Sudakov Form Factor and Colour-Kinematics Duality at Five Loops

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Based on:

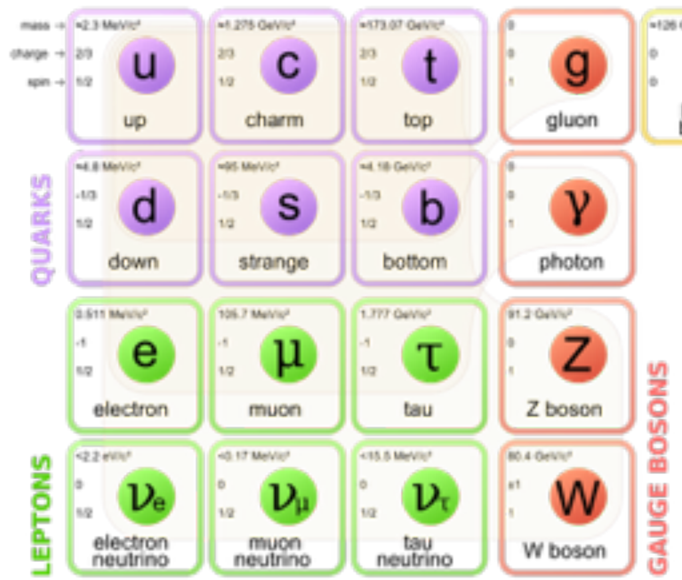
- JHEP 1302 (2013) 063 with R. Boels, B. Kniehl, O. Tarasov
- PRL 117 (2016) 27
- also related work: 1705.03444 with R. Boels, T. Huber

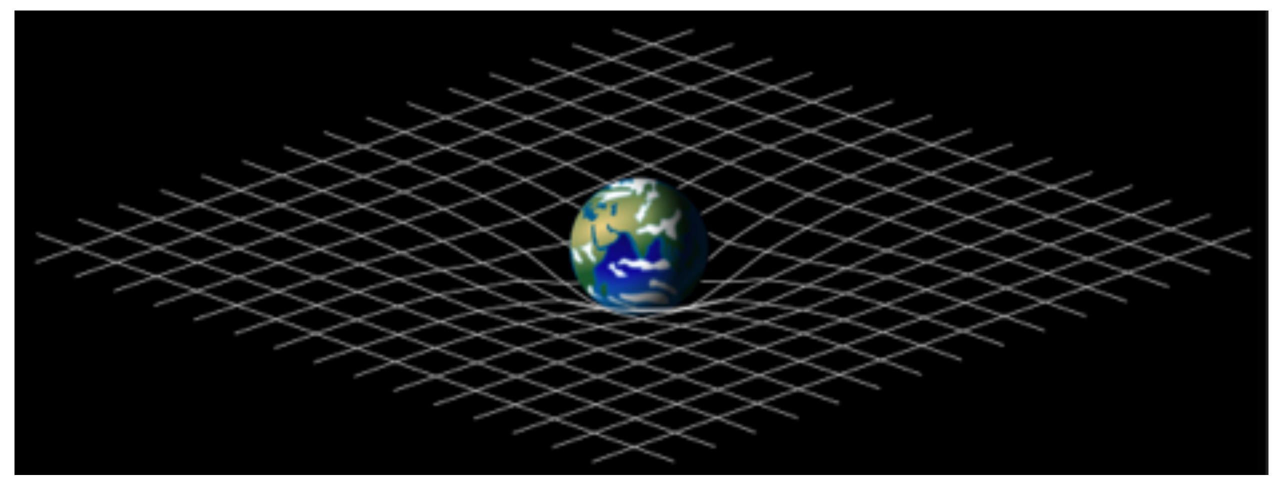
Amplitudes
July 10-14, 2017

Higgs Centre
Edinburgh

gauge theory

gravity





Confinement

Quantisation

Challenge of high loop computations

Infrared divergences

QCD factorisation

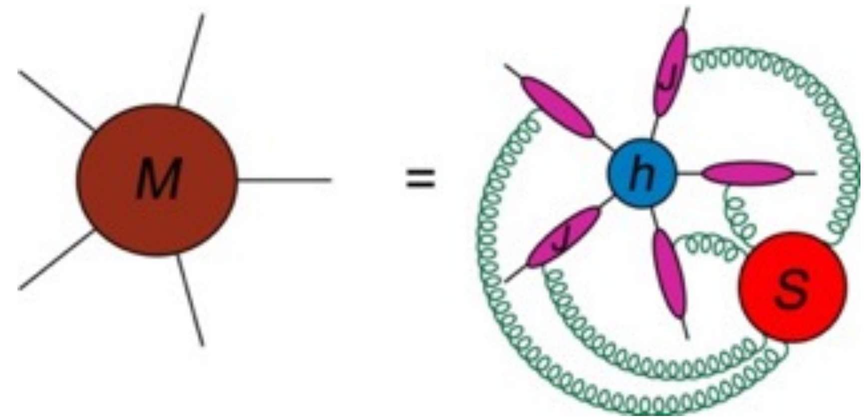


figure from L.Dixon 1105.0771

IR structure of amplitudes: **see e.g. Gardi, Magnea; Becher, Neubert 2009; … and Vernazza's talk**

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon\right)\right]^{1/2} \times h_n\left(k_i, \mu, \alpha_s, \epsilon\right)$$

Sudakov form factor $= \exp\left[-\frac{1}{4} \sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2}\right)^{l\epsilon} \left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon} \right) + \text{finite} \right]$

Leading IR singularity -> Cusp anomalous dimension

Non-planar correction starts at 4 loop!

see Boels' talk tomorrow

Ultraviolet divergences

A general belief:
gravity theories are non-renormalisable and must be UV divergent from certain loops.

This is based on power counting of loop momentum of individual Feynman diagrams.

Surprising UV finiteness! **[see Roiban's talk]**

Is N=8 SUGRA UV finite? **Bern, Dixon, Roiban 2006**

Bets on UV divergences

Ongoing bets: [Bossard, Howe, Stelle; Green, Russo, Vanhove; Green, Bjornsson; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Bossard, Hillmann, Nicolai; Ramond, Kallosh; …]

- At 5-loop in D=24/5, does N=8 supergravity diverge?
- At 7-loop in D=4, does N=8 supergravity diverge?

5 loops

Kelly Stelle:
English wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

7 loops

David Gross:
California wine
"It will diverge"

Zvi Bern:
California wine
"It won't diverge"

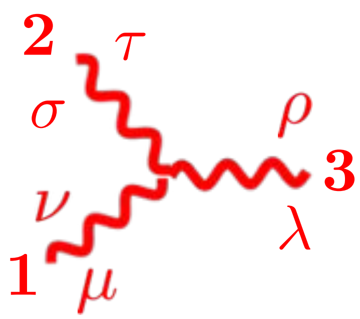
pictures from Z.Bern's talk at MHV30

Feynman diagram?

by JAMES O'BRIEN FOR QUANTA MAGAZINE

Feynman diagram method works in principle, but the complexity grows extremely fast with increasing number of external legs / loops.

Feynman diagram?



more than
100 terms

[DeWitt 1967]

$$
\begin{aligned}
&\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \quad\rightarrow\quad 2\eta^{\mu\tau}\eta^{\nu\sigma}{k_1}^{\lambda}{k_1}^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}{k_1}^{\lambda}{k_1}^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}{k_1}^{\lambda}{k_1}^{\rho} + \\
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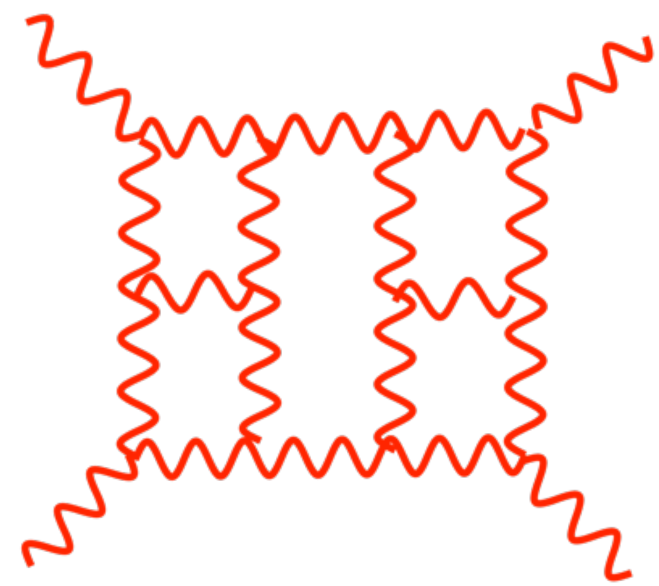
\end{aligned}
$$

Gravity



a single 5-loop diagram

more than 10^{30} terms !

Planar (large N_c) Limit

Remarkable progress in planar N=4 SYM:

possible for 6-loop and beyond [see e.g. Papathanasiou's talk]

Some observables even known to all order: CAD, …

[Beisert, Eden, Staudacher 2006; …]

Dual conformal symmetry, Integrability, AdS/CFT correspondence

Challenge: generalisation to non-planar

Non-planar is necessary

QCD: $N_c = 3$

Gravity is intrinsically non-planar

Colour-kinematics duality [Bern, Carrasco, Johansson 2008]

We provide a five-loop construction for Sudakov form factor in N=4 SYM.

Colour-Kinematics duality

A duality between:

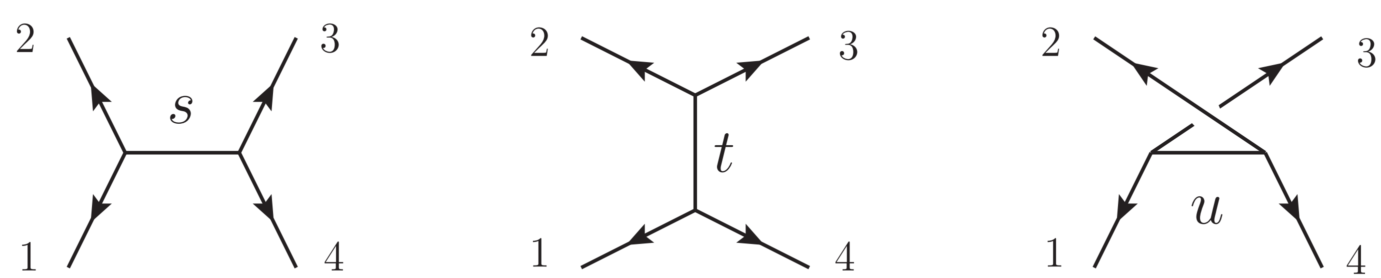
[Bern, Carrasco, Johansson 2008]

colour factor $\longleftrightarrow$ momentum numerator

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c) \qquad s_{ij} = (p_i + p_j)^2$$

Claim (conjecture):

There is a cubic graph representation of amplitudes such that colour and kinematics satisfy the same algebraic equations.



$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity — dual Jacobi relation

Power of the duality

Obtain **non-planar** from **planar** "for free"!

Obtain **gravity** from **YM** for free!
(Once having the CK duality) [Bern, Carrasco, Johansson 2008]

$$c_k\, n_k \longrightarrow n_k\, n_k$$

similar spirit in: [Kawai, Lewellen, Tye 1986]
[Cachazo, He, Yuan 2013]

Proof case by case

It is **still a conjecture** at loop level, and relies on explicit construction.

examples in various gauge and gravity theories, including QCD @2-loop

[see Roiban's talk]

Challenge in studying new cases:
without a underlying principle, it is a priori NOT guaranteed to work $\longrightarrow$ effort + luck

Five-loop construction

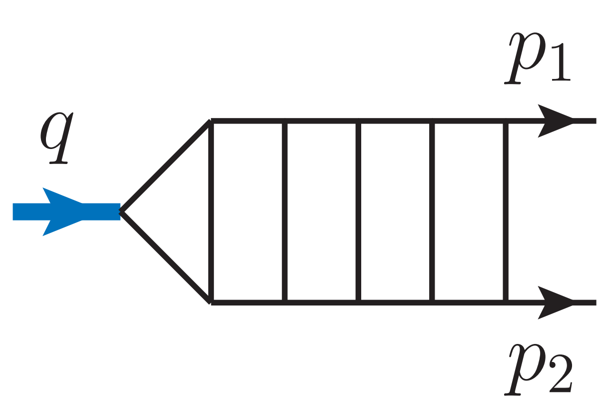
Sudakov form factor in N=4 SYM

Operator in the stress tensor supermultiplet

$\longrightarrow$ UV finite at four dimensions

e.g. $\langle \phi(p_1)\phi(p_2)|\mathrm{Tr}(\phi^2)|0\rangle$ [see Brandhuber's talk]

A five-loop graph:



$$p_1^2 = p_2^2 = 0, \quad q^2 = (p_1+p_2)^2 \neq 0$$

Strategy

Colour-kinematic duality

See e.g.:
Bern, Carrasco, Dixon, Johansson, Roiban 2012;
Boels, Kniehl, Tarasov, GY 2012;
Carrasco 2015

provides an **ansatz** of the integrand

Unitarity

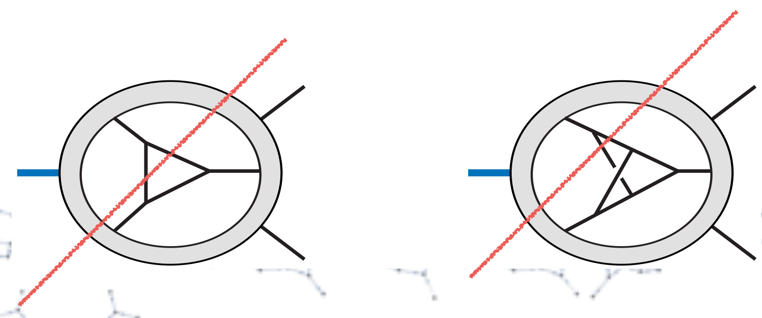
provides physical constraints as well as checks

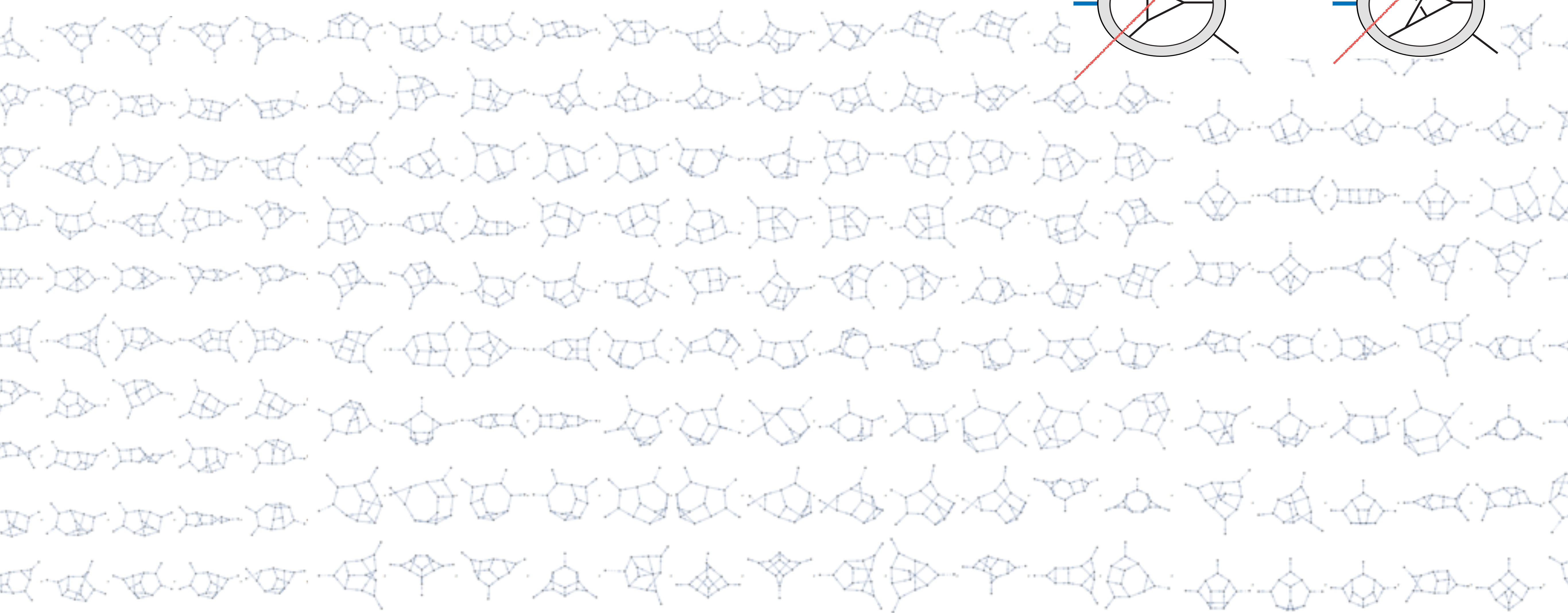
- A linear algebra problem
- Integrand results in a compact form

Cubic graphs

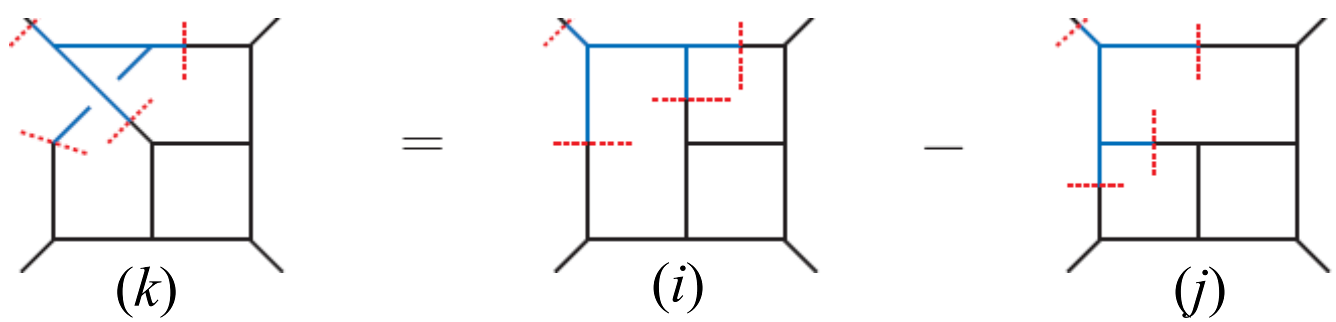
There are 306 trivalent topologies to consider.

For N=4 SYM: exclude those containing tadpole, bubble and triangle one-loop subgraphs.





Dual Jacobi relations



$$c_k = c_i - c_j \longrightarrow n_k = n_i - n_j$$

Jacobi identity

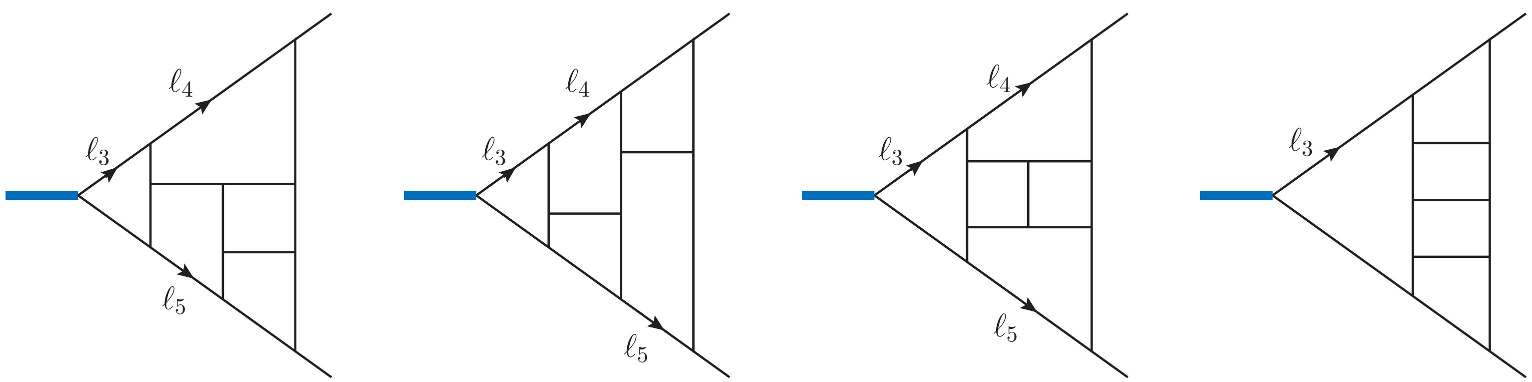
dual Jacobi relation

Every propagator provides such a relation.

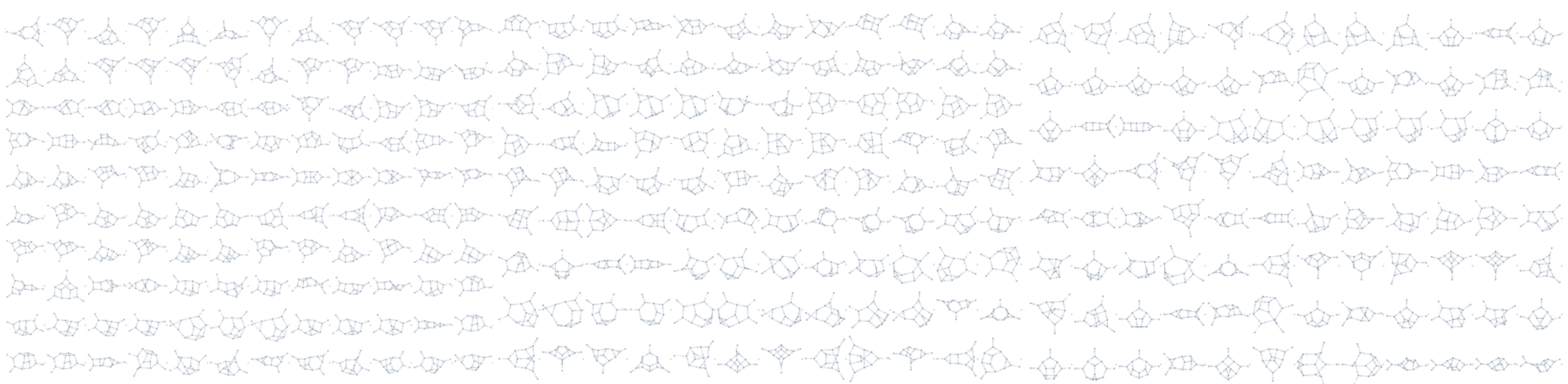
A highly constrained set of equations for the numerators.

Master graphs

Four master graphs obtained via dual Jacobi relation:



All other graphs can be generated from the master graphs by using dual Jacobi relations.



Statistics

Boels, Kniehl, Tarasov, GY 2012;
GY 2016

TABLE I. Number of cubic and master graphs up to five loops.

L loops	L=1	L=2	L=3	L=4	L=5
# of topologies	1	2	6	34	306
# of planar masters	1	1	1	2	4

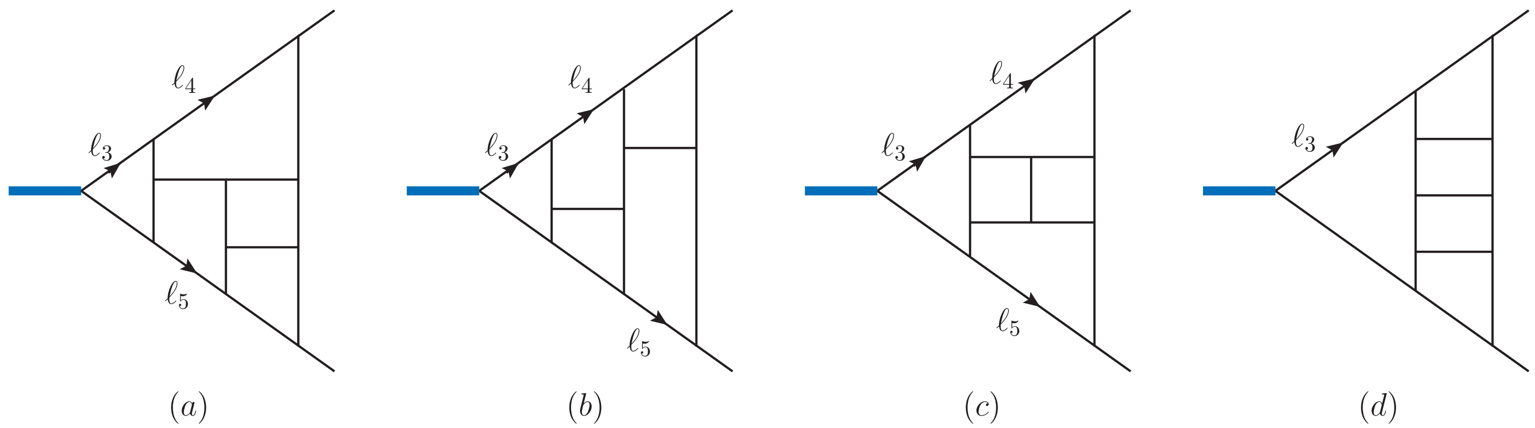
Compare to four-point amplitude in N=4 SYM:

L loops	L=1	L=2	L=3	L=4
# of topologies	1	2	12	85
# of masters	1	1	1	2

Bern, Carrasco, Johansson 2008;
Bern, Carrasco, Dixon, Johansson, Roiban 2012

Ansatz of master graphs

Power counting property for N=4 SYM:



$$M^{(\mathrm{a})} = \{\tau_{i3}\tau_{j4}\tau_{k5},\ s_{12}\tau_{i3}\tau_{45},\ s_{12}\tau_{i4}\tau_{35},\ s_{12}\tau_{i5}\tau_{34},\ s_{12}\tau_{i3}\tau_{j4},\ s_{12}\tau_{i3}\tau_{j5},\ s_{12}\tau_{i4}\tau_{j5},\ s_{12}^2\tau_{34},\ s_{12}^2\tau_{35},\ s_{12}^2\tau_{45},\ s_{12}^2\tau_{i3},\ s_{12}^2\tau_{i4},\ s_{12}^2\tau_{i5}, s_{12}^3\}$$

$$N^{(a)} = \sum_{j=1}^{36} a_j M_j^{(a)}$$

parameters
5 master -> in total 162

$\tau_{ij} = 2k_i \cdot \ell_j$ or $2\ell_i \cdot \ell_j$

With dual Jacobi equations: $N_s = N_t + N_u$

we get numerators of all other graphs

Full ansatz

$$\mathcal{F}_2^{5\text{-loop}} = s_{12}^2\, F_2^{\text{tree}} \sum_{\sigma_2} \sum_{i=1}^{306} \int \prod_j^L d^D \ell_j \, \frac{1}{S_i} \, \frac{C_i \, N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

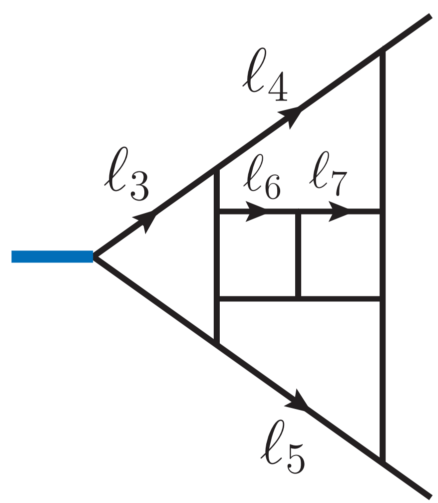
The numerators depend linearly on 162 parameters

We need to fix these parameters:

- Automorphism symmetry
- Unitarity constraints

Automorphism symmetry

Example:



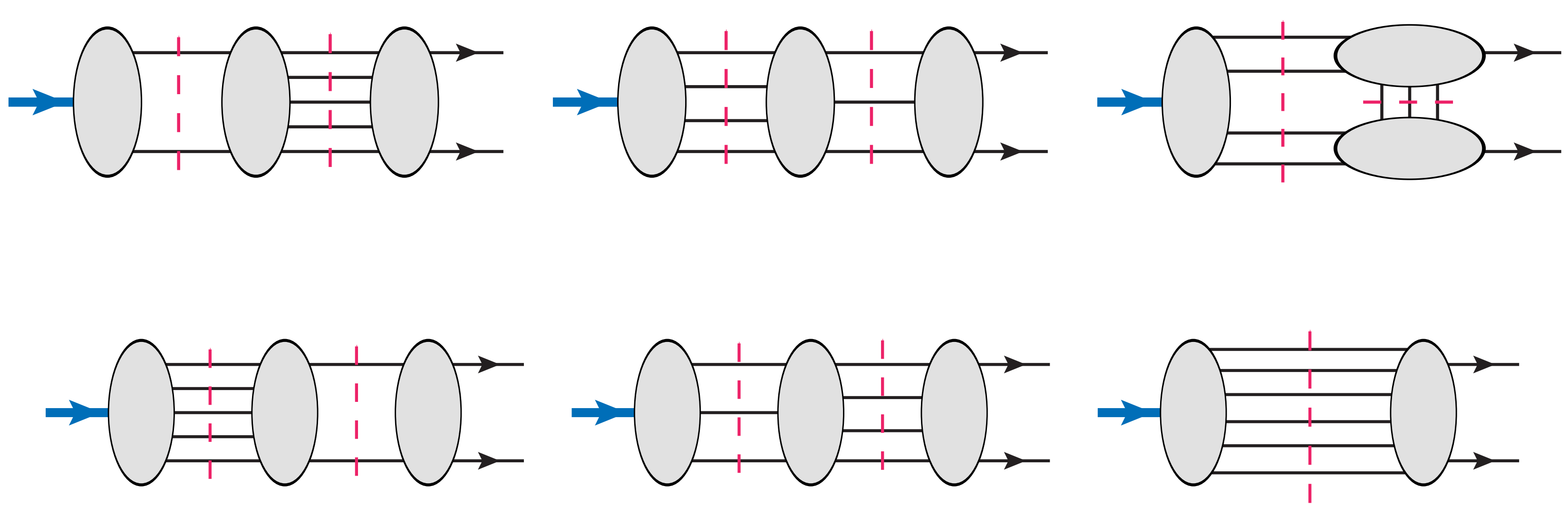
$$\{\ell_3 \to p_1 + p_2 - \ell_3, \quad \ell_4 \leftrightarrow \ell_5,$$
$$\ell_6 \to p_1 + p_2 - \ell_4 - \ell_5 - \ell_6,$$
$$\ell_7 \to p_1 + p_2 - \ell_4 - \ell_5 - \ell_7\}.$$

Fix 115 parameters!

Unitarity checks

$$\text{cut}\big(\sum \text{cubic graphs}\big) = \sum_{\text{states}} F^{\text{tree}} \prod_I A_I^{\text{tree}}$$

from the ansatz



Fixing parameters

Starting point: 162

$\downarrow$

Automorphism symmetry:

47

$\downarrow$

Simple maximal cuts:

20

$\downarrow$

All dual Jacobi relations:

10

$\downarrow$

Non-trivial cuts:

$\downarrow$

3

Local poles:

$\downarrow$

1

Final result

[GY 2016]

$$\mathcal{F}_2^{\text{5-loop}} = s_{12}^2 \, F_2^{\text{tree}} \sum_{\sigma_2} \sum_{i=1}^{306} \int \prod_j^L d^D \ell_j \, \frac{1}{S_i} \, \frac{C_i \, N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

Numerators saturate the same finiteness bound for N=4 SYM amplitudes:

$$D < 4 + \frac{6}{L} \,, \qquad L > 1$$

Double copy indicate possible UV divergences at D=22/5.

(possible enhanced cancellation after integration?)

Six loops?

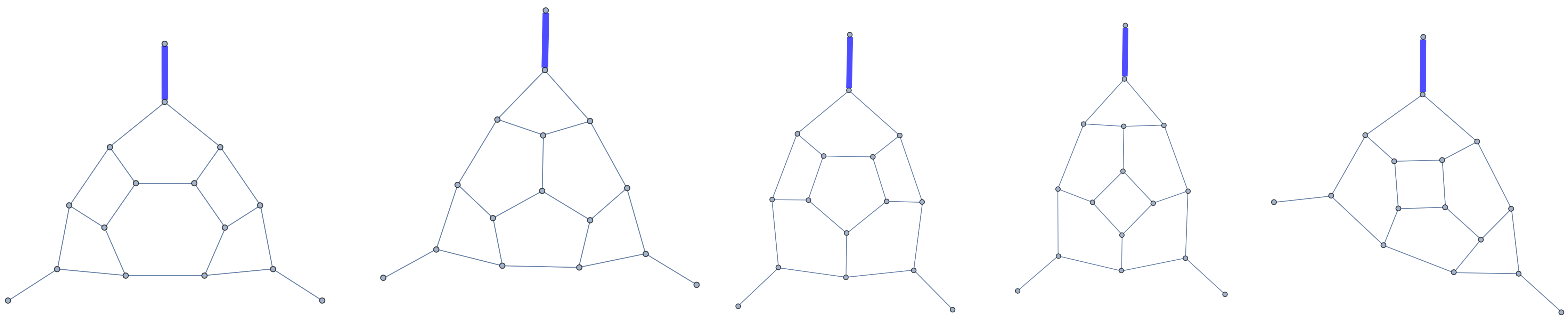
Interesting: contain 5-loop four-point amplitude!

Six loops

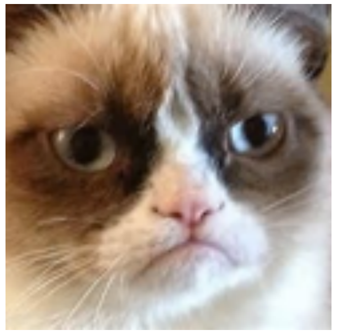
Try as in the lower loops:

~**3000** trivalent topologies

5 masters, ansatz with ~**1400** parameters



fail even for some maximal cuts



(more efforts needed)

Summary

- First five-loop realisation of the colour-kinematics duality
- Compact four/five-loop integrand of Sudakov form factor in N=4 SYM
- Potential implication to five-loop UV property of N=8 SUGRA (via double copy)

Outlook

- Integration: UV -> Enhanced cancellation?
  IR -> non-planar cusp AD

  see Rutger Boels' talk tomorrow

- Interpretation for double-copy for form factor?
- Six-loop?
- Generalisation to other operators/theories(QCD)?

What is the underlying principle of the duality?

Thank you for your attention!