The duality of Wilson Loop form factors

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Based on work arXiv:1612.05197 in collaboration with
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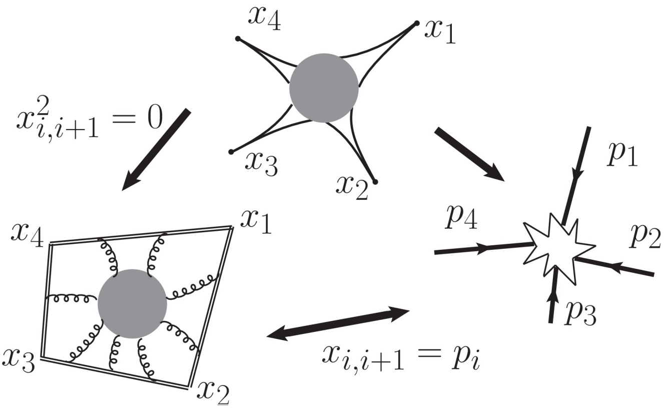
Outline

- $\mathcal{N} = 4$ SYM at weak coupling in the planar limit
- Duality between Correlation Functions, Wilson Loops, and Scattering Amplitudes
- Duality of the Wilson Loop form factors (two polygonal light-like contours)
- MHV example of the duality
- Supersymmetric extension of the duality
- LHC (Lorentz Harmonic Chiral) superspace formulation of the Wilson Loop
- Diagrammatic proof of the duality

Duality between N = 4 SYM observables

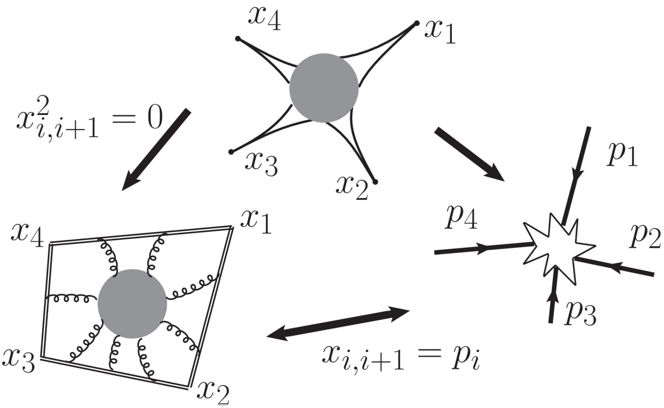


Alday, Maldacena '07;
Drummond, Henn, Korchemsky, Sokatchev '07 '08;
Alday, Eden, Korchemsky, Maldacena, Sokatchev '10;
Mason, Skinner '10;
Caron-Huot '10;
Adamo, Bullimore, Mason, Skinner '11;
Brandhuber, Heslop, Travaglini '07

- Correlators of half-BPS operators $\mathcal{O} = \mathrm{tr}\ \phi^2$. **UV & IR finite**. Conformal symmetry!!!

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$$

Duality between N = 4 SYM observables



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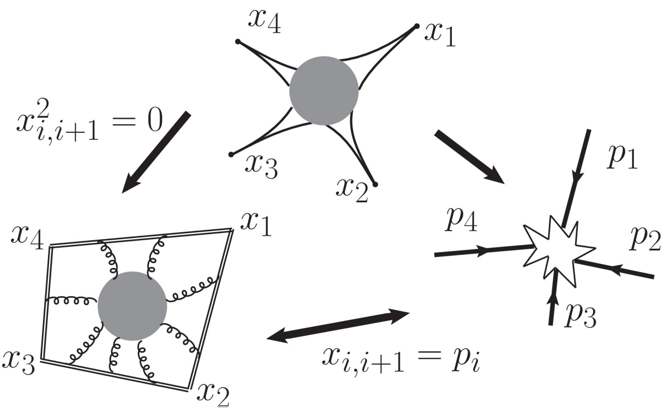
- Polygonal light-like Wilson Loops on the contour $C_n = p_1 \cup \ldots \cup p_n$. **UV div**

$$W[C_n] = \left\langle \mathrm{tr}\ \mathrm{Pexp}\left(ig \oint_{C_n} dx \cdot \mathcal{A} \right) \right\rangle$$

Duality between N = 4 SYM observables



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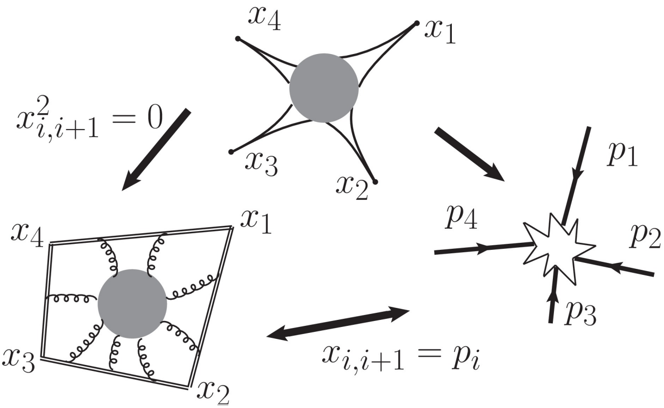
- Loop corrections of the MHV amplitudes. **IR div**

$$\mathcal{A}_n^{\rm MHV}(p_1^+, \dots p_i^-, \dots p_j^-, \dots p_n^+)$$

Duality between N = 4 SYM observables



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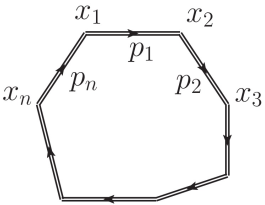
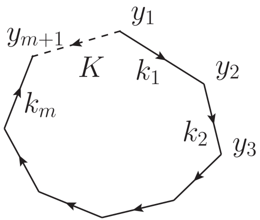
- Conformal and dual conformal symmetries are anomalous.
- Correlator $\leftrightarrow$ Amplitude works at the level of INTEGRANDS
- Dualities work for supersymmetric observables

"MHV" Wilson loop form factors and their duality

Polygonal light-like contours





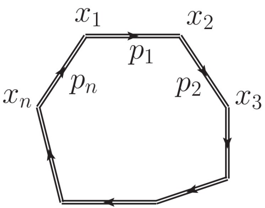
$p_i^2 = 0\,, \qquad \sum_i p_i = 0$

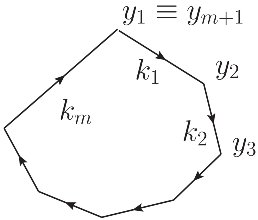
$k_i^2 = 0\,, \quad \sum_i k_i \equiv K = y_1 - y_{m+1} \neq 0$

$$\left\langle \mathrm{tr\ Pexp}\left(ig \oint_{p_1 \cup \ldots \cup p_n} dx \cdot \mathcal{A} \right) \middle| k_1^+ k_2^+ \ldots k_m^+ \right\rangle$$

"MHV" Wilson loop form factors and their duality

Polygonal light-like contours





$p_i^2 = 0\,, \quad \sum_i p_i = 0$ $\qquad$ $k_i^2 = 0\,, \quad \sum_i k_i = \mathbf{0}$ **Constraint!**

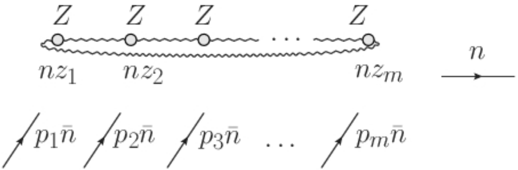
$$\left\langle \mathrm{tr\ Pexp}\left(ig \oint_{p_1 \cup \ldots \cup p_n} dx \cdot \mathcal{A} \right) \middle| k_1^+ k_2^+ \ldots k_m^+ \right\rangle \sim \left\langle \mathrm{tr\ Pexp}\left(ig \oint_{k_1 \cup \ldots \cup k_m} dx \cdot \mathcal{A} \right) \middle| p_1^+ p_2^+ \ldots p_n^+ \right\rangle$$

in the planar limit

Form factors of $SL(2)$ light-ray operators

[Derkachov, Korchemsky, Manashov '13]

$\mathbb{O}(z_1, \ldots, z_m) =$



$n^2 = 0$

$\bar{n}^2 = 0$

$n \cdot \bar{n} \neq 0$

$$f(z_1, \ldots, z_m; y_1, \ldots, y_m) \equiv \langle \mathbb{O}(z_1, \ldots, z_m) | p_1 \bar{n}, \ldots, p_m \bar{n} \rangle$$

where $$\sum_{i=1}^{m} p_i = 0 \quad , \quad p_i \equiv y_i - y_{i+1}$$

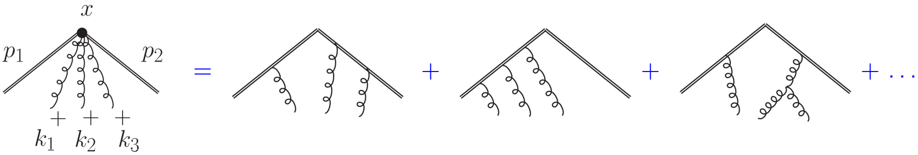
One-loop integrability of the dilatation operator implies

$$f(z_1, \ldots, z_m; y_1, \ldots, y_m) = f(y_1, \ldots, y_m; z_1, \ldots, z_m)$$

Now we unfold contour and consider form factors of polygonal Wilson Loop

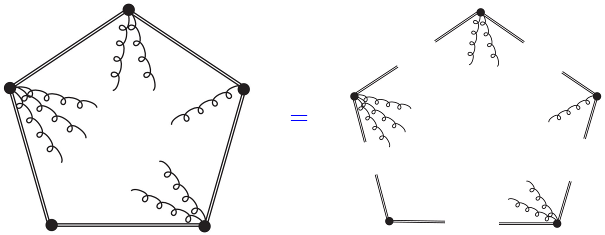
Tree level "MHV" Wilson Loop form factors

Color ordered MHV form factor of the "wedge"



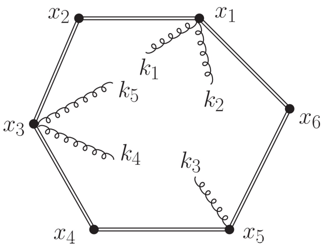
$$= g^3 \frac{\langle p_1 p_2 \rangle}{\langle p_1 k_1 \rangle \langle k_1 k_2 \rangle \langle k_2 k_3 \rangle \langle k_3 p_2 \rangle} \exp\left(ix(k_1 + k_2 + k_3)\right)$$

Factorization of the Wilson Loop form factor

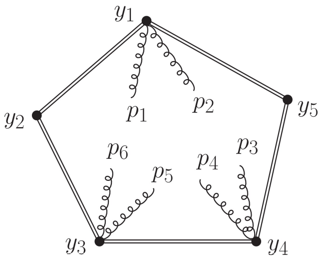


Tree level "MHV" Wilson Loop form factors. Example

[S. Caron-Huot]

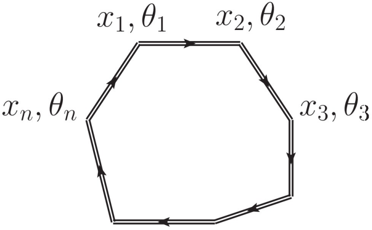


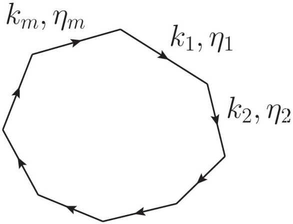
$=$



$$\exp\Big(ix_1 \underbrace{(k_1+k_2)}_{y_{13}} + ix_3 \underbrace{(k_4+k_5)}_{y_{41}} + ix_5 \underbrace{(k_3)}_{y_{34}}\Big) = \exp\Big(iy_1 \underbrace{(p_1+p_2)}_{x_{13}} + iy_3 \underbrace{(p_5+p_6)}_{x_{51}} + iy_4 \underbrace{(p_3+p_4)}_{x_{35}}\Big)$$

Supersymmetric Wilson Loop form factors





Light-like chiral contour. θ^A, $A = 1, \ldots, 4$

Superconnections $\mathcal{A}_{\alpha\dot\alpha}(x,\theta)$, $\mathcal{A}_{\alpha A}(x,\theta)$

$$x_i - x_{i+1} \equiv p_i \quad , \quad \theta_i^A - \theta_{i+1}^A \equiv |p_i\rangle\, \omega_i^A$$

$$\sum_i p_i = 0 \quad , \quad \sum_i |p_i\rangle\, \omega_i = 0$$

Superstate (vector multiplet: $G_\pm$, $\Gamma_\pm$, S)

$$|k,\eta\rangle = G_+ + \eta_A \Gamma_+^A + \ldots + (\eta)^4 G_-$$

$$y_i - y_{i+1} \equiv k_i \quad , \quad \psi_{i,A} - \psi_{i+1,A} \equiv |k_i\rangle\, \eta_{i,A}$$

$$\sum_i k_i = 0 \quad , \quad \sum_i |k_i\rangle\, \eta_i = 0$$ **Constraint!**

$$W_{n,m}(\{x,\theta\}\{y,\psi\}) \equiv \prod_{i=1}^{m} \langle k_i k_{i+1}\rangle \cdot \left\langle \mathrm{tr\,Pexp}\; ig \oint_{p_1\cup\ldots\cup p_n} \left(dx^{\alpha\dot\alpha}\mathcal{A}_{\alpha\dot\alpha} + d\theta^{\alpha A}\mathcal{A}_{\alpha A}\right) \middle| k_1,\eta_1;\ldots;k_m,\eta_m \right\rangle$$

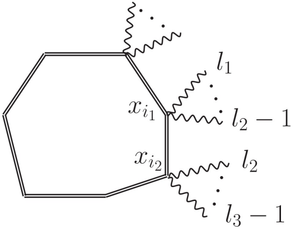
Duality: $$\boxed{W_{n,m}(\{x,\theta\}\{y,\psi\}) = W_{m,n}(\{y,\psi\}\{x,\theta\})}$$

Grassmann structure of the Wilson Loop form factors in Born approximation

Recall: **Vacuum** expectation value of the super Wilson Loop

$$W_n = \sum_{\kappa=0}^{n-4} W_n^{(\kappa)} , \qquad \text{where} \quad W_n^{(\kappa)} \sim (\theta)^{4\kappa}$$

Form factors: Grassmann variables θ^A, ψ_A $\Rightarrow$ $SU(4)$ invariants: $(\theta)^4$, $(\psi)^4$, $(\theta \cdot \psi)$



$$W_{n,m} = \sum_{\text{clusters}} e^{ix_{i_1} y_{l_1 l_2} + \dots} \, e^{i\langle \theta_{i_1} \psi_{l_1 l_2} \rangle + \dots} \, \widehat{W}_{n,m}$$

Grassmann expansion $\quad \widehat{W}_{n,m} = \sum_{\kappa,\sigma \geq 0} W_{n,m}^{(\kappa,\sigma)}$

$$W_{n,m}^{(\kappa,\sigma)} \sim (\theta)^{4\kappa} (\eta)^{4\sigma} \quad — \quad \text{N}^{\kappa}\text{MHV} \times \text{N}^{\sigma}\text{MHV}$$

Duality: $\text{N}^{\kappa}\text{MHV} \times \text{N}^{\sigma}\text{MHV} \quad \leftrightarrow \quad \text{N}^{\sigma}\text{MHV} \times \text{N}^{\kappa}\text{MHV}$

How to describe supersymmetric Wilson Loop?

Superconnections (at $\bar{\theta} \to 0$)

$$\nabla_{\alpha\dot{\alpha}} = \partial/\partial x^{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}}(x,\theta) \quad , \quad \nabla^{\alpha}_{A} = \partial/\partial\theta^{A}_{\alpha} + \mathcal{A}^{A}_{\alpha}(x,\theta)$$

Defining constraints of $\mathcal{N} = 4$ SYM

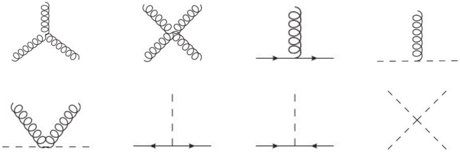
$$\{\nabla^{\alpha}_{A}, \nabla^{\beta}_{B}\} = \epsilon^{\alpha\beta} W_{AB} + \cancel{W^{(\alpha\beta)}_{AB}} \qquad , \quad \dots$$

put all fields **on shell**

- Complicated expressions for $\mathcal{A}_{\alpha\dot{\alpha}}(x,\theta)$ and $\mathcal{A}^{\alpha}_{A}(x,\theta)$ in terms of standard fields

$$\mathcal{A}_{\alpha\dot{\alpha}}(x,\theta) = \mathcal{A}_{\alpha\dot{\alpha}}(x) + \theta_{\alpha} \cdot \bar{\psi}_{\dot{\alpha}} + \dots \quad , \quad \mathcal{A}_{\alpha A}(x,\theta) = \phi_{AB}(x)\, \theta^{B}_{\alpha} + \dots$$

- Component field interactions produce plenty of Feynman graphs



How to describe supersymmetric Wilson Loop?

Superconnections (at $\bar{\theta} \to 0$)

$$\nabla_{\alpha\dot{\alpha}} = \partial/\partial x^{\alpha\dot{\alpha}} + \mathcal{A}_{\alpha\dot{\alpha}}(x,\theta) \quad , \quad \nabla_A^\alpha = \partial/\partial\theta_\alpha^A + \mathcal{A}_\alpha^A(x,\theta)$$

Defining constraints of $\mathcal{N}=4$ SYM

$$\{\nabla_A^\alpha, \nabla_B^\beta\} = \epsilon^{\alpha\beta} W_{AB} + \cancel{W_{AB}^{(\alpha\beta)}} \qquad , \quad \ldots$$

put all fields **on shell**

- Complicated expressions for $\mathcal{A}_{\alpha\dot{\alpha}}(x,\theta)$ and $\mathcal{A}_A^\alpha(x,\theta)$ in terms of standard fields

$$\mathcal{A}_{\alpha\dot{\alpha}}(x,\theta) = A_{\alpha\dot{\alpha}}(x) + \theta_\alpha \cdot \bar{\psi}_{\dot{\alpha}} + \ldots \quad , \quad \mathcal{A}_{\alpha A}(x,\theta) = \phi_{AB}(x)\,\theta_\alpha^B + \ldots$$

- On-shell superfields. Chiral anomaly of the Wilson Loop [Belitsky, Korchemsky, Sokatchev '11]
- We need chiral SUSY **off shell**
- Twistor [Witten '04; Mason '05; Mason & Skinner '10; Adamo, Boels, Bullimore, Mason, Skinner]
  vs Lorentz Harmonic [Sokatchev '95; D.C. & Sokatchev '16]
  approach to **off-shell** formulation of $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM in the Lorentz Harmonic Chiral superspace

Harmonic variables $u^{\pm}{}_{\alpha}$ on $SU(2)/U(1) \sim S^2$, a pair of spinors

Rosly '83; Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev '84; Sokatchev '87

$$(u^{+}{}_{\alpha}, u^{-}{}_{\alpha}) \in SU(2) \quad , \quad u^{+\alpha}u^{-}_{\alpha} = 1$$

A harmonic field of given $U(1)$ charge contains an infinite number of irrep., e.g.

$$f^{++}(x,u) = f^{(\alpha\beta)}(x)u^{+}_{\alpha}u^{+}_{\beta} + f^{(\alpha\beta\gamma)}(x)u^{+}_{\alpha}u^{+}_{\beta}u^{-}_{\gamma} + f^{(\alpha\beta\gamma\delta)}(x)u^{+}_{\alpha}u^{+}_{\beta}u^{-}_{\gamma}u^{-}_{\delta} + \dots$$

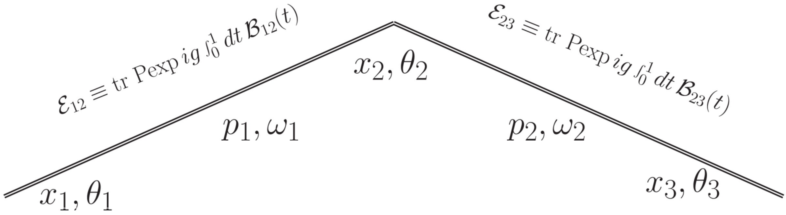
	τ-frame		Analytic frame
Superfields	$A(x,\theta)$		$A(x,\theta,u)$
Gauge transform	$\nabla \to \left(e^{\tau(x,\theta)}\right)_{\rm Adj} \nabla$		$\nabla \to \left(e^{\Lambda(x,\theta^{+},u)}\right)_{\rm Adj} \nabla$
Connections	$\nabla^{++} = \partial^{++} \equiv u^{+}\dfrac{\partial}{\partial u^{-}}$ $\nabla^{+}_{A} = u^{+}_{\alpha}\nabla^{\alpha}_{A},\ \{\nabla^{+}_{A},\nabla^{+}_{B}\} = 0$	generalized gauge transform $\xrightarrow[h(x,\theta,u)]{}$	$\nabla^{++} = \partial^{++} + A^{++}$ $\nabla^{+}_{A} = u^{+}_{\alpha}\dfrac{\partial}{\partial\theta^{A}_{\alpha}}$

Off-shell fields of the theory: gauge connections covariantizing ∂^{++} and $\partial^{+}_{\dot{\alpha}} \equiv u^{+\alpha}\partial_{\alpha\dot{\alpha}}$

$$A^{++}(x,\theta^{+},u) \quad , \quad A^{+}_{\dot{\alpha}}(x,\theta^{+},u)$$

Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson Loop $\quad \mathcal{W}_n = \mathrm{tr}\, \mathcal{E}_{12}\, \mathcal{E}_{23}\, \ldots\, \mathcal{E}_{n1}$

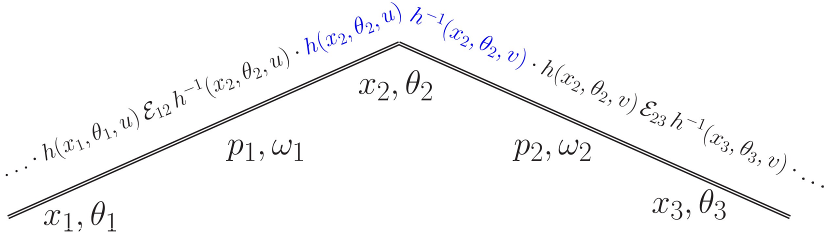


$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}}(x(t), \theta(t)) + \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t), \theta(t)) \xrightarrow[u^+ \,=\, |p_i\rangle]{\text{generalized gauge transform}} B_{i,i+1}(t) \equiv p_i^{\dot{\alpha}} A^+_{\dot{\alpha}}(x(t), \langle p_i \theta_i \rangle, p_i^\alpha)$$

where $x(t) = x_i - t\, p_i, \ldots$

Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson Loop $\mathcal{W}_n = \mathrm{tr}\, \mathcal{E}_{12}\, \mathcal{E}_{23}\ \ldots\ \mathcal{E}_{n1}$



$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\alpha\dot\alpha} \mathcal{A}_{\alpha\dot\alpha}(x(t),\theta(t)) + \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t),\theta(t))$$

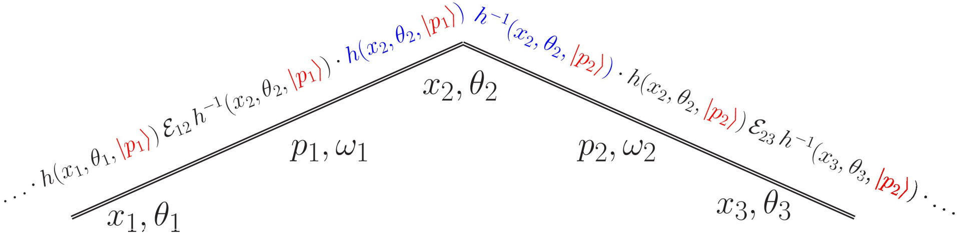
generalized gauge transform, $u^+ = |p_i\rangle$:

$$B_{i,i+1}(t) \equiv p_i^{\dot\alpha} A^+_{\dot\alpha}(x(t), \langle p_i \theta_i\rangle, p_i^\alpha)$$

where $x(t) = x_i - t\, p_i, \ \ldots$

Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson Loop $\qquad \mathcal{W}_n = \mathrm{tr}\, \mathcal{E}_{12}\, \mathcal{E}_{23} \;\ldots\; \mathcal{E}_{n1}$



$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}}(x(t), \theta(t)) + \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t), \theta(t)) \quad \xrightarrow[u^+ \;=\; |p_i\rangle]{\text{generalized gauge transform}} \quad B_{i,i+1}(t) \equiv p_i^{\dot{\alpha}} A^+_{\dot{\alpha}}(x(t), \langle p_i \theta_i \rangle, p_i^\alpha)$$

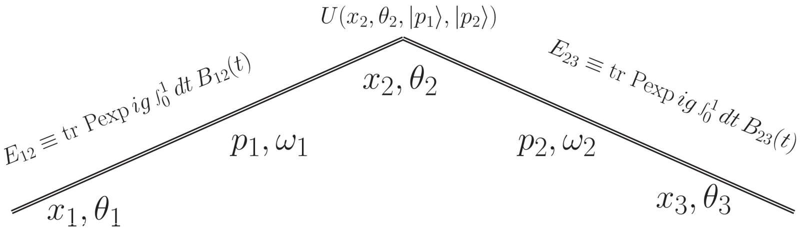
where $x(t) = x_i - t\, p_i$, ...

Wilson Loop in the Lorentz Harmonic Chiral superspace

Classical Wilson Loop $\quad \mathcal{W}_n = \mathrm{tr}\, \mathcal{E}_{12}\, \mathcal{E}_{23}\, \ldots\, \mathcal{E}_{n1}$

$$= \mathrm{tr}\, E_{12}\, U_2\, E_{23}\, U_3\, \ldots\, E_{n1}\, U_1$$

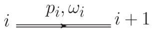
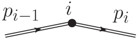
gauge invariant & completes the twistor formula [Mason & Skinner '10]



$$\mathcal{B}_{i,i+1}(t) \equiv p_i^{\alpha\dot{\alpha}} \mathcal{A}_{\alpha\dot{\alpha}}(x(t), \theta(t)) + \omega_i^A p_i^\alpha \mathcal{A}_{\alpha A}(x(t), \theta(t)) \xrightarrow[u^+ = |p_i\rangle]{\text{generalized gauge transform}} B_{i,i+1}(t) \equiv p_i^{\dot{\alpha}} A^+_{\dot{\alpha}}(x(t), \langle p_i \theta_i \rangle, p_i^\alpha)$$

where $x(t) = x_i - t\, p_i, \ldots$

Classical Wilson Loop $\qquad \mathcal{W}_n = \mathrm{tr}\, E_{12}\ U_2\ E_{23}\ U_3\ \ldots\ E_{n1}\ U_1$

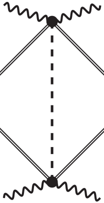
$E_{i,i+1} = E_{i,i+1}[A^+_{\dot\alpha}]$	sits on edge 	Wilson line	Nonlocal in x-space
$U_i = U_i[A^{++}]$	sits at vertex 	Transform between two analytic frames $u^+ = \|p_{i-1}\rangle \to u^+ = \|p_i\rangle$	Nonlocal in harmonic space

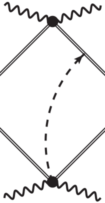
Quantization of the theory in the light-cone gauge

$$\langle A^{++} A^{++} \rangle \quad , \quad \langle A^{++} A^+_{\dot\alpha} \rangle \quad , \quad \langle 0 | A^{++} | k, \eta \rangle$$

The main role is played by **vertex graphs**

Edge graphs restore gauge invariance

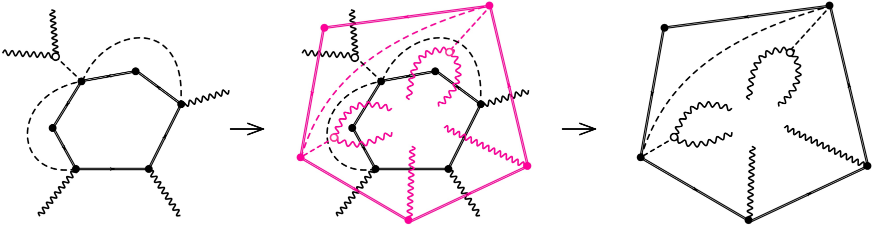




$N^2MHV \times NMHV$ is dual to $NMHV \times N^2MHV$

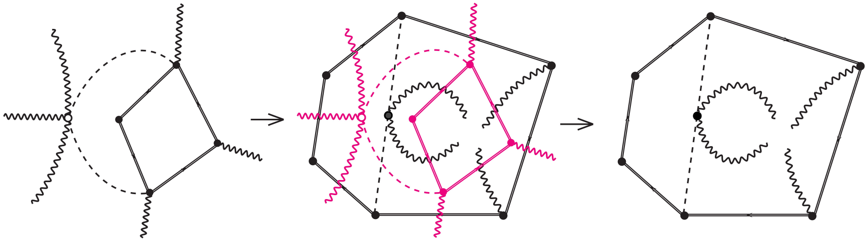
Drop E-pieces of the Wilson Loop. Then the duality works graph by graph.

Duality of Wilson Loop form factors $\sim$ Duality of planar graphs



$$\widehat{W}_{6,5}^{(2,1)}(\{x,\theta\}\{y,\psi\}) = \widehat{W}_{5,6}^{(1,2)}(\{y,\psi\}\{x,\theta\})$$

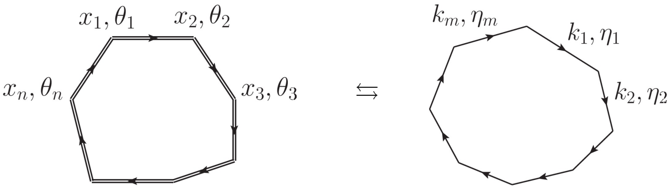
One loop MHV form factor of the bosonic Wilson Loop



UV and IR divergences, but the duality works at the level of the INTEGRAND !

Summary

- Supersymmetric Wilson Loop form factors
- New duality which exchanges two polygonal light-like contours



- Lorentz Harmonic superspace formulation of the Wilson Loop
- Deep reason for the duality?
- Superconformal properties of the Wilson Loop form factors?