The Steinmann Cluster Bootstrap for $\mathcal{N} = 4$ SYM Amplitudes

Georgios Papathanasiou

DESY

Centre for Theoretical Physics Higgs

Amplitudes
July 10-14 2017

The University of Edinburgh

1412.3763 w/ Drummond,Spradlin
1612.08976 w/ Dixon,Drummond,Harrington,McLeod,Spradlin
+ in progress w/ Caron-Huot,Dixon,McLeod,von Hippel

Outline

Motivation: Why Planar $\mathcal{N} = 4$ Amplitudes?

The Bootstrap Philosophy

Cluster Algebra Upgrade
- The 3-loop MHV Heptagon

Steinmann Upgrade
- The 3-loop NMHV/4-loop MHV Heptagon

New Developments

Conclusions & Outlook

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N}=4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Would amount to "solving" an interacting 4D gauge theory...

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N}=4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

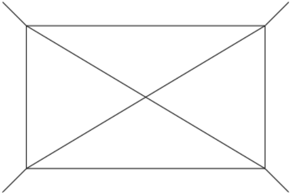
Would amount to "solving" an interacting 4D gauge theory...

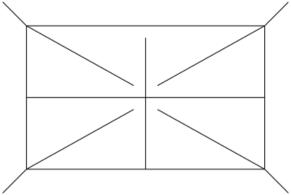
Ambitious, but promising in 't Hooft limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Would amount to "solving" an interacting 4D gauge theory...

Ambitious, but promising in 't Hooft limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:



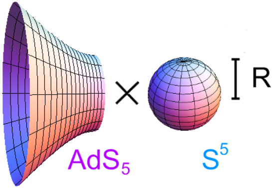


- Perturbatively, only planar diagrams contribute

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N}=4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Would amount to "solving" an interacting 4D gauge theory...

Ambitious, but promising in 't Hooft limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:

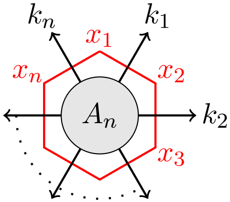


- Perturbatively, only planar diagrams contribute
- Planar $\mathcal{N}=4$ SYM $\Leftrightarrow$ Free type IIB superstrings on $AdS_5 \times S^5$
  strongly coupled $\Leftrightarrow$ weakly coupled

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N}=4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Would amount to "solving" an interacting 4D gauge theory...

Ambitious, but promising in 't Hooft limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:



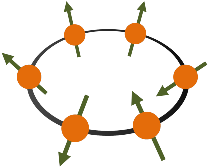
$k_i = x_{i+1} - x_i$

- Perturbatively, only planar diagrams contribute
- Planar $\mathcal{N}=4$ SYM $\Leftrightarrow$ Free type IIB superstrings on $AdS_5 \times S^5$
  strongly coupled $\Leftrightarrow$ weakly coupled
- Amplitudes $\Leftrightarrow$ Wilson Loops; Dual Conformal Symmetry
  [Alday,Maldacena] [Drummond,Henn,Korchemsky,Sokatchev] [Brandhuber,Heslop,Travaglini]

Aim: Can we compute scattering amplitudes in $SU(N)$ $\mathcal{N}=4$ super Yang Mills theory to all loops, for any multiplicity and quantum numbers of the external particles?

Would amount to "solving" an interacting 4D gauge theory...

Ambitious, but promising in 't Hooft limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:



- Perturbatively, only planar diagrams contribute
- Planar $\mathcal{N}=4$ SYM $\Leftrightarrow$ Free type IIB superstrings on $AdS_5 \times S^5$
  strongly coupled $\Leftrightarrow$ weakly coupled
- Amplitudes $\Leftrightarrow$ Wilson Loops; Dual Conformal Symmetry
  [Alday,Maldacena] [Drummond,Henn,Korchemsky,Sokatchev] [Brandhuber,Heslop,Travaglini]
- Integrable structures $\Rightarrow$ All loop quantities! [Beisert,Eden,Staudacher]

Practical significance

Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Practical significance

Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

Practical significance

Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

- Generalised Unitarity [Bern,Dixon,Dunbar,Kosower...]

Practical significance

Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

- Generalised Unitarity [Bern,Dixon,Dunbar,Kosower...]
- Method of Symbols [Goncharov,Spradlin,Vergu,Volovich]

Practical significance

Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

- Generalised Unitarity [Bern,Dixon,Dunbar,Kosower...]
- Method of Symbols [Goncharov,Spradlin,Vergu,Volovich]

leading to significant practical applications!

Practical significance

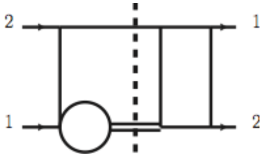
Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

- Generalised Unitarity [Bern,Dixon,Dunbar,Kosower...]
- Method of Symbols [Goncharov,Spradlin,Vergu,Volovich]

leading to significant practical applications! For example,

$|gg \to Hg|^2$ for N^{3}LO Higgs cross-section [Anastasiou,Duhr,Dulat,Herzog,Mistlberger]



Practical significance

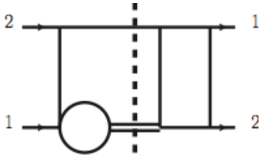
Hopefully I've convinced you that this aim is theoretically interesting and possibly within reach.

Along the way, it is very likely that new computational methods will also be developed, as prompted by earlier successes,

- Generalised Unitarity [Bern,Dixon,Dunbar,Kosower...]
- Method of Symbols [Goncharov,Spradlin,Vergu,Volovich]

leading to significant practical applications! For example,

$|gg \to Hg|^2$ for N^{3}LO Higgs cross-section [Anastasiou,Duhr,Dulat,Herzog,Mistlberger]



or more recently the 3-loop QCD soft anomalous dimension.
[Almelid,Duhr,Gardi,McLeod,White]

So which part of this journey are we at?

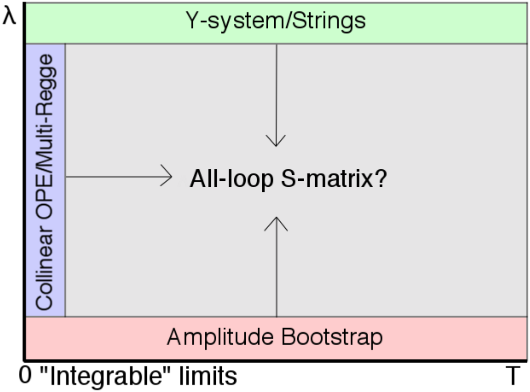
So which part of this journey are we at?

Amplitudes with $n = 4, 5$ particles already known to all loops! Captured by the Bern-Dixon-Smirnov ansatz $\mathcal{A}_n^{\text{BDS}}$.

So which part of this journey are we at?

Amplitudes with $n = 4, 5$ particles already known to all loops! Captured by the Bern-Dixon-Smirnov ansatz $\mathcal{A}_n^{\mathrm{BDS}}$.

More generally,



The Amplitude Bootstrap

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

B. Fix the coefficients of the ansatz by imposing consistency conditions (e.g. known near-collinear or multi-Regge limiting behavior)

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

B. Fix the coefficients of the ansatz by imposing consistency conditions (e.g. known near-collinear or multi-Regge limiting behavior)

First applied very successfully for the first nontrivial, 6-particle amplitude through 5 loops. [Dixon,Drummond,Henn] [Dixon,Drummond,Hippel/Duhr,Pennington] [(Caron-Huot,)Dixon,McLeod,von Hippel]

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

B. Fix the coefficients of the ansatz by imposing consistency conditions (e.g. known near-collinear or multi-Regge limiting behavior)

First applied very successfully for the first nontrivial, 6-particle amplitude through 5 loops. [Dixon,Drummond,Henn] [Dixon,Drummond,Hippel/Duhr,Pennington] [(Caron-Huot,)Dixon,McLeod,von Hippel]

Motivated by this progress, we upgraded this procedure for $n = 7$, with information from the cluster algebra structure of the kinematical space. Surprisingly, more powerful than $n = 6$! [Drummond,GP,Spradlin]

What are the right functions?

Multiple polylogarithms (MPLs)

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential may be written as a finite linear combination

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)} \, d\log\phi_\alpha$$

over some set of ϕ_α, where $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

What are the right functions?

Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential may be written as a finite linear combination

$$df_k = \sum_{\alpha} f_{k-1}^{(\alpha)}\, d\log\phi_\alpha$$

over some set of ϕ_α, where $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ [See Brandhuber's talk] encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\dots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\dots,\alpha_k)}\,(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k})\,.$$

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential may be written as a finite linear combination

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)}\, d\log\phi_\alpha$$

over some set of ϕ_α, where $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ [See Brandhuber's talk] encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\dots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\dots,\alpha_k)}\,(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k})\,.$$

Collection of ϕ_α : **symbol alphabet** | $f_0^{(\alpha_1,\dots,\alpha_k)}$ rational

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential may be written as a finite linear combination

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)} \, d\log\phi_\alpha$$

over some set of ϕ_α, where $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ [See Brandhuber's talk] encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \left(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k}\right).$$

Collection of ϕ_α : **symbol alphabet** | $f_0^{(\alpha_1,\ldots,\alpha_k)}$ rational

Empeirical evidence: L-loop amplitudes=MPLs of weight $k = 2L$
[Duhr,Del Duca,Smirnov] [Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [GP]

What are the right variables?

What are the right variables?

More precisely, what is the symbol alphabet? [See talk by Volovich]

What are the right variables?

More precisely, what is the symbol alphabet? [See talk by Volovich]

- For $n = 6$, 9 letters, motivated by analysis of relevant integrals

What are the right variables?

More precisely, what is the symbol alphabet? [See talk by Volovich]

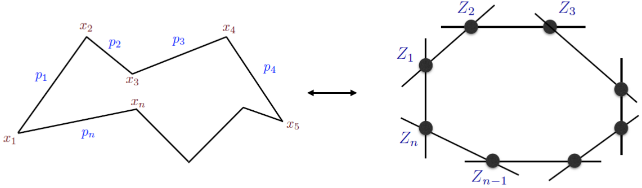
- For $n = 6$, 9 letters, motivated by analysis of relevant integrals
- More generally, strong motivation from *cluster algebra structure* of *kinematical configuration space* $\mathrm{Conf}_n(\mathbb{P}^3)$
  [Golden,Goncharov,Spradlin,Vergu,Volovich]

What are the right variables?

More precisely, what is the symbol alphabet? [See talk by Volovich]

- For $n = 6$, 9 letters, motivated by analysis of relevant integrals
- More generally, strong motivation from *cluster algebra structure* of *kinematical configuration space* $\mathrm{Conf}_n(\mathbb{P}^3)$ [Golden,Goncharov,Spradlin,Vergu,Volovich]

The latter is a collection of n ordered *momentum twistors* Z_i on $\mathbb{P}^3$, (an equivalent way to parametrise massless kinematics), modulo dual conformal transformations. [Hodges][See talks by Arkani-Hammed,Bai,Ferro]



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

Cluster algebras [Fomin,Zelevinsky]

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers.

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra

$$a_1 \longrightarrow a_2 \longrightarrow a_3$$

Initial Cluster

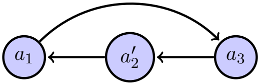
Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra

Initial Cluster



Mutate a_2: New cluster

General rule for mutation at node k:

1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra

$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[r] & a_3 \end{tikzcd}$$

Initial Cluster

$$\begin{tikzcd} a_1 \arrow[rr, bend left] & a_2' \arrow[l] & a_3 \arrow[l] \end{tikzcd}$$

Mutate a_2: New cluster

$$a_2' = (a_1 + a_3)/a_2$$

and so on...

General rule for mutation at node k:

1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.
2. In new quiver/cluster, $a_k \to a_k' = \Big(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j \Big)/a_k$

Connection to the kinematic space

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed...]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed...]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed...]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \ldots i_k \rangle$

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed. . .]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \ldots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed...]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \dots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates
- Crucial observation: For all known cases, symbol alphabet of n-point amplitudes for $n = 6, 7$ are $Gr(4,n)$ cluster variables (also known as $\mathcal{A}$-coordinates) [Golden,Goncharov,Spradlin,Vergu,Volovich]

Connection to the kinematic space

The latter is closely related to a Graßmannian: [See talks by Arkani-Hammed...]

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \dots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates
- Crucial observation: For all known cases, symbol alphabet of n-point amplitudes for $n = 6, 7$ are $Gr(4,n)$ cluster variables (also known as $\mathcal{A}$-coordinates) [Golden,Goncharov,Spradlin,Vergu,Volovich]

Fundamental assumption of "cluster bootstrap"

Symbol alphabet is made of cluster $\mathcal{A}$-coordinates on $\mathrm{Conf}_n(\mathbb{P}^3)$. For the heptagon, 42 of them.

Heptagon Symbol Letters

Multiply $\mathcal{A}$-coordinates with suitable powers of $\langle i\, i+1\, i+2\, i+3\rangle$ to form conformally invariant cross-ratios,

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

where

$$\langle ijkl\rangle \equiv \langle Z_i Z_j Z_k Z_l\rangle = \det(Z_i Z_j Z_k Z_l)$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle\,,$$

together with a_{ij} obtained from a_{i1} by cyclically relabeling $Z_m \to Z_{m+j-1}$.

Back to contructing and constraining function space

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})
2. Integrability: For given $\mathcal{S}$, ensures $\exists$ function with given symbol

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} \, d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} = 0 \quad \forall j\,.$$

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})
2. Integrability: For given $\mathcal{S}$, ensures $\exists$ function with given symbol

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} \; d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} = 0 \quad \forall j \,.$$

3. Dual superconformal symmetry $\Rightarrow$ constrains last symbol entry of amplitudes (MHV 7-pts: a_{2j}, a_{3j}) [Caron-Huot,He]

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})
2. Integrability: For given $\mathcal{S}$, ensures $\exists$ function with given symbol

$$\sum_{\alpha_1,\dots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\dots,\alpha_k)} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} \; d\log \phi_{\alpha_j} \wedge d\log \phi_{\alpha_{j+1}} = 0 \quad \forall j\,.$$

3. Dual superconformal symmetry $\Rightarrow$ constrains last symbol entry of amplitudes (MHV 7-pts: a_{2j}, a_{3j}) [Caron-Huot,He]
4. Collinear limit: Bern-Dixon-Smirnov ansatz $\mathcal{A}_n^{\mathsf{BDS}}$ contains all IR divergences $\Rightarrow$ Constraint on $B_n \equiv \mathcal{A}_n/\mathcal{A}_n^{\mathsf{BDS}}$: $\lim_{i+1\|i} B_n = B_{n-1}$

Back to contructing and constraining function space

1. Locality: Amplitudes may only have singularities when intermediate particles go on-shell $\Rightarrow$ constrains first symbol entry (7-pts: a_{1j})
2. Integrability: For given $\mathcal{S}$, ensures $\exists$ function with given symbol

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} \; d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} = 0 \quad \forall j\,.$$

3. Dual superconformal symmetry $\Rightarrow$ constrains last symbol entry of amplitudes (MHV 7-pts: a_{2j}, a_{3j}) [Caron-Huot,He]
4. Collinear limit: Bern-Dixon-Smirnov ansatz $\mathcal{A}_n^{\text{BDS}}$ contains all IR divergences $\Rightarrow$ Constraint on $B_n \equiv \mathcal{A}_n/\mathcal{A}_n^{\text{BDS}}$: $\lim_{i+1\|i} B_n = B_{n-1}$

Define ***n*-gon symbol**: A symbol of the corresponding n-gon alphabet, obeying 1 & 2.

Results [Drummond,GP,Spradlin]

Weight k =	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
with both of the previous two	0	0	0	1	0	1

Table: Heptagon symbols and their properties.

Results [Drummond,GP,Spradlin]

Weight k =	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
with both of the previous two	0	0	0	1	0	1

Table: Heptagon symbols and their properties.

The symbol of the three-loop seven-particle MHV amplitude is the only weight-6 heptagon symbol which satisfies the last-entry condition and which is finite in the $7 \parallel 6$ collinear limit.

Comparison with the hexagon case

Weight k =	1	2	3	4	5	6
Number of hexagon symbols	3	9	26	75	218	643
well-defined (vanish) in the $6 \parallel 5$ limit	0	2	11	44	155	516
well-defined (vanish) for all $i{+}1 \parallel i$	0	0	2	12	68	307
with MHV last entries	0	3	7	21	62	188
with both of the previous two	0	0	1	4	14	59

Table: Hexagon symbols and their properties.

Surprisingly, heptagon bootstrap more powerful than hexagon one! Fact that $\lim_{7\parallel 6} R_7^{(3)} = R_6^{(3)}$, as well as discrete symmetries such as cyclic $Z_i \to Z_{i+1}$, flip $Z_i \to Z_{n+1-i}$ or parity symmetry **follow for free**, not imposed a priori.

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

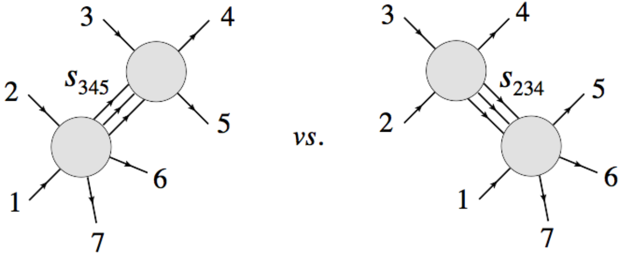
Dramatically simplify n-gon function space

[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Dramatically simplify n-gon function space
[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Double discontinuities vanish for any set of overlapping channels

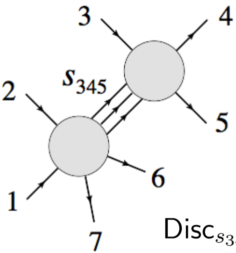


Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

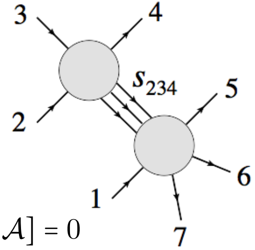
Dramatically simplify n-gon function space

[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Double discontinuities vanish for any set of overlapping channels



vs.



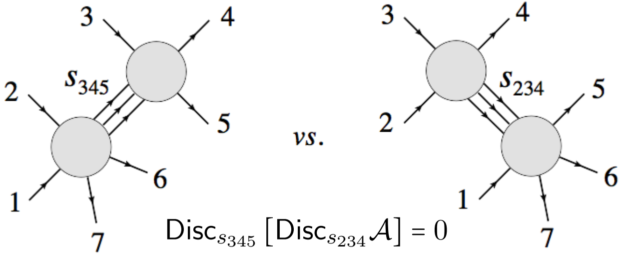
$$\mathrm{Disc}_{s_{345}}\left[\mathrm{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

- Channel labelled by Mandelstam invariant we analytically continue

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Dramatically simplify n-gon function space
[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Double discontinuities vanish for any set of overlapping channels



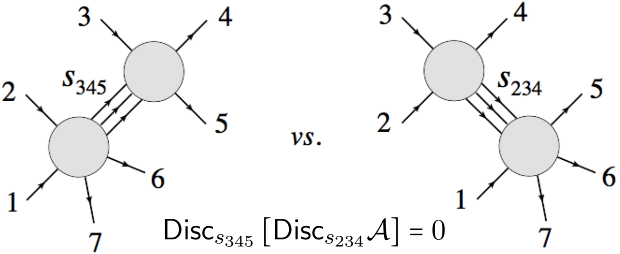
$$\mathrm{Disc}_{s_{345}}\left[\mathrm{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

- Channel labelled by Mandelstam invariant we analytically continue
- Channels overlap if they divide particles in 4 nonempty sets.
  Here: $\{2\}$, $\{3,4\}$, $\{5\}$, and $\{6,7,1\}$

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Dramatically simplify n-gon function space
[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Double discontinuities vanish for any set of overlapping channels



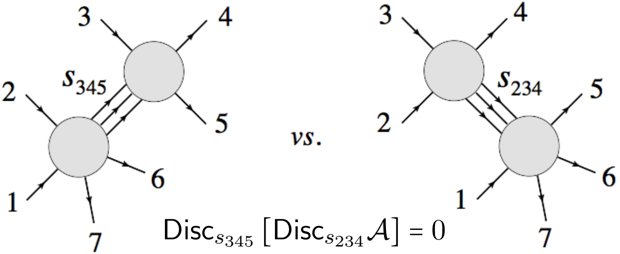
$$\mathrm{Disc}_{s_{345}}\left[\mathrm{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

- Channel labelled by Mandelstam invariant we analytically continue
- Channels overlap if they divide particles in 4 nonempty sets. Here: $\{2\}$, $\{3,4\}$, $\{5\}$, and $\{6,7,1\}$
- Focus on $s_{i-1,i,i+1} \propto a_{1i}$ (s_{i-1i} more subtle)

Upgrade II: Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Dramatically simplify n-gon function space
[Caron-Huot,Dixon,McLeod,von Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Double discontinuities vanish for any set of overlapping channels



$$\text{Disc}_{s_{345}}\left[\text{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

- Channel labelled by Mandelstam invariant we analytically continue
- Channels overlap if they divide particles in 4 nonempty sets. Here: $\{2\}$, $\{3,4\}$, $\{5\}$, and $\{6,7,1\}$
- Focus on $s_{i-1,i,i+1} \propto a_{1i}$ (s_{i-1i} more subtle)

Heptagon: No $a_{1,i\pm1}, a_{1,i\pm2}$ after $a_{1,i}$ on second symbol entry

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

Results: Steinmann Heptagon symbols

Weight $k =$	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

1. Compare with 7, 42, 237, 1288, 6763 non-Steinmann heptagon symbols

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

1. Compare with 7, 42, 237, 1288, 6763 non-Steinmann heptagon symbols
2. $\frac{28}{42} = \frac{6}{9} = \frac{2}{3}$ reduction at weight 2

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

1. Compare with 7, 42, 237, 1288, 6763 non-Steinmann heptagon symbols
2. $\frac{28}{42} = \frac{6}{9} = \frac{2}{3}$ reduction at weight 2
3. Increase by a factor of ~ 3 instead of ~ 5 at each weight

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
parity −, flip −	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

Table: Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7. All of them are organized with respect to the discrete symmetries $Z_i \to Z_{i+1}$, $Z_i \to Z_{8-i}$ of the MHV amplitude.

1. Compare with 7, 42, 237, 1288, 6763 non-Steinmann heptagon symbols
2. $\frac{28}{42} = \frac{6}{9} = \frac{2}{3}$ reduction at weight 2
3. Increase by a factor of ~ 3 instead of ~ 5 at each weight
4. E.g. 6-fold reduction already at weight 5!

In this manner, obtained 3-loop NMHV and 4-loop MHV heptagon

New Developments I

New Developments I

The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

New Developments I

The 6-loop, 6-particle N+MHV amplitude
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

New Developments I

The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

1. Exorcising Elliptic Beasts

New Developments I

The 6-loop, 6-particle N+MHV amplitude
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

1. Exorcising Elliptic Beasts

Elliptic generalizations of MPLs needed starting at 2 loops
[See talks by Adams,Broadhurst,Vanhove]

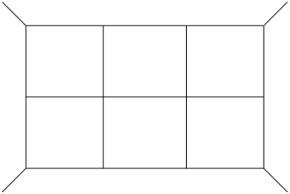
New Developments I

The 6-loop, 6-particle N+MHV amplitude [Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

1. Exorcising Elliptic Beasts

Elliptic generalizations of MPLs needed starting at 2 loops
[See talks by Adams,Broadhurst,Vanhove]



By analyzing its cuts, arguments that following integral, potentially contributing to 6-loop NMHV, is elliptic. [Bourjaily,Parra Martinez]

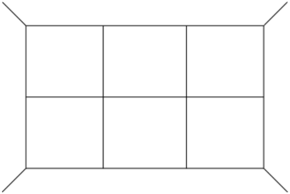
New Developments I

The 6-loop, 6-particle N+MHV amplitude
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

1. Exorcising Elliptic Beasts

Elliptic generalizations of MPLs needed starting at 2 loops
[See talks by Adams,Broadhurst,Vanhove]



By analyzing its cuts, arguments that following integral, potentially contributing to 6-loop NMHV, is elliptic. [Bourjaily,Parra Martinez]

> Our result is purely MPL, thus lending no support to this claim.

New Developments I

The 6-loop, 6-particle N+MHV amplitude
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

2. Application of heptagon ideas simplifying construction of function bases

The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

2. Application of heptagon ideas simplifying construction of function bases

New alphabet: $\{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}$, where

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}, \quad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle} \text{ \& cyclic}$$

New Developments I

The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

2. Application of heptagon ideas simplifying construction of function bases

New alphabet: $\{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}$, where

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}, \quad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$ & cyclic

Simplest formulation of Steinmann relations for the amplitude:

$\boxed{\text{No } b, c \text{ can appear after } a \text{ in } 2^{\text{nd}} \text{ symbol entry \& cyclic}}$

The 6-loop, 6-particle N+MHV amplitude
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

2. Application of heptagon ideas simplifying construction of function bases

New alphabet: $\{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}$, where

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}, \quad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle} \text{ \& cyclic}$$

3. Expose *extended* Steinmann relations for the amplitude:

No b, c can appear after a in *any* symbol entry & cyclic

New Developments I

The 6-loop, 6-particle N+MHV amplitude

[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

Significance:

2. Application of heptagon ideas simplifying construction of function bases

New alphabet: $\{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\}$, where

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle}, \quad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle} \text{ \& cyclic}$$

3. Expose *extended* Steinmann relations for the amplitude:

$$\boxed{\text{No } b, c \text{ can appear after } a \text{ in } \textit{any} \text{ symbol entry \& cyclic}}$$

Observed empirically at first, must be consequence of original Steinmann holding not just in the Euclidean region, but also on other Riemann sheets.

New Developments II

Double penta-ladders to all orders

New Developments II

Double penta-ladders to all orders

Can we construct n-gon function space without solving large linear systems?

New Developments II

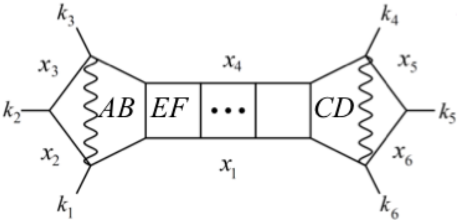
Double penta-ladders to all orders

Can we construct n-gon function space without solving large linear systems?

At least for $n = 6$ subspace spanned by double penta-ladder integrals, yes!
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot, Trnka]

[Drummond,Henn,Trnka]



$$\Omega^{(L)}(u, v, w)$$

New Developments II

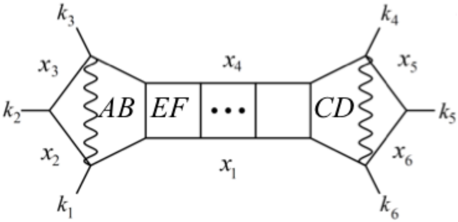
Double penta-ladders to all orders

Can we construct n-gon function space without solving large linear systems?

At least for $n = 6$ subspace spanned by double penta-ladder integrals, yes!
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot, Trnka]

[Drummond,Henn,Trnka]



$$\Omega^{(L)}(u, v, w)$$

E.g. $\Omega^{(2)} \equiv \int \frac{d^4 Z_{AB} d^4 Z_{CD} (i\pi^2)^{-2} \langle AB13\rangle\langle CD46\rangle\langle 2345\rangle\langle 5612\rangle\langle 3461\rangle}{\langle AB61\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD45\rangle\langle CD56\rangle\langle CD61\rangle}$

New Developments II

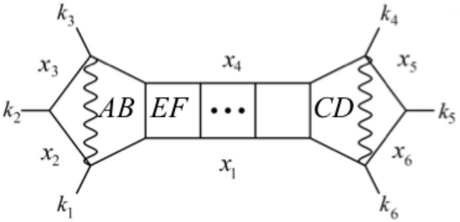
Double penta-ladders to all orders

Can we construct n-gon function space without solving large linear systems?

At least for $n = 6$ subspace spanned by double penta-ladder integrals, yes!
[Caron-Huot,Dixon,McLeod,GP,von Hippel;to appear]

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot, Trnka]

[Drummond,Henn,Trnka]



$$\Omega^{(L)}(u,v,w)$$

E.g. $\Omega^{(2)} \equiv \int \frac{d^4 Z_{AB} d^4 Z_{CD} (i\pi^2)^{-2} \langle AB13\rangle\langle CD46\rangle\langle 2345\rangle\langle 5612\rangle\langle 3461\rangle}{\langle AB61\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABCD\rangle\langle CD34\rangle\langle CD45\rangle\langle CD56\rangle\langle CD61\rangle}$

Can in fact resum $\Omega \equiv \sum \lambda^L \Omega^{(L)}$ in terms of a simple integral.

Beyond seven particles

Beyond seven particles

For $N \geq 8$, $Gr(N,8)$ cluster algebra becomes infinite

$$
\begin{tikzcd}
\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} \arrow[r] \arrow[d] & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \arrow[d] \\
\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle} \arrow[r] \arrow[d] & \frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle} \arrow[r] \arrow[d] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle} \arrow[d] \arrow[ul] \\
\frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle} \arrow[r] & \frac{\langle 1256\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1245\rangle\langle 1567\rangle\langle 3456\rangle} \arrow[r] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \frac{\langle 12\,N-2\,N-1\rangle\langle 1\,N-4\,N-3\,N-2\rangle\langle N-3\,N-2\,N-1\,N\rangle}{\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle\langle N-4\,N-3\,N-2\,N-1\rangle} \arrow[ul]
\end{tikzcd}
$$

Beyond seven particles

For $N \geq 8$, $Gr(N,8)$ cluster algebra becomes infinite

$$
\begin{tikzcd}
\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} \arrow[r] \arrow[d] & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \arrow[d] \\
\xcancel{\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle}} \arrow[r] \arrow[d] & \xcancel{\frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle}} \arrow[r] \arrow[d] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \xcancel{\frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle}} \arrow[d] \arrow[ul] \\
\frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle} \arrow[r] & \frac{\langle 1256\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1245\rangle\langle 1567\rangle\langle 3456\rangle} \arrow[r] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \frac{\langle 12\,N-2\,N-1\rangle\langle 1\,N-4\,N-3\,N-2\rangle\langle N-3\,N-2\,N-1\,N\rangle}{\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle\langle N-4\,N-3\,N-2\,N-1\rangle} \arrow[ul]
\end{tikzcd}
$$

- However, in multi-Regge limit, $Gr(N,8) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

Beyond seven particles

For $N \geq 8$, $Gr(N,8)$ cluster algebra becomes infinite

$$
\begin{tikzcd}
\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} \arrow[r] \arrow[d] & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \arrow[d] \\
\xcancel{\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle}} \arrow[r] \arrow[d] & \xcancel{\frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle}} \arrow[r] \arrow[d] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \xcancel{\frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle}} \arrow[d] \arrow[ul] \\
\xcancel{\frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle}} \arrow[r] & \xcancel{\frac{\langle 1256\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1245\rangle\langle 1567\rangle\langle 3456\rangle}} \arrow[r] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \xcancel{\frac{\langle 12\,N-2\,N-1\rangle\langle 1\,N-4\,N-3\,N-2\rangle\langle N-3\,N-2\,N-1\,N\rangle}{\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle\langle N-4\,N-3\,N-2\,N-1\rangle}} \arrow[ul]
\end{tikzcd}
$$

- However, in multi-Regge limit, $Gr(N,8) \to A_{N-5} \times A_{N-5}$: finite! [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]
- The two A_{N-5} factors not independent: Related by single-valuedness

Therefore multi-Regge limit important stepping stone towards bootstrapping higher-point amplitudes, and also closely related to integrability & collinear OPE limit. [Basso,Caron-Huot,Sever] [Drummond,Papathanasiou]

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Our improved understanding of the latter has led to two major upgrades:

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Our improved understanding of the latter has led to two major upgrades:

- Cluster algebras are instrumental in identifying the function space (arguments) in which the amplitude "lives"

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Our improved understanding of the latter has led to two major upgrades:

- Cluster algebras are instrumental in identifying the function space (arguments) in which the amplitude "lives"
- (Extended) Steinmann relations massively reduce the size of this space $\Rightarrow$ much simpler to single it out

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Our improved understanding of the latter has led to two major upgrades:

- Cluster algebras are instrumental in identifying the function space (arguments) in which the amplitude "lives"
- (Extended) Steinmann relations massively reduce the size of this space $\Rightarrow$ much simpler to single it out

This has led a wealth of results for $n = 6, 7$ amplitudes, with the power of the method, surprisingly, increasing with n. More to come, $n \geq 8$, QCD...

Conclusions & Outlook

In this presentation, we talked about the bootstrap program for constructing $\mathcal{N} = 4$ SYM amplitudes at fixed-order/general kinematics, by exploiting their analytic properties.

Our improved understanding of the latter has led to two major upgrades:

- Cluster algebras are instrumental in identifying the function space (arguments) in which the amplitude "lives"
- (Extended) Steinmann relations massively reduce the size of this space ⇒ much simpler to single it out

This has led a wealth of results for $n = 6, 7$ amplitudes, with the power of the method, surprisingly, increasing with n. More to come, $n \geq 8$, QCD...

> Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

Momentum Twistors Z^I [Hodges]

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$
- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline\end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

- Can show
$$(x-x')^2 \propto 2X\cdot X' = \epsilon_{IJKL}Z^I\tilde{Z}^J Z'^K\tilde{Z}'^L = \det(Z\tilde{Z}Z'\tilde{Z}') \equiv \langle Z\tilde{Z}Z'\tilde{Z}'\rangle$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

- Can show
$$(x-x')^2 \propto 2X\cdot X' = \epsilon_{IJKL}Z^I\tilde{Z}^J Z'^K\tilde{Z}'^L = \det(Z\tilde{Z}Z'\tilde{Z}') \equiv \langle Z\tilde{Z}Z'\tilde{Z}'\rangle$$

- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1} \wedge Z_i$

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4,n)$.

$Gr(k,n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix
- Under $GL(k)$ transformations, basis vectors change, but still span the same plane.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\dots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix
- Under $GL(k)$ transformations, basis vectors change, but still span the same plane.

Comparing the two matrices,

$$\boxed{\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4, n)/(C^*)^{n-1}}$$

Imposing Constraints: Integrable Words

Imposing Constraints: Integrable Words

Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}\, d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} \underbrace{(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j}\otimes\phi_{\alpha_{j+1}}} = 0\,,$$

$\forall j \in \{1,\ldots,k-1\}$. These are necessary and sufficient conditions for a function f_k with symbol $\mathcal{S}$ to exist.

Imposing Constraints: Integrable Words

Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \; d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} = 0\,,$$

$\forall j \in \{1,\ldots,k-1\}$. These are necessary and sufficient conditions for a function f_k with symbol $\mathcal{S}$ to exist.

Example: $(1 - xy) \otimes (1 - x)$ with x, y independent.

Imposing Constraints: Integrable Words

Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}\; d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} = 0\,,$$

$\forall j \in \{1,\ldots,k-1\}$. These are necessary and sufficient conditions for a function f_k with symbol $\mathcal{S}$ to exist.

Example: $(1-xy) \otimes (1-x)$ with x, y independent.

$$\begin{aligned} d\log(1-xy) \wedge d\log(1-x) &= \frac{-ydx - xdy}{1-xy} \wedge \frac{-dx}{1-x} \\ &= \frac{x}{(1-xy)(1-x)} dy \wedge dx \end{aligned}$$

Imposing Constraints: Integrable Words

Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

$$\sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \, d\log\phi_{\alpha_j} \wedge d\log\phi_{\alpha_{j+1}} \underbrace{(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k})}_{\text{omitting } \phi_{\alpha_j} \otimes \phi_{\alpha_{j+1}}} = 0\,,$$

$\forall j \in \{1,\ldots,k-1\}$. These are necessary and sufficient conditions for a function f_k with symbol $\mathcal{S}$ to exist.

Example: $(1-xy) \otimes (1-x)$ with x, y independent.

$$\begin{aligned} d\log(1-xy) \wedge d\log(1-x) &= \frac{-ydx - xdy}{1-xy} \wedge \frac{-dx}{1-x} \\ &= \frac{x}{(1-xy)(1-x)} dy \wedge dx \end{aligned}$$

Not integrable

Imposing Constraints: Physical Singularities

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Singularities of multiple polylogarithm functions are encoded in the first entry of their symbols.

First-entry condition: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Singularities of multiple polylogarithm functions are encoded in the first entry of their symbols.

First-entry condition: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

Particularly for $n = 7$, this restricts letters of the first entry to a_{1j}.

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Singularities of multiple polylogarithm functions are encoded in the first entry of their symbols.

First-entry condition: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

Particularly for $n = 7$, this restricts letters of the first entry to a_{1j}.

Define a **heptagon symbol**: An integrable symbol with alphabet a_{ij} that obeys first-entry condition.

MHV Constraints: Yangian anomaly equations

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry [Drummond,Henn,Korchemsky,Sokatchev]

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry
  [Drummond,Henn,Korchemsky,Sokatchev]
- Combination of two symmetries gives rise to a Yangian
  [Drummond,Henn,Plefka] [Drummond,Ferro]

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry [Drummond,Henn,Korchemsky,Sokatchev]
- Combination of two symmetries gives rise to a Yangian [Drummond,Henn,Plefka] [Drummond,Ferro]
- Although broken at loop level by IR divergences, Yangian anomaly equations governing this breaking have been proposed [Caron-Huot,He]

Consequence for MHV amplitudes: Their differential is a linear combination of $d\log\langle i\, j{-}1\, j\, j{+}1\rangle$, which implies

> **Last-entry condition**: Only $\langle i\, j{-}1\, j\, j{+}1\rangle$ may appear in the last entry of the symbol of any MHV amplitude.

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry [Drummond,Henn,Korchemsky,Sokatchev]
- Combination of two symmetries gives rise to a Yangian [Drummond,Henn,Plefka] [Drummond,Ferro]
- Although broken at loop level by IR divergences, Yangian anomaly equations governing this breaking have been proposed [Caron-Huot,He]

Consequence for MHV amplitudes: Their differential is a linear combination of $d\log\langle i\, j{-}1\, j\, j{+}1\rangle$, which implies

Last-entry condition: Only $\langle i\, j{-}1\, j\, j{+}1\rangle$ may appear in the last entry of the symbol of any MHV amplitude.

Particularly here: Only the 14 letters a_{2j} and a_{3j} may appear in the last symbol entry of R_7.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS normalized n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1 \| i} R_n^{(L)} = R_{n-1}^{(L)} \, .$$

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS normalized n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1 \parallel i} R_n^{(L)} = R_{n-1}^{(L)} \,.$$

For $n = 7$, taking this limit in the most general manner reduces the 42-letter heptagon symbol alphabet to 9-letter hexagon symbol alphabet, plus nine additional letters.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS normalized n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1 \parallel i} R_n^{(L)} = R_{n-1}^{(L)} \,.$$

For $n = 7$, taking this limit in the most general manner reduces the 42-letter heptagon symbol alphabet to 9-letter hexagon symbol alphabet, plus nine additional letters.

A function has a well-defined $i+1 \parallel i$ limit only if its symbol is independent of all nine of these letters.

Computing Heptagon Symbols

Computing Heptagon Symbols

Step 1 (Straightforward)

Form linear combination of all length-k symbols made of a_{ij} obeying initial/Steinmann (+final) entry conditions, with unknown coefficients grouped in vector X.

Computing Heptagon Symbols

Step 1 (Straightforward)

Form linear combination of all length-k symbols made of a_{ij} obeying initial/Steinmann (+final) entry conditions, with unknown coefficients grouped in vector X.

Step 2 (Challenging)

Solve integrability constraints, which take the form

$$A \cdot X = 0\,.$$

Namely all weight-k heptagon functions will be the *right nullspace* of rational matrix A.

Computing Heptagon Symbols

Step 1 (Straightforward)

Form linear combination of all length-k symbols made of a_{ij} obeying initial/Steinmann (+final) entry conditions, with unknown coefficients grouped in vector X.

Step 2 (Challenging)

Solve integrability constraints, which take the form

$$A \cdot X = 0 \,.$$

Namely all weight-k heptagon functions will be the *right nullspace* of rational matrix A.

"Just" linear algebra, however for e.g. 4-loop MHV hexagon A boils down to a size of 941498×60182. Tackled with fraction-free variants of Gaussian elimination that bound the size of intermediate expressions, implemented in `Integer Matrix Library` and `Sage`. [Storjohann]

BDS versus BDS-like normalized amplitudes

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

$$\mathcal{A}_7^{\textsf{BDS-like}} \equiv \mathcal{A}_7^{\textsf{BDS}} \exp\left[\frac{\Gamma_{\textsf{cusp}}}{4} Y_7\right]$$

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

$$\mathcal{A}_7^{\text{BDS-like}} \equiv \mathcal{A}_7^{\text{BDS}} \exp\left[\frac{\Gamma_{\text{cusp}}}{4} Y_7\right]$$

$$Y_7 = -\sum_{i=1}^{7}\left[\mathrm{Li}_2\left(1 - \frac{1}{u_i}\right) + \frac{1}{2}\log\left(\frac{u_{i+2}u_{i-2}}{u_{i+3}u_i u_{i-3}}\right)\log u_i\right],$$

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

$$\mathcal{A}_7^{\text{BDS-like}} \equiv \mathcal{A}_7^{\text{BDS}} \exp\left[\frac{\Gamma_{\text{cusp}}}{4} Y_7\right]$$

$$Y_7 = -\sum_{i=1}^{7}\left[\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)+\frac{1}{2}\log\left(\frac{u_{i+2}u_{i-2}}{u_{i+3}u_iu_{i-3}}\right)\log u_i\right],$$

$$u_i = \frac{x_{i+1,i+5}^2\, x_{i+2,i+4}^2}{x_{i+1,i+4}^2\, x_{i+2,i+5}^2}\,, \qquad \Gamma_{\text{cusp}} = 4g^2 - \frac{4\pi^2}{3}g^4 + \mathcal{O}(g^6)\,,$$

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

$$\mathcal{A}_7^{\text{BDS-like}} \equiv \mathcal{A}_7^{\text{BDS}} \exp\left[\frac{\Gamma_{\text{cusp}}}{4} Y_7\right]$$

$$Y_7 = -\sum_{i=1}^{7}\left[\text{Li}_2\left(1 - \frac{1}{u_i}\right) + \frac{1}{2}\log\left(\frac{u_{i+2}u_{i-2}}{u_{i+3}u_i u_{i-3}}\right)\log u_i\right],$$

$$u_i = \frac{x_{i+1,i+5}^2\, x_{i+2,i+4}^2}{x_{i+1,i+4}^2\, x_{i+2,i+5}^2}\,, \qquad \Gamma_{\text{cusp}} = 4g^2 - \frac{4\pi^2}{3}g^4 + \mathcal{O}(g^6)\,,$$

This way, $\text{Disc}_{s_{i-1,i,i+1}}\mathcal{A}_7 = \mathcal{A}_7^{\text{BDS-like}}\text{Disc}_{s_{i-1,i,i+1}}\left[\mathcal{A}_7/\mathcal{A}_7^{\text{BDS-like}}\right]$

BDS versus BDS-like normalized amplitudes

- BDS ansatz: Essentially the exponentiated 1-loop amplitude
- Contains 3-particle invariants $s_{i-1,i,i+1}$
- BDS-like: Remove $s_{i-1,i,i+1}$ from BDS in conformally invariant fashion

$$\mathcal{A}_7^{\text{BDS-like}} \equiv \mathcal{A}_7^{\text{BDS}} \exp\left[\frac{\Gamma_{\text{cusp}}}{4} Y_7\right]$$

$$Y_7 = -\sum_{i=1}^{7}\left[\text{Li}_2\left(1 - \frac{1}{u_i}\right) + \frac{1}{2}\log\left(\frac{u_{i+2}u_{i-2}}{u_{i+3}u_i u_{i-3}}\right)\log u_i\right],$$

$$u_i = \frac{x_{i+1,i+5}^2\, x_{i+2,i+4}^2}{x_{i+1,i+4}^2\, x_{i+2,i+5}^2}\,, \qquad \Gamma_{\text{cusp}} = 4g^2 - \frac{4\pi^2}{3}g^4 + \mathcal{O}(g^6)\,,$$

This way, $\text{Disc}_{s_{i-1,i,i+1}}\mathcal{A}_7 = \mathcal{A}_7^{\text{BDS-like}}\text{Disc}_{s_{i-1,i,i+1}}\left[\mathcal{A}_7/\mathcal{A}_7^{\text{BDS-like}}\right]$

BDS-like normalized amplitudes obey Steinmann relations, BDS normalized ones do not!

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N} = 4$ SYM.

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N} = 4$ SYM.

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A \eta^B S_{AB} + \frac{1}{3!}\eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D + \frac{1}{4!}\eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-$$

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N}=4$ SYM.

$$\Phi = G^{+} + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^{-}$$

$$\mathcal{A}_n^{\mathrm{MHV}} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{\mathrm{MHV}}(1^+\dots j^-\dots k^-\dots n^+) + \dots,$$

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N} = 4$ SYM.

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-$$

$$\mathcal{A}_n^{\mathrm{MHV}} = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^n p_i\Big) \sum_{1\le j<k\le n} (\eta_j)^4 (\eta_k)^4 A_n^{\mathrm{MHV}}(1^+ \dots j^- \dots k^- \dots n^+) + \dots,$$

$$E \equiv \frac{\mathcal{A}_7^{\mathsf{NMHV}}}{\mathcal{A}_7^{\mathsf{BDS\text{-}like}}} = \mathcal{P}^{(0)}\, E_0 + \Big[(12)\, E_{12} + (14)\, E_{14} + \text{ cyclic}\Big].$$

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N} = 4$ SYM.

$$\Phi = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\bar{\Gamma}^{D} + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}$$

$$\mathcal{A}_n^{\mathrm{MHV}} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{\mathrm{MHV}}(1^+\ldots j^-\ldots k^-\ldots n^+) + \ldots,$$

$$E \equiv \frac{\mathcal{A}_7^{\mathsf{NMHV}}}{\mathcal{A}_7^{\mathsf{BDS\text{-}like}}} = \mathcal{P}^{(0)}\,E_0 + \Big[(12)\,E_{12} + (14)\,E_{14} + \text{ cyclic}\Big].$$

- E_0, E_{12}, E_{14} the transcendental functions we wish to determine

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N} = 4$ SYM.

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-$$

$$\mathcal{A}_n^{\mathrm{MHV}} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{\mathrm{MHV}}(1^+\ldots j^-\ldots k^-\ldots n^+) + \ldots,$$

$$E \equiv \frac{\mathcal{A}_7^{\text{NMHV}}}{\mathcal{A}_7^{\text{BDS-like}}} = \mathcal{P}^{(0)}\,E_0 + \Big[(12)\,E_{12} + (14)\,E_{14} + \text{ cyclic}\Big].$$

- E_0, E_{12}, E_{14} the transcendental functions we wish to determine
- $\mathcal{P}_7^{(0)} = \frac{3}{7}\,(12) + \frac{1}{7}\,(13) + \frac{2}{7}\,(14)$ + cyclic the tree-level superamplitude

NMHV (super)amplitudes

Beyond MHV, amplitudes most efficiently organized by exploiting the (dual) superconformal symmetry of $\mathcal{N}=4$ SYM.

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-$$

$$\mathcal{A}_n^{\rm MHV} = (2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^n p_i\Big)\sum_{1\le j<k\le n}(\eta_j)^4(\eta_k)^4 A_n^{\rm MHV}(1^+\ldots j^-\ldots k^-\ldots n^+)+\ldots,$$

$$E \equiv \frac{\mathcal{A}_7^{\textsf{NMHV}}}{\mathcal{A}_7^{\textsf{BDS-like}}} = \mathcal{P}^{(0)}\,E_0 + \Big[(12)\,E_{12} + (14)\,E_{14} + \text{ cyclic}\Big].$$

- E_0, E_{12}, E_{14} the transcendental functions we wish to determine
- $\mathcal{P}_7^{(0)} = \frac{3}{7}\,(12) + \frac{1}{7}\,(13) + \frac{2}{7}\,(14)$ + cyclic the tree-level superamplitude
- $(67) = (76) \equiv [12345]$ *Dual superconformal* R-invariants, with

$$[abcde] = \frac{\delta^{0|4}\big(\chi_a\langle bcde\rangle + \text{cyclic}\big)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}\,, \quad \chi_i^A = \sum_{j=1}^{i-1}\langle ji\rangle\eta_j^A\,.$$

NMHV final entry conditions

[Caron-Huot]

$$(34)\log a_{21},\quad (14)\log a_{21},\quad (15)\log a_{21},\quad (16)\log a_{21},\quad (13)\log a_{21},\quad (12)\log a_{21},$$
$$(45)\log a_{37},\quad (47)\log a_{37},\quad (37)\log a_{37},\quad (27)\log a_{37},\quad (57)\log a_{37},\quad (67)\log a_{37},$$
$$(45)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{11}a_{24}}{a_{46}},\quad (14)\log\frac{a_{14}a_{31}}{a_{34}},$$
$$(24)\log\frac{a_{44}}{a_{42}},\qquad (56)\log a_{57},\qquad (12)\log a_{57},\qquad (16)\log\frac{a_{67}}{a_{26}},$$
$$(13)\log\frac{a_{41}}{a_{26}a_{33}}+((14)-(15))\log a_{26}-(17)\log a_{26}a_{37}+(45)\log\frac{a_{22}}{a_{34}a_{35}}-(34)\log a_{33}\,,$$

Results: 3-loop NMHV Heptagon

Loop order $L =$	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

1. Independent R-invariants × functions

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	15×3192
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

1. Independent R-invariants × functions
2. Relations between 15×42 R-invariants × final entries [Caron-Huot]

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	15×3192
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

1. Independent R-invariants × functions
2. Relations between 15×42 R-invariants × final entries [Caron-Huot]
3. Cyclic: $i \to i+1$ on all twistor labels and letters
   Flip: $i \to 8-i$ on all twistor labels and letters, except $a_{2i} \leftrightarrow a_{3,8-i}$

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

1. Independent R-invariants × functions
2. Relations between 15×42 R-invariants × final entries [Caron-Huot]
3. Cyclic: $i \to i+1$ on all twistor labels and letters
   Flip: $i \to 8-i$ on all twistor labels and letters, except $a_{2i} \leftrightarrow a_{3,8-i}$
4. We also need collinear limit of R-invariants

Results: 4-loop MHV Heptagon

Loop order $L =$	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

For last step, we need to convert BDS-like normalized amplitude F to BDS normalized one $\mathcal{F}$,

$$\mathcal{F} = F e^{\frac{\Gamma_{\text{cusp}}}{4} Y_7} \xrightarrow[\Gamma_{\text{cusp}} \to 4g^2]{\text{symbol}} \mathcal{F}^{(L)} = \sum_{k=0}^{L} F^{(k)} \frac{Y_n^{L-k}}{(L-k)!} \,.$$

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

For last step, we need to convert BDS-like normalized amplitude F to BDS normalized one $\mathcal{F}$,

$$\mathcal{F} = F e^{\frac{\Gamma_{\text{cusp}}}{4} Y_7} \xrightarrow[\Gamma_{\text{cusp}} \to 4g^2]{\text{symbol}} \mathcal{F}^{(L)} = \sum_{k=0}^{L} F^{(k)} \frac{Y_n^{L-k}}{(L-k)!} \, .$$

Independence of $\lim_{i+1\|i} \mathcal{F}$ on 9 additional letters no longer a homogeneous constraint, fixes amplitude completely!

Results: 4-loop MHV Heptagon

Loop order L =	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
Well-defined collinear	0	0	0	0

For last step, we need to convert BDS-like normalized amplitude F to BDS normalized one $\mathcal{F}$,

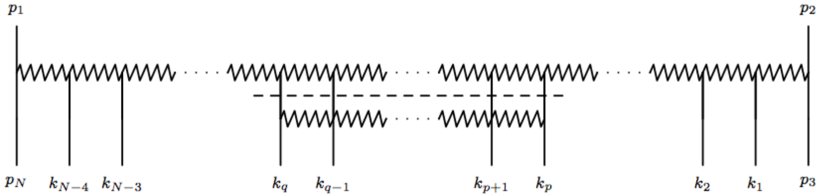
$$\mathcal{F} = F e^{\frac{\Gamma_{\text{cusp}}}{4} Y_7} \xrightarrow[\Gamma_{\text{cusp}} \to 4g^2]{\text{symbol}} \mathcal{F}^{(L)} = \sum_{k=0}^{L} F^{(k)} \frac{Y_n^{L-k}}{(L-k)!}\,.$$

Independence of $\lim_{i+1\|i} \mathcal{F}$ on 9 additional letters no longer a homogeneous constraint, fixes amplitude completely!

Strong tension between collinear properties and Steinmann relations.

Further checks: Multi-Regge limit

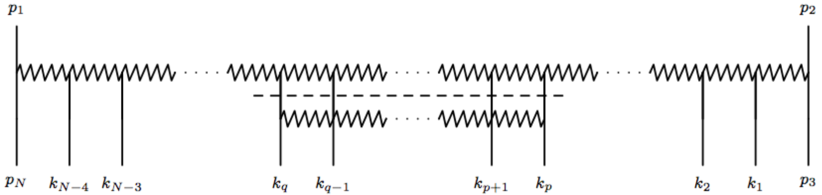
Phenomenologically relevant high-energy gluon scattering



$$s_{12} \gg s_{3\cdots N-1}, s_{4\cdots N} \gg s_{3\cdots N-2}, s_{4\cdots N-1}, s_{5\cdots N} \gg \cdots$$
$$\ldots \gg s_{34}, \ldots, s_{N-1N} \gg -t_1, \cdots, -t_{N-3}\,.$$

Further checks: Multi-Regge limit

Phenomenologically relevant high-energy gluon scattering



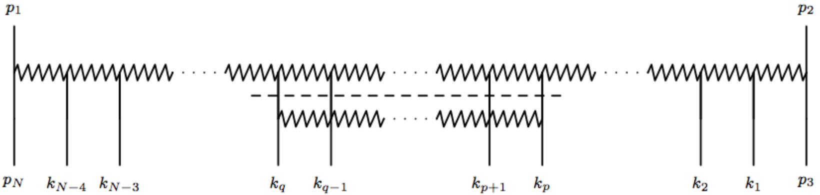
$$s_{12} \gg s_{3\cdots N-1}, s_{4\cdots N} \gg s_{3\cdots N-2}, s_{4\cdots N-1}, s_{5\cdots N} \gg \cdots$$
$$\ldots \gg s_{34}, \ldots, s_{N-1N} \gg -t_1, \cdots, -t_{N-3}\,.$$

Actively studied at weak and strong coupling [Bartels,Kormilitzin,Lipatov(Prygarin)] [Bartels,Schomerus,Sprenger] [Bargheer,Papathanasiou,Schomerus] [Bargheer]

- To obtain nontrivial result, necessary to analytically continue the energies of $k_{p+1}, \ldots k_q$

Further checks: Multi-Regge limit

Phenomenologically relevant high-energy gluon scattering



$$s_{12} \gg s_{3\cdots N-1}, s_{4\cdots N} \gg s_{3\cdots N-2}, s_{4\cdots N-1}, s_{5\cdots N} \gg \cdots$$

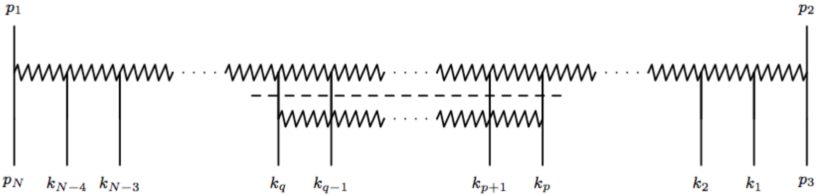
$$\ldots \gg s_{34}, \ldots, s_{N-1N} \gg -t_1, \cdots, -t_{N-3}\,.$$

Actively studied at weak and strong coupling [Bartels,Kormilitzin,Lipatov(Prygarin)] [Bartels,Schomerus,Sprenger] [Bargheer,Papathanasiou,Schomerus] [Bargheer]

- To obtain nontrivial result, necessary to analytically continue the energies of $k_{p+1}, \ldots k_q$
- Compared limit of heptagon to results on the leading logarithmic approximation (LLA) [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

Further checks: Multi-Regge limit

Phenomenologically relevant high-energy gluon scattering



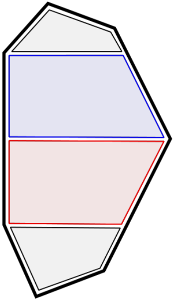
$$s_{12} \gg s_{3\cdots N-1}, s_{4\cdots N} \gg s_{3\cdots N-2}, s_{4\cdots N-1}, s_{5\cdots N} \gg \cdots$$
$$\ldots \gg s_{34}, \ldots, s_{N-1N} \gg -t_1, \cdots, -t_{N-3}\,.$$

Actively studied at weak and strong coupling [Bartels,Kormilitzin,Lipatov(Prygarin)] [Bartels,Schomerus,Sprenger] [Bargheer,Papathanasiou,Schomerus] [Bargheer]

- To obtain nontrivial result, necessary to analytically continue the energies of $k_{p+1}, \ldots k_q$
- Compared limit of heptagon to results on the leading logarithmic approximation (LLA) [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]
- Obtained new results for all terms beyond LLA

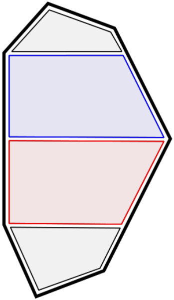
Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.

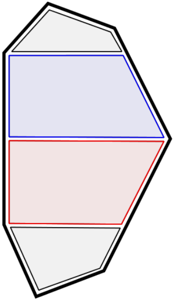


Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g.[Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times$$
$$\times\, e^{-\tau_1\gamma_1 + ip_1\sigma_1 - \tau_2\gamma_2 + ip_2\sigma_2}\,.$$

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



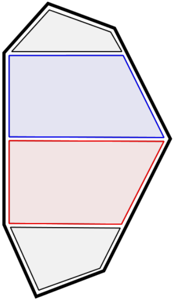
Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g.[Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v)\times \\ \times\, e^{-\tau_1\gamma_1+ip_1\sigma_1-\tau_2\gamma_2+ip_2\sigma_2}\,.$$

1. Computed its weak-coupling expansion to 3 loops, employing the technology of Z-sums [Moch,Uwer,Weinzierl] [GP'13] [GP'14]

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



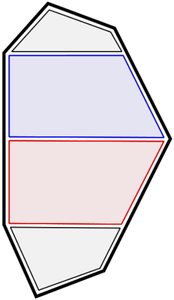
Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g.[Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times \\ \times\, e^{-\tau_1\gamma_1 + ip_1\sigma_1 - \tau_2\gamma_2 + ip_2\sigma_2}\,.$$

1. Computed its weak-coupling expansion to 3 loops, employing the technology of Z-sums [Moch,Uwer,Weinzierl] [GP'13] [GP'14]
2. Expanded our symbol for $R_7^{(3)}$ in the same kinematics, relying on [Dixon,Drummond,Duhr,Pennington]

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g. [Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times \\ \times\; e^{-\tau_1\gamma_1+ip_1\sigma_1-\tau_2\gamma_2+ip_2\sigma_2}\,.$$

Perfect match, currently computing 4 loops

1. Computed its weak-coupling expansion to 3 loops, employing the technology of Z-sums [Moch,Uwer,Weinzierl] [GP'13] [GP'14]
2. Expanded our symbol for $R_7^{(3)}$ in the same kinematics, relying on [Dixon,Drummond,Duhr,Pennington]