From gravity on-shell diagrams to scattering amplitudes

Enrico Herrmann

Based on: arXiv:1604.03479 (+ work in progress)

in collaboration with:

Jaroslav Trnka (+ James Stankowicz)

[see also: Heslop,Lipstein,Farrow: arXiv:1604.03046,1705.07087]

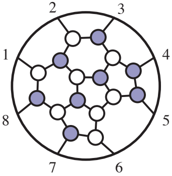
July 13, 2017

Caltech

Motivation

planar $\mathcal{N}=4$ sYM $\Rightarrow$ Hydrogen atom of the 21^{st} century!

- Dual Conformal Invariance [Drummond, Henn, Smirnov, Sokatchev, Korchemsky,...]
- Relation to Wilson loops and Correlation Functions [Mason, Skinner, Caron-Huot, Alday, Eden, Korchemsky, Maldacena, Sokatchev, ...]
- Yangian Invariance and Integrability [Drummond, Henn, Plefka, Beisert, Staudacher, Alday, Viera, Basso,...]



Mathematical structures beyond planar $\mathcal{N}=4$ sYM?

non-planar $\mathcal{N}=4$ sYM	$\mathcal{N}=8$ SUGRA

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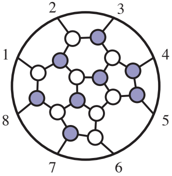

Mathematical structures beyond planar $\mathcal{N}=4$ sYM?

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Outline

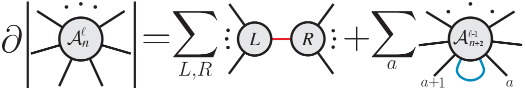
On-Shell methods, Cuts of loop amplitudes and Generalized Unitarity

[Britto, Cachazo, Feng, Witten; Bern, Dixon, Kosower; ...]

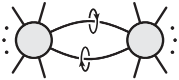
- core idea: on-shell amplitudes break up into products of simpler amplitudes on all factorization channels
- amplitudes are **fixed** from their singularities

locality: $\frac{1}{P^2}$ propagators

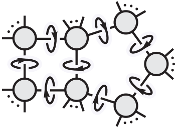
unitarity: factorization on poles



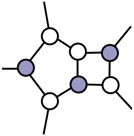
iterative cuts:



$\Rightarrow$



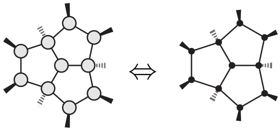
Maximal Cut

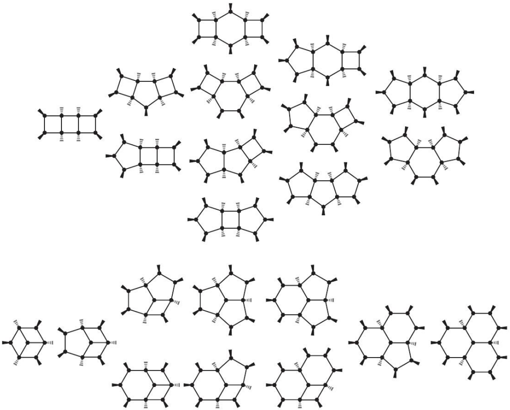


OS-Diagram

OS-diagrams: reference data for local loop integrands

[Bourjaily,EH,Trnka]





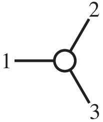
- Integrals tailored to match QFT on associated OS-functions
- 3-loop formula for arbitrary number of external points

Three-Point Amplitudes

On-shell conditions have two solutions:

$\overline{\text{MHV}},\ k=1$

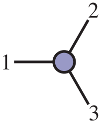
$\lambda_1 \sim \lambda_2 \sim \lambda_3$



$$\mathcal{A}_3^{\overline{\text{MHV}}} = \frac{\delta^4(P)\delta^4([12]\widetilde{\eta}_3+[23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2)}{[12][23][31]}$$

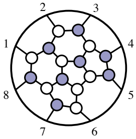
$\text{MHV},\ k=2$

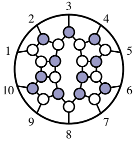
$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$

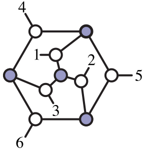


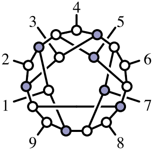
$$\mathcal{A}_3^{\text{MHV}} = \frac{\delta^4(P)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

use 3pt-amplitudes as building blocks for more complicated On-Shell diagrams









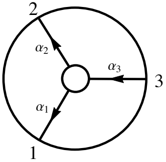
Grassmannian Formulation for On-Shell Diagrams

[Arkani-Hamed,Cachazo,Cheung,Kaplan][Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka]

- deep connection: On-Shell Diagrams $\Leftrightarrow$ Submanifold $C \subset G(k,n)$
- planar OS-diags $\Rightarrow$ positive Grassmannian $G_+(k,n) \rightarrow$ connection to combinatorics $\&$ algebraic geometry

  [Lusztig,Postnikov,Speyer,Williams,Knutson,Lam,...]
- Motivation from physics: linearize momentum conservation!

Encode linear relations in terms of $(k\times n)$-matrix C mod $GL(k)$.



$$\delta(P) \equiv \delta(\lambda\cdot\widetilde{\lambda}) \rightarrow \delta(C\cdot\widetilde{\lambda})\delta(C^\perp\cdot\lambda)$$

$C^\perp$: orthogonal matrix to C

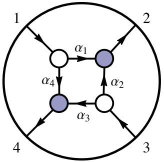
$\Updownarrow$

$$C = \begin{pmatrix}\alpha_1\alpha_3 & \alpha_2\alpha_3 & 1\end{pmatrix}$$

$$\mathcal{A}_3^{\overline{\text{MHV}}} = \frac{1}{\text{vol}(GL(1))}\int \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\delta(C\cdot\widetilde{\lambda})\delta(C^\perp\cdot\lambda)\delta(C\cdot\widetilde{\eta})$$

Grassmannian Formulation for On-Shell Diagrams

- glue 3pt-Grassmannians $\rightarrow$ build bigger on-shell diagrams
- encode momentum conservation by $(k \times n)$-matrix C
- n: # external legs, $k = 2n_B + n_W - n_I$: MHV-degree



$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}$$

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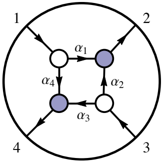

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math

positroid varieties, cluster algebras, pos. Grassmannian

physics

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\delta(C\cdot Z)$$

$$\delta(C\cdot Z) \equiv \delta(C\cdot\widetilde{\lambda})\delta(C^\perp\cdot\lambda)\delta(C\cdot\widetilde{\eta})$$

$$\Omega = \prod \mathcal{A}_3$$

- $d\log$-measure! $\to$ special for Yang-Mills
- holds for nonplanar YM, positivity of C lost $\Rightarrow G(k,n)$

relation to amplitudes: hidden symmetries, geometric formulation,...

A first look at gravity on-shell diagrams

- 3pt-amplitudes: squaring relation

$$A_3 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \to M_3 = \left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right)^2$$

- general on-shell diagram (product of 3pt amplitudes)

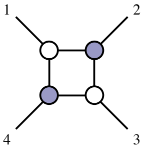
$$(\text{YM})^2 = (\text{GR}) \times (\phi^3)$$

"don't square the propagators"

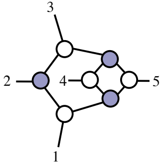
- (ϕ^3) factor changes expressions drastically

A first look at gravity on-shell diagrams

study some data (MHV leading singularities)



$$\frac{[13][24]}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle}$$



$$\frac{[12][23][45]^2}{\langle 12\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 34\rangle\langle 35\rangle}$$

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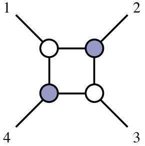

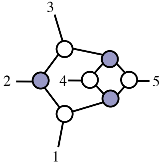

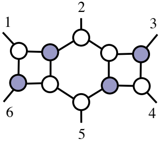

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$$\frac{\langle 5|Q_{16}|2]\langle 2|Q_{34}|5][16]^2[34]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 25\rangle^2}$$

- nontrivial numerators!
- higher power poles possible! (unlike YM)

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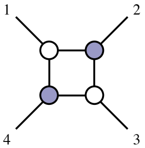

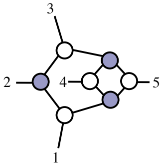

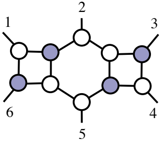

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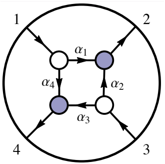
Detailed analysis

- nontrivial numerators! $\rightarrow$ collinearity condition in the vertices
- higher power poles possible! (unlike YM) $\rightarrow$ infinite momenta

Grassmannian Formula for Gravity



$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & \alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}$$

Inspired by the explicit data, can "discover" the gravity formula:

- Yang–Mills: $\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\,\mathcal{J}^{\mathcal{N}-4}\,\delta(C\cdot Z)$
- Gravity: $\Omega = \frac{d\alpha_1}{\alpha_1^3}\frac{d\alpha_2}{\alpha_2^3}\frac{d\alpha_3}{\alpha_3^3}\frac{d\alpha_4}{\alpha_4^3}\,\left(\prod_v \Delta_v\right)\mathcal{J}^{\mathcal{N}-4}\,\delta(C\cdot Z)$

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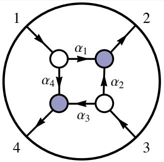

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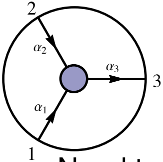
- special numerator Δ_v for each vertex
- α_i^3 poles

Can motivate this formula by looking at 3pt amplitudes

Works for arbitrary $\#$ of supersymmetry, incl. $\mathcal{N}=0$!

Grassmannian Formula for Gravity



$$C = \begin{pmatrix} 1 & 0 & \alpha_1\alpha_3 \\ 0 & 1 & \alpha_2\alpha_3 \end{pmatrix}, \quad C^{\perp} = \begin{pmatrix} -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & 1 \end{pmatrix}$$

- Need to modify measure by some dimensionful, permutation invariant object Δ
- $\delta(C^{\perp}\cdot\lambda) \Rightarrow -\overbrace{\alpha_1\lambda_1}^{\lambda_A} - \overbrace{\alpha_2\lambda_2}^{\lambda_B} + \overbrace{\frac{1}{\alpha_3}\lambda_3}^{\lambda_E} = 0 \Rightarrow \Delta = \langle AB\rangle = \langle BE\rangle = \langle EA\rangle$

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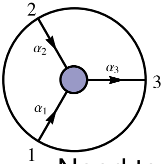

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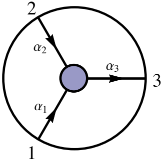
Ansatz:

$$\Omega_3^{\text{MHV}} = \kappa \frac{\Delta^\rho d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1^{\sigma_1}\alpha_2^{\sigma_2}\alpha_3^{\sigma_3}} \delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{1\times 2}(C^\perp\cdot\lambda)\delta^{2\times\mathcal{N}}(C\cdot\widetilde{\eta})$$

Impose: a) maximal SUSY $\mathcal{N} = 4s$ b) permutation invariance
c) independence of α_3 $(\Omega \sim \frac{d\alpha_3}{\alpha_3})$

$$\rho = s - 1\,, \qquad \sigma_1 = \sigma_2 = \sigma_3 = 2s - 1$$

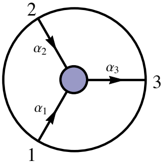
Grassmannian Formula for Gravity



$$\Omega_3^{\text{MHV},s} = \kappa \frac{\Delta^{s-1} d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1^{2s-1}\alpha_2^{2s-1}\alpha_3^{2s-1}} \overbrace{\delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{1\times 2}(C^{\perp}\cdot\lambda)\delta^{2\times 4s}(C\cdot\widetilde{\eta})}^{=\delta(C\cdot Z)}$$

What does this mean for $s > 2$?

Grassmannian Formula for Gravity



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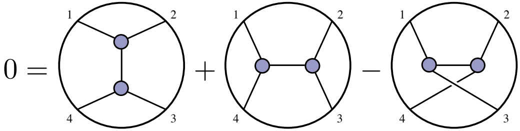
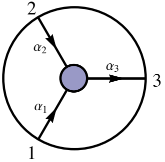
What does this mean for $s > 2$?

OS-diags in $\mathcal{N} = 4$ sYM satisfy equivalence moves!



$$\Gamma_s(\langle 12\rangle\langle 34\rangle)^{s-1} + \Gamma_t(\langle 14\rangle\langle 23\rangle)^{s-1} = \Gamma_u(\langle 13\rangle\langle 24\rangle)^{s-1}$$

Grassmannian Formula for Gravity



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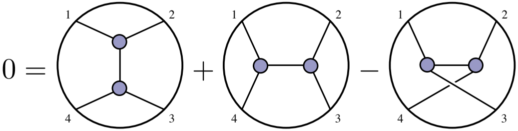


$$\Gamma_s(\langle 12\rangle\langle 34\rangle)^{s-1} + \Gamma_t(\langle 14\rangle\langle 23\rangle)^{s-1} = \Gamma_u(\langle 13\rangle\langle 24\rangle)^{s-1}$$

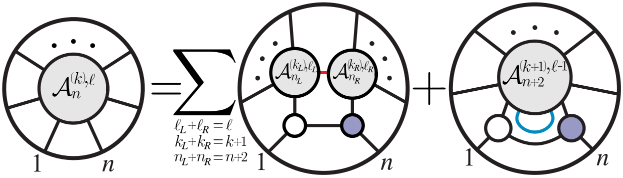
Two solutions:

1. $s = 1$: $\Gamma_s + \Gamma_t = \Gamma_u \Rightarrow$ Jacobi identity
   $\Gamma_s = f^{12a}f^{34a}$, $\Gamma_t = f^{14a}f^{23a}$, $\Gamma_u = f^{13a}f^{24a}$
2. $s = 2$: $\Gamma_s = \Gamma_t = \Gamma_u \Rightarrow$ Shouten identity
   universality of gravitational coupling

From On-Shell Diagrams to Amplitudes; YM

planar $\mathcal{N} = 4$ sYM: all-loop recursion relation for integrand

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka]



Integrand inherits properties of individual OS-diags:

- Dual (super-) Conformal Invariance, Yangian Invariance, no poles at $\ell \to \infty$
- Logarithmic Singularities (for N$^{<3}$MHV amplitudes directly in momentum space)

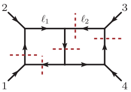
From On-Shell Diagrams to Amplitudes; YM

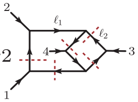
- emerging idea: what is minimal set of conditions to fix amplitude?
  - tree-level [Arkani-Hamed, Rodina, Trnka]
  - loop-level [Bern, EH, Litsey, Stankowicz, Trnka]
    - universal homogeneous constraints: vanishing on "wrong factorizations"
    - for $\mathcal{N} = 4$sYM: set of "defining" homogeneous constraints, no poles at infinity, no double poles $\Rightarrow$ fix answer up to overall scale

From On-Shell Diagrams to Amplitudes; YM

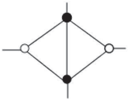
- emerging idea: what is minimal set of conditions to fix amplitude?
  - tree-level [Arkani-Hamed, Rodina, Trnka]
  - loop-level [Bern, EH, Litsey, Stankowicz, Trnka]
    - universal homogeneous constraints: vanishing on "wrong factorizations"
    - for $\mathcal{N} = 4$sYM: set of "defining" homogeneous constraints, no poles at infinity, no double poles $\Rightarrow$ fix answer up to overall scale

Two-loop amplitude

$$M_2 = \sum_{\sigma} \; a_1$$


$$+ \; a_2$$


$$k = 1$$



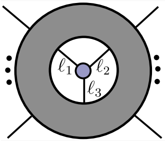
homogeneous constraints fix nontrivial numerators and relative coefficients

$$a_1 = a_2$$

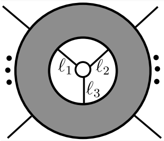
From On-Shell Diagrams to Amplitudes; GR

- what are the defining properties of gravity?
- dual formulation for gravity on-shell functions $\Rightarrow$ hint: collinear vanishing conditions

OS-diagrams suggest special collinear behavior of amplitudes on cut:



$\sim \langle \ell_1 \ell_2 \rangle$



$\sim [\ell_1 \ell_2]$

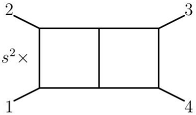
For special case of external legs ($k_1 || k_2$), known collinear limit of amplitudes (c.f. splitting functions): [Bern, Dixon, Perelstein, Rozowsky]

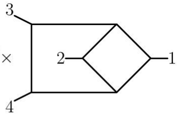
$$\mathcal{M} \overset{\langle 12\rangle \to 0}{\longrightarrow} \frac{[12]}{\langle 12\rangle} \cdot \mathcal{R}, \qquad \mathcal{M} \overset{[12] \to 0}{\longrightarrow} \frac{\langle 12\rangle}{[12]} \cdot \overline{\mathcal{R}}$$

$\mathcal{R}$, $\overline{\mathcal{R}}$ regular in $\langle 12\rangle$, $[12]$

Collinear Behavior of Gravity Amplitudes

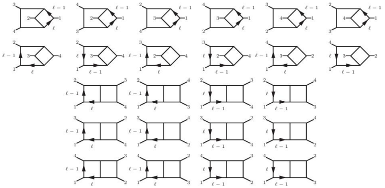
1) study theoretical data

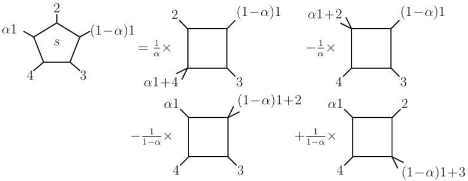
$I^{(P)}_{(1234)} = s^2 \times$ 

$I^{(NP)}_{(1234)} = s^2 \times$ 

$$\mathcal{M}_4^2 = \frac{\mathcal{K}_8}{4} \sum_{\sigma \in \mathfrak{S}_4} \left[I_\sigma^{(P)} + I_\sigma^{(NP)} \right],$$

go directly to collinear region $\ell = \alpha p_1$ $:\Rightarrow$ $\mathcal{M}_4^2 \to 0$:





2) use collinear vanishing as homogeneous conditions to construct GR amps? $\Rightarrow$ work in progress [/w Stankowicz, Trnka]

Comments-Conclusion

- Extended Grassmannian formulation for On-Shell Diagrams to gravity
- Formulas for $\mathcal{N} < 8$ SUGRA only slightly modified
- presence of poles at infinity
- collinear vanishing of on-shell diagrams

Interesting further directions

- relation to ambitwistor string $\Rightarrow$ [see talk by A.Lipstein]
- recursion relations
- relation to properties of amplitudes
- UV-structure of gravity