From 4d Ambitwistor Strings to On-Shell Diagrams and Back

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Overview

- In recent years, several new approaches have been developed which compute amplitudes more efficiently and reveal new mathematical structure.

- In this talk, will explore the relationship between 4d ambitwistor string theory and on-shell diagrams for N=4 super-Yang-Mills (SYM) and N=8 supergravity (SUGRA).

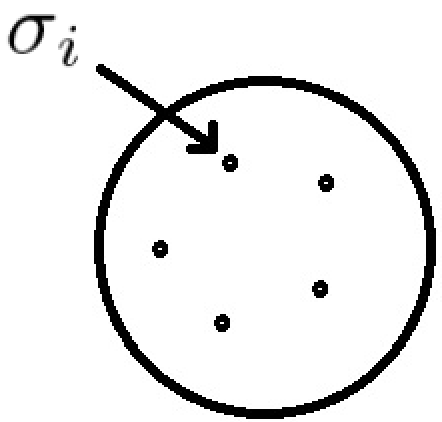
- We obtain new formulae at tree-level and 1-loop.

Scattering Equations

external momentum

$$\sum_{i \neq j} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

point on 2-sphere



- First discovered by Fairlie and Roberts.
- Gross,Mende: arise from the tensionless limit of string amplitudes
- Cachazo,He,Yuan: underlie the scattering amplitudes of massless particles in any dimension!

Ambitwistor Strings

- Mason,Skinner: Amplitudes of massless point particles can be computed using a chiral, infinite tension limit of string theory:

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{e}{2} P_\mu P^\mu + \ldots$$

- Critical in d=26 (bosonic) and d=10 (superstring)
- Tree-level correlators reproduce the CHY formulae!
- Recent progress for higher genus (Adamo,Casali,Skinner/ Geyer,Mason,Monteiro,Tourkine/Baadsgaard,Bjerrum-Bohr, Bourjaily,Caron-Huot,Damgaard,Feng)

4d Ambitwistor Strings

- Action: (Berkovits/Witten/Skinner)

$$\mathcal{L} = W_A \bar{\partial} Z^B + \tilde{\rho}_A \bar{\partial} \rho^A$$

$$Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \\ \chi^a \end{pmatrix}, \quad W_A = \begin{pmatrix} \tilde{\mu}^\alpha \\ \tilde{\lambda}_{\dot{\alpha}} \\ \tilde{\chi}_a \end{pmatrix}, \quad \rho^A = \begin{pmatrix} \rho_\alpha \\ \rho^{\dot{\alpha}} \\ \omega^a \end{pmatrix}, \quad \tilde{\rho}_A = \begin{pmatrix} \tilde{\rho}^\alpha \\ \tilde{\rho}_{\dot{\alpha}} \\ \tilde{\omega}_a \end{pmatrix}$$

- Whereas twistor strings give rise to tree-level amplitudes as integrals over curves in twistor space (Roiban,Spradlin,Volovich), ambitwistor strings give rise to formulae supported on scattering equations refined by MHV degree (Geyer,Lipstein,Mason)
- Recent generalization to form factors (He,Liu,Zhang/Brandhuber,Hughes, Panerai,Spence,Travaglini)

Tree Amplitudes

- N=4 SYM:

$$\mathcal{A}_{n,k}^{(0)} = \int \frac{1}{GL(2)} \prod_{i=1}^{n} \frac{d^2\sigma_i}{(i\,i{+}1)} \prod_l \delta^{2|4}\left(\tilde{\lambda}_l - \sum_r \frac{\tilde{\lambda}_r}{(lr)}\right) \prod_r \delta^2\left(\lambda_r - \sum_l \frac{\lambda_l}{(rl)}\right)$$

- N=8 SUGRA

$$\mathcal{M}_{n,k}^{(0)} = \int \frac{\prod_{i=1}^{n} d^2\sigma_i}{GL(2)} \det{}'H \ \det{}'\tilde{H} \prod_l \delta^{2|8}\left(\tilde{\lambda}_l - \sum_r \frac{\tilde{\lambda}_r}{(lr)}\right) \prod_r \delta^2\left(\lambda_r - \sum_l \frac{\lambda_l}{(rl)}\right)$$

where $\sigma_i^\alpha = t_i^{-1}(1, \sigma_i)$

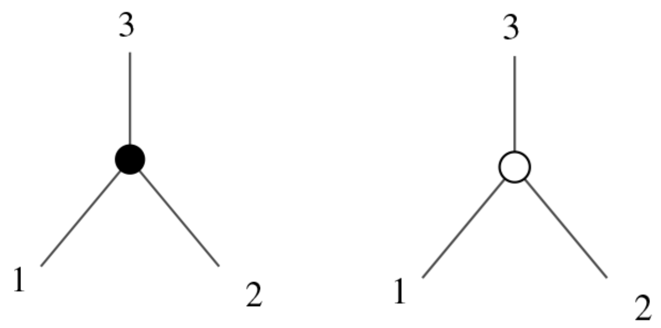
$(ij) = \sigma_i^\alpha \sigma_j^\beta \epsilon_{\alpha\beta}$

$l \in \{1, ..., k\}$ and $r \in \{k+1, ..., n\}$

On-Shell Diagrams

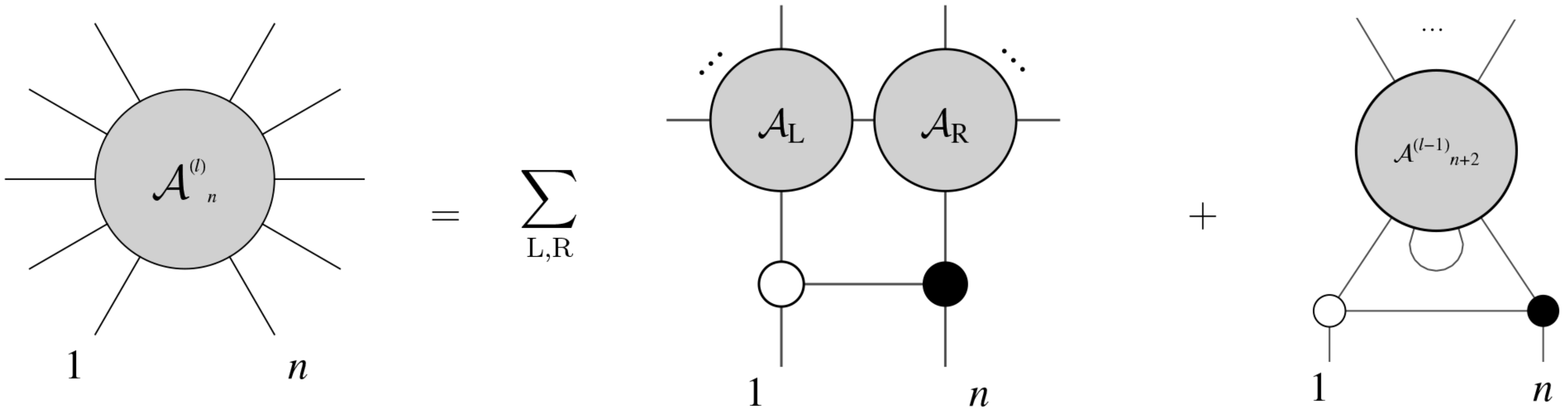
- First developed for planar N=4 SYM (Arkani-Hamed,Bourjaily,Cachazo, Goncharov,Postnikov,Trnka) and recently generalized to N=8 SUGRA (Heslop,Lipstein/Herrmann,Trnka)

- Bipartite graphs built recursively out of 3-point amplitudes:



- No virtual particles!

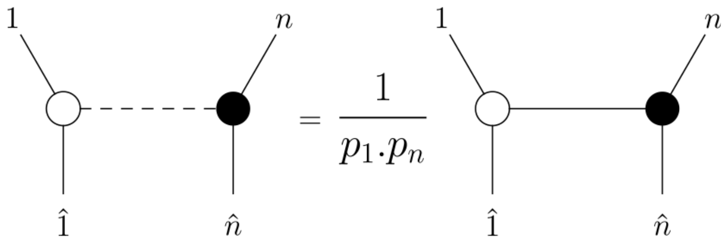
N=4 On-Shell Diagrams



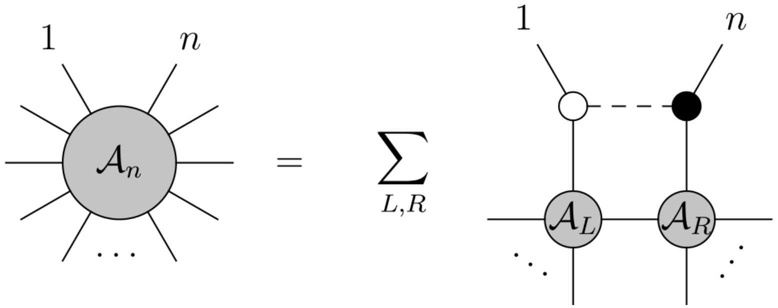
- First term encodes tree-level BCFW recursion (Britto,Cachazo,Feng,Witten).
- Second term seeds loop-level recursion (Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka)

N=8 On-Shell Diagrams

- Naïve BCFW bridge doesn’t work; need to decorate it!



- Tree-level recursion:



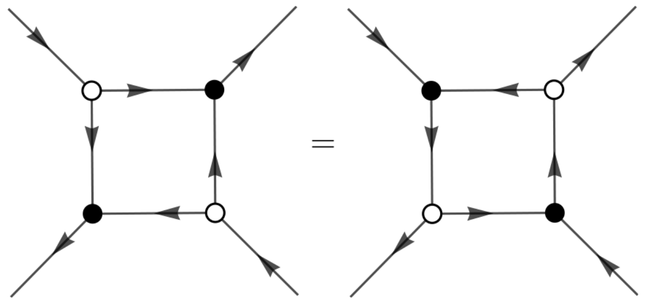
where the sum is over all partitions of particles {2,…,n-1} into L,R.

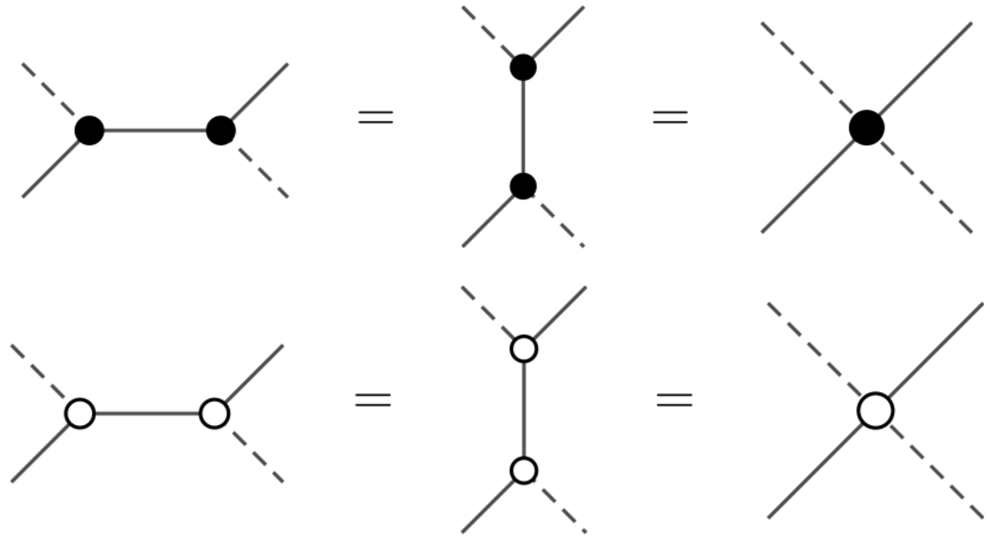
- Can modify the bridge decoration to incorporate bonus relations.

Relation to N=4

- The N=8 recursion will generally give non-planar diagrams, but it is possible to restrict it to a planar sector and sum the final answer over permutations.

- N=8 on-shell diagrams enjoy similar equivalence relations, notably the square move and decorated mergers:





Grassmannians

- On-shell diagrams naturally give rise to formulas for N^{k-2}MHV amplitudes in the form of integrals over k-planes in n dimensions, or Gr(k,n).
- Given a k-plane C, define $C^{\perp}$ to be the orthogonal (n-k) plane whose minors satisfy

$$(i_{k+1}\ldots i_n)^{\perp} = (i_1 i_2 \ldots i_k)\epsilon^{i_1 i_2 \ldots i_k}{}_{i_{k+1}\ldots i_n}$$

- The formulas are supported on

$$\delta^{k\times(2|\mathcal{N})}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^{2\times(n-k)}\left(C^{\perp}\cdot\lambda\right)$$

- Suggest a new geometric interpretation of scattering amplitudes as the volume of an “Amplituhedron” (Arkani-Hamed,Trnka)

4d Ambitwistor Strings vs On-Shell Diagrams

- Choose local coordinates on Gr(k,n): (link variables)

$$C = \left(\begin{array}{cc} \mathbb{I}_{k\times k} & c_{k\times(n-k)} \end{array} \right)$$

then

$$\delta^{k\times(2|\mathcal{N})}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^{2\times(n-k)}\left(C^{\perp}\cdot\lambda\right)$$

has the same form as the refined scattering equations if

$$c_{lr}\rightarrow\frac{1}{(lr)}$$

- For non-MHV amplitudes, the mapping is not so simple because one must specify a contour in the Grassmannian.

MHV Amplitudes

- To map the ambitwistor string formulae to a Grassmannian integral formula, insert

$$1 = \int \Pi_{l=1}^{k} \Pi_{r=k+1}^{n} \mathrm{d}c_{lr} \delta \left(c_{lr} - \frac{1}{(lr)} \right)$$

and integrate out the worldsheet coordinates, leaving an integral over local coordinates of the Grassmannian.

- For N=4 SYM, this gives the well-known formula

$$\mathcal{A}_{n,2}^{(0)} = \int \frac{d^{2\times n}C}{GL(2)} \frac{1}{(12)...(n1)} \delta^{2\times(2|4)} \left(C \cdot \lambda\right) \delta^{(n-2)\times 2} \left(\lambda \cdot C^{\perp}\right)$$

first obtained by Arkani-Hamed,Cachazo,Cheung,Kaplan

- For N=8 MHV amplitudes, this procedure gives

$$\mathcal{M}_n^{k=2} = \int \frac{d^{2\times n}C}{GL(2)} \frac{\langle pq\rangle}{(pq)} \frac{\det \tilde{H}}{(ab)^2(bc)^2(ca)^2} \delta^{4|16}\left(C\cdot\tilde{\lambda}\right)\delta^{2(n-2)}\left(\lambda\cdot C^{\perp}\right)$$

where H is an (n-3)x(n-3) matrix first obtained by Hodges:

$$\tilde{H}_{ij} = \frac{[ij]}{(ij)}, \quad i \neq j \qquad \tilde{H}_{ii} = \sum_{j=1, j\notin\{a,b,i\}}^{n} \frac{[ij]}{(ij)} \frac{(aj)(bj)}{(ai)(bi)} \qquad i,j \notin \{a,b,c\}$$

- Note that $\frac{\langle pq\rangle}{(pq)}$ is independent of p,q and ensures GL(2) symmetry.

non-MHV Amplitudes

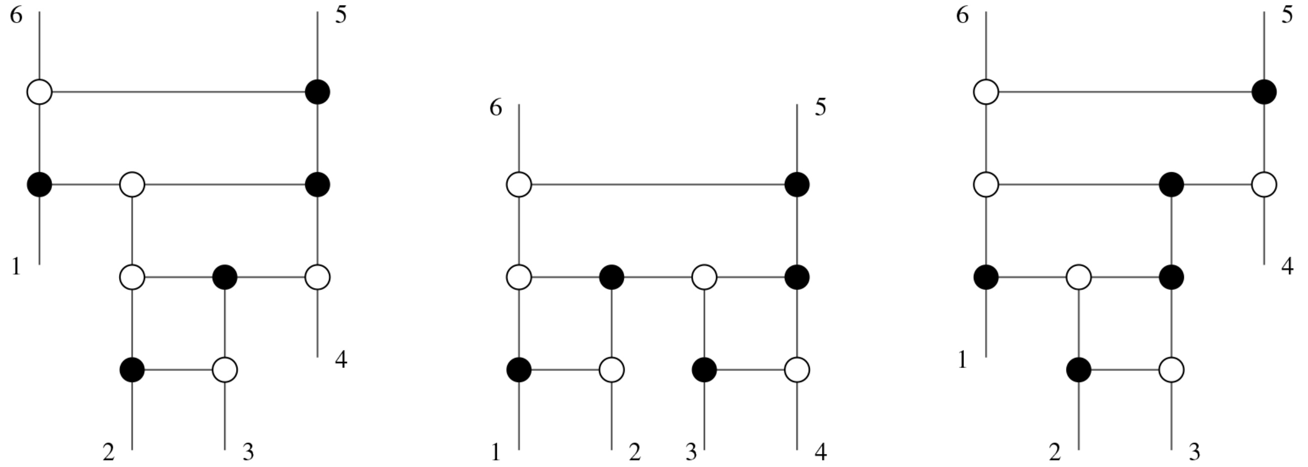
- Grassmannian formulas for N^{k-2}MHV amplitudes contain k(n-k) integrals and 2n-4 bosonic delta functions, so one must specify a (k-2)(n-k-2)-dimensional contour in order for the integrals to be well-defined.

- A contour can be deduced from on-shell diagrams or 4d ambitwistor string theory.

- For N=4 SYM, the two contours are related by global residue theorems (Spradlin,Volovich,Nandan,Wen/Dolan,Goddard/Arkani-Hamed,Bourjaily, Cachazo,Trnka), but the situation is more complicated for N=8 SUGRA.

N=4 6-pt NMHV

- Recursion relation gives three on-shell diagrams. Remarkably, each can be written as a residue of a single top-form:

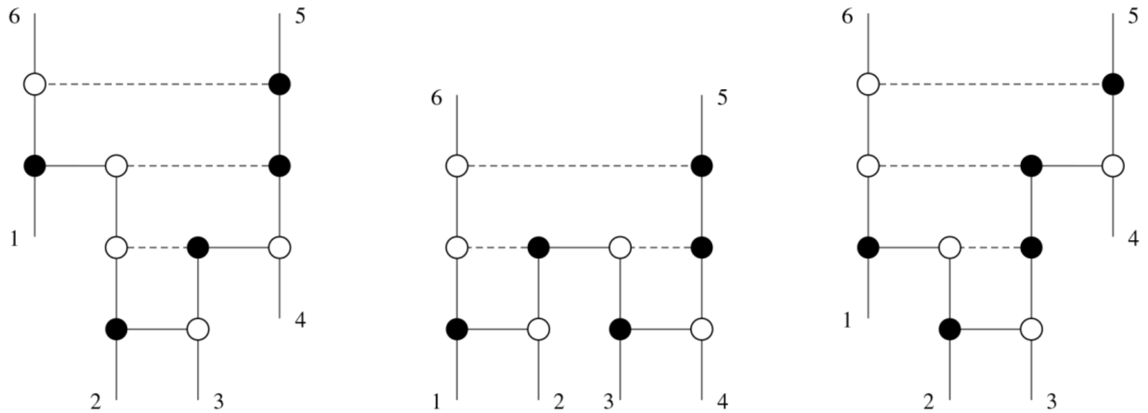


$$= \left(\underset{(234)=0}{\mathrm{Res}} + \underset{(456)=0}{\mathrm{Res}} + \underset{(612)=0}{\mathrm{Res}} \right) \int d^{3\times 6}\Omega_4$$

where $d^{k\times n}\Omega_{\mathcal{N}} := \dfrac{d^{k\times n}C}{\mathrm{Vol}(\mathrm{GL}(k))} \dfrac{\delta^{k\times(2|\mathcal{N})}(C\cdot\tilde{\lambda}|C\cdot\eta)\,\delta^{(n-k)\times 2}(\lambda\cdot C^{\perp})}{\prod_{i=1}^{n}(i\,\ldots\,i{+}k{-}1)}$

N=8 6-pt NMHV

- Restricting the recursion to a planar sector gives three decorated on-shell diagrams, from which the full amplitude can be obtained by summing over permutations of legs 1 to 4:

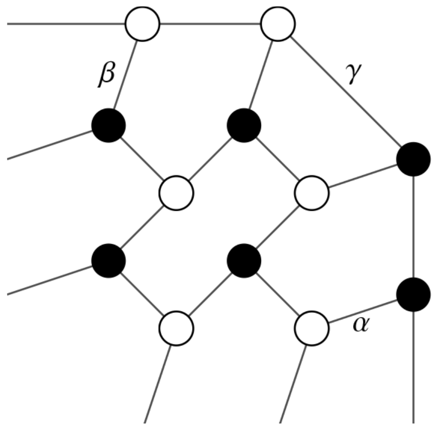


- These three diagrams do not correspond to residues of a single top-form:

$$\mathcal{A}_{6,3}^{(0)} = \left(\underset{(234)=0}{\mathrm{Res}} + \underset{(456)=0}{\mathrm{Res}} \right) \int d^{3\times 6}\Omega_8 \frac{\langle 16\rangle[45]\langle 23\rangle[32]}{(123)(561)(146)(236)} + \underset{(612)=0}{\mathrm{Res}} \int d^{3\times 6}\Omega_8 \frac{\langle 16\rangle[45]\langle 34\rangle[12]}{(123)(561)(346)^2}$$

Postnikov Diagrams

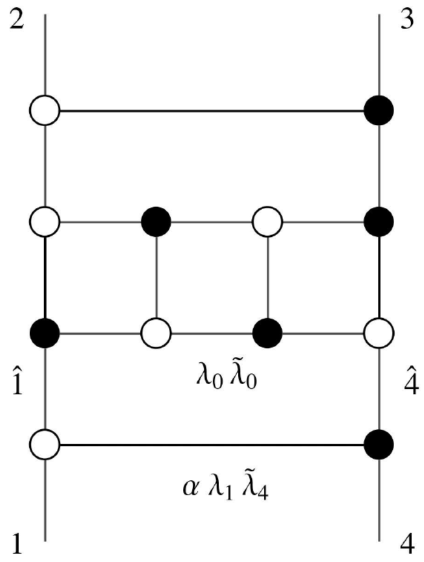
- The three N=4 contributions can be encoded in a single on-shell diagram (Postnikov/Ferro,Lukowski,Meneghelli,Plefka,Staudacher):



- On the other hand, it is not possible to decorate this diagram in such a way that it encodes the three planar N=8 contributions.

1-Loop N=4

- Using loop-level BCFW and equivalence relations, the 1-loop 4-point amplitude can be computed from the following on-shell diagram:



- Mapping link variables into worldsheet coordinates then gives:

$$\mathcal{A}_{4,2}^{(1)} = \int \frac{d^4 l}{l^2} \frac{1}{GL(2)} \prod_{i=0}^{5} \frac{d^2\sigma_i}{(i\, i{+}1)} \frac{(14)(05)}{(15)(04)} \delta^2(\tilde{S}_0)\delta^2\left(S_0\right) \prod_l \delta^{2|4}\left(S_l\right) \prod_r \delta^2\left(S_r\right)$$

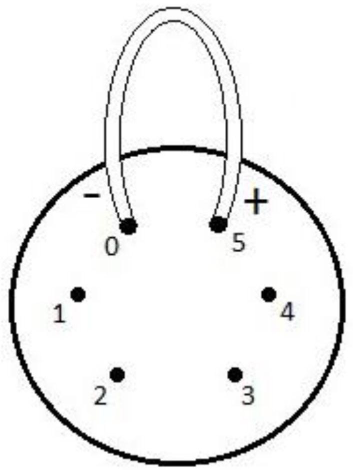
where

$$\tilde{S}_0 = \tilde{\lambda}_0 - \sum_r \frac{\lambda_r}{(0r)}, \quad S_0 = \lambda_0 - \sum_l \frac{\lambda_l}{(5l)}$$

$$S_l = \hat{\tilde{\lambda}}_l - \sum_r \tilde{\lambda}_r \left(\frac{1}{(lr)} + \frac{1}{(l5)(0r)} \right), \quad S_r = \hat{\lambda}_r - \sum_l \lambda_l \left(\frac{1}{(rl)} - \frac{1}{(r0)(5l)} \right)$$

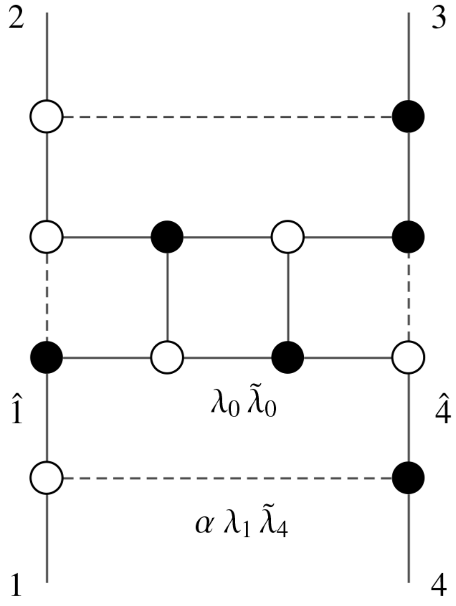
are 1-loop scattering equations refined by MHV degree.

- Note that $\displaystyle \prod_{i=0}^{5} \frac{d^2\sigma_i}{(i\, i{+}1)} \frac{(14)(0\,5)}{(1\,5)(04)} = \prod_{i=0}^{5} \frac{d^2\sigma_i}{(i\, i{+}1)} + (0 \leftrightarrow 5)$



1-Loop N=8

- Although loop-level BCFW is not known for N=8 SUGRA, the one-loop 4-point amplitude can be obtained by summing the following decorated on-shell diagram over permutations of the external legs: (Heslop,Lipstein)



- Following similar steps to N=4, we obtain the following 1-loop 4-point amplitude:

$$\mathcal{M}_{4,2}^{(1)} = \sum_{\text{perms}\{1,2,3,4\}} \int \frac{d^4 l}{l^2} \frac{1}{GL(2)} \prod_{i=0}^{5} \frac{d^2\sigma_i}{(i\, i{+}1)} \frac{(14)(05)}{(15)(04)} \frac{\prod_{i=1}^{4}(0i)(5i)}{1-(05)^2} \det H \det \tilde{H}$$

$$\times \delta^2(\tilde{S}_0)\delta^2(S_0) \prod_l \delta^{2|8}(S_l) \prod_r \delta^2(S_r) \, .$$

where

$$H = \begin{pmatrix} -\frac{\langle 10\rangle}{(10)} - \frac{\langle 12\rangle}{(12)} & \frac{\langle 12\rangle}{(12)} \\ \frac{\langle 12\rangle}{(12)} & -\frac{\langle 20\rangle}{(20)} - \frac{\langle 21\rangle}{(21)} \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} -\frac{[30]}{(35)} - \frac{[34]}{(34)} & \frac{[34]}{(34)} \\ \frac{[34]}{(34)} & -\frac{[40]}{(45)} - \frac{[43]}{(43)} \end{pmatrix}$$

and the 1-loop scattering equations are same as N=4.

Conclusion

- 4d ambitwistor string theory and on-shell diagrams are intimately related.
- We obtain new tree-level Grassmannian integral formulae for N=8 SUGRA.
- We also find new worldsheet formulae for 1-loop 4-point amplitudes in N=4 SYM and N=8 SUGRA supported on scattering equations refined by MHV degree.
- These formulae can be extended to more complicated amplitudes in N=4 SYM using loop-level BCFW recursion.

Future Directions

- Gravituhedron?
- Higher loop worldsheet formulae in N=4 and N=8
- Worldsheet formulae for 1-loop form factors
- Regulate loop integrals
- Derive 4d loop amplitudes from a worldsheet theory

Thank You