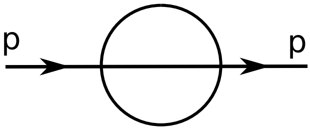
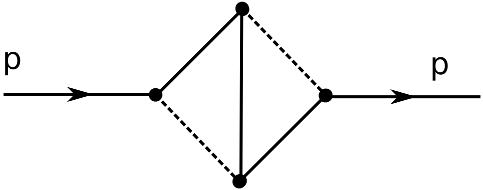
An elliptic generalisation of polylogarithms for the sunrise and kite integrals





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Outline of the talk

1. Short introduction to the mathematical background

2. The sunrise integral around $D = 2 - 2\epsilon$
   (i) Order ϵ^0 (excursus: Modular forms)
   (ii) Higher ϵ-orders

3. The kite integral

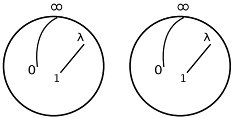
4. Conclusions

I. Introduction

Elliptic Curves and Elliptic Integrals

An elliptic curve can be written with the help of the Weierstrass equation:

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

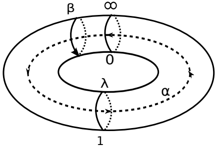


and is topologically equivalent to a torus;

two integrals along the paths α, β are in the first homology group of the torus $\Rightarrow$ periods ω_1, ω_2

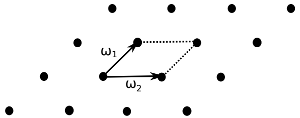
elliptic integral = path integral along elliptic curve E; consider for example elliptic integral

$$\int_{\infty}^{x} \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}$$



Integral well-defined?
$\rightarrow$ branch cuts along 0 to ∞ and 1 to λ and put them together to a Riemann surface $\rightarrow$ **torus**

this integral is only well-defined
mod $\Lambda = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$ which defines a lattice



$E(\mathbb{C}) \xrightarrow{\text{elliptic integrals}}$ torus $\mathbb{C}/\Lambda$:

elliptic integral gives an isomorphism from $E(\mathbb{C})$ to $\mathbb{C}/\Lambda$

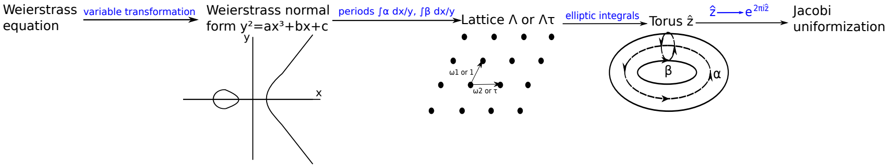
modified lattice Λ_τ generated by 1 and $\tau = \omega_2/\omega_1$:

$$\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \qquad \text{and it is} \qquad \Lambda_\tau = \Lambda_{\tau+k}, k \in \mathbb{Z}$$

Under the exponential map $J : \mathbb{C} \to \mathbb{C}^*, z \to e^{2\pi i z} = w$ the lattice Λ_τ in $\mathbb{C}$ is mapped to $q^{\mathbb{Z}}$ in $\mathbb{C}^*$ $\quad \to$ analytic isomorphism $E_\tau = \mathbb{C}/\Lambda_\tau \to \mathbb{C}^*/q^{\mathbb{Z}}$

The representation of the elliptic curve in $\mathbb{C}^*/q^{\mathbb{Z}}$ is called the Jacobi uniformization of the curve.

Chain of mappings between the different representations of an elliptic curve

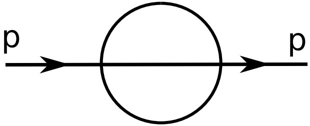


side remark:
the (doubly periodic) elliptic Weierstrass function provides an isomorphism from the torus representation $\mathbb{C}/\Lambda$ to the Weierstrass representation $E(\mathbb{C})$:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda,\omega\neq 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

and is in that sense "inverse" to elliptic integrals

II. The sunrise integral



In D space-time dimensions the two-loop sunrise integral family reads

$$S_{\nu_1\nu_2\nu_3}(D,p^2,m_1,m_2,m_3) =$$
$$= (\mu^2)^{\nu-D}\int \frac{d^Dk_1}{i\pi^{\frac{D}{2}}}\int \frac{d^Dk_2}{i\pi^{\frac{D}{2}}}\frac{1}{(-k_1^2+m_1^2)^{\nu_1}(-k_2^2+m_2^2)^{\nu_2}(-(p-k_1-k_2)^2+m_3^2)^{\nu_3}}$$

In its Feynman parameterisation the integral reads $\left(\nu=\nu_1+\nu_2+\nu_3\right)$:

$$S_{\nu_1\nu_2\nu_3}(D,t) = \frac{\Gamma(\nu-D)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}(\mu^2)^{\nu-D}\int\limits_\sigma \omega\; x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\nu_3-1}\frac{\mathcal{U}^{\nu-\frac{3}{2}D}}{\mathcal{F}^{\nu-D}}$$

with the differential two-form

$$\omega = x_1\; dx_2\wedge dx_3 + x_2\; dx_3\wedge dx_1 + x_3\; dx_1\wedge dx_2$$

and the integration region

$$\sigma = \{[x_1 : x_2 : x_3]\in\mathbb{P}^2 | x_i\geq 0, i=1,2,3\}$$

and the first and second graph polynomial $(t=p^2)$
$\mathcal{U} = x_1x_2+x_2x_3+x_1x_3,\; \mathcal{F} = -x_1x_2x_3t+(x_1m_1^2+x_2m_2^2+x_3m_3^2)(x_1x_2+x_2x_3+x_3x_1).$

The (equal mass) sunrise integral $S_{111}(D,t)$ satisfies for $\boxed{D = 2 - 2\epsilon}$ the following differential equation:

$$\left\{ L_2^{(0)} + \epsilon L_2^{(1)} + \epsilon^2 L_2^{(2)} \right\} S_{111}(2-2\epsilon, t) = -6\mu^2 \Gamma(1+\epsilon)^2 \left(\frac{m^2}{\mu^2} \right)^{-2\epsilon}$$

[Müller-Stach, Weinzierl, Zayadeh, 2011]

where the Laurent contributions $L_2^{(i)}$'s are of order 0, 1 and 2, respectively:

$$L_2^{(0)} = p_2^{(0)} \frac{d^2}{dt^2} + p_1^{(0)} \frac{d}{dt} + p_0^{(0)}, \qquad L_2^{(1)} = p_1^{(1)} \frac{d}{dt} + p_0^{(1)}, \qquad L_2^{(2)} = p_0^{(2)}.$$

The $p_i^{(j)}$'s are polynomials in t and the mass m.

Let us now study the solution of this differential equation order by order.

Let us start with the order ϵ^0 and the case of arbitrary masses:

$$\left\{ p_2^{(0)} \frac{d^2}{dt^2} + p_1^{(0)} \frac{d}{dt} + p_0^{(0)} \right\} S_{111}^{(0)}(2,t) = I^{(0)}$$

where $S_{111}^{(0)}(2,t) = (\mu^2) \int_{x_i \geq 0} \omega \frac{1}{\mathcal{F}}$.

The equation

$$\mathcal{F} = -x_1 x_2 x_3 t + (x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)(x_1 x_2 + x_2 x_3 + x_3 x_1) = 0$$

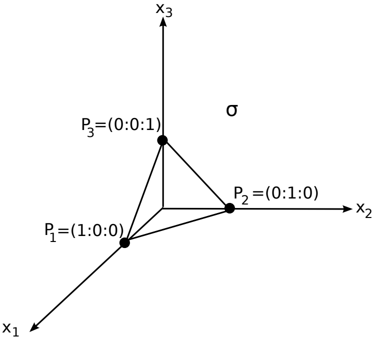
defines together with the choice of an origin $\mathcal{O}$ an elliptic curve.

Which point should be chosen?

Two regions of interest:

- integration region σ
- algebraic variety defined by $\mathcal{F} = 0$

$\rightarrow$ choose one of the intersection points as origin



With a convenient variable transformation mapping

$$(x_1, x_2, x_3) \longrightarrow (x, y, z = 1)$$

we can compute the Weierstrass normal form of the curve:

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3) \qquad \text{with} \qquad e_1 + e_2 + e_3 = 0$$

where the roots e_i depend on t and the masses.
In a next step we find for two periods of this elliptic curve

$$\Psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \frac{4\mu^2}{D^{\frac{1}{4}}} K(k), \qquad \Psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i\mu^2}{D^{\frac{1}{4}}} K(k')$$

with the complete elliptic integral of the first kind

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

and the (complementary) modulus $k^{(\prime)}$:

$$k = \sqrt{\frac{e_3 - e_2}{e_1 - e_2}}, \qquad k' = \sqrt{1-k^2} = \sqrt{\frac{e_1 - e_3}{e_1 - e_2}}.$$

We find: The periods are the solutions of the homogeneous differential equation

$$\boxed{L_2^{(0)}\Psi_i = 0} \qquad !$$

After performing the change of variables

$$t \ (= p^2) \qquad \longrightarrow \qquad \text{nome } q := e^{i\pi\tau} \qquad \text{with} \qquad \tau = \frac{\Psi_2}{\Psi_1}$$

we can express the special inhomogeneous solution as

$$S^{(0)}_{111,special}(2, q) = -\mu^2 \frac{\Psi_1}{\pi^2} \int_0^q \frac{dq_1}{q_1} \int_0^{q_1} \frac{dq_2}{q_2} \frac{p_3(q_2)\Psi_1^3(q_2)}{p_2(q_2)W^2(q_2)}$$

with the Wronskian $W = \Psi_1 \frac{d}{dt}\Psi_2 - \Psi_2 \frac{d}{dt}\Psi_1$.

How can we express this special solution in terms of the homogeneous solutions (elliptic integrals) and a suitable generalisation of polylogarithms?

For the equal mass case $m_1 = m_2 = m_3 = m$ we find a closed q-expression for the integrand

$$\frac{\mu^2}{\pi} \frac{p_3(q)\Psi_1^3(q)}{p_2(q)W^2(q)} = -3\sqrt{3}\frac{\eta(\tau)^{11}\eta(3\tau)^7}{\eta(\frac{\tau}{2})^5\eta(2\tau)^5\eta(\frac{3\tau}{2})\eta(6\tau)}$$
$$= -\frac{3}{i}\left[\mathrm{ELi}_{0;-2}(r_3;-1;-q) - \mathrm{ELi}_{0;-2}(r_3^{-1};-1;-q)\right]$$

with

$$\mathrm{ELi}_{n;m}(x,y;q) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i}{i^n}\frac{y^j}{j^m}q^{ij} \qquad \text{and} \qquad r_3 = e^{\frac{2\pi i}{3}}$$

similar to: [Brown, Levin; 2013]

in terms of the Dedekind η-function $\eta(\tau) = e^{\frac{i\pi\tau}{12}}\prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})$.

We end up with

$$S_{111}^{(0)}(2,t) = 3\frac{\Psi_1}{i\pi}\left\{\frac{1}{2}\left[\mathrm{Li}_2(r_3) - \mathrm{Li}_2(r_3^{-1})\right] + \mathrm{ELi}_{2;0}(r_3,-1;-q) - \mathrm{ELi}_{2;0}(r_3^{-1},-1;-q)\right\}.$$

compare to: [Bloch, Vanhove; 2013], [Bloch, Kerr, Vanhove; 2014]

Excursus: Modular forms in the sunrise integral

For the equal mass case we have a closed expression relating the variable t and the nome $q = e^{i\pi\tau}$ $(\tau' = \tau/2)$:

$$t = -9m^2 \frac{\eta(2\tau')^4 \eta(3\tau')^4 \eta(12\tau')^4}{\eta(\tau')^4 \eta(4\tau')^4 \eta(6\tau')^4}$$

which helps us finding this η-expression for the integration kernel:

$$\frac{\mu^2}{\pi} \frac{p_3(q)\Psi_1^3(q)}{p_2(q)W^2(q)} = -3\sqrt{3} \frac{\eta(2\tau')^{11}\eta(6\tau')^7}{\eta(\tau')^5\eta(4\tau')^5\eta(3\tau')\eta(12\tau')}$$

This η-quotient is a modular form of weight 3 and expressible as a linear combination of so-called generalised Eisenstein series (ϕ, ψ are Dirichlet characters):

$$E_k(\tau;\phi,\psi) = a_0 + \sum_{m=1}^{\infty} \left(\sum_{d|m} \psi(d) \cdot \phi(m/d) \cdot d^{k-1} \right) e^{2\pi i \tau m/M}$$

for $k = 3$ and the Kronecker symbols $\phi(n) = \left(\frac{-3}{n}\right), \psi(n) = \left(\frac{1}{n}\right)$.

$\Rightarrow$ iterative integration in higher ϵ-orders leads to iterated integrals over modular forms as solution for the sunrise integral

How does this result change for the arbitrary mass case?

We compute the images of the intersection points P_1, P_2 and P_3 in the Jacobi uniformization of our elliptic curve finding

$$P_i \longrightarrow w_i = e^{i\pi \frac{F(u_i,k)}{K(k)}} \qquad (i = 1,2,3)$$

with $u_i = \sqrt{\frac{e_1 - e_2}{x_{j,k} - e_2}}$ and the incomplete elliptic integral of the first kind

$$F(z,x) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}.$$

The full solution reads then

$$S_{111}^{(0)}(2,t) = \frac{\Psi_1}{\pi} \sum_{i=1}^{3} \mathrm{E}_{2;0}(w_i(q), -1; -q)$$

with

$$E_{n;m}(x,y;q) =$$
$$\begin{cases} \frac{1}{\mathrm{i}} \left[\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x,y;q) - \mathrm{ELi}_{n;m}(x^{-1},y^{-1};q) \right], & n+m \text{ even} \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x,y;q) + \mathrm{ELi}_{n;m}(x^{-1},y^{-1};q), & n+m \text{ odd.} \end{cases}$$

Let us now turn to **higher ϵ-orders** (for the equal mass case).

With a slight transformation of the sunrise integral:

$$S_{111}(2-2\epsilon,t) = \Gamma(\epsilon+1)^2 \left(\frac{3\mu^4\sqrt{t}}{m(t-m^2)(t-9m^2)}\right)^{\epsilon} \tilde{S}_{111}(2-2\epsilon,t)$$

we can bring the differential equation into an integrable form:

$$\left\{ \underbrace{\tilde{L}_2^{(0)}}_{=L_2^{(0)}} + \epsilon^2 \tilde{L}_2^{(2)} \right\} \tilde{S}_{111}(2-2\epsilon,t) = -6\mu^2 \left(\frac{(t-m^2)(t-9m^2)}{3m^3\sqrt{t}}\right)^{\epsilon}.$$

Rewriting this equation for every order ϵ^j yields:

$$L_2^{(0)} \tilde{S}_{111}^{(j)}(2,t) = \underbrace{-\frac{6\mu^2}{j!}\log^j\left(\frac{(t-m^2)(t-9m^2)}{3m^3\sqrt{t}}\right)}_{=:I_a^{(j)}(t)} + \underbrace{\frac{(t+3m^2)^4}{4t(t-m^2)(t-9m^2)}\tilde{S}_{111}^{(j-2)}(2,t)}_{=:I_b^{(j)}(t)}$$

Our solution has the structure $\tilde{S}^{(j)}_{111}(2,q) = \frac{\Psi_1}{\pi}\tilde{E}^{(j)}_{111}(2,q)$.

Once we have expressed the inhomogeneous terms $I^{(j)}_a(t)$ and $I^{(j)}_b(t)$ as closed q-expressions

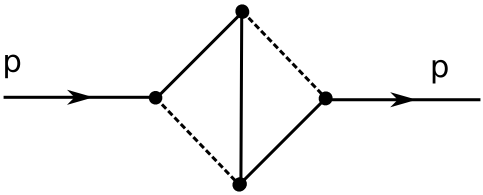
$$I^{(j)}_a(t) \propto \log(-q), \mathrm{ELi}_{1;0}(-1,1;-q), \mathrm{ELi}_{1;0}(r_3,\pm 1;-q), \mathrm{ELi}_{1;0}(r_3^{-1},\pm 1;-q)$$
$$I^{(j)}_b(t) \propto \left\{\mathrm{ELi}_{0;0}(r_3,1;-q), \mathrm{ELi}_{0;0}(r_3^{-1},1;-q\right\} \times \tilde{E}^{(j-2)}_{111}(2,q)$$

in terms of ELi-functions, we are able to compute order-by-order by iterative integration.

We integrate over products of ELi-functions and are therefore able to express every order within the following class of functions:

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) =$$
$$= \sum_{j_1=1}^{\infty}\cdots\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_l=1}^{\infty}\frac{x_1^{j_1}}{j_1^{n_1}}\cdots\frac{x_l^{j_l}}{j_l^{n_l}}\frac{y_1^{k_1}}{k_1^{m_1}}\cdots\frac{y_l^{k_l}}{k_l^{m_l}}\frac{q^{j_1k_1+\cdots+j_lk_l}}{\prod_{i=1}^{l-1}(j_ik_i+\cdots+j_lk_l)^{o_i}}.$$

II. The kite integral



In D-dimensional Minkowski space the equal mass kite integral family is given by

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = (-1)^{\nu_{12345}} (\mu^2)^{\nu_{12345}-D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

with the two massless and three massive propagators

$$D_1 = k_1^2 - m^2, D_2 = k_2^2, D_3 = (k_1 - k_2)^2 - m^2, D_4 = (k_1 - p)^2, D_5 = (k_2 - p)^2 - m^2,$$

the quantity $\nu_{12345} = \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5$ and the internal momenta k_i and the external momentum p.
We can choose a linear combination of the integrals

$$I_{20200}, I_{20210}, I_{02210}, I_{02120}, I_{21012}, I_{10101}, I_{20101}, I_{11111}$$

as a basis. [Remiddi, Tancredi, 2016]

Please note that the integrals I_{10101} and I_{20101} correspond to the two (equal mass) master integrals for the sunrise family.

With this choice of basis we find the following differential equation of Fuchsian type:

$$\mu^2 \frac{d}{dt}\vec{I} = \left[\frac{\mu^2}{t} A_0 + \frac{\mu^2}{t-m^2} A_1 + \frac{\mu^2}{t-9m^2} A_9 \right] \vec{I}$$

which shows the following closed subsystems:

- first subsystem yields (only) multiple polylogarithms: $\{I_1, \dots, I_5\}$

$$\frac{d}{dt}\begin{pmatrix} I_1 \\ \vdots \\ I_5 \end{pmatrix} = \epsilon \left[\frac{\mu^2}{t} A_0^{(5\times5)} + \frac{1}{t-m^2} A_1^{(5\times5)} \frac{1}{t-9m^2} A_9^{(5\times5)} \right] \begin{pmatrix} I_1 \\ \vdots \\ I_5 \end{pmatrix}$$

$\rightsquigarrow$ constant matrix entries

- second (sunrise) subsystem yields elliptic stuff: $\{I_1, I_6, I_7\}$:

$$\frac{d}{dt}\begin{pmatrix} I_1 \\ I_6 \\ I_7 \end{pmatrix} = \left[\frac{\mu^2}{t} A_0^{(3\times3)} + \frac{1}{t-m^2} A_1^{(3\times3)} \frac{1}{t-9m^2} A_9^{(3\times3)} \right] \begin{pmatrix} I_1 \\ I_6 \\ I_7 \end{pmatrix}$$

$\rightsquigarrow$ matrix entries linear in ϵ

Let us solve the differential equation for the kite integral I_8 at every order ϵ:

$$\mu^2 \frac{d}{dt} I_8^{(j)}(4,t) = \frac{\mu^2}{t}\left[-I_5^{(j)}(4,t) - 2I_5^{(j-1)}(4,t) - 3I_6^{(j-1)}(4,t) + I_8^{(j-1)}(4,t)\right] + \frac{\mu^2}{t-m^2}\left[\frac{1}{2}I_1^{(j)}(4,t) + I_1^{(j-1)}(4,t) - I_3^{(j)}(4,t) - 2I_1^{(j-1)}(4,t) + \frac{8}{3}I_6^{(j-1)}(4,t) - 2I_8^{(j-1)}(4,t)\right]$$

This equation leads directly to the following procedure:

1.) Solve the simpler subtopologies in terms of multiple polylogarithms by iterative integration of the differential equation

2.) Consider the sunrise topology (already solved) in terms of ELi-functions:

$$I_6(D,t) = (D-4)(D-5)\frac{t}{\mu^2} I_{10101}(D-2,t)$$

with

$$I_{10101}(D-2,t) = S_{111}(D-2,t) = \frac{\Psi_1}{\pi} E_{111}(D-2,q)$$

3.) Put everything together according to the kite differential equation

For the first step, the solution of the blue subtopology in terms of multiple polylogarithms we have in principle two different possibilities:

- integration w.r.t. the variable t $\rightsquigarrow$ yields multiple polylogarithms
- integration w.r.t. the nome q $\rightsquigarrow$ yields ELi-functions

since we can express the integration kernels of the r.h.s. of the DE in terms of ELi-functions:

$$\frac{1}{i\pi}\frac{\Psi_1^2}{W} \times \left\{ \frac{1}{t}, \frac{1}{t-m^2}, \frac{1}{t-9m^2} \right\} \to \text{ELi}_{0;-1}(r_3, \pm 1; -q), \text{ELi}_{0;-1}(r_3^{-1}, \pm 1; -q), \\ \text{ELi}_{0;-1}(\pm 1, 1; -q)$$

We find a relation between the polylogs in this special case and the ELi-functions:

$$\begin{aligned}
G(1; y) &\rightsquigarrow \text{ELi}_{1;0}(-1; 1; -q), \text{ELi}_{1;0}(r_6\ (r_6^{-1}); 1; -q) \\
G(0,1; y) &\rightsquigarrow \text{ELi}_{2;1}(-1; 1; -q), \text{ELi}_{2;1}(r_6(r_6^{-1}); 1; -q), \\
&\quad \text{ELi}_{0,1;-1,0;2}(r_3(r_3^{-1}), -1; -1, 1; -q), \\
&\quad \text{ELi}_{0,1;-1,0;2}(r_3(r_3^{-1}), r_6(r_6^{-1}); -1, 1; -q) \\
G(1,1; y) &\rightsquigarrow \text{ELi}_{0,1;-1,0;2}(-1, -1; 1, 1; -q), \text{ELi}_{0,1;-1,0;2}(-1, r_6(r_6^{-1}); 1, 1; -q), \\
&\quad \text{ELi}_{0,1;-1,0;2}(r_6(r_6^{-1}), -1; 1, 1; -q), \\
&\quad \text{ELi}_{0,1;-1,0;2}(r_6(r_6^{-1}), r_6(r_6^{-1}); 1, 1; -q), \qquad \ldots
\end{aligned}$$

We can take our sunrise results from the previous section and end up with the following results for the first few terms of the Laurent expansion of the kite integral:

$$
\begin{aligned}
I_8^{(0)}(4,t) &= 2G(1;y),\\
I_8^{(1)}(4,t) &= 2\left[G(0,1;y) - 4G(1,1;y) - 2G(1;y)\log\left(\frac{m^2}{\mu^2}\right) + 2G(1;y)\right],\\
I_8^{(2)}(4,t) &= 32G(1,1,1;y) - 8G(1,0,1;y) - 16G(0,1,1;y) + 2G(0,0,1;y)\\
&\quad - 16G(1,1;y) + 4G(0,1;y) + 16G(1,1;y)\log\left(\frac{m^2}{\mu^2}\right)\\
&\quad - 4G(0,1;y)\log\left(\frac{m^2}{\mu^2}\right) + \left(\frac{\pi^2}{3} - 8\log\left(\frac{m^2}{\mu^2}\right) + 4\log^2\left(\frac{m^2}{\mu^2}\right)\right)G(1;y)\\
&\quad - 108\,\mathrm{Cl}_2\left(\frac{2\pi}{3}\right)\ \bar{\mathrm{E}}_{1;-1}(r_3;1;-q) - 108\ \bar{\mathrm{E}}_{0,2;-2,0;2}(r_3,r_3;1,-1;-q)\\
&\quad \ldots
\end{aligned}
$$

Conclusions

- the sunrise integral is the simplest integral which **cannot be expressed in terms of multiple polylogarithms**

- around $D = 2 - 2\epsilon$ we can express the leading order ϵ^0 in terms of an **elliptic generalisation** of polylogarithms (also for the arbitrary mass case)

- the homogeneous solutions of the differential equation are the **periods of the elliptic curve** underlying the sunrise integral

- the higher ϵ-orders can be computed by iterative integration over products of ELi-functions yielding an **extended class of ELi-functions**

- the **four-dimensional case** can be obtained by the use of **Tarasov's dimensional shift relations**; it depends on the two-dimensional ϵ^0- and ϵ^1-solution

- with a convenient basis choice the equal mass **kite integral** with two massless and three massive propagators depends on simpler subtopologies (which can be completely be solved in terms of multiple polylogs) and the two-dimensional sunrise integral

- also the **polylogs from the easier subtopologies** can be expressed **in terms of ELi-functions** when we perform the integration w.r.t. the nome q

Thank you very much for your attention!