Soft Photon and Graviton Theorems in Effective Field Theory

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Based on work with Henriette Elvang and Callum R. T. Jones (UM)
[arXiv:1611.07534, PRL 118 (2017) 231601]

SOFT PHOTON THEOREMS

Consider the amplitude for n particles and a photon:

$$A_{n+1}^{\text{ph}}(p_1, \cdots, p_n, \epsilon p_s)$$

When the photon goes soft ($\epsilon \to 0$), the amplitude approaches

$$A_{n+1}^{\text{ph}} = \left[\frac{S^{(0)}}{\epsilon} + S^{(1)}\right] A_n + O(\epsilon)$$

where A_n is the amplitude without the photon and

$$S^{(0)} = \sum_{k=1}^{n} Q_k \frac{\varepsilon_s \cdot p_k}{p_s \cdot p_k}, \qquad S^{(1)} = \sum_{k=1}^{n} Q_k \frac{\varepsilon_s^{\mu} p_s^{\nu} J_{\mu\nu}^{(k)}}{p_s \cdot p_k}$$

The subleading $\mathcal{O}(\epsilon^0)$ contribution is fixed by the leading contribution and gauge invariance

[Low 1954, Gell-Mann and Goldberger 1954, Weinberg 1965, Burnett and Kroll 1968]

SOFT GRAVITON THEOREMS

Consider the amplitude for n particles and a graviton:

$$A_{n+1}^{\text{grav}}(p_1, \cdots, p_n, \epsilon p_s)$$

When the graviton goes soft ($\epsilon \to 0$), the amplitude approaches

$$A_{n+1}^{\text{grav}} = \left[\frac{\mathcal{S}^{(0)}}{\epsilon} + \mathcal{S}^{(1)} + \epsilon\, \mathcal{S}^{(2)}\right] A_n + O(\epsilon^2)$$

where A_n is the amplitude without the graviton and

$$\mathcal{S}^{(0)} = \kappa \sum_{k=1}^{n} \frac{\varepsilon_s^{\mu\nu}(p_k)_\mu (p_k)_\nu}{p_s \cdot p_k}\,, \qquad \mathcal{S}^{(1)} = \kappa \sum_{k=1}^{n} \frac{\varepsilon_s^{\mu\nu}(p_k)_\mu (p_s)^\rho J_{\nu\rho}^{(k)}}{p_s \cdot p_k}\,,$$

$$\mathcal{S}^{(2)} = \kappa \sum_{k=1}^{n} \frac{\varepsilon_s^{\mu\nu}(p_s)^\rho (p_s)^\sigma J_{\mu\rho}^{(k)} J_{\nu\sigma}^{(k)}}{p_s \cdot p_k}$$

The leading $\mathcal{O}(1/\epsilon)$ contribution is due to [Weinberg 1965] .
The subleading $\mathcal{O}(\epsilon^0)$ [Gross Jackiw 1968] and subsubleading $\mathcal{O}(\epsilon)$ [Cachazo Strominger 1404.4091, Broedel deLeeuw Plefka Rosso 1406.6574, Bern Davies DiVecchia Nohle 1406.6987] contributions are fixed by the leading contribution and general coordinate invariance (tree level).

HOLOMORPHIC SOFT LIMIT: PHOTONS

Restrict to four dimensions and massless particles $\Longrightarrow$ $p_k = -|k\rangle[k|$.

Introduce *holomorphic soft limit* [Cachazo and Strominger 1404.4091]

$$p_s \to \epsilon p_s \qquad |s\rangle \to \epsilon|s\rangle, \qquad |s] \to |s], \qquad \epsilon \to 0$$

Use BCFW to derive the soft limit for positive-helicity photons

$$A_{n+1}^{\text{ph}} = \left[\frac{S^{(0)}}{\epsilon^2} + \frac{S^{(1)}}{\epsilon}\right] A_n + O(\epsilon^0)$$

with all divergent terms coming from a small subset of diagrams.

The soft operators become, in spinor-helicity language,

$$S^{(0)} = \sum_k Q_k \frac{\langle xk\rangle}{\langle xs\rangle\langle sk\rangle}\,, \qquad S^{(1)} = \sum_k Q_k \frac{D_{sk}}{\langle sk\rangle}$$

where $D_{sk} \equiv |s]\partial_{|k]}$ and $|x\rangle$ = arbitrary reference spinor.

HOLOMORPHIC SOFT LIMIT: GRAVITONS

Similarly, for a positive-helicity graviton,

$$A_{n+1}^{\text{grav}} = \left[\frac{\mathcal{S}^{(0)}}{\epsilon^3} + \frac{\mathcal{S}^{(1)}}{\epsilon^2} + \frac{\mathcal{S}^{(2)}}{\epsilon}\right] A_n + O(\epsilon^0)$$

where all divergent terms come from a small subset of diagrams and

$$\mathcal{S}^{(0)} = \kappa \sum_k \frac{[sk]\langle xk\rangle\langle yk\rangle}{\langle sk\rangle\langle xs\rangle\langle ys\rangle} ,$$

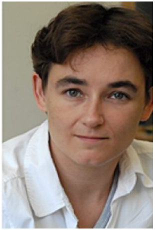
$$\mathcal{S}^{(1)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk\rangle}\left(\frac{\langle xk\rangle}{\langle xs\rangle} + \frac{\langle yk\rangle}{\langle ys\rangle}\right) D_{sk} ,$$

$$\mathcal{S}^{(2)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk\rangle} D_{sk}^2$$

Cachazo and Strominger first identified the subsubleading term $\mathcal{S}^{(2)}$.

Soft graviton theorem derived from Ward identities associated with asymptotic (BMS) symmetries [Strominger lectures 1703.05448] .

SOFT THEOREMS IN EFFECTIVE FIELD THEORY



[Henriette Elvang, Callum R. T. Jones, SN,
1611.07534, PRL 118 (2017) 231601]

We address the effects of higher-dimension operators on the structure of soft theorems for photons and gravitons in four dimensions. These operators could arise in an effective field theory by integrating out massive particles in loops.

We use on-shell methods and employ a two-parameter holomorphic shift of the momenta.

Our findings:

- a very small subset of effective operators can modify the soft theorems, and they do so in a well-defined universal way
- a new derivation of the subleading soft theorems
- obtain constraints on allowed three-particle interactions involving massless particles

MODIFIED SOFT PHOTON THEOREM

In particular, we find that higher-dimension operators modify the soft photon theorem to

$$A^{\text{ph}}_{n+1} = \left[\frac{S^{(0)}}{\epsilon^2} + \frac{S^{(1)}}{\epsilon}\right] A_n + \frac{\tilde{S}^{(1)}}{\epsilon}\tilde{A}_n + O(\epsilon^0)$$

where the universal subleading correction is

$$\tilde{S}^{(1)}\tilde{A}_n = \sum_k g_k \frac{[sk]}{\langle sk\rangle}\tilde{A}^{(k)}_n$$

and the coupling g_k of the higher-dimension operator has mass dimension -1. The dimension-five operators that give rise to this modification are

$$\overline{\chi}\gamma^{\mu\nu}F_{\mu\nu}\chi\,,\quad \phi F_{\mu\nu}F^{\mu\nu}\,,\quad \phi F_{\mu\nu}\tilde{F}^{\mu\nu}\,,\quad \overline{\psi}_\mu F_{\nu\rho}\gamma^{\mu\nu\rho}\chi\,,\quad hF^2$$

The tilde signifies that the kth particle of $\tilde{A}^{(k)}_n$ differs from the kth particle of A^{ph}_{n+1}.

MODIFIED SOFT GRAVITON THEOREM

We find that higher-dimension operators modify the soft graviton theorem to

$$A_{n+1}^{\text{grav}} = \left[\frac{\mathcal{S}^{(0)}}{\epsilon^3} + \frac{\mathcal{S}^{(1)}}{\epsilon^2} + \frac{\mathcal{S}^{(2)}}{\epsilon}\right] A_n + \frac{\tilde{\mathcal{S}}^{(2)}}{\epsilon} \tilde{A}_n + O(\epsilon^0)$$

There is no modification at subleading $\mathcal{O}(1/\epsilon^2)$ order.

The universal subsubleading $\mathcal{O}(1/\epsilon)$ modification is

$$\tilde{\mathcal{S}}^{(2)} \tilde{A}_n = \sum_k g_k \frac{[sk]^3}{\langle sk \rangle} \tilde{A}_n^{(k)}$$

The coupling g_k of the higher-dimension operator has mass dimension -3. The dimension-seven operators that give rise to this modification are

$$\phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \,, \quad R^{\mu\nu\rho\sigma} \overline{\psi}_\rho \gamma_{\mu\nu} \partial_\sigma \chi \,, \quad R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

The modification due to the operator ϕR^2 was previously noted by [Bianchi He Huang Wen 1406.5155, Di Vecchia Marotta Mojaza 1604.03355]

Recent generalization to massive particles and arbitrary number of dimensions by [Laddha Sen 1706.00759]

ON-SHELL METHODS

Complexify the momenta

$$\hat{p}_k = p_k + z q_k$$

while preserving momentum conservation ($\sum_k q_k = 0$) and on-shellness of external momenta ($q_k \cdot p_k = q_k^2 = 0$).

Tree-level amplitude becomes a meromorphic function $\hat{A}(z)$ whose poles arise from diagrams in which intermediate momenta $\hat{P}_I$ go on-shell.

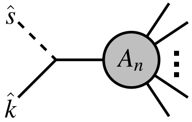
Complex analysis $\implies$ amplitude is a sum over poles

$$\hat{A}(z) = \sum \frac{R_I}{z - z_I} + A_\infty$$

Residues of the poles R_I are given by products of on-shell subamplitudes.

SOFT SHIFT

In the soft theorems, the divergent (in ϵ) behavior arises from the diagrams in which the soft particle is attached to an external leg



This can seen by performing a *soft shift* [Cheung Shen Trnka 1502.05057]

$$|\hat{s}\rangle = \epsilon|s\rangle, \qquad |\hat{i}] = |i] - \epsilon\frac{\langle js\rangle}{\langle ji\rangle}|s]\,, \qquad |\hat{j}] = |j] - \epsilon\frac{\langle is\rangle}{\langle ij\rangle}|s]$$

The momentum in the propagator satisfies

$$\hat{P}_{sk}^2 = (\hat{p}_k + \hat{p}_s)^2 = \epsilon[sk]\langle sk\rangle$$

and goes on-shell as $\epsilon \to 0$. This gives rise to *degenerate* poles at $\epsilon = 0$ in $\hat{A}_{n+1}$.

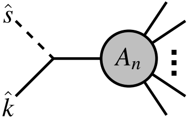
Solution: resolve these poles by introducing a second parameter z.

TWO-PARAMETER SHIFT

Combine a soft shift with BCFW shifts $[i,s\rangle$ and $[j,s\rangle$ (with parameters z_1 and z_2) to obtain (letting $z|X\rangle = z_1|i\rangle + z_2|j\rangle$)

$$|\hat{s}\rangle = \epsilon|s\rangle - z|X\rangle\,, \quad |\hat{i}] = |i] - \epsilon\frac{\langle js\rangle}{\langle ji\rangle}|s] + z\frac{\langle jX\rangle}{\langle ji\rangle}|s]\,, \quad |\hat{j}] = |j] - \epsilon\frac{\langle is\rangle}{\langle ij\rangle}|s] + z\frac{\langle iX\rangle}{\langle ij\rangle}|s]$$

with no other spinors shifted. Then

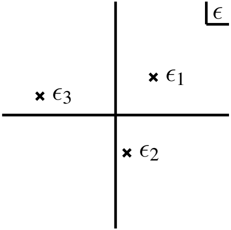


has intermediate momentum

$$\hat{P}_{sk}^2 = (\hat{p}_k + \hat{p}_s)^2 = (\epsilon - \epsilon_k)[sk]\langle sk\rangle\,, \qquad \epsilon_k = z\frac{\langle Xk\rangle}{\langle sk\rangle}$$

For fixed z, write amplitude as a sum over poles in the ϵ plane:

$$\hat{A}_{n+1}(z,\epsilon) = \sum_{k=1}^{n} \frac{R_k(z)}{\epsilon - \epsilon_k} + \text{other poles} + A_\infty$$



As $z \to 0$, the simple poles at $\epsilon = \epsilon_k = z\frac{\langle Xk\rangle}{\langle sk\rangle}$ coalesce to give higher-order poles at $\epsilon = 0$:

$$\sum_{k=1}^{n} \frac{R_k(z)}{\epsilon - \epsilon_k} \quad \longrightarrow \quad \frac{c_0}{\epsilon^{\mathtt{a}}} + \frac{c_1}{\epsilon^{\mathtt{a}-1}} + \cdots + \mathcal{O}(\epsilon^0)$$

The other poles are $\mathcal{O}(\epsilon^0)$ as $z \to 0$, so the unshifted amplitude is

$$A_{n+1} = \lim_{z\to 0} \quad \sum_{k=1}^{n} \frac{R_k(z)}{\epsilon - \epsilon_k} + \mathcal{O}(\epsilon^0)$$

CONSTRAINTS

The coalescence of simple poles into higher-order poles in ϵ

$$\sum_{k=1}^{n} \frac{R_k(z)}{\epsilon - \epsilon_k} \quad \longrightarrow \quad \frac{c_0}{\epsilon^{\mathtt{a}}} + \frac{c_1}{\epsilon^{\mathtt{a}-1}} + \cdots + \mathcal{O}(\epsilon^0)$$

requires individual residues $R_k(z)$ to have poles in z. Simple example:

$$\frac{1}{\epsilon^2 - z^2} = \left[\frac{(1/2z)}{\epsilon - z} - \frac{(1/2z)}{\epsilon + z} \right]$$

But the absence of z poles in the full amplitude imposes constraints on the residues, e.g.

$$\sum_{k=1}^{n} R_k(z) \bigg|_{1/z^{\mathtt{a}-1}} = 0$$

We will use these constraints to prove that $\mathtt{a} \leq 3$ for soft gravitons and $\mathtt{a} \leq 2$ for soft photons. This restricts possible three-point couplings.

MASTER FORMULA

Let's calculate the residues

$$\hat{A}_{n+1}(z,\epsilon) = \sum_{k,h_P} \hat{A}_3\left(\hat{s},\hat{k},\hat{P}_{sk}\right) \frac{1}{(\epsilon-\epsilon_k)[sk]\langle sk\rangle} \hat{A}_n^{(k)}(z) + O(\epsilon^0)$$

The three-point amplitude $\hat{A}_3[h_s, h_k, h_P]$ is fixed by little-group scaling

$$\hat{A}_3 \quad \propto \quad [s\hat{k}]^{h_s+h_k-h_P}[\hat{k}\hat{P}_{sk}]^{h_k+h_P-h_s}[\hat{P}_{sk}s]^{h_P+h_s-h_k}$$

Evaluating the shifted spinors at the poles $\epsilon = \epsilon_k$, we obtain

$$\hat{A}_3 = g_{k,\mathtt{a}}\,[sk]^{2h_s+1-\mathtt{a}} \left(\frac{\langle sk\rangle}{z\langle Xs\rangle}\right)^{\mathtt{a}-1} , \qquad \mathtt{a} \equiv h_s - h_k - h_P + 1$$

Thus we obtain our master formula

$$\boxed{\hat{A}_{n+1}(z,\epsilon) = \sum_{k,\mathtt{a}} g_{k,\mathtt{a}} \frac{[sk]^{2h_s-\mathtt{a}}\langle sk\rangle^{\mathtt{a}-2}\langle Xs\rangle^{1-\mathtt{a}}}{z^{\mathtt{a}-1}\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \hat{A}_n^{(k)}(z) + O(\epsilon^0)}$$

Sum over all external legs k and all possible three-point couplings $g_{k,\mathtt{a}}$. The dimension of the effective coupling g_k is $[g_{k,\mathtt{a}}] = \mathtt{a} - 2h_s$.

CONSTRAINTS FROM ABSENCE OF z POLES

Note that individual terms in the expression

$$\hat{A}_{n+1}(z,\epsilon) = \sum_{k,\mathtt{a}} g_{k,\mathtt{a}} \frac{[sk]^{2h_s-\mathtt{a}} \langle sk\rangle^{\mathtt{a}-2} \langle Xs\rangle^{1-\mathtt{a}}}{z^{\mathtt{a}-1}\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \hat{A}_n^{(k)}(z) + O(\epsilon^0)$$

have z poles. In a Laurent expansion in z, the coefficients of $1/z^{\mathtt{a}-1}$, $1/z^{\mathtt{a}-2}$, etc. must vanish. This condition leads to constraints on allowed three-particle effective operators. Specifically:

- There are no allowed couplings with $\mathtt{a} > 3$. This prohibits gravitational couplings to particles with $|h| > 2$.
- Photons can only have couplings with $\mathtt{a} = 2$ (standard soft photon theorem) or $\mathtt{a} = 1$ (subleading modification).
- Gravitons can only have couplings with $\mathtt{a} = 3$ (standard soft graviton theoreom) or $\mathtt{a} = 1$ (subsubleading modification). Absence of $\mathtt{a} = 2$ couplings implies no subleading modifications.
- Couplings with $\mathtt{a} \leq 0$ may also be allowed but do not contribute to the soft theorems (all terms vanish as $z \to 0$).

$a = 2$ FOR $h_s = 1 \implies$ SOFT PHOTON THEOREM

For photons ($h_s = 1$), three-point couplings with $\mathrm{a} = 2$ are of the form

$$A_3[1, h_k, -h_k] \qquad \text{with} \qquad [g_{k,2}] = 0$$

Minimal coupling of a photon to a field of helicity h_k with coupling $g_{k,2}$ given by the electric charge Q_k.

The master formula specializes to

$$\hat{A}^{\text{ph}}_{n+1}(z,\epsilon) = \left[\sum_k Q_k \frac{\langle Xs\rangle^{-1}}{z\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)}\right]\hat{A}_n(z) + O(\epsilon^0)\,.$$

Observe that $\hat{A}^{(k)}_n(0)$ is independent of k, so the amplitude factorizes.

The coefficient of the $1/z$ pole vanishes by charge conservation.

Laurent expanding in z and setting $z = 0$ gives

$$A^{\text{ph}}_{n+1} = \frac{1}{\epsilon^2}\sum_k Q_k \frac{\langle Xk\rangle}{\langle Xs\rangle\langle sk\rangle} A_n + \frac{1}{\epsilon}\sum_k Q_k \frac{1}{\langle Xs\rangle}\partial_z \hat{A}_n(z)\Big|_{z=0} + O(\epsilon^0)\,.$$

which is precisely the standard subleading soft photon theorem.

$a = 1$ FOR $h_s = 1 \implies$ MODIFIED SOFT THEOREM

For photons ($h_s = 1$), three-point couplings with $\mathtt{a} = 1$ are of the form

$$A_3[1, h_k, 1 - h_k] \qquad \text{with} \qquad [g_{k,1}] = -1$$

These arise from the dimension-five operators

$$\overline{\chi}\gamma^{\mu\nu}F_{\mu\nu}\chi\,, \quad \phi F_{\mu\nu}F^{\mu\nu}\,, \quad \phi F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \quad \overline{\psi}_{\mu}F_{\nu\rho}\gamma^{\mu\nu\rho}\chi\,, \quad hF^2$$

The master formula specializes to

$$\hat{A}^{\text{ph}}_{n+1}(z,\epsilon) = \sum_k g_k \frac{[sk]\langle sk\rangle^{-1}}{\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \hat{A}^{(k)}_n(z) + O(\epsilon^0)\,.$$

There are no z poles, so no constraints.

The $z = 0$ limit gives the universal form for the modification of the subleading soft photon theorem

$$\frac{1}{\epsilon}\sum_k g_k \frac{[sk]}{\langle sk\rangle} A^{(k)}_n$$

MODIFIED SOFT PHOTON THEOREM

To summarize, the modified soft photon theorem is

$$A_{n+1}^{\text{ph}} = \left[\frac{S^{(0)}}{\epsilon^2} + \frac{S^{(1)}}{\epsilon}\right]A_n \; + \; \frac{\tilde{S}^{(1)}}{\epsilon}\tilde{A}_n + O(\epsilon^0)$$

where $\mathtt{a} = 2$ (minimal) couplings give

$$S^{(0)} = \sum_k Q_k \frac{\langle xk \rangle}{\langle xs \rangle \langle sk \rangle}\,, \qquad S^{(1)} = \sum_k Q_k \frac{D_{sk}}{\langle sk \rangle}$$

and $\mathtt{a} = 1$ (dimension-five) couplings give the modification

$$\tilde{S}^{(1)}\tilde{A}_n = \sum_k g_k \frac{[sk]}{\langle sk \rangle}\tilde{A}_n^{(k)}$$

$a = 3$ FOR $h_s = 2 \implies$ SOFT GRAVITON THEOREM

For gravitons ($h_s = 2$), three-point couplings with $a = 3$ are of the form

$$A_3[2, h_k, -h_k] \qquad \text{with} \qquad [g_{k,3}] = -1$$

GR coupling of a graviton to a field of helicity h_k with coupling $g_{k,3}$ given universally by Newton's constant κ (equivalence principle).

The master formula specializes to

$$\hat{A}^{\text{grav}}_{n+1}(z,\epsilon) = \left[\sum_k \kappa \frac{[sk]\langle sk\rangle \langle Xs\rangle^{-2}}{z^2\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \right] \hat{A}_n(z) + O(\epsilon^0)\,.$$

Observe that $\hat{A}_n^{(k)}(0)$ is independent of k, so the amplitude factorizes.

The coefficient of the $1/z^2$ pole is proportional to $\sum_k [sk]\langle sk\rangle$ and vanishes by momentum conservation. Equivalently, the absence of the $1/z^2$ pole implies the equivalence principle.

In the Laurent expansion in z, it can shown that the $1/z$ pole also vanishes.

SOFT GRAVITON THEOREM

Laurent expanding the master formula

$$\hat{A}^{\text{grav}}_{n+1}(z,\epsilon) = \left[\sum_k \kappa \frac{[sk]\langle sk\rangle \langle Xs\rangle^{-2}}{z^2\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)}\right]\hat{A}_n(z) + O(\epsilon^0)$$

and setting $z = 0$ gives

$$\hat{A}^{\text{grav}}_{n+1}(z,\epsilon) = \frac{1}{\epsilon^3}\sum_k \kappa \frac{[sk]\langle Xk\rangle^2}{\langle sk\rangle\langle Xs\rangle^2}A_n + \frac{1}{\epsilon^2}\sum_k \kappa \frac{[sk]\langle Xk\rangle}{\langle Xs\rangle^2}\partial_z \hat{A}_n(z)\Big|_{z=0}$$
$$+ \frac{1}{\epsilon}\sum_k \frac{\kappa}{2}\frac{[sk]\langle sk\rangle}{\langle Xs\rangle^2}\partial_z^2 \hat{A}_n(z)\Big|_{z=0} + O(\epsilon^0)\,.$$

With some algebra, one may show that this is precisely the soft graviton theorem derived in [Cachazo and Strominger 1404.4091] .

Since $\mathrm{a} = 2$ couplings are forbidden for $h_s = 2$ by the constraint that the $1/z$ pole be absent, there are no modifications to this theorem through subleading ($1/\epsilon^2$) order.

$a = 1$ FOR $h_s = 2 \implies$ MODIFIED SOFT THEOREM

For gravitons ($h_s = 2$), three-point couplings with $a = 1$ are of the form

$$A_3[2, h_k, 2 - h_k] \qquad \text{with} \qquad [g_{k,1}] = -3$$

These arise from the dimension-seven operators

$$\phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\,, \quad R^{\mu\nu\rho\sigma}\overline{\psi}_\rho \gamma_{\mu\nu}\partial_\sigma \chi\,, \quad R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

The master formula specializes to

$$\hat{A}^{\text{grav}}_{n+1}(z, \epsilon) = \sum_{k, h_P} g_k \frac{[sk]^3 \langle sk\rangle^{-1}}{\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \hat{A}_n^{(k)}(z) + O(\epsilon^0)\,.$$

There are no z poles, so no constraints.

The $z = 0$ limit gives the universal form for the modification of the subleading soft graviton theorem

$$\frac{1}{\epsilon} \sum_{k, h_P} g_k \frac{[sk]^3}{\langle sk\rangle} A_n^{(k)}$$

MODIFIED SOFT GRAVITON THEOREM

To summarize, the modified soft graviton theorem is

$$A_{n+1}^{\text{grav}} = \left[\frac{\mathcal{S}^{(0)}}{\epsilon^3} + \frac{\mathcal{S}^{(1)}}{\epsilon^2} + \frac{\mathcal{S}^{(2)}}{\epsilon}\right] A_n \; + \; \frac{\tilde{\mathcal{S}}^{(2)}}{\epsilon} \tilde{A}_n + O(\epsilon^0)$$

where $\mathtt{a} = 3$ (GR) couplings give

$$\mathcal{S}^{(0)} = \kappa \sum_k \frac{[sk]\langle xk\rangle\langle yk\rangle}{\langle sk\rangle\langle xs\rangle\langle ys\rangle} ,$$

$$\mathcal{S}^{(1)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk\rangle} \left(\frac{\langle xk\rangle}{\langle xs\rangle} + \frac{\langle yk\rangle}{\langle ys\rangle} \right) D_{sk} ,$$

$$\mathcal{S}^{(2)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk\rangle} D_{sk}^2$$

and $\mathtt{a} = 1$ (dimension-seven) couplings give the modification

$$\tilde{\mathcal{S}}^{(2)} \tilde{A}_n = \sum_k g_k \frac{[sk]^3}{\langle sk\rangle} \tilde{A}_n^{(k)}$$

CONCLUSIONS

To summarize:

- We have used a two-complex-parameter holomorphic shift of the momenta to obtain a new derivation of the soft photon and graviton theorems (for massless particles in four dimensions).
- We have shown that corrections to the standard soft theorems can only arise from a small subset of higher-dimension three-point operators of massless fields, and that these corrections have a universal form.
- The soft photon theorem can receive corrections at subleading order, but the soft graviton theorem can only receive corrections at subsubleading order.

Thanks for listening!

EXTRA SLIDES

$\mathtt{a} > 3$ COUPLINGS ARE FORBIDDEN

Recall the master formula

$$\hat{A}_{n+1}(z,\epsilon) = \sum_{k,\mathtt{a}} g_{k,\mathtt{a}} \frac{[sk]^{2h_s-\mathtt{a}} \langle sk\rangle^{\mathtt{a}-2} \langle Xs\rangle^{1-\mathtt{a}}}{z^{\mathtt{a}-1}\left(\epsilon - z\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} \hat{A}_n^{(k)}(z) + O(\epsilon^0)$$

The coefficient of the leading pole $1/z^{\mathtt{a}-1}$ is

$$\sum_{k=1}^{n} g_{k,\mathtt{a}} [sk]^{2h_s-\mathtt{a}} \langle sk\rangle^{\mathtt{a}-2} \, A_n^{(k)} \tag{1}$$

Applying the operator $|p\rangle\partial_{|s\rangle}$ gives (for $\mathtt{a} > 2$)

$$\sum_{k=1}^{n} g_{k,\mathtt{a}} [sk]^{2h_s-\mathtt{a}} \langle sk\rangle^{\mathtt{a}-3} \langle pk\rangle \, A_n^{(k)} \tag{2}$$

This sum must vanish for the amplitude not to have a $1/z^{\mathtt{a}}$ pole.

The constraint is thus

$$\sum_{k=1}^{n} g_{k,\mathtt{a}}[sk]^{2h_s-\mathtt{a}}\langle sk\rangle^{\mathtt{a}-3}\langle pk\rangle\, A_n^{(k)} = 0 \tag{3}$$

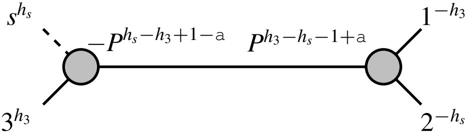
Suppose that $\mathtt{a} > 3$. Since $|s\rangle$ and $|p\rangle$ are arbitrary, we can use them to remove two of the terms in the sum. E.g. if $|s\rangle = |1\rangle$ and $|p\rangle = |2\rangle$

$$\sum_{k=3}^{n} g_{k,\mathtt{a}}[sk]^{2h_s-\mathtt{a}}\langle 1k\rangle^{\mathtt{a}-3}\langle 2k\rangle\, A_n^{(k)} = 0 \tag{4}$$

Now consider the four-point amplitude

$$A_4(s^{h_s}, 1^{-h_3}, 2^{-h_s}, 3^{h_3}).$$

for which the equation above has only one term, from



proportional to $|g_{k,\mathtt{a}}|^2$. The constraint thus implies $g_{k,\mathtt{a}} = 0$. Thus, any non-vanishing three-point coupling must obey $\mathtt{a} \leq 3$.

$\mathtt{a} = 3$ COUPLINGS ARE FORBIDDEN FOR $h_s = 1$

For $\mathtt{a} = 3$, the constraint from the absence of z poles is

$$\sum_{k=1}^{n} g_{k,\mathtt{a}} [sk]^{2h_s - 3} \langle pk \rangle \, A_n^{(k)} = 0 \qquad (5)$$

Set $|p\rangle = |1\rangle$ to remove one of the terms in the sum

$$\sum_{k=2}^{n} g_{k,\mathtt{a}} [sk]^{2h_s - 3} \langle 1k \rangle \, A_n^{(k)} = 0 \qquad (6)$$

For photons ($h_s = 1$), three-point couplings with $\mathtt{a} = 3$ are of the form

$$A_3[1, h_k, -1 - h_k] \qquad \text{with} \qquad [g_{k,3}] = 1$$

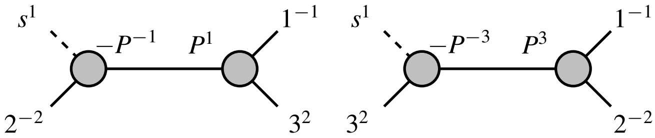
The only possibilities are

$$A_3[1, 1, -2], \qquad A_3[1, 1/2, -3/2], \qquad A_3[1, 0, -1], \qquad A_3[1, -1/2, -1/2]$$

To show that the coupling for $A_3[1, 1, -2]$ vanishes, consider the four-point amplitude

$$A_4(s^1, 1^{-1}, 2^{-2}, 3^2).$$

The two channels still appearing in the constraint sum are



but a state of helicity 3 cannot appear in an intermediate channel, so only the first term contributes to the constraint. Since that term is proportional to $|g_{k,\mathtt{a}}|^2$, the coupling must vanish.

The other $h_s = 1$, $\mathtt{a} = 3$ couplings can similarly be shown to vanish.

$\mathtt{a} = 2$ COUPLINGS ARE FORBIDDEN FOR $h_s = 2$

For $\mathtt{a} = 2$, the constraint from the absence of z poles is

$$\sum_{k=1}^{n} g_{k,\mathtt{a}} [sk]^{2h_s - 2} A_n^{(k)} = 0 \tag{7}$$

For gravitons ($h_s = 2$), three-point couplings with $\mathtt{a} = 2$ are of the form

$$A_3[2, h_k, 1 - h_k] \qquad \text{with} \qquad [g_{k,2}] = -2$$

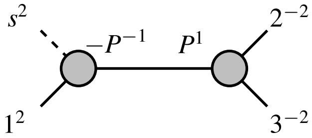
The only possibilities are

$$A_3[2, 2, -1], \qquad A_3[2, 3/2, -1/2], \qquad A_3[2, 1, 0], \qquad A_3[2, 1/2, 1/2]$$

To show that the coupling for $A_3[2, 2, -1]$ vanishes, consider the four-point amplitude

$$A_4(s^2, 1^2, 2^{-2}, 3^{-2}).$$

One of the channels is



while the other two involve the exchange of a helicity 3 particle and thus vanish. Thus the constraint requires the channel shown to vanish as well, implying the vanishing of the coupling for $A_3[2, 2, -1]$.

The other $h_s = 2$, $\mathtt{a} = 2$ couplings can similarly be shown to vanish.