Symmetries of the amplituhedron

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Amplitudes 2017

Higgs Center, The University of Edinburgh, 10.07.2017

Based on:

J. Phys. A: Math. Theor. 50 294005 &

10.1007 JHEP03(2016)014

with T. Lukowski, A. Orta, M. Parisi

Outline

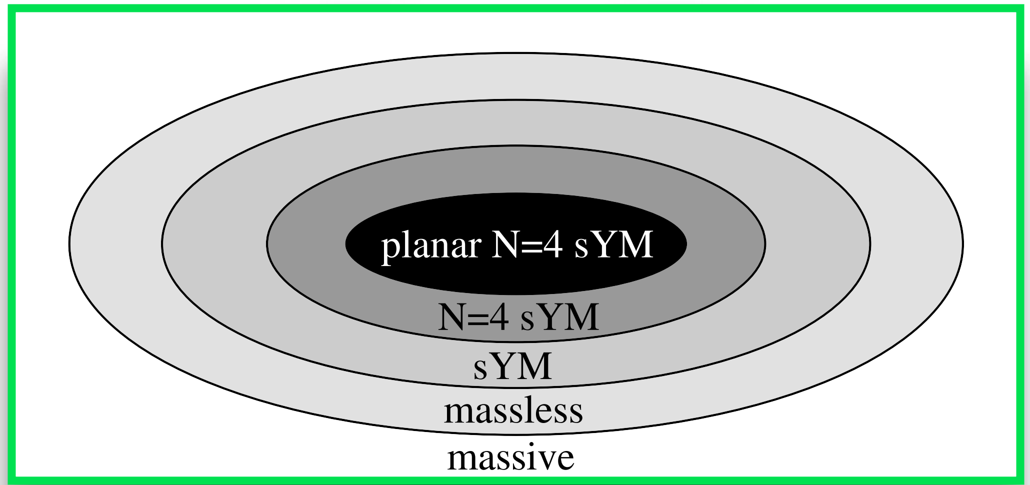
- Introduction
- Symmetries of amplitudes in planar N=4
    - Yangian symmetry - map to spin chain
- Symmetries of the amplituhedron
    - Capelli differential eqs
    - Map to spin chain - Yangian symmetry
    - New on-shell diagrammatics
- Conclusions and open questions

Scattering amplitudes in N=4 sYM

Standard methods for computing ampls involved

symmetries and good (geometric) formalism help

Maximally supersymmetric Yang-Mills theory



(picture of L. Dixon)

Interacting 4d QFT with highest degree of symmetry

Symmetries and ampls in N=4 sYM

Important in discovering the characteristic of ampls

Superconformal symm. expected [Witten]

$$j_a \mathcal{A}_n^{\text{tree}} = 0, \quad j_a \in \mathfrak{psu}(2,2|4)$$

Dual superconformal symm. (planar) hidden [DHKS]

$$J'_a \mathcal{A}_n^{\text{tree}} = 0, \quad J'_a \in \mathfrak{psu}(2,2|4)^{\text{dual}}$$

Yangian symmetry [Drummond,Henn Plefka]

infinite number of „levels" of generators

- level-zero generators: $[j_a^{(0)}, j_b^{(0)}] = f_{ab}{}^c j_c^{(0)}$
- level-one generators: $[j_a^{(0)}, j_b^{(1)}] = f_{ab}{}^c j_c^{(1)}$
- + Serre relations

The Yangian Y(g) of the Lie algebra g is generated by $j^{(0)}$ and $j^{(1)}$

- tree-level collinear singularities
- broken at loop level

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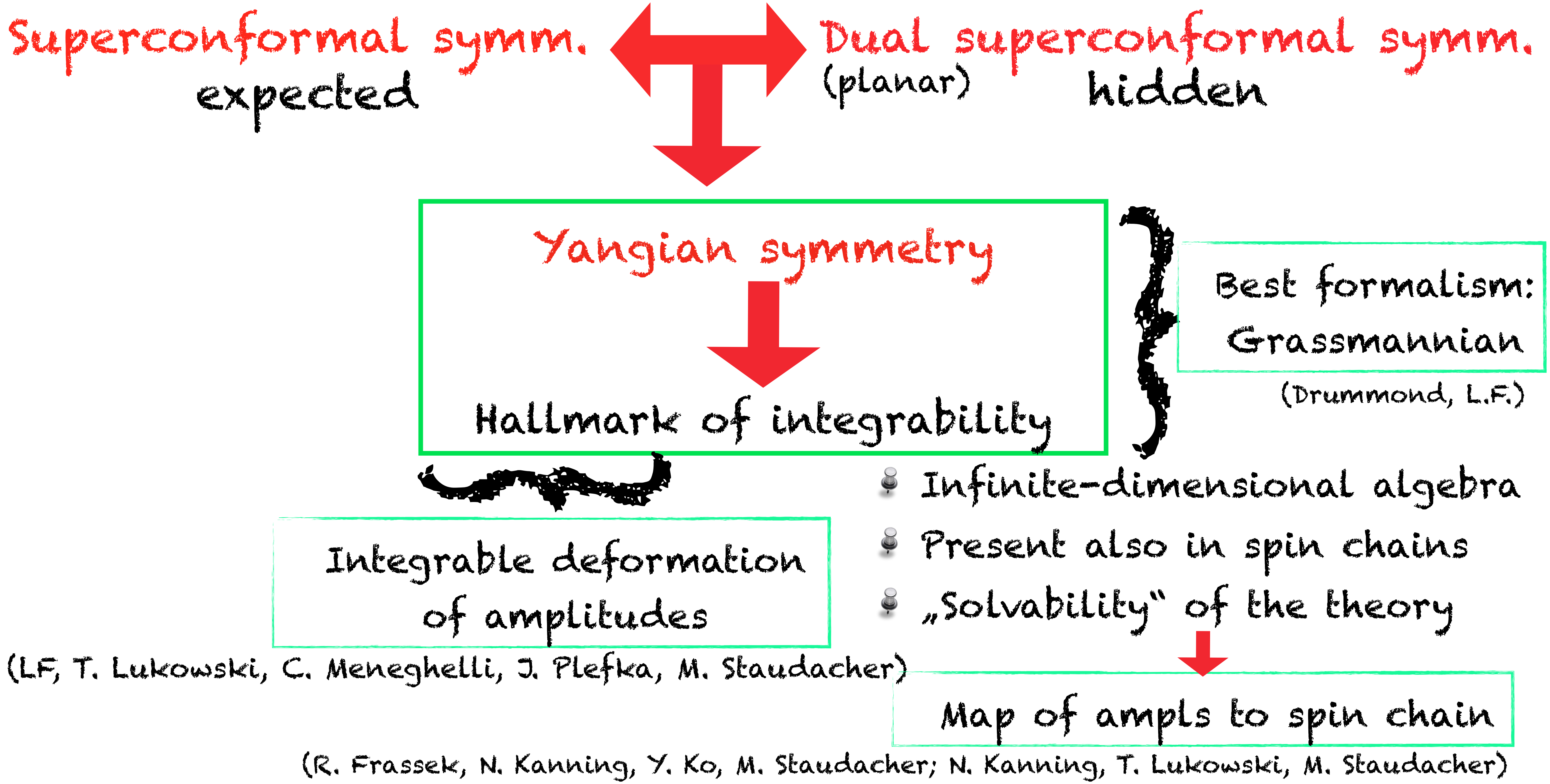
Symmetries and ampls in N=4 sYM

Important in discovering the characteristic of ampls



Superconformal symm.
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Dual superconformal symm.
(planar) hidden

Yangian symmetry

Hallmark of integrability

Best formalism:
Grassmannian

(Drummond, L.F.)

- Infinite-dimensional algebra
- Present also in spin chains
- „Solvability" of the theory

Integrable deformation
of amplitudes

(LF, T. Lukowski, C. Meneghelli, J. Plefka, M. Staudacher)

Map of ampls to spin chain

(R. Frassek, N. Kanning, Y. Ko, M. Staudacher; N. Kanning, T. Lukowski, M. Staudacher)

Grassmannian

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan,Postnikov,Trnka)(Mason,Skinner)

In momentum twistor space: $\mathcal{Z}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \tilde{\mu}_i^{\dot{\alpha}}, \chi_i^{\mathcal{A}})$

$$A_{n,k}^{\text{tree}}(\mathcal{Z}) = \oint \frac{d^{k\times n}c}{(12...k)(23...k+1)...(n1...n+k-1)\text{GL}(k)} \prod_{\alpha=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{\alpha i}\mathcal{Z}_i\right)$$

- c's: complex parameters forming a kxn matrix
- (i i+1...i+k-1): determinant of kxk submatrix of c's
- space of k-planes in n dimensions = Gr(k,n)
- Yangian invariant:

$$\begin{cases} J^{(0)A}{}_{B} = \displaystyle\sum_{i=1}^{n} \mathcal{Z}_i^{A} \frac{\partial}{\partial \mathcal{Z}_i^{B}} \\ J^{(1)A}{}_{B} = \displaystyle\sum_{i<j} \mathcal{Z}_i^{A} \frac{\partial}{\partial \mathcal{Z}_i^{C}} \mathcal{Z}_j^{C} \frac{\partial}{\partial \mathcal{Z}_i^{B}} - (i \leftrightarrow j) \end{cases}$$

Map to spin chain

(R. Frassek, N. Kanning, Y. Ko, M. Staudacher)

- Tree amplitudes: states of an integrable spin chain
- Quantum space = space of functions (distributions) of n copies of momentum supertwistors

$$\boxed{V = V_1 \otimes \ldots \otimes V_n}$$

- This spin chain is integrable
- How to find Yangian invariants?

Quantum Inverse Scattering Method

Map to spin chain

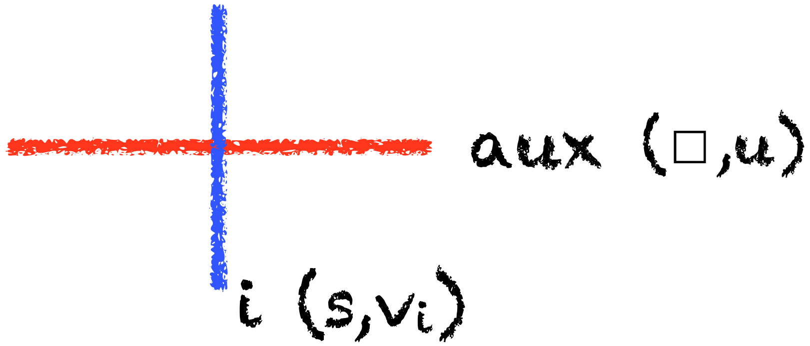
- * Introduce auxiliary space $V_{aux}(\square)$

- * Define Lax operators

$$L_i(u,v_i)^A{}_B = \delta^A{}_B + (u-v_i)^{-1} J_i^{(0)A}{}_B$$

inhomogeneity

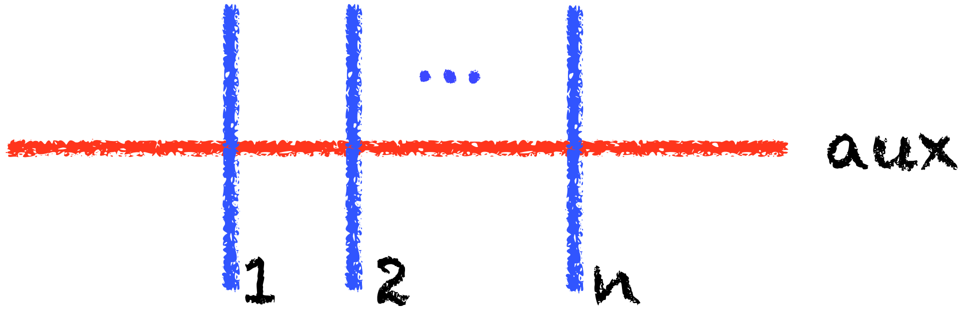
level-zero generators



- * Define a monodromy matrix

$$M(u)^A{}_B = L_1(u)^A L_2(u)\dots L_n(u)_B$$

spectral parameter



Map to spin chain

Yangian invariants

$$M(u)^A{}_B \mathcal{A}_{n,k} = \delta^A{}_B \mathcal{A}_{n,k}$$

Expand to see generators

$$M(u)^A{}_B = \delta^A{}_B + \frac{1}{u} J^{(0)A}{}_B + \frac{1}{u^2} J^{(1)A}{}_B + \dots$$

$$\begin{cases} J^{(0)A}{}_B = \sum_{i=1}^{n} \mathcal{Z}_i^A \frac{\partial}{\partial \mathcal{Z}_i^B} \\ J^{(1)A}{}_B = \sum_{i<j} \mathcal{Z}_i^A \frac{\partial}{\partial \mathcal{Z}_i^C} \mathcal{Z}_j^C \frac{\partial}{\partial \mathcal{Z}_i^B} - (i \leftrightarrow j) \end{cases}$$

Explicit construction of Yangian invariants

Map to spin chain

(N. Kanning, T. Lukowski, M. Staudacher)

Explicit construction of Yangian invariants

Ingredients:

- ☑ vacuum of V, $|0\rangle$: k $\delta^{4|4}(\mathcal{Z}_i)$'s, n-k 1's
- ☑ permutation σ and its decomposition in adjacent transpositions $\sigma = \tau_1 \circ \ldots \circ \tau_p$
- ☑ bridge operator

$$\mathcal{B}_{ij}(v) = (\mathcal{Z}_j \cdot \frac{\partial}{\partial \mathcal{Z}_i})^v = \int \frac{d\alpha}{\alpha^{1+v}} e^{\alpha \mathcal{Z}_j \cdot \frac{\partial}{\partial \mathcal{Z}_i}}$$

- ☑ act according to decompositions

Map to spin chain

Explicit construction of Yangian invariants

n=4, k=2

vacuum: $|0\rangle = \delta(\mathcal{Z}_1)\delta(\mathcal{Z}_2)1_3 1_4$

permutation: $\sigma_{4,2} = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{array} \right) = (23)(34)(12)(23)$

$$\mathcal{A}_{4,2} = \mathcal{B}_{23}(0)\mathcal{B}_{12}(0)\mathcal{B}_{34}(0)\mathcal{B}_{23}(0)|0\rangle$$

Grassmannian

$$= \int \frac{d^{2\times 4}c}{(12)(23)(34)(41)\,\mathrm{GL}(2)} \prod_k \delta^{4|4}(c\cdot\mathcal{Z})$$

Map to spin chain

Explicit construction of Yangian invariants

n=4,k=2

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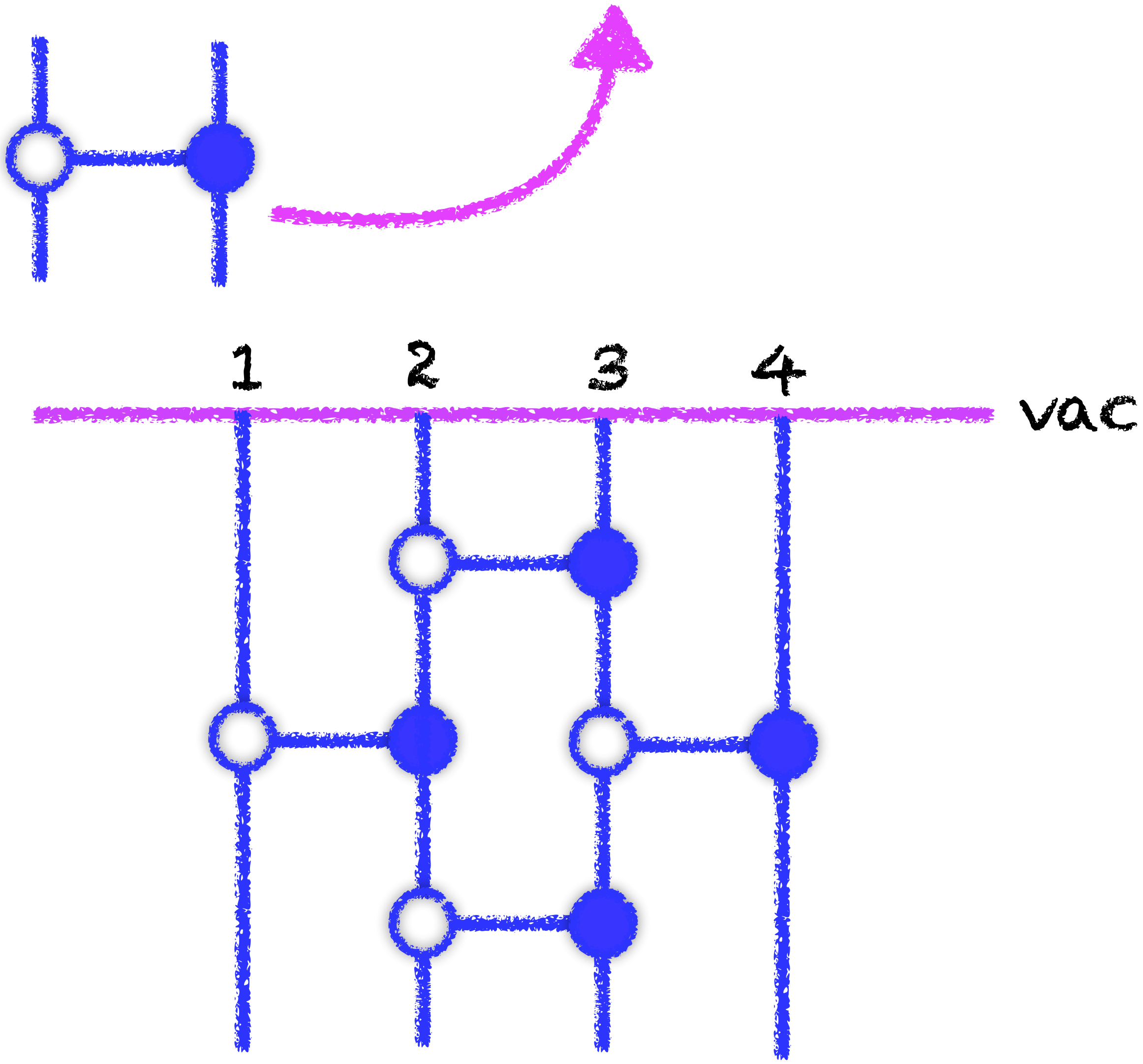

Map to spin chain

Explicit construction of Yangian invariants

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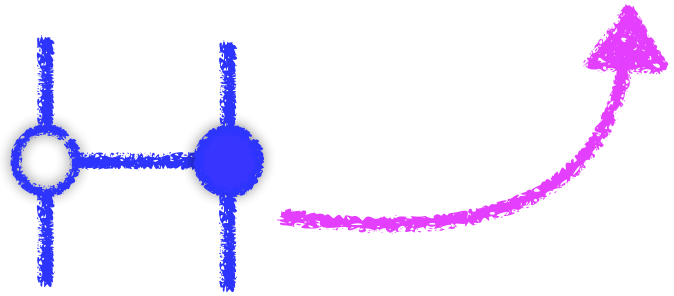

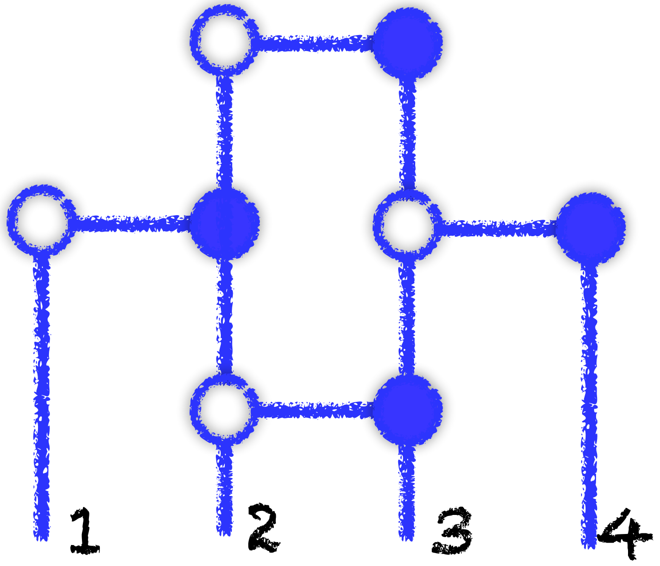

Map to spin chain

Explicit construction of Yangian invariants

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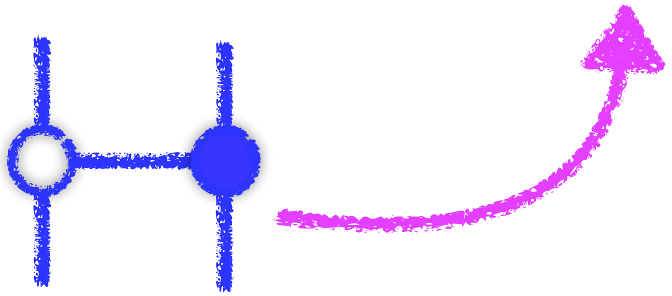

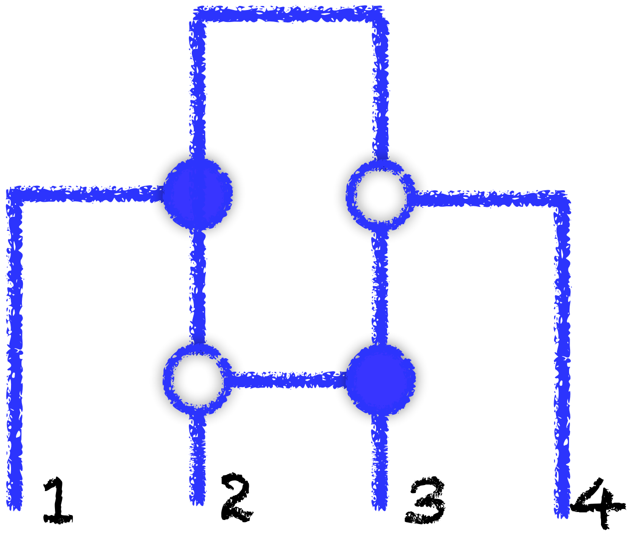

Map to spin chain

Explicit construction of Yangian invariants

n=4, k=2

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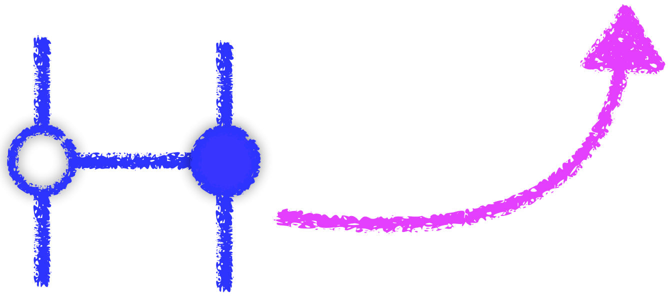

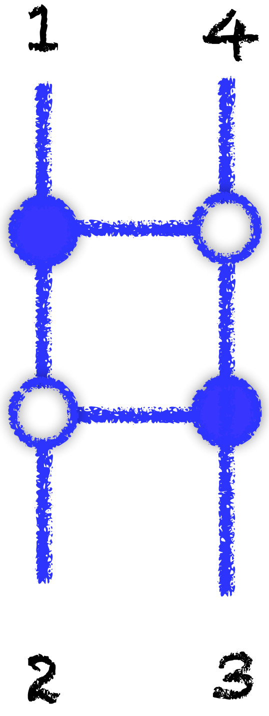

Symmetries and ampls in N=4 sYM

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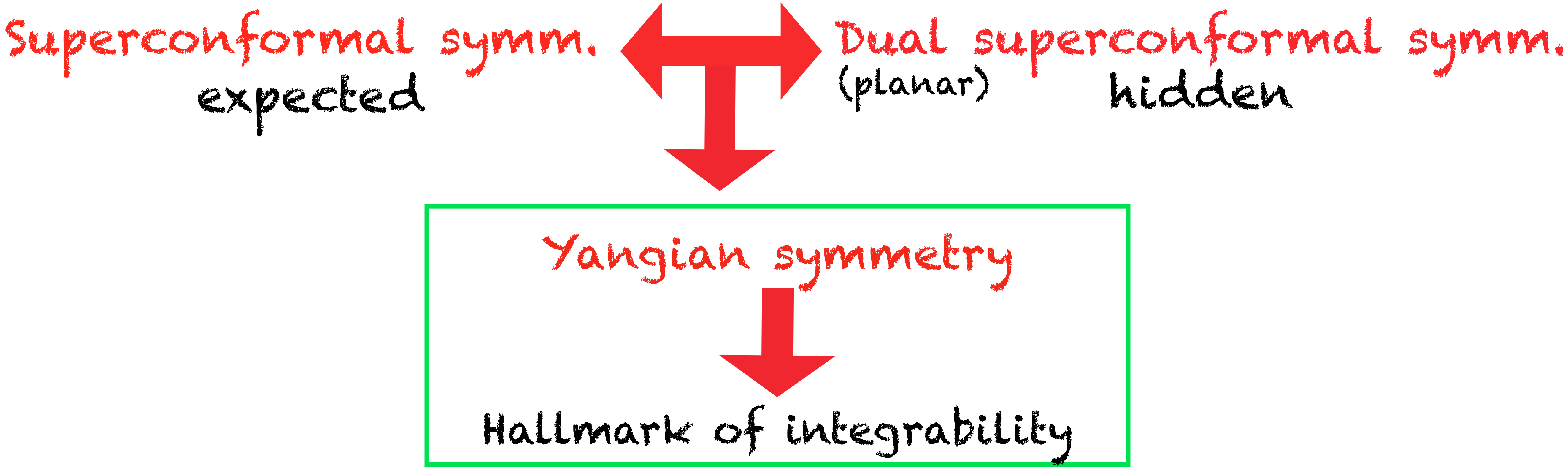


Superconformal symm.
expected

Dual superconformal symm.
(planar) hidden

Yangian symmetry

Hallmark of integrability

What about the Amplituhedron?

non-trivial realization!

help from spin chain

Amplituhedron

(picture of A. Gilmore)

amplitude in planar N=4 sYM

=

through canonical form on the amplituhedron space

=

volume of the dual amplituhedron

(N. Arkani-Hamed, J. Trnka; N. Arkani-Hamed, Y. Bai, T. Lam)

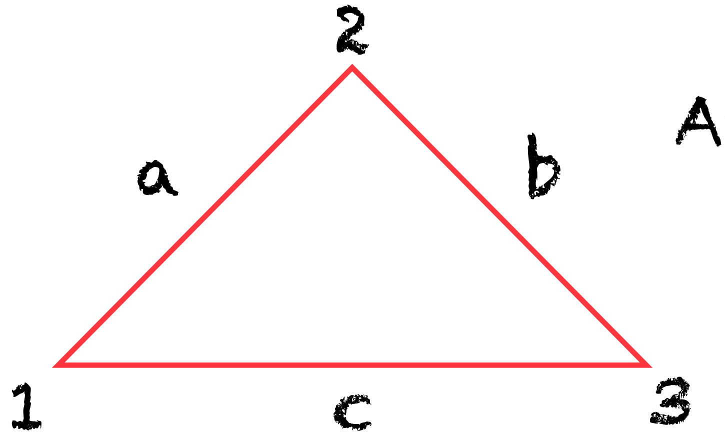
Amplituhedron

(picture of A. Gilmore)

amplitude in planar N=4 sYM
=
through canonical form on the amplituhedron space
=
volume of the dual amplituhedron

(N. Arkani-Hamed, J. Trnka; N. Arkani-Hamed, Y. Bai, T. Lam)

The idea: NMHV tree-level amplitudes = volume of polytope in dual momentum twistor space (Hodges)



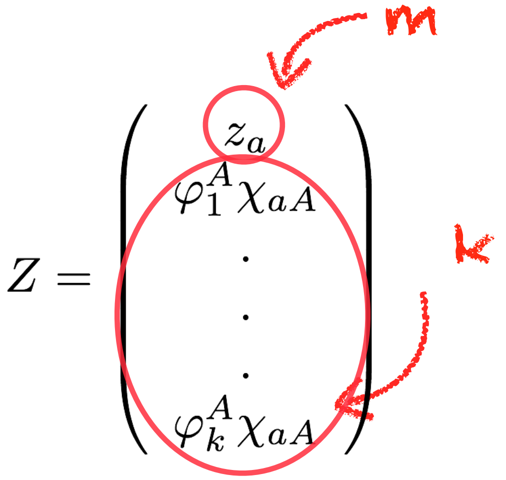
Area = $\frac{1}{2}\frac{\langle abc\rangle^2}{\langle 0bc\rangle\langle 0ab\rangle\langle 0ca\rangle} := [abc]$

„R-invariants"

„Amplitude"= area of the triangle in dual space

Amplituhedron

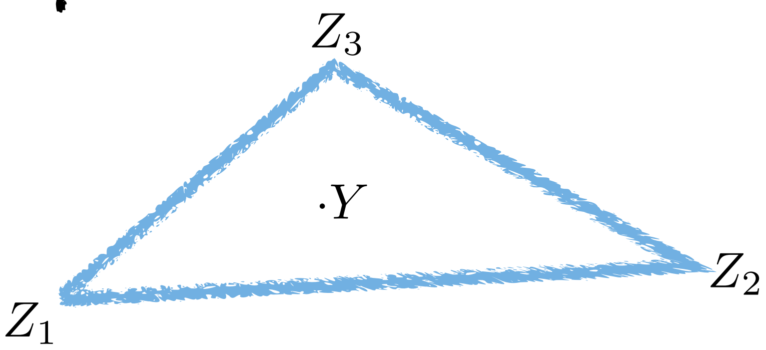
Generalization of triangle in projective space:



$$Z = \begin{pmatrix} z_a \\ \varphi_1^A \chi_{aA} \\ \vdots \\ \varphi_k^A \chi_{aA} \end{pmatrix}$$

Bosonized momentum twistors

$$Y_\alpha^I = \sum_{a=1}^{n} c_{\alpha a} Z_a^I$$



$I = 1, 2, ..., k + m$

$\alpha = 1, 2, ..., k$

Geometric requirements {

„Internal" positivity (c) = interior

„External" positivity (Z) = convexity

ordered minors > 0

- physics: m=4
- tree: k=1 polytope, k>1 more complicated object (which?)
- loops: similar, more complicated formulae

Amplituhedron

Generalization of triangle in projective space:

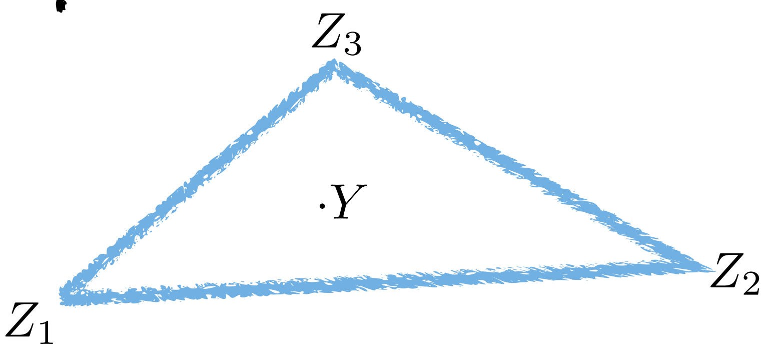
m

$$Z = \begin{pmatrix} z_a \\ \varphi_1^A \chi_{aA} \\ \cdot \\ \cdot \\ \cdot \\ \varphi_k^A \chi_{aA} \end{pmatrix}$$

k

Bosonized momentum twistors

$$Y_\alpha^I = \sum_{a=1}^{n} c_{\alpha a} Z_a^I$$



$I = 1, 2, ..., k+m$

$\alpha = 1, 2, ..., k$

Geometric requirements {

„Internal" positivity (c) = interior

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ordered minors > 0

How to compute the volume directly in dual space?

Amplituhedron

$$A^{\rm tree}_{n,k}(\mathcal{Z}) = \int d^{m}\varphi_1 \ldots d^{m}\varphi_k \int \delta^{mk}(Y,Y_0) \int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots n+k-1)} \underbrace{\prod_{\alpha=1}^{k} \delta^{k+m}\Big(Y_\alpha - \sum_a c_{\alpha a} Z_a\Big)}_{\Omega^{(m)}_{n,k}}$$

volume function

Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

- GL(m+k) covariance
- GL(1) invariance for Z's and GL(k) covariance for Y's
- Capelli differential equations

$$\det\left(\frac{\partial}{\partial W^{A_\nu}_{a_\mu}}\right)_{\substack{1\le\nu\le k+1\\1\le\mu\le k+1}} \Omega^{(m)}_{n,k}(Y,Z)=0$$

(k+1)th order

k=1:

$$\begin{cases} \dfrac{\partial^2\Omega}{\partial Z_i^A\partial Z_j^B} = \dfrac{\partial^2\Omega}{\partial Z_i^B\partial Z_j^A} \\[2ex] \dfrac{\partial^2\Omega}{\partial Z_i^A\partial Y_j^B} = \dfrac{\partial^2\Omega}{\partial Z_i^B\partial Y_j^A} \end{cases}$$

Amplituhedron

$$A^{\text{tree}}_{n,k}(\mathcal{Z}) = \int d^{m}\varphi_1 \ldots d^{m}\varphi_k \int \delta^{mk}(Y,Y_0) \int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots} \underbrace{\!\!\!\!\!\!\!\frac{}{(n1\ldots n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+m}\Big(Y_\alpha - \sum_a c_{\alpha a} Z_a\Big)}_{\Omega^{(m)}_{n,k}}$$

volume function

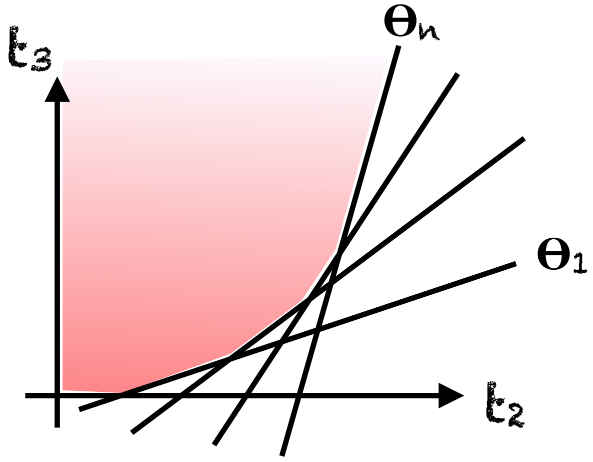
Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

- ☆GL(m+k) covariance
- ☆GL(1) invariance for Z's and GL(k) covariance for Y's
- ☆Capelli differential equations

k=1

$$\boxed{\Omega^{(m)}_{n,1} = \int_0^{+\infty} \left(\prod_{A=2}^{m+1} dt_A\right) \frac{m!}{(t\cdot Y)^{m+1}} \prod_{i=m+2}^{n} \theta\left(t\cdot Z_i\right)}$$



Amplituhedron

$$A^{\text{tree}}_{n,k}(\mathcal{Z}) = \int d^{\color{red}m}\varphi_1 \ldots d^{\color{red}m}\varphi_k \int \delta^{{\color{red}m}k}(Y, Y_0) \underbrace{\int \frac{d^{k\times n}c}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+{\color{red}m}}(Y_\alpha - \sum_a c_{\alpha a} Z_a)}_{\Omega^{(m)}_{n,k}}$$

volume function

Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

★GL(m+k) covariance

★GL(1) invariance for Z's and GL(k) covariance for Y's

★Capelli differential equations

☑ m=4 <-> physics

☑ no need to think about triangulation

☑ directly in dual space

Amplituhedron

$$A_{n,k}^{\text{tree}}(\mathcal{Z}) = \int d^m\varphi_1 ... d^m\varphi_k \int \delta^{mk}(Y, Y_0) \int \underbrace{\frac{d^{k\times n}c}{(12...k)(23...k+1)...(n1...n+k-1)} \prod_{\alpha=1}^{k} \delta^{k+m}(Y_\alpha - \sum_a c_{\alpha a} Z_a)}_{\Omega_{n,k}^{(m)}}$$

volume function

Symmetries of Ω

(LF, T. Lukowski, A. Orta, M. Parisi)

☆GL(m+k) covariance

☆GL(1) invariance for Z's and GL(k) covariance for Y's

☆Capelli differential equations

$\longrightarrow$ Higher helicity

☐ integrand not fully fixed

☐ can Yangian symmetry help us?

Map to spin chain

(LF, T. Lukowski, A. Orta, M. Parisi)

- Yangian symmetry: long-standing problem
- Use spin chain formalism

Lax operator

Monodromy matrix

- New ingredients

-> bosonised twistors Z

-> auxiliary k-plane Y

Change of vacuum

$$\prod_{i=1}^{k} \delta^{m|m}(\mathcal{Z}_i) \longrightarrow S_k^{(m)}$$ seed

$$S_k^{(m)} = \int \frac{d^{k\times k}\beta}{(\det\beta)^k} \prod_{\alpha=1}^{k} \delta^{m+k}\left(Y_\alpha^A - \sum_{i=1}^{k} \beta_{\alpha i} Z_i^A\right)$$

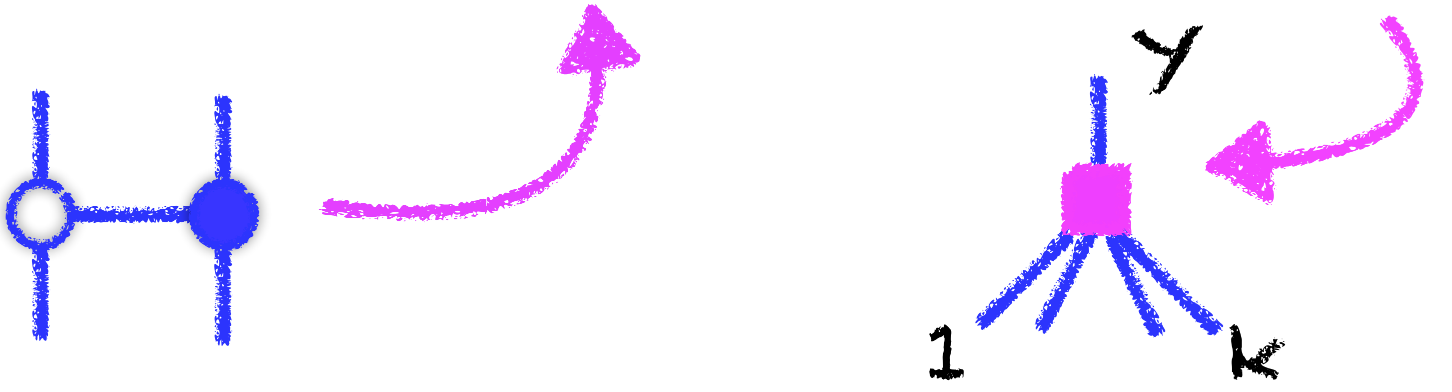
Map to spin chain

- Yangian symmetry: long-standing problem
- Use spin chain formalism

$$\boxed{\Omega_{n,k}^{(m)} = \prod_{l=1}^{k(n-k)} \mathcal{B}_{i_l j_l}(0) S_k^{(m)}}$$

Example n=4,k=2

$$\Omega_{4,2}^{(m)} = \mathcal{B}_{23}(0)\mathcal{B}_{12}(0)\mathcal{B}_{34}(0)\mathcal{B}_{23}(0)S_2^{(m)}$$



Map to spin chain

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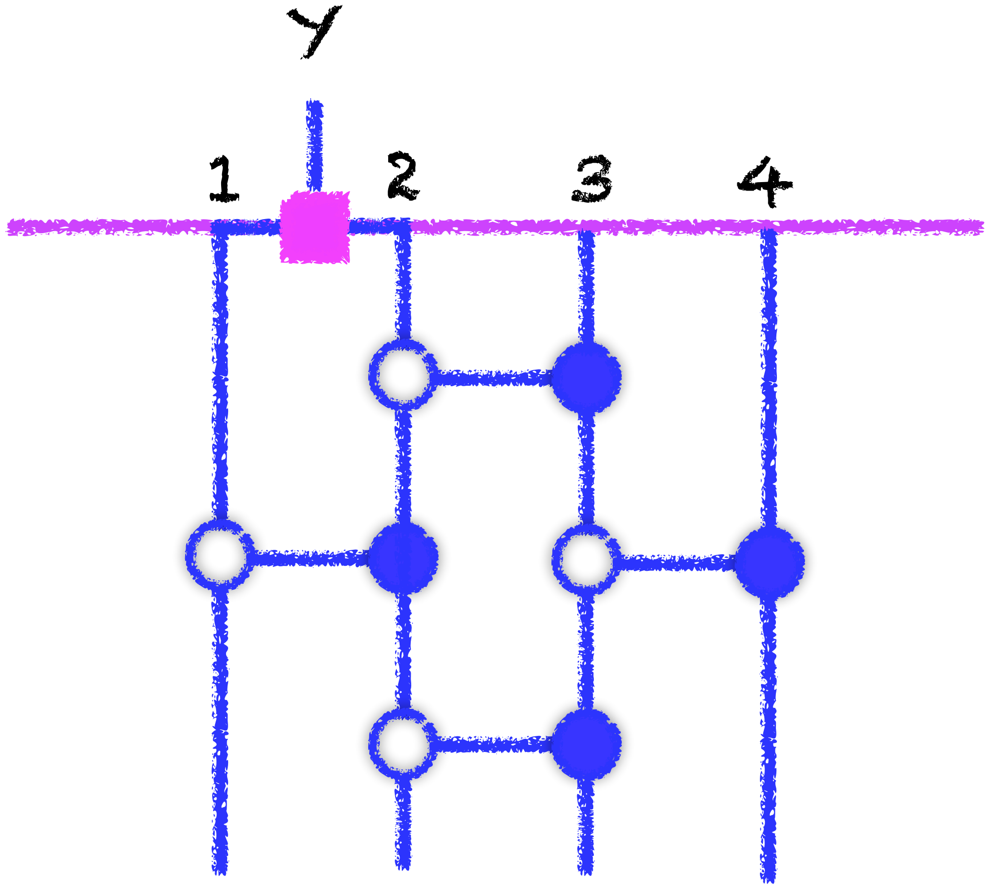

Map to spin chain

Example n=4,k=2

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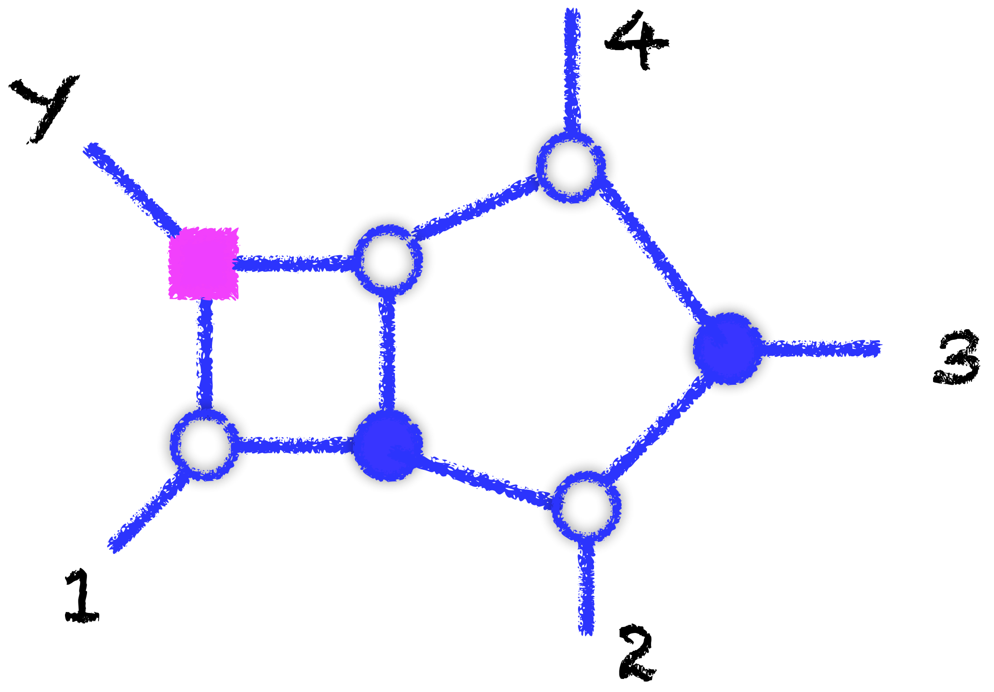
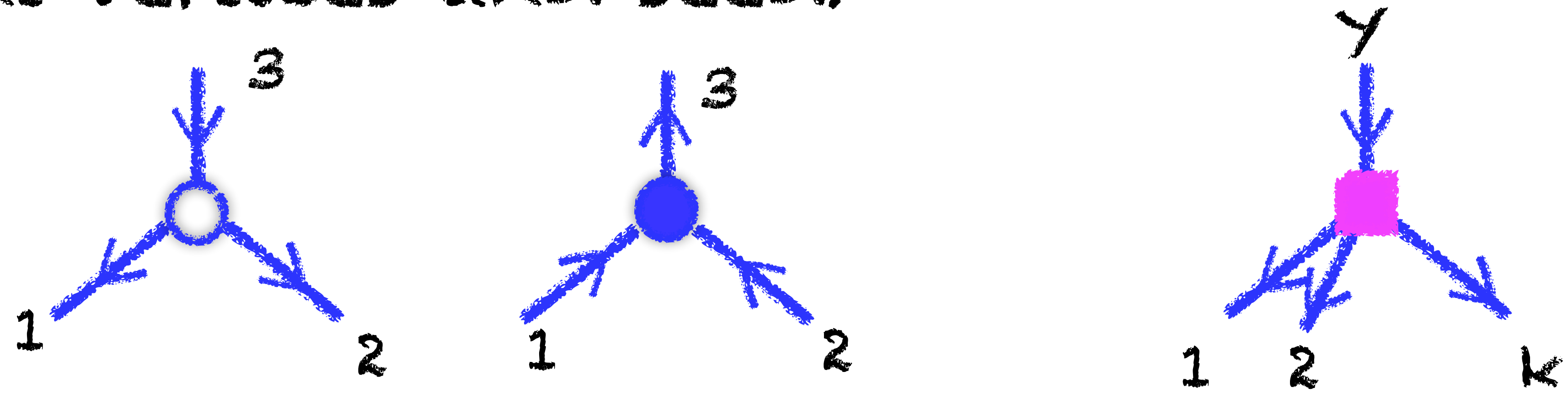


New on-shell diagrams transformations

On-shell diagrammatics

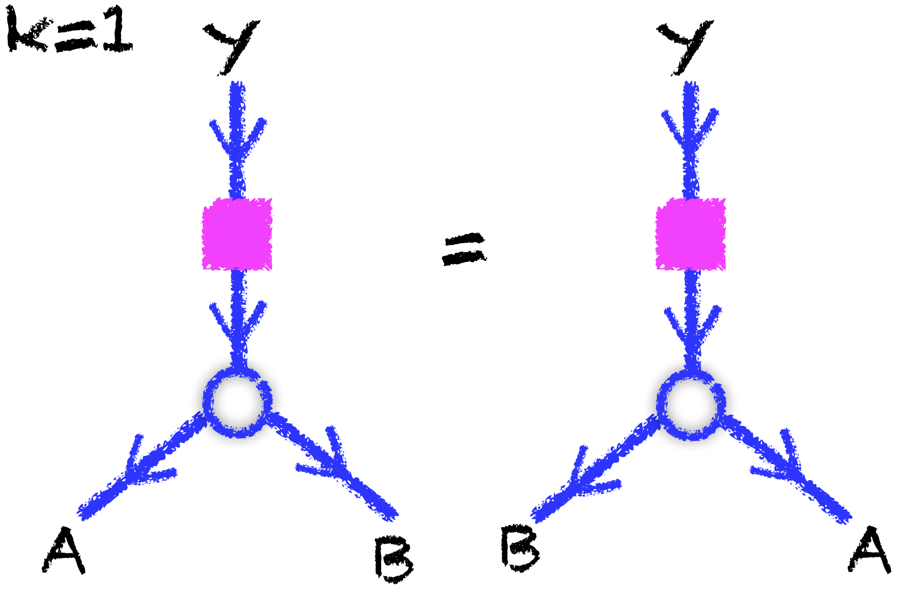
New on-shell diagrams transformations

Trivalent vertices and seed:

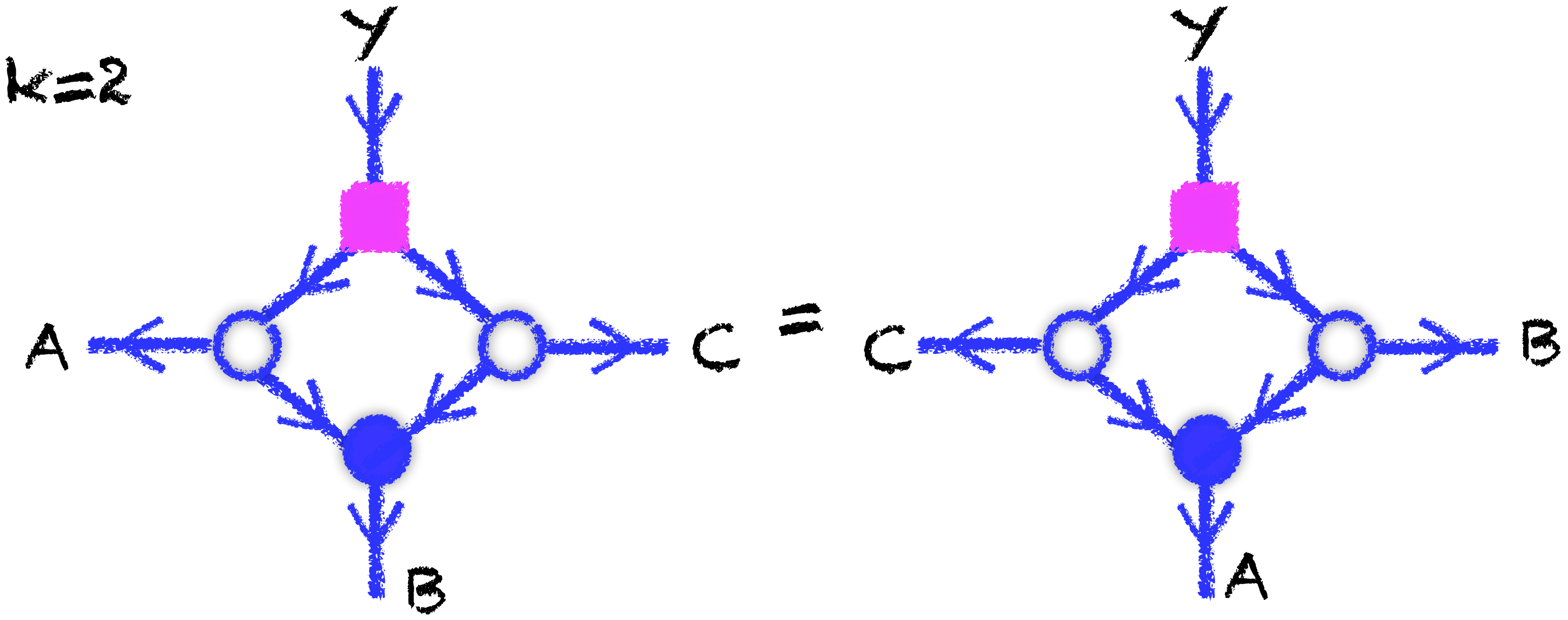


Mutations for the seed:

k=1



k=2



...hold for any k

Yangian symmetry

- Yangian symmetry: long-standing problem
- Use spin chain formalism

Define a set of functions

$$\Omega^A{}_B = (J_Y)^A{}_B\, \Omega^{(m)}_{n,k}$$

$$(J_Y)^A{}_B = \sum_{\alpha=1}^{k} Y^A_\alpha \frac{\partial}{\partial Y^B_\alpha} + k\delta^A_B$$

$$\Big\Downarrow$$

$$M(u)^C{}_B\, \Omega^A{}_C(Y,Z) = \Omega^A{}_B(Y,Z)$$

Yangian generators

$$M(u)^A{}_B = \delta^A{}_B + \frac{1}{u} J^{(0)A}{}_B + \frac{1}{u^2} J^{(1)A}{}_B + \dots$$

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$$(J_Y)^A{}_B = \sum_{\alpha=1}^{k} Y^A_\alpha \frac{\partial}{\partial Y^B_\alpha} + k\delta^A_B$$

$$\boxed{\begin{array}{l} (J^{(0)})^C{}_B \, \Omega^A{}_C = 0 \\ (J^{(1)})^C{}_B \, \Omega^A{}_C = 0 \end{array}} \quad \text{with} \quad \left\{ \begin{array}{l} J^{(0)A}{}_B = \displaystyle\sum_{i=1}^{n} Z_i^A \frac{\partial}{\partial Z_i^B} + k\delta^A{}_B \\[2ex] J^{(1)A}{}_B = \displaystyle\sum_{i<j} Z_i^A \frac{\partial}{\partial Z_i^C} Z_j^C \frac{\partial}{\partial Z_i^B} - (i \leftrightarrow j) \end{array} \right.$$

generators of Y(gl(m+k))

Yangian symmetry

- Yangian symmetry: long-standing problem
- Use spin chain formalism!

Define a set of functions

$$\Omega^A{}_B = (J_Y)^A{}_B \, \Omega^{(m)}_{n,k}$$

$$(J_Y)^A{}_B = \sum_{\alpha=1}^{k} Y^A_\alpha \frac{\partial}{\partial Y^B_\alpha} + k\delta^A_B$$

$$\boxed{\begin{aligned} (J^{(0)})^C{}_B \, \Omega^A{}_C &= 0 \\ (J^{(1)})^C{}_B \, \Omega^A{}_C &= 0 \end{aligned}}$$

with

$$\begin{cases} J^{(0)A}{}_B = \displaystyle\sum_{i=1}^{n} Z^A_i \frac{\partial}{\partial Z^B_i} + k\delta^A{}_B \\ J^{(1)A}{}_B = \displaystyle\sum_{i<j} Z^A_i \frac{\partial}{\partial Z^C_i} Z^C_j \frac{\partial}{\partial Z^B_i} - (i \leftrightarrow j) \end{cases}$$

Matrices $\Omega^A{}_B$ are Yangian invariant

Conclusions

Amplituhedron gives a geometric interpretation of the amplitudes for planar N=4 sYM

- ☑ volume function possesses interesting symmetries
- ☑ described via spin chain
- ☑ suitable modification invariant under Y(gl(m+k))
- ☐ how to evaluate volume for k>1 and for loops?
- ☐ can Yangian symmetry fix it?

A lot of work still has to be done!