Positive Geometries and Canonical Forms

Scattering Amplitudes and the Associahedron

Yuntao Bai

with N. Arkani-Hamed & T. Lam arXiv:1703.04541;
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- We introduce the concept of **positive geometries and canonical forms** as a new framework for thinking about a class of scattering amplitudes.
- Loosely speaking, a positive geometry A is a closed geometry with boundaries of all co-dimensions (e.g. polytopes).
- **Each positive geometry has a unique differential form** (A) **called its canonical form** defined by the following properties:
  1. It has logarithmic (i.e. $d\log z$-like) singularities on the boundary of A.
  2. Its singularities are recursive: At every boundary B, we have $\text{Res}_B \ (A) = \ (B)$.

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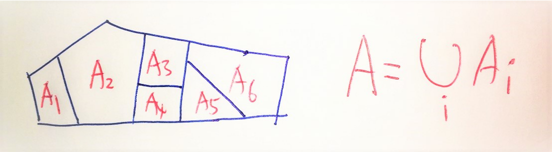

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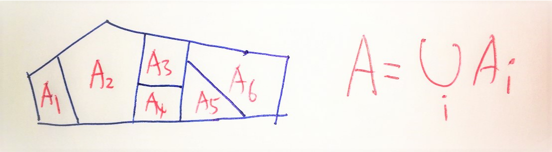

- **Pushforward**: Given a diffeomorphism mapping A to B, the map pushes (A) to (B).

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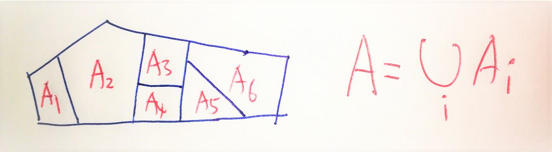

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- **For positive geometries that appear in physics, the canonical form is a physical observable!**

Positive Geometries and Canonical Forms

- For instance, the amplituhedron $A(k; n; L)$ is a positive geometry. The canonical form $\quad(A(k; n; L))$ is conjectured to be the n-particle N^kMHV tree level amplitude for $L = 0$ and the L-loop integrand for $L > 0$.

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- Slight novelty: The amplitude is a **differential form** on the underlying geometry.
- **Our focus today:** The $(n \quad 3)$-dimensional associahedron A_n is a positive geometry, and its canonical form (A_n) is the n-particle tree level scattering amplitude of planar bi-adjoint scalar theory with identical ordering. We will refer to these simply as “bi-adjoint amplitudes”.

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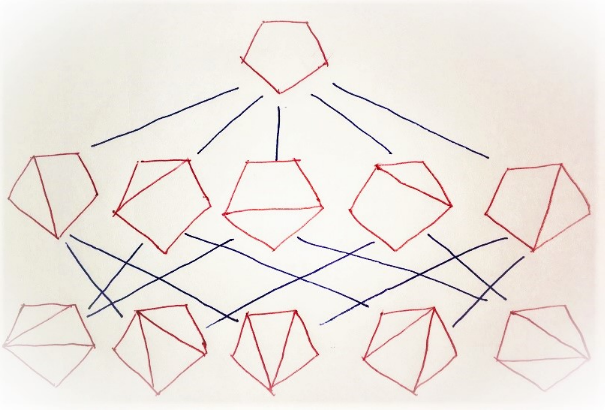
  Positive Geometry ! Canonical Form = Physical Observable

- We can therefore say that the associahedron is the amplituhedron of the bi-adjoint theory.
- There are other instances where this pattern has emerged, so we anticipate that it is relevant for many other theories.

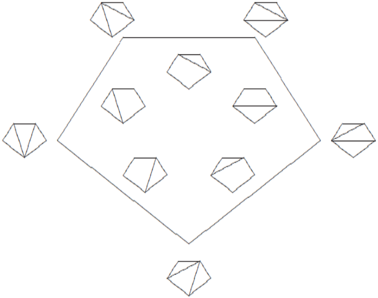
The Associahedron

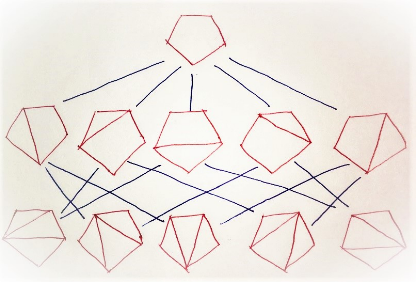
- A partial triangulation of the (regular) *n*-gon is a set of non-intersecting diagonals.
  The set of all partial triangulations of the *n*-gon can be organized in a hierarchical web:



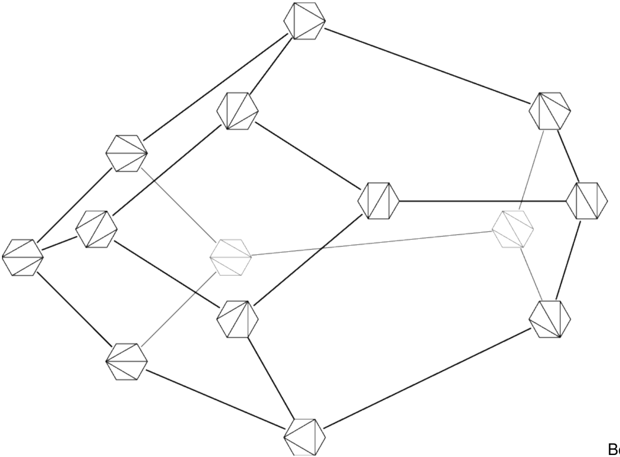
The Associahedron

- The associahedron of dimension (n 3) is a polytope whose codimension d faces are in 1-1 correspondence with the partial triangulations with d diagonals. And the lines connecting partial triangulations tell us how the faces are glued together.





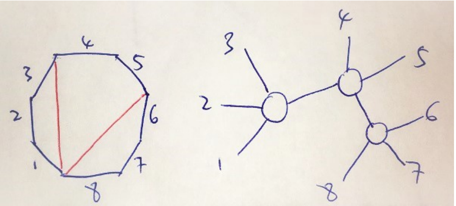
Left: Marni Sheppeard. Arcadian Functor. "M Theory Lesso 294." (Sep 11, 2009)
http://kea-monad.blogspot.co.id/2009/09/m-theory-lesson-294.html



Bowman, Douglas, and Alon Regev. "Counting symmetry classes of dissections of a convex regular polygon." Advances in Applied Mathematics 56 (2014): 35-55. Figure 1

The Associahedron

- Recall that partial triangulations are dual to cuts on planar cubic diagrams, with each diagonal corresponding to a cut. So the codimenion *d* faces of the associahedron are dual to *d*-cuts.



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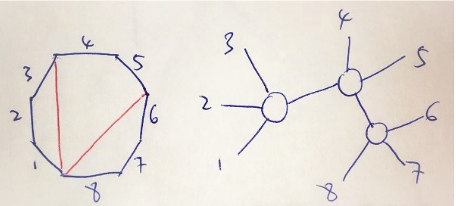

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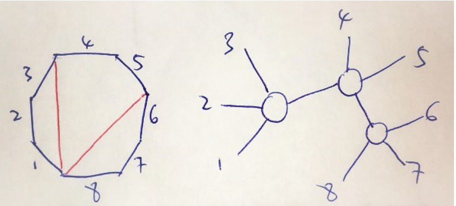

- The faces of the associahedron are therefore dual to the singularities of a cubic scattering amplitude.
- It appears that the associahedron knows about the structure of planar cubic amplitudes. It is therefore natural to look for an explicit construction of an associahedron within kinematic space.

The Associahedron in Mandelstam Space

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- More generally, we have $s_{i_1 \ldots i_m} = (k_{i_1} + \cdots + k_{i_m})^2$.
- There are n kinematic constraints: $\sum_j s_{ij} = 0$ for each i. So Mandelstam space has dimension $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$.

Cutting Out the Associahedron

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- We set the variables s_{ij} for all non-adjacent index pairs $1 \quad i < j \quad n \quad 1$ to be negative constants. Namely, $s_{ij} \quad c_{ij}$.

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- We set the variables s_{ij} for all non-adjacent index pairs $1 \leq i < j \leq n-1$ to be negative constants. Namely, $s_{ij} = -c_{ij}$.
- The number of constraints is: $(n-3) + (n-2) + \cdots + 1 = \frac{(n-2)(n-3)}{2}$. This cuts the space down to the required dimension: $\frac{n(n-3)}{2} - \frac{(n-2)(n-3)}{2} = n-3$.

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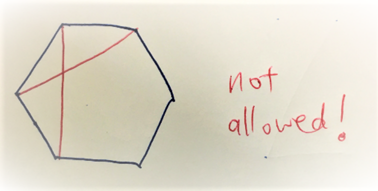
- **The intersection between the big simplex and these equations is an associahedron!**
- To show this, we argue that the faces of the polytope correspond to cuts on planar cubic diagrams. But this is true by construction, since the faces are defined using cuts.
- However, we need to argue that there are no faces given by non-planar cuts (i.e. diagram with self-intersecting diagonals). This follows from the negative constants.



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- The faces of the polytope, of all codimension, are in one-to-one correspondence with cuts on planar cubic diagrams. The polytope must be an associahedron!

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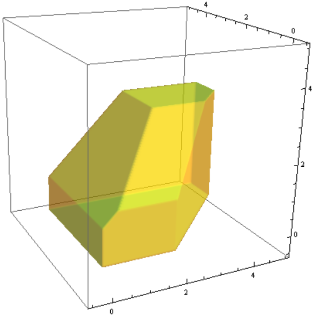
- Here we show a numerical plot (left) for $n = 6$. The figure has 9 faces, 21 edges and 14 vertices, and is equivalent to the 3D associahedron (right).



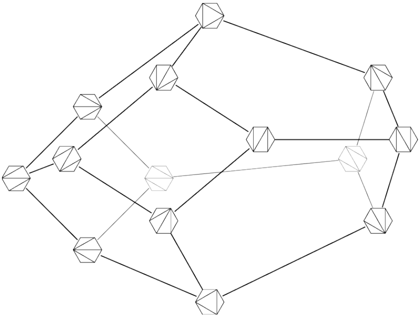


Image created using Mathematica 9

The Canonical Form of the Associahedron

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- We discover that **the canonical form of the associahedron is the bi-adjoint amplitude!**

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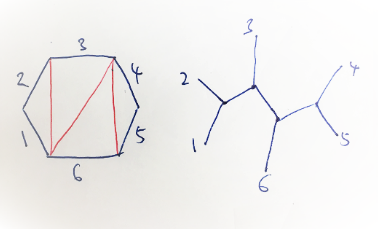
- For a simple polytope of dimension D, the canonical form is:

$$\sum_{\text{vertex } P} \bigwedge_{i=1}^{D} d\log(E_{i;P})$$

where $E_{i;P} = 0$ are the equations of the facets adjacent to P, ordered by orientation.

The Canonical For of the Associahedron

- The canonical form of the associahedron is therefore a sum over vertices. But each vertex is labeled by a triangulation of the *n*-gon, or equivalently, a planar cubic diagram:



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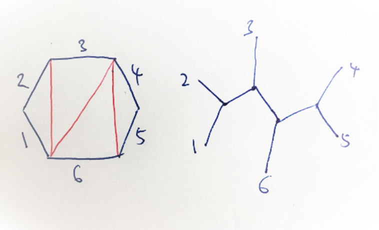

- The canonical form is therefore a sum over planar cubic diagrams. This looks like an amplitude!

The Canonical Form of the Associahedron

The expression for each vertex is the product of d-log of the equations for the $n-3$ adjacent facets. But the facets are given by the propagators $s_{g_i} = 0$ on the diagram. So the expression is just the d-log of the propagators.

$$= \prod_{i=1}^{n-3} \frac{ds_{g_i}}{s_{g_i}}$$

$$= d\log s_{23} \; d\log s_{123} \; d\log s_{45}$$

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\begin{aligned}
&= \prod_{i=1}^{n-3} \frac{ds_{g_i}}{s_{g_i}} \\
&= d\log s_{23}\, d\log s_{123}\, d\log s_{45} \\
&= \frac{ds_{23}\; ds_{123}\; ds_{45}}{s_{23}\; s_{123}\; s_{45}}
\end{aligned}
$$

The numerator $\prod_{i=1}^{n-3} ds_{g_i}$ is the same for each diagram when pulled back onto the $(n-3)$-plane containing the associahedron.

The Canonical Form of the Associahedron

The expression for each diagram is therefore:

$$\text{Planar cubic diagram} = \left(\prod_{i=1}^{n-3} \frac{1}{s_{g_i}} \right) d^{n-3}s$$

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The terms add to form the bi-adjoint amplitude:

$$\begin{aligned} \text{Canonical form} &= \sum \text{Planar cubic diagram} \\ &= (\text{Amplitude})\, d^{n-3}s \end{aligned}$$

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It is crucial that we pull back to the $(n-3)$-plane where the form reduces to a top form, otherwise the cubic diagrams do not add in a physically meaningful way.

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  Intersecting the big simplex with the plane gives us an associahedron.
- We then introduced positive geometries and canonical forms.
  Every positive geometry A has a unique form $(\ A)$ called its canonical form.
- The associahedron is a positive geometry whose canonical form is the bi-adjoint amplitude!

An Example for $n = 4$

For $n = 4$, we have $s; t; u$ satisfying $s + t + u = 0$.
We impose $s; t$ 0 and $u < 0$ constant (hence $ds =$ dt).
The associahedron is a line segment (red).
The canonical form is the 4 point amplitude.

$$\text{Canonical form} = \frac{ds}{s} \quad \frac{dt}{t} = \quad \frac{1}{s} + \frac{1}{t} \quad ds = (\text{ 4pt Amplitude}) \, ds$$

Scattering Equations as Pushforwards

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So there are two associahedra: one in moduli space, the other in Mandelstam space.

The two spaces are related by the scattering equations: $E_d(\{\sigma_a\}; \{s_{bc}\}) = 0$. So it is natural to expect that the two associahedra are also related in some way.

Scattering Equations as a Diffeomorphism

Observation: If the kinematic variables are on the interior of the Mandelstam associahedron, then there exists exactly one solution on the interior of the moduli space associahedron.

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Hence the scattering equations act as a diffeomorphism from the moduli space associahedron to the Mandelstam associahedron.

Scattering Equations as a Diffeomorphism

Mandelstam space

Moduli space:

$(\ _1; \ _4; \ _5) = (0 \, ; 1; 1 \ )$

$0 < \ _2 < \ _3 < 1$

Scattering equations as a diffeo. $f \ _i g \,!\, f \ s_{jk}\, g$:

$s_{12} = \ \frac{2}{3}(s_{13} + \ s_{14} \ \ 3)$

$s_{123} = \frac{1}{1 \ \ \ 2}(s_{24} \ \ 2 \ \ \ (s_{14} + s_{24}) \ \ 3 + s_{14} \ \ 2 \ \ 3)$

Image created using Mathematica 9

Scattering Equations as a Diffeomorphism

Recall: Diffeomorphisms push canonical forms to canonical forms.

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$$(\ \text{moduli space assoc.}) \xrightarrow{\text{pushforward by}} \quad (\ \text{Mandelstam assoc.})$$

$$\text{Hence,} \quad \frac{d^n \ = \mathrm{Vol}\ SL(2)}{\prod_{i=1}^{n}(\ _i \quad _{i+1})} \xrightarrow{\text{pushforward by}} \quad (\text{Amplitude})d^{n\ \ 3}s$$

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Hence, $\frac{d^n = \text{Vol}\, SL(2)}{\prod_{i=1}^{n} (_i \quad _{i+1})} \xrightarrow{\text{pushforward by}} (\text{Amplitude}) d^{n-3} s$

The scattering equations push the Parke-Taylor form to the amplitude form!

CHY Formula as a Pushforward

In order words,

$$\sum_{\text{sol.}} \frac{d^n\sigma / \text{Vol}\, SL(2)}{\prod_{i=1}^{n}(\sigma_i - \sigma_{i+1})} = (\text{Amplitude})\, d^{n-3}s$$

where the sum over solutions $\sigma_a(\{s_{ab}\})$ is required by the pushforward.

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where the sum over solutions $_a(\{s_{cb}\})$ is required by the pushforward.

This is equivalent to the CHY formula for the bi-adjoint amplitude:

$$\int \frac{d^n \quad / \text{Vol}\, SL(2)}{\prod_{i=1}^{n} (\quad_i \qquad _{i+1})^2} \prod_i{}' \quad \left(\sum_{j \neq i} \frac{s_{ij}}{\quad_i \qquad _j} \right) = \text{Amplitude}$$

CHY Formula as a Pushforward

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where the sum over solutions $\sigma_a(\{s_{cb}\})$ is required by the pushforward.

This is equivalent to the CHY formula for the bi-adjoint amplitude:

$$\int \frac{d^n\sigma / \text{Vol}\, SL(2)}{\prod_{i=1}^{n}(\sigma_i - \sigma_{i+1})^2} \prod_i{}' \delta\left(\sum_{j \neq i} \frac{s_{ij}}{\sigma_i - \sigma_j}\right) = \text{Amplitude}$$

We have deduced the CHY formula above purely as a consequence of geometry!

Summary

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- The associahedron is a positive geometry and therefore has a canonical form, which is the bi-adjoint amplitude.
- Furthermore, the scattering equations act as a diffeomorphism from the moduli space associahedron to the Mandelstam associahedron, and the CHY formula is the corresponding pushforward.

Outlook

The story is similar for other orderings in the bi-adjoint theory. The geometry for each ordering is obtained by taking an associahedron and sending to in nity vertices corresponding to inadmissible cubic graphs.

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- The story is similar for other orderings in the bi-adjoint theory. The geometry for each ordering is obtained by taking an associahedron and sending to infinity vertices corresponding to inadmissible cubic graphs.
- **Long term goal**: Find all theories whose physical observables can be reformulated as the canonical form of some positive geometry.

Positive Geometry ! Canonical Form = Physical Observable