tree}}(1,2,\ldots,n,\phi^*_{-\ell},\phi_\ell)$$

can export tree-level amplitude relations.

- KK-relations $\Rightarrow$ $(n{-}2)!$-“basis” $\{A^{\text{tree}}(1,\rho(2,\ldots,n{-}1),n),\ \rho\in S_{n-2}\}$

$$\Longrightarrow \text{ non-planar } a(\ldots,+,\ldots,-) \text{ determined by } a(\ldots,-,+)$$

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$\Longrightarrow$ non-planar $a(\ldots,+,\ldots,-)$ determined by $a(\ldots,-,+)$

- from BCJ-relations $0=\sum_{j=2}^{n-1}(k_1\cdot k_{23\ldots j})A^{\text{tree}}(2,3,\ldots,j,1,j+1,\ldots,n)$

$$\boxed{0 \;=\; \sum_{i=1}^{n-1}(\ell\cdot k_{12\ldots i})\; a(1,2,\ldots,i,+,i+1,\ldots,n,-)}$$

at most $(n-1)!$ independent $a(\ldots)$, with SUSY-dependent degeneracy

# II. 1 KLT amplitude relations

1st double-copy formula: tree-level KLT relations **[Kawai, Lewellen, Tye 1986]**

$$M^{\text{tree}}_{\text{grav}} = \sum_{\rho,\tau\in S_{n-3}} A^{\text{tree}}(1,\rho(2,3,\ldots,n-2),n,n-1)$$

$$\times\, S[\rho\,|\,\tau]_1\, \tilde{A}^{\text{tree}}(1,\tau(2,3,\ldots,n-2),n-1,n)$$

kernel $S[\rho|\tau]_1 \sim s_{ij}^{n-3}$ recursively determined by $S[j|j]_i = k_i \cdot k_j$ and

$$S[A,j\,|\,B,j,C]_i \;=\; k_j\cdot(k_i{+}k_B)\,S[A\,|\,\underbrace{B,C}]_i$$

$A$: $a_1, a_2, \ldots, a_x$ $\qquad$ $B,C$: $b_1, b_2, \ldots, b_y, c_1, \ldots, c_z$

**[Bern, Dixon, Perelstein, Rozowsky 1998]**

**[Bjerrum-Bohr, Damgaard, Feng, Sondergaard, Vanhove 2010]**

$M^{\text{tree}}_{\text{grav}}$ is "secretly" permutation invariant by BCJ relations among $A^{\text{tree}}(\ldots)$.

Identify $\sum_{\text{states}} A^{\text{tree}}(\ldots, \phi^*_{-\ell}, \ldots, \phi_\ell) \leftrightarrow a(\ldots, -, \ldots, +)$

in both gauge-theory copies of $(n+2)$-point tree-level KLT formula

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d \ell}{\ell^2} \sum_{\rho,\tau \in S_{n-1}} a(+, \rho, n, -)\, S[\rho \,|\, \tau]_\ell\, \tilde{a}(+, \tau, -, n)$$

with e.g. $\rho = \rho(1, 2 \ldots, n-1)$, same for $\tau$ and tree-level kernel $S[\rho \,|\, \tau]_\ell$.

Identify $\sum_{\text{states}} A^{\text{tree}}(\ldots,\phi^*_{-\ell},\ldots,\phi_\ell) \leftrightarrow a(\ldots,-,\ldots,+)$

in both gauge-theory copies of $(n{+}2)$-point tree-level KLT formula

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_{n-1}} a(+,\rho,n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau,-,n)$$

with e.g. $\rho=\rho(1,2\ldots,n{-}1)$, same for $\tau$ and tree-level kernel $S[\rho\,|\,\tau]_\ell$.

e.g. 4pt supergravity amplitude @ maximal SUSY

$$M^{\text{1-loop}}_{\text{max}} = \big|t_8(1,2,3,4)\big|^2 \int \frac{\mathrm{d}^d\ell}{\ell^2} \left\{ \frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} + \text{perm}(1,2,3,4) \right\}$$

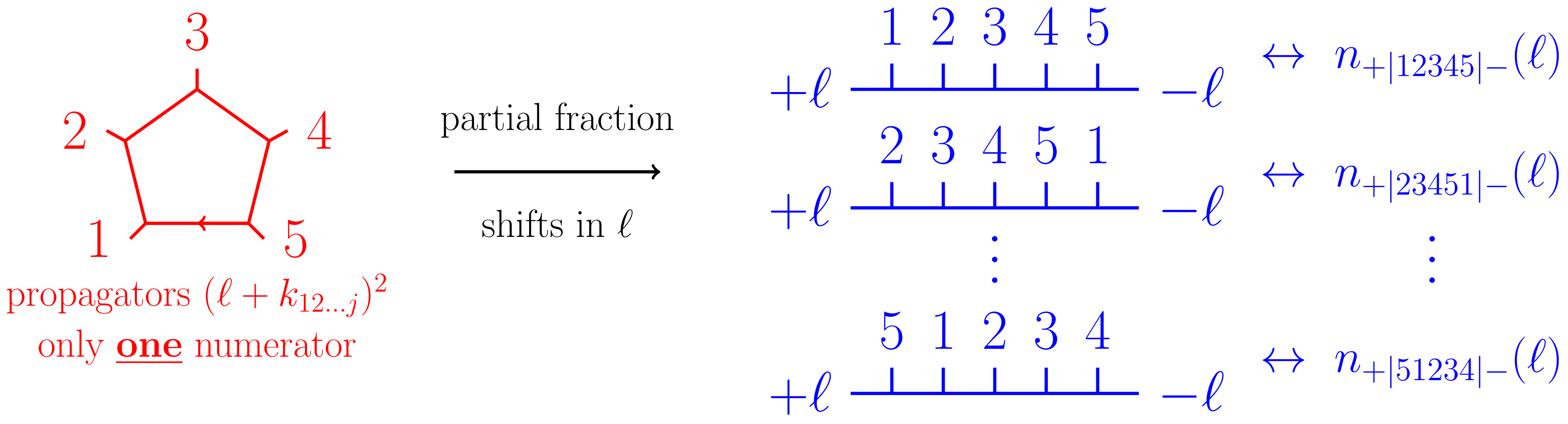
is reproduced from KLT formula

$$\int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau\in S_3} a(+,\rho(1,2,3),4,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau(1,2,3),-,4)$$

with $a(+,1,2,3,4,-)=\frac{t_8(1,2,3,4)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}$ & 4-term expression for $a(+,1,2,3,-,4)$.

# II. 2 Cubic-graph equivalent of one-loop KLT

Rearranged Feynman integrals $\Rightarrow$ more cubic diagrams, e.g.



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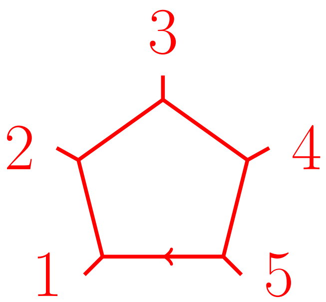
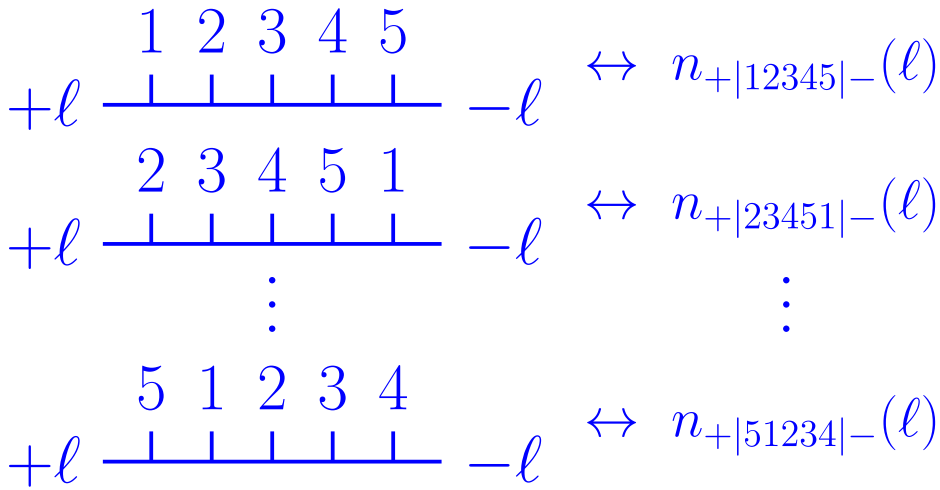



propagators $(\ell + k_{12\ldots j})^2$
only **one** numerator

partial fraction $\longrightarrow$ shifts in $\ell$




One-loop KLT $\Leftrightarrow$ separate squaring of the 5 numerators on the right

$$M^{\text{1-loop}}_{\text{grav}} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{\rho,\tau \in S_{n-1}} a(+,\rho,n,-)\, S[\rho\,|\,\tau]_\ell\, \tilde{a}(+,\tau,-,n)$$

$$= \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i \in \Gamma^{n+2}_{\text{tree}}} \frac{n_i(\ell)\, \tilde{n}_i(\ell)}{\prod_{\text{edge } \alpha_i} s_{\alpha_i}(\ell)} \quad \begin{array}{l} \longleftarrow \text{ kin. Jacobi rel's} \\ \longleftarrow \text{ linear in } \ell \end{array}$$

$\exists$ all-multiplicity method for BCJ numerators **[He, OS, Zhang 1706.00640]**

# II. 3 EYM relations at one loop

E.g. $n$ external gluons, one ext. graviton $\{e_p, p\}$ & internal gauge states:

notion of partial integrand carries over to EYM:

$$A^{(1)}_{\text{EYM}}(1,2,\ldots,n\,|\,p) = \int \frac{\mathrm{d}^d\ell}{\ell^2} \sum_{i=1}^{n} \underbrace{a_{\text{EYM}}(1,2,\ldots,i,-,+,i{+}1,\ldots,n\,|\,p)}$$

- all propagators linear in $\ell$
- forward limit of $A^{\text{tree}}_{\text{EYM}}(\ldots\,|\,p)$

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- all propagators linear in $\ell$
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Recycle tree-level relations such as

$$A^{\rm tree}_{\rm EYM}(1,2,\ldots,n\,|\,p) \ = \ \sum_{j=1}^{n-1} (e_p \cdot k_{12\ldots j})\, A^{\rm tree}(1,2,\ldots,j,p,j{+}1,\ldots,n)$$

**[Stieberger, Taylor 1606.09616]**

$\exists$ analogous relations for multiple ext. gravitons and color traces.

**[Nandan, Plefka, OS, Wen; OS; Du, Feng, Fu, Huang; Chiodaroli, Günaydin, Johansson, Roiban; Feng, Teng]**

One-loop amplitude relation among partial integrands via forward limit

$$A^{\text{tree}}_{\text{EYM}}(1,2,\ldots,n\,|\,p) \;=\; \sum_{j=1}^{n-1}(e_p\cdot k_{12\ldots j})\,A^{\text{tree}}(1,2,\ldots,j,p,j{+}1,\ldots,n)$$

$$\Downarrow$$

$$a_{\text{EYM}}(+,1,2,\ldots,n,-\,|\,p) \;=\; -\,(e_p\cdot\ell)\,a(+,1,2,\ldots,n,-,p) + \sum_{j=1}^{n-1}(e_p\cdot k_{12\ldots j})\,a(+,1,2,\ldots,j,p,j{+}1,\ldots,n,-)$$

Gauge invariance under $e_p\to p$ follows from BCJ relations among $a(\ldots)$.

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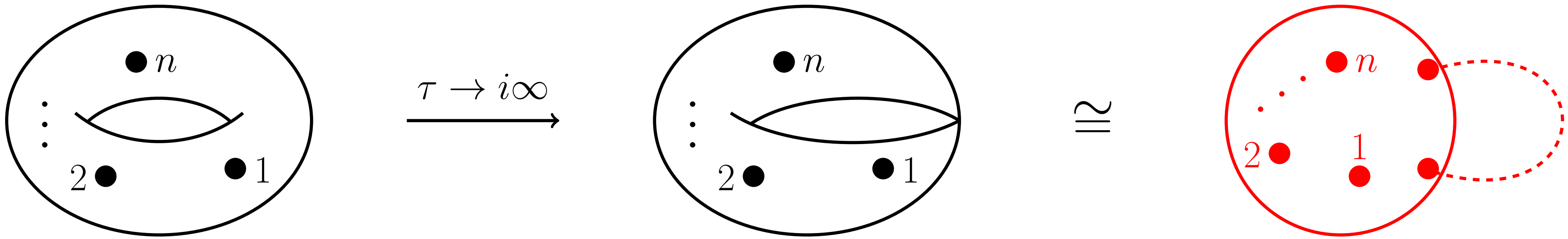
Example for recombination to Feynman integral: $n=3$ @ max. SUSY:

$$A^{(1),\text{max}}_{\text{EYM}}(1,2,3;p) \;=\; t_8(1,2,3,p) \times \int \frac{\mathrm{d}^d\ell \;\; (e_p\cdot\ell)}{\ell^2\,(\ell{+}k_{123})^2\,(\ell{+}k_{23})^2\,(\ell{+}k_3)^2} + \text{cyc}(1,2,3)$$

# III. CHY derivation of one-loop amplitude relations

Ambitwistor 1-loop prescription can be localized on nodal Riemann spheres




[Lionel Mason's talk; Geyer, Mason, Monteiro, Tourkine 1507.00321 & 1511.06315]

Integrands factorize into halves "$L$ & $R$" for color or kinematics

$$M^{(1)}_{L\otimes R} = \int \frac{d^d\ell}{\ell^2} \int \prod_{i=2}^{n} d\sigma_i \, \delta\left( \frac{\ell\cdot k_i}{\sigma_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i\cdot k_j}{\sigma_{ij}} \right) \widehat{I}_L(\ell)\, \widehat{I}_R(\ell)$$

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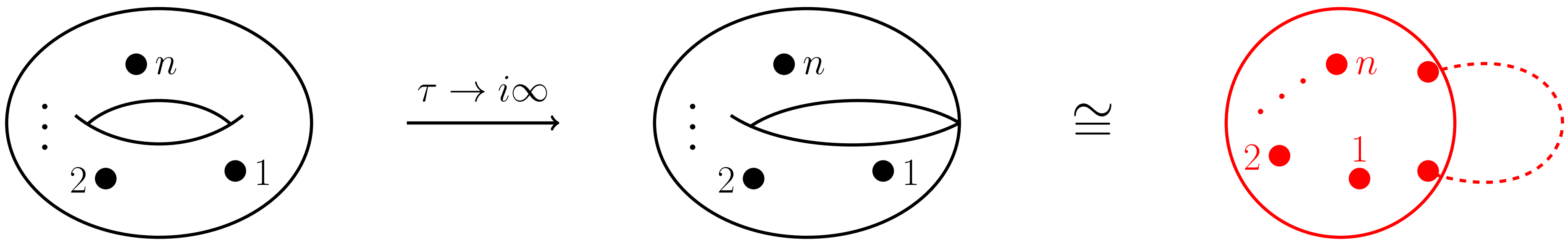


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Recognize as $\mathrm{SL}_2$-fixed expression $(\sigma_+, \sigma_-, \sigma_1) = (0, \infty, 1)$ with $k_\pm = \pm\ell$

$$M^{(1)}_{L\otimes R} = \int \frac{\mathrm{d}^d\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \int \mathrm{d}\mu^{\mathrm{tree}}_{n+2}\, I_L(\ell)\, I_R(\ell)$$

with tree-level measure $\mathrm{d}\mu^{\mathrm{tree}}_{n+2}$ of the CHY formalism.

1-loop amplitudes of ambitwistor string $\leftrightarrow$ tree-level CHY measure $\mathrm{d}\mu_{n+2}^{\text{tree}}$

$$M_{L\otimes R}^{(1)} = \int \frac{\mathrm{d}^d \ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int \mathrm{d}\mu_{n+2}^{\text{tree}} \, I_L(\ell) \, I_R(\ell)$$

- tree-level KLT formula encoded in factorization of integrand $I_L(\ell) \, I_R(\ell)$

  and properties of $\mathrm{d}\mu_{n+2}^{\text{tree}}$ (yields doubly partial amplitudes of $\phi^3$ thy.)

  **[Lionel Mason's talk; Cachazo, He, Yuan 1306.6575, 1307.2199, 1309.0885]**

- for supersymmetric integrands $I_{L,R}(\ell) \to I_{\text{SUSY}}(\ell)$ ($\geq$ 4 supercharges),

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  $\Rightarrow$ proof of tree-level KLT carries over to one loop!

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  **[He, OS, Zhang 1706.00640]**

- explicit BCJ master numerators for $n$-gons: expand $I_{L,R}(\ell)$ in terms of

  $n!$ Parke–Taylor factors $\{\text{PT}(-, \rho(1,2,\ldots,n), +) \, , \ \rho \in S_n\}$

# Conclusions & Outlook

- partial-fraction representation of Feynman integrals identifies

  partial integrands $a(\ldots)$ as gauge-invariant seeds @ one loop

- $a(\ldots) \cong$ forward limit of trees $\Rightarrow$ natural one-loop uplift

  of tree-level BCJ, KLT and EYM amplitude relations

- CHY formalism $\Rightarrow$ concise derivation of one-loop amplitude relations

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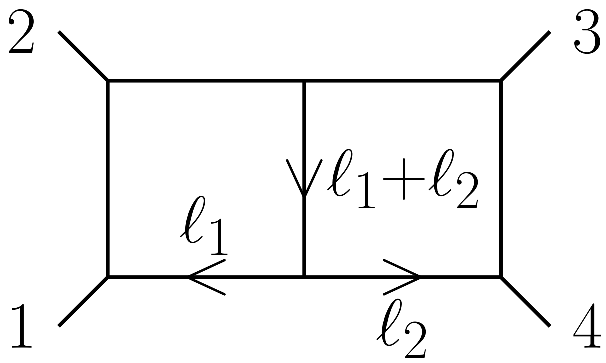
- CHY formalism $\Rightarrow$ concise derivation of one-loop amplitude relations

- How to systematically undo partial-fractioning of Feynman integrals?

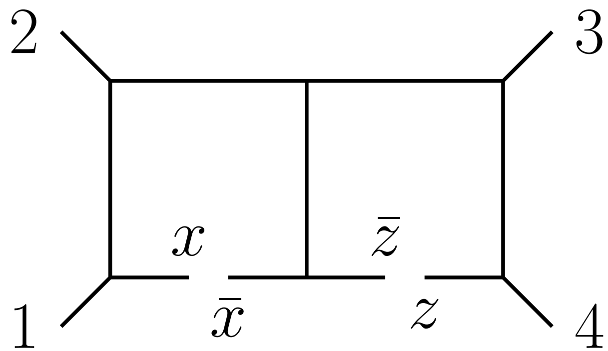
- Desirable to develop integration techniques for propagators linear in $\ell$.

**[Arkani-Hamed, Yuan: in progress]**

- need suitable higher-loop partial integrands $a^{(g)}$ for $g$-loop KLT rel's




$A^{(2)}(1,2,3,4)$




$a^{(2)}(1,2,3,4,z,\bar{z},\bar{x},x)$

$$\stackrel{?}{\Longrightarrow} \quad M^{(g)}_{\text{grav}} = \prod_{j=1}^{g} \int \frac{\mathrm{d}^d \ell_j}{\ell_j^2} \sum_{\rho,\tau\in\ldots} a^{(g)}(\ldots)\, S[\ldots|\ldots]\, \tilde{a}^{(g)}(\ldots)$$

**starting points in [Geyer, Mason, Monteiro, Tourkine 2016; He, OS, Zhang 2017]**

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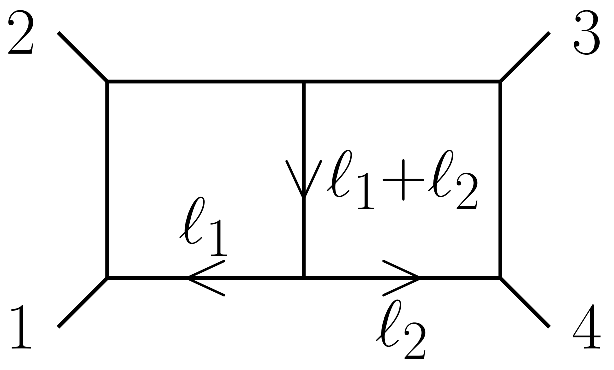


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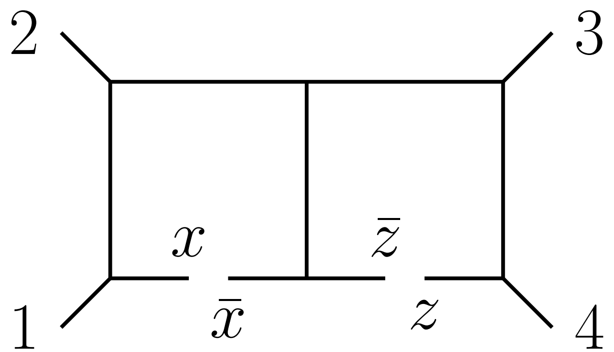


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- loop-level KLT in string theory? connections with monodromy rel's?

**[Piotr's talk; Vanhove, Tourkine 1608.01665; Hohenegger, Stieberger 1702.04963]**

**Thank you for your attention !**