Feynman integrals, beyond polylogs, up to 22 loops

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The L-series of a **modular form** of weight 4 appears in the 4-loop QED contributions to the magnetic moment of the electron. I shall report on **L-series** that similarly result from Feynman diagrams with up to **22 loops**. A salient feature is the existence of intricate **quadratic** relations between Feynman integrals, encoded by **Betti** and **de Rham** matrices.

1. QED beyond polylogs: the 4-loop **Laporta** frontier
2. **Modular forms**, up to 6 loops [!]
3. **Betti** and **de Rham** matrices, for all loops [∗]
4. **L-series**, up to 22 loops [∗]

[!] With very recent **proofs** by **Yajun Zhou**, Princeton and Beijing.
[∗] Joint work with **David P. Roberts**, University of Minnesota Morris.

1 QED beyond polylogs: the 4-loop Laporta frontier

The **magnetic moment** of the electron, in Bohr magnetons, has quantum electrodynamic contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L=4$ loops by

$$\begin{aligned}
a_0 &= \quad 1 \qquad [\textbf{Genesis}: \text{Dirac, 1928}]\\
a_1 &= \quad 0.5 \quad\; [\textbf{Exodus}: \text{Schwinger, 1947}]\\
a_2 &= -0.32847896557919378458217281696489239241111929867962\ldots\\
a_3 &= \quad 1.18124145658720000627475398221287785336878939093213\ldots\\
a_4 &= -1.91224576492644557415264716743983005406087339065872\ldots
\end{aligned}$$

Leviticus: in 1957, Petermann and Sommerfield obtained

$$a_2 = \frac{197}{144} + \frac{\zeta(2)}{2} + \frac{3\zeta(3) - 2\pi^2 \log 2}{4}\,.$$

Numbers: in 1996, Laporta and Remiddi encountered the multiple polylog $U_{3,1} := \sum_{m>n>0}(-1)^{m+n}/(m^3 n)$ in

$$\begin{aligned}
a_3 \;=\; & \frac{28259}{5184} + \frac{17101\zeta(2)}{135} + \frac{139\zeta(3) - 596\pi^2 \log 2}{18}\\
& - \frac{39\zeta(4) + 400U_{3,1}}{24} - \frac{215\zeta(5) - 166\zeta(3)\zeta(2)}{24}\,.
\end{aligned}$$

Deuteronomy: the first non-polylog

A **Bessel moment** $B := \sqrt{3}E_{4a}$ occurs at weight 4 in the breath-taking evaluation by **Stefano Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals. U comes from 6 light-by-light master integrals, still under investigation.

The weight-4 **non-polylogarithm** at 4 loops has $N = 6$ Bessel functions:

$$\begin{aligned} B &= -\int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t) K_0^5(t) \mathrm{d}t \\ &= -1483.68505914882529459059985184510836700500152630607810\ldots \end{aligned}$$

with 5 instances of $K_0(t)$, from 5-fermion intermediate states. The sibling of $K_0(t)$ is $I_0(t) = \sum_{k\geq 0}((t/2)^k/k!)^2$, resulting from Fourier transformation. The powers of t in B are easy to interpret in $D = 2$ spacetime dimensions.

2 Modular forms up to 6 loops

With $N = a + b$ **Bessel** functions and $c \geq 0$, I define **moments**

$$M(a,b,c) := \int_0^\infty I_0^a(t) K_0^b(t) t^c \mathrm{d}t$$

that converge for $b > a \geq 0$. For $b = a = N/2$, we have convergence for $b > c + 1$. The L-loop on-shell **sunrise** diagram in $D = 2$ dimensions gives

$$2^L M(1, L+1, 1) = \int_0^\infty \ldots \int_0^\infty \frac{\Pi_{k=1}^L \mathrm{d}x_k / x_k}{(1 + \Sigma_{i=1}^L x_i)(1 + \Sigma_{j=1}^L 1/x_j) - 1}$$

as an integral over Schwinger parameters. $M(2, L, 1)$ is obtained by cutting an internal line. To obtain $M(1, L+1, 3)$ and $M(2, L, 3)$, we differentiate w.r.t. an external momentum, before taking the **on-shell** limit.

Very recently, in [arXiv:1706.08308], **Yajun Zhou** gave a complete **proof**, indicated by $\stackrel{!}{=}$, of my 10-year-old conjecture on the 5-Bessel matrix:

$$\mathcal{M}_5 := \left[\begin{array}{ll} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{array} \right] \stackrel{!}{=} \left[\begin{array}{ll} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{array} \right].$$

The **determinant** $\det \mathcal{M}_5 = 2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C := \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4 = \frac{1}{240\sqrt{5}\pi^2}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right)$$

with Γ values from the **square** of an elliptic integral [arXiv:0801.0891] at the 15th singular value. The **L-series** for $N=5$ Bessel functions comes from a **modular form** of weight 3 and level 15 [arXiv:1604.03057]:

$$\begin{aligned}
\eta_n &:= q^{n/24}\prod_{k>0}(1-q^{nk})\\
f_{3,15} &:= (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n\\
L_5(s) &:= \sum_{n>0}\frac{A_5(n)}{n^s} \quad \text{for } s>2\\
\Lambda_5(s) &:= \left(\frac{15}{\pi^2}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_5(s) = \Lambda_5(3-s)\\
L_5(1) &= \sum_{n>0}\frac{A_5(n)}{n}\left(2+\frac{\sqrt{15}}{2\pi n}\right)\exp\left(-\frac{2\pi n}{\sqrt{15}}\right)\\
&\stackrel{!}{=} 5C \stackrel{!}{=} \frac{5}{\pi^2}\int_0^\infty I_0(t)K_0^4(t)t\mathrm{d}t\,.
\end{aligned}$$

2.1 The Laporta frontier at $N = 6$

Here the **modular form**, found with **Francis Brown** in 2010, is

$$f_{4,6} := (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and **Yajun Zhou** proved that

$$2M(3,3,1) \stackrel{!}{=} 3L_6(2), \quad 2M(2,4,1) \stackrel{!}{=} 3L_6(3), \quad 2M(1,5,1) \stackrel{!}{=} \pi^2 L_6(2).$$

It is notable that the hypergeometric series in

$$L_6(3) = \frac{\pi^2}{15}\,{}_4F_3\left(\left.\begin{array}{cccc}\frac{1}{3}, & \frac{1}{2}, & \frac{1}{2}, & \frac{2}{3}\\ \frac{5}{6}, & 1, & \frac{7}{6} & \end{array}\right| 1\right)$$

does **not** appear in Laporta's final result, though $A_3 := 20L_6(3)/3$ appeared at intermediate stages of his calculation. Thus 4-loop QED engages only the **first** row of the **determinant** [arXiv:1604.03057]

$$\det\begin{bmatrix} M(1,5,1) & M(1,5,3)\\ M(2,4,1) & M(2,4,3)\end{bmatrix} = \frac{5\zeta(4)}{32}\,.$$

2.2 Kloosterman sums over finite fields

For $a \in \mathbf{F}_q$, with $q = p^k$, we define Kloosterman sums

$$K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right)$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c_N(q) := -\frac{1 + S_N(q)}{q^2}, \quad S_N(q) := \sum_{a \in \mathbf{F}_q^*} \sum_{k=0}^{N} [g(a)]^k [h(a)]^{N-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$. Then

$$Z_N(p, T) = \exp\left(-\sum_{k>0} \frac{c_N(p^k)}{k} T^k\right)$$

is a polynomial in T. For $N < 8$ and $s > (N-1)/2$, the L-series is

$$L_N(s) = \prod_p \frac{1}{Z_N(p, p^{-s})}.$$

With $N=7$ Bessel functions, the **local** factors at the **primes** in

$$L_7(s)=\prod_p \frac{1}{Z_7(p,p^{-s})} \quad \text{for } s>3$$

are given, for the **good** primes p coprime to 105, by the **cubic**

$$Z_7(p,T)=\left(1-\left(\frac{p}{105}\right)p^2T\right)\left(1+\left(\frac{p}{105}\right)(2p^2-|\lambda_p|^2)T+p^4T^2\right)$$

where $(\frac{p}{105})=\pm 1$ is a **Kronecker** symbol and λ_p is a Hecke eigenvalue of a weight-3 newform with level 525. For the primes of **bad** reduction, I obtained **quadratics** from **Kloosterman** moments in **finite fields**:

$$Z_7(3,T)=1-10T+3^4T^2,\ Z_7(5,T)=1-5^4T^2,\ Z_7(7,T)=1+70T+7^4T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s):=\left(\frac{105}{\pi^3}\right)^{s/2}\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_7(s)=\Lambda_7(5-s)$$

that was validated at high precision and gave us the empirical result

$$24M(2,5,1)=5\pi^2L_7(2).$$

2.3 Subtleties at $N=8$

With $N=8$ Bessel functions, the L-series comes from the **modular form**

$$f_{6,6} := \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and **Yajun Zhou** proved that

$$M(4,4,1) \stackrel{!}{=} L_8(3), \quad 4M(3,5,1) \stackrel{!}{=} 9L_8(4), \quad 4M(2,6,1) \stackrel{!}{=} 27L_8(5),$$

and $4M(1,7,1) \stackrel{!}{=} 9\pi^2 L_8(4)$ for the **6-loop sunrise** integral.

There are **two subtleties**. Kloosterman moments at $N=8$ do **not** deliver the local factors directly: in $L_8(s) = \prod_p Z_4(p,p^{2-s})/Z_8(p,p^{-s})$ we remove factors from $N=4$. Secondly, there is an infinite family of **sum rules**:

$$a(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left(\pi^2 I_0^2(t) - K_0^2(t)\right) I_0(t) K_0^5(t)\, (2t)^{2n-1} \mathrm{d}t$$

delivers the **integers** of `http://oeis.org/A262961` as was recently proved by **Yajun Zhou** in [arXiv:1706.01068].

2.4 Vacuum integrals and non-critical modular L-series

In the **modular** cases $N = 5, 6, 8$, L-series **outside** the critical strip are empirically related to **determinants** that contain **vacuum** integrals:

$$
\begin{aligned}
\det \int_0^\infty K_0^3(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{45}{8\pi^2} L_5(4) \\
\det \int_0^\infty K_0^4(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{27}{4\pi^2} L_6(5) \\
\det \int_0^\infty K_0^6(t) \left[\begin{array}{cc} K_0^2(t) & t^2(1-2t^2) K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2) I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{6075}{128\pi^2} L_8(7)\,.
\end{aligned}
$$

2.5 Signpost

In work at $N > 8$ with **David Roberts** these features are notable:
local factors from **Kloosterman** moments, sometimes with adjustment;
guesses of Γ factors, signs and conductors in **functional equations**;
empirical fits of L-series to **determinants** of Feynman integrals;
quadratic relations between Bessel moments; **sum rules** when $4|N$.

3 Betti and de Rham matrices for all loops

Construction: Let v_k and w_k be the rational numbers **generated** by

$$
\begin{aligned}
\frac{J_0^2(t)}{C(t)} &= \sum_{k\geq 0}\frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{17t^2}{54}+\frac{3781t^4}{186624}+\ldots\\
\frac{2J_0(t)J_1(t)}{tC(t)} &= \sum_{k\geq 0}\frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1-\frac{41t^2}{216}+\frac{325t^4}{186624}+\ldots
\end{aligned}
$$

where $J_0(t)=I_0(\mathrm{i}t)$, $J_1(t)=-J_0'(t)$ and

$$
C(t) := \frac{32(1-J_0^2(t)-tJ_0(t)J_1(t))}{3t^4} = 1-\frac{5t^2}{27}+\frac{35t^4}{2304}-\frac{7t^6}{9600}+\ldots
$$

We construct rational bivariate polynomials by the **recursion**

$$
\begin{aligned}
H_s(y,z) &= zH_{s-1}(y,z)-(s-1)yH_{s-2}(y,z)\\
&- \sum_{k=1}^{s-1}\binom{s-1}{k}\left(v_kH_{s-k}(y,z)-w_kzH_{s-k-1}(y,z)\right)
\end{aligned}
$$

for $s>0$, with $H_0(y,z)=1$. We use these to define

$$
d_s(N,c) := \frac{H_s(3c/2,N+2-2c)}{4^s s!}.
$$

Matrices: We construct rational **de Rham** matrices, with elements

$$D_N(a,b) := \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}$$

and a and b running from 1 to $k = \lceil N/2-1 \rceil$.
We act on those, on the left, with **period** matrices whose elements are

$$\begin{aligned} P_{2k+1}(u,a) &:= \frac{(-1)^{a-1}}{\pi^u} M(k+1-u, k+u, 2a-1) \\ P_{2k+2}(u,a) &:= \frac{(-1)^{a-1}}{\pi^{u+1/2}} M(k+1-u, k+1+u, 2a-1) \end{aligned}$$

and on the right with their `transposes`, to define **Betti** matrices

$$B_N := P_N D_N P_N^{\mathtt{tr}}.$$

Conjecture: The Betti matrices have **rational** elements given by

$$\begin{aligned} B_{2k+1}(u,v) &= (-1)^{u+k}2^{-2k-2}(k+u)!(k+v)!Z(u+v) \\ B_{2k+2}(u,v) &= (-1)^{u+k}2^{-2k-3}(k+u+1)!(k+v+1)!Z(u+v+1) \\ Z(m) &= \frac{1+(-1)^m}{(2\pi)^m}\zeta(m). \end{aligned}$$

4 L-series up to 22 loops

Let $\Omega_{a,b}$ be the **determinant** of the $r \times r$ matrix with $M(a,b,1)$ at top left, size $r = \lceil (a+b)/4 - 1 \rceil$, powers of t^2 increasing to the right and powers of $I_0^2(t)$ increasing downwards. Thus $\Omega_{1,23}$ is a 5×5 determinant with the **22-loop sunrise** integral $M(1,23,1)$ at **top left** and $M(9,15,9)$ at bottom right. With $N = 4r+4$ Bessel functions, we discovered that

$$\begin{aligned}
L_8(4) &\stackrel{!}{=} \frac{2^2\,\Omega_{1,7}}{3^2\pi^2} \equiv \frac{4}{9\pi^2}\int_0^\infty I_0(t)K_0^7(t)t\mathrm{d}t \\
L_{12}(6) &= \frac{2^6\,\Omega_{1,11}}{3^4 \times 5\pi^6} \\
L_{16}(8) &= \frac{2^{14}\,\Omega_{1,15}}{3^7 \times 5^2 \times 7\pi^{12}} \\
L_{20}(10) &= \frac{2^{22} \times 11 \times \mathbf{131}\,\Omega_{1,19}}{3^{11} \times 5^6 \times 7^3\pi^{20}} \qquad \text{to 44 digits} \\
L_{24}(12) &= \frac{2^{29} \times \mathbf{12558877}\,\Omega_{1,23}}{3^{19} \times 5^9 \times 7^3 \times 11\pi^{30}} \qquad \text{to 19 digits,}
\end{aligned}$$

where boldface highlights **primes** greater than N. We used **Kloosterman sums** over finite fields $\mathbf{F}_q$ with $q < 280000$. **30 GHz-years** of work, on 50 cores, gave 44-digit **precision** for $L_{20}(10)$. $L_{24}(12)$ agrees up to 19 digits.

With a **cut** of a line in the diagram at top left of the matrix, we found

$$\begin{aligned}
L_8(5) &\stackrel{!}{=} \frac{2^2\,\Omega_{2,6}}{3^3} \equiv \frac{4}{27}\int_0^\infty I_0^2(t)K_0^6(t)t\mathrm{d}t\\
L_{12}(7) &= \frac{2^5\times 11\,\Omega_{2,10}}{3^6\times 5^2\pi^2}\\
L_{16}(9) &= \frac{2^{14}\times 13\,\Omega_{2,14}}{3^9\times 5^3\times 7^2\pi^6}\\
L_{20}(11) &= \frac{2^{19}\times 17\times 19\times \mathbf{23}\,\Omega_{2,18}}{3^{13}\times 5^7\times 7^3\pi^{12}}\\
L_{24}(13) &= \frac{2^{27}\times 17\times 19^2\times 23^2\times \mathbf{46681}\,\Omega_{2,22}}{3^{23}\times 5^{12}\times 7^4\times 11^2\pi^{20}}\,.
\end{aligned}$$

At $N=12,16,20$, with an **odd** sign in the functional equation, we found

$$\begin{aligned}
-L_{12}'(5) &= \frac{2^4\left(2^6\times \mathbf{29}\,\widehat{\Omega}_{2,10}+3\,\Omega_{2,10}\log 2\right)}{3^2\times 7\pi^6}\\
-L_{16}'(7) &= \frac{2^9\left(2^7\times \mathbf{83}\,\widehat{\Omega}_{2,14}+3\times 11\,\Omega_{2,14}\log 2\right)}{3^5\times 5\pi^{12}}\\
-L_{20}'(9) &= \frac{2^{17}\times 17\times 19\left(2^9\times 7\times \mathbf{101}\,\widehat{\Omega}_{2,18}+5\times 13\,\Omega_{2,18}\log 2\right)}{3^8\times 5^4\times 7^2\times 11\pi^{20}}
\end{aligned}$$

for **central derivatives**, using **enlarged** determinants $\widehat{\Omega}_{2,4r+2}$ of size $r+1$ with **regularization** of $M(2r+2,2r+2,2r+1)$ at bottom right.

In the cases with $N = 4r + 2$, we obtained

$$\begin{aligned}
L_6(2) &\stackrel{!}{=} \frac{2\,\Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2}\int_0^\infty I_0(t)K_0^5(t)t\mathrm{d}t \quad \text{[present in a}_4\text{]}\\
L_6(3) &\stackrel{!}{=} \frac{2\,\Omega_{2,4}}{3} \equiv \frac{2}{3}\int_0^\infty I_0^2(t)K_0^4(t)t\mathrm{d}t \quad \text{[absent from a}_4\text{]}\\
L_{10}(4) &= \frac{2^7\,\Omega_{1,9}}{3^2\pi^6}\\
L_{10}(5) &= \frac{2^4\,\Omega_{2,8}}{3\times 5\pi^2}\\
L_{14}(6) &= 0\\
L_{14}(7) &= \frac{2^{10}\times 11\times 13\,\Omega_{2,12}}{3^6\times 5^2\times 7\pi^6}\\
L_{18}(8) &= \frac{2^{21}\times 17\times \mathbf{19}\,\Omega_{1,17}}{3^5\times 5^4\times 7\pi^{20}}\\
L_{18}(9) &= \frac{2^{12}\times 13\times 17\times \mathbf{41}\,\Omega_{2,16}}{3^8\times 5^3\times 7^2\pi^{12}}\\
L_{22}(10) &= 0\\
L_{22}(11) &= \frac{2^{23}\times 17\times 19\times \mathbf{11621}\,\Omega_{2,20}}{3^{14}\times 5^7\times 7^3\pi^{20}}
\end{aligned}$$

with central vanishing from an odd sign at $N = 14$ and $N = 22$.

For cases with odd N, we obtained

$$\begin{aligned}
L_5(2) &\stackrel{!}{=} \frac{2^2\,\Omega_{2,3}}{3} \equiv \frac{4}{3}\int_0^\infty I_0^2(t)K_0^3(t)t\mathrm{d}t\\
L_7(2) &= \frac{2^3\times 3\,\Omega_{2,5}}{5\pi^2} \equiv \frac{24}{5\pi^2}\int_0^\infty I_0^2(t)K_0^5(t)t\mathrm{d}t\\
L_9(4) &= \frac{2^6\,\Omega_{2,7}}{3\times 5\pi^2}\\
L_{11}(4) &= \frac{2^8\times 5\,\Omega_{2,9}}{3\times 7\pi^6}\\
L_{13}(6) &= \frac{2^7\times \mathbf{149}\,\Omega_{2,11}}{3^3\times 5\times 7\pi^6}\\
L_{15}(6) &= \frac{2^8\times 7\times \mathbf{53}\,\Omega_{2,13}}{3^2\times 5\pi^{12}} && \text{to 43 digits}\\
L_{17}(8) &= \frac{2^{15}\times \mathbf{29}\,\Omega_{2,15}}{3^5\times 5^2\times 7\pi^{12}} && \text{to 23 digits}\\
L_{19}(8) &= \frac{2^{14}\times \mathbf{1093}\times \mathbf{13171}\,\Omega_{2,17}}{3^4\times 5^4\times 7\times 11\pi^{20}} && \text{to 14 digits.}
\end{aligned}$$

Comment: We also have results relating Bessel moments $M(a,b,c)$ with **even** c to L-series from Kloosterman moments with a quadratic **twist**.

Summary

1. **QED** at 4 loops involves **Bessel** moments and a weight-4 **L-series**.
2. The L-series for 5, 6 and 8 Bessel functions are **modular**. This seems to be necessary for relating **vacuum** integrals to **non**-critical L-series.
3. There are **quadratic** relations of the form $P_N D_N P_N^{\rm tr} = B_N$ with **period**, **de Rham** and **Betti** matrices that we have specified.
4. Relations between **determinants** of Feynman integrals and **L-series** have been discovered up to **22 loops** and presumably go on for **ever**.

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