Surprising simplicity of massive scattering amplitudes

Johannes M. Henn

based on

[JHEP 1406 (2014)114] [PRL 113 (2014)16] with S. Caron-Huot
and work in progress with R. Brüser and S. Caron-Huot

Talk at Amplitudes 2017, Edinburgh

Cluster of Excellence PRISMA
Precision Physics, Fundamental Interactions and Structure of Matter
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Outline

- Introduction - hidden symmetries
- Massive amplitudes in N=4 sYM
- Cusp anomalous dimension

- Simple structure of subleading Regge behavior
- Conjecture for an exact high-energy cross section

Introduction

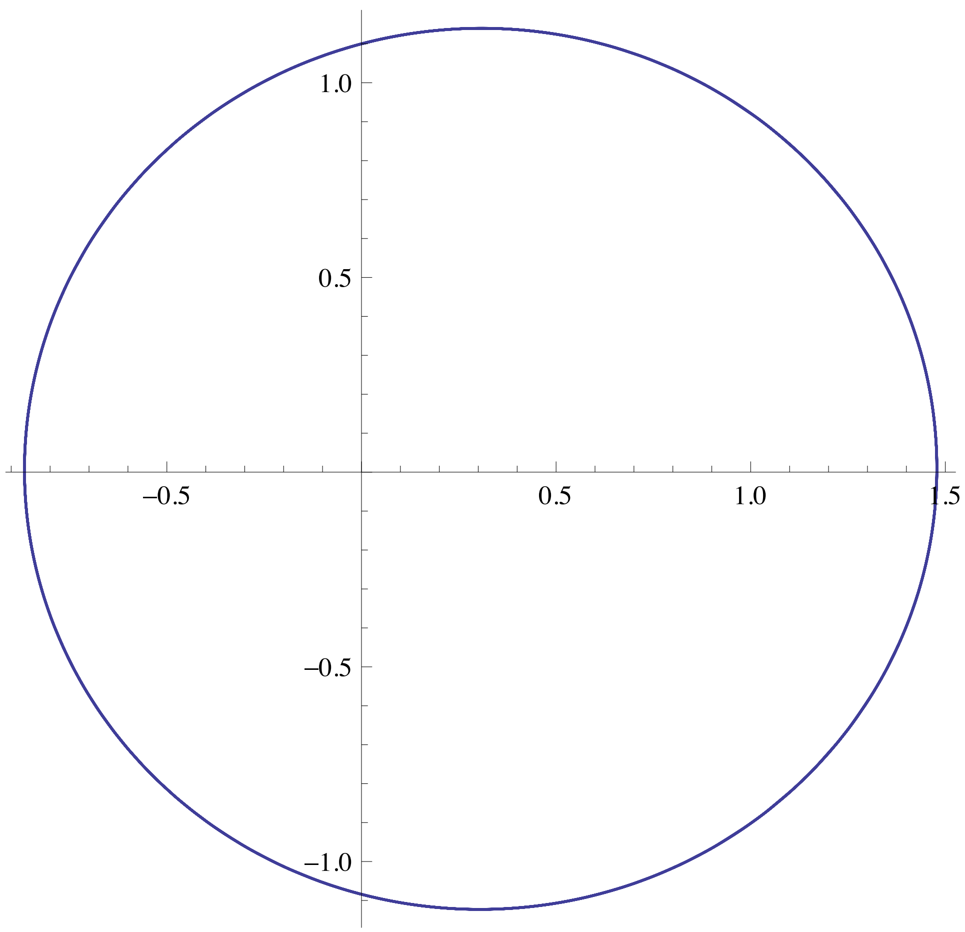
- Kepler problem and hydrogen atom are important classical and quantum mechanics problems that can be exactly solved

  they have a hidden symmetry

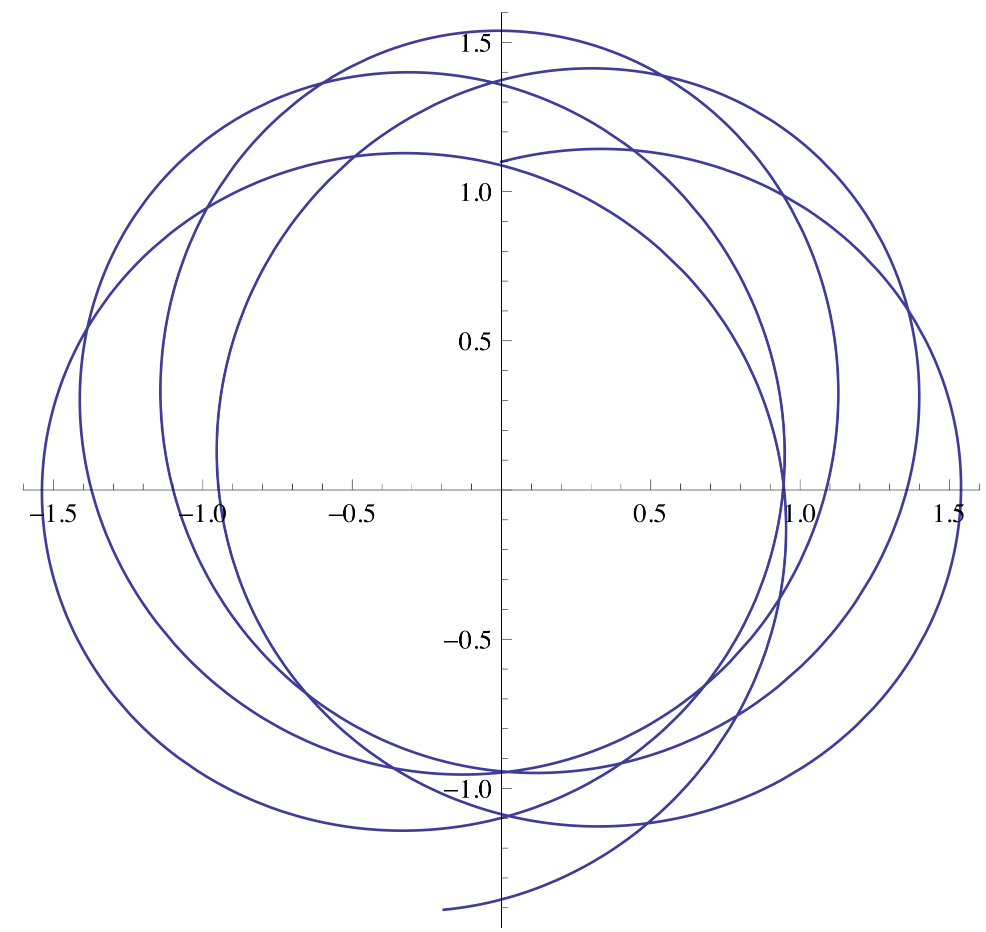
- will show that N=4 super Yang-Mills is a natural QFT analogue of these systems

Kepler problem

$V = 1/r$



$V = 1/r^{0.9}$



- orbits do not precess
- conservation of Laplace-Runge-Lenz vector

$$\vec{A} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$

Hydrogen atom

- Hamiltonian $$H = \frac{1}{2m}p^2 - \frac{k}{r}$$

- hidden symmetry:
Laplace-Runge-Lenz-Pauli vector

- conserved quantity in quantum mechanics

$$[H, L_i] = 0 \qquad [H, A_i] = 0$$

$$[A_i, A_i] = -i\hbar\epsilon_{ijk}L_k\frac{2}{m}H$$

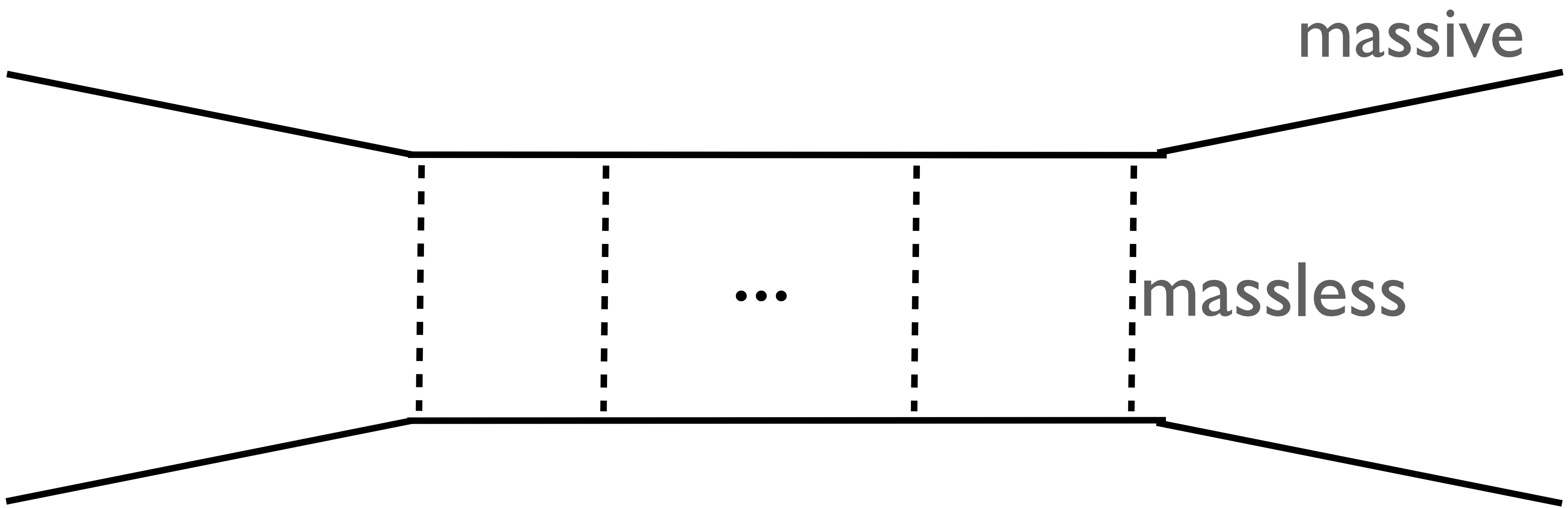
- operator algebra allows to find spectrum

$$E_n = -\frac{mk^2}{2\hbar^2}\frac{1}{n^2} \qquad n = 1, 2, \ldots$$

- degeneracy n^2

extension to a relativistic QFT

- Wick and Cutkowsky considered the following model:



- This is the ladder approximation to $ep \to ep$, ignoring the spin of the photon

- In the non-relativistic limit, this reduces to the hydrogen Hamiltonian

SO(4) symmetry of Wick-Cutcosky model

- This model possesses an exact O(4) symmetry, even *away* from the NR limit
- Consider just one rung

$$\cdots \int \frac{d^4\ell_2}{(\ell_2-\ell_1)^2\left[(\ell_2-p_1)^2+m^2\right]\left[(\ell_2+p_2)^2+m^2\right](\ell_2-\ell_3)^2} \cdots$$

- The symmetry is non-obvious in this form, and is a conformal symmetry in momentum space

- The symmetry becomes evident if we use Dirac's embedding formalism

- Rewrite each vector as a 6-vector, with $L^2=0$:

$$L_i^a \equiv \begin{pmatrix} \ell_i^\mu \\ L_i^+ \\ L_i^- \end{pmatrix} = \begin{pmatrix} \ell_i^\mu \\ \ell_i^2 \\ 1 \end{pmatrix}$$

and similarly for the external regions:

$$Y_1^a = \begin{pmatrix} p_1^\mu \\ p_1^2 + m^2 \\ 1 \end{pmatrix}, \qquad Y_3^a = \begin{pmatrix} -p_2^\mu \\ p_2^2 + m^2 \\ 1 \end{pmatrix}$$

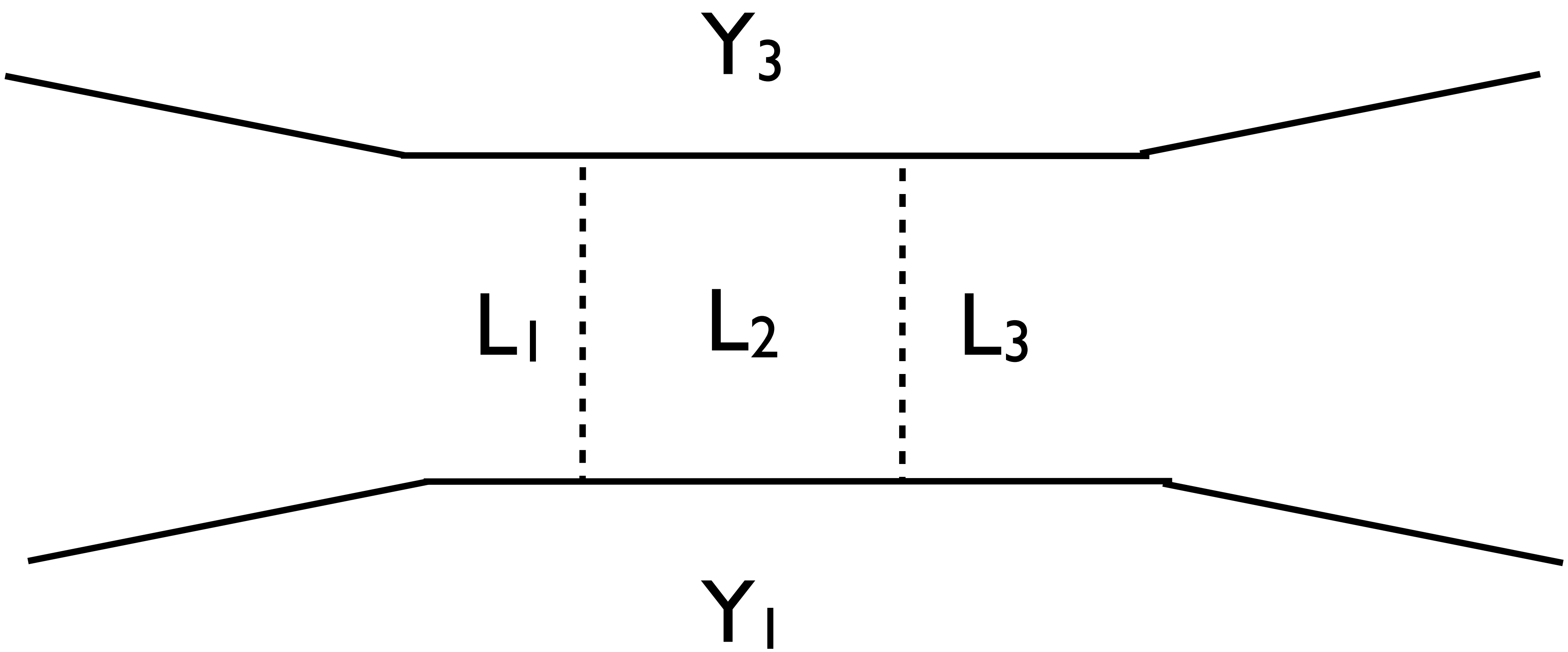
- The 6D vector product gives:

$$L_i \cdot L_j = (\ell_i - \ell_j)^2$$

$$L_i \cdot Y_1 = (\ell_i - p_1)^2 + m^2$$

$$L_i \cdot Y_3 = (\ell_i + p_2)^2 + m^2$$

- The L's and Y's 'live' in regions of the planar graph



$$\cdots \int \text{``}d^4 L_2\text{''} \frac{1}{(L_1 \cdot L_2)(L_2 \cdot Y_1)(L_2 \cdot Y_3)(L_2 \cdot L_3)} \cdots$$

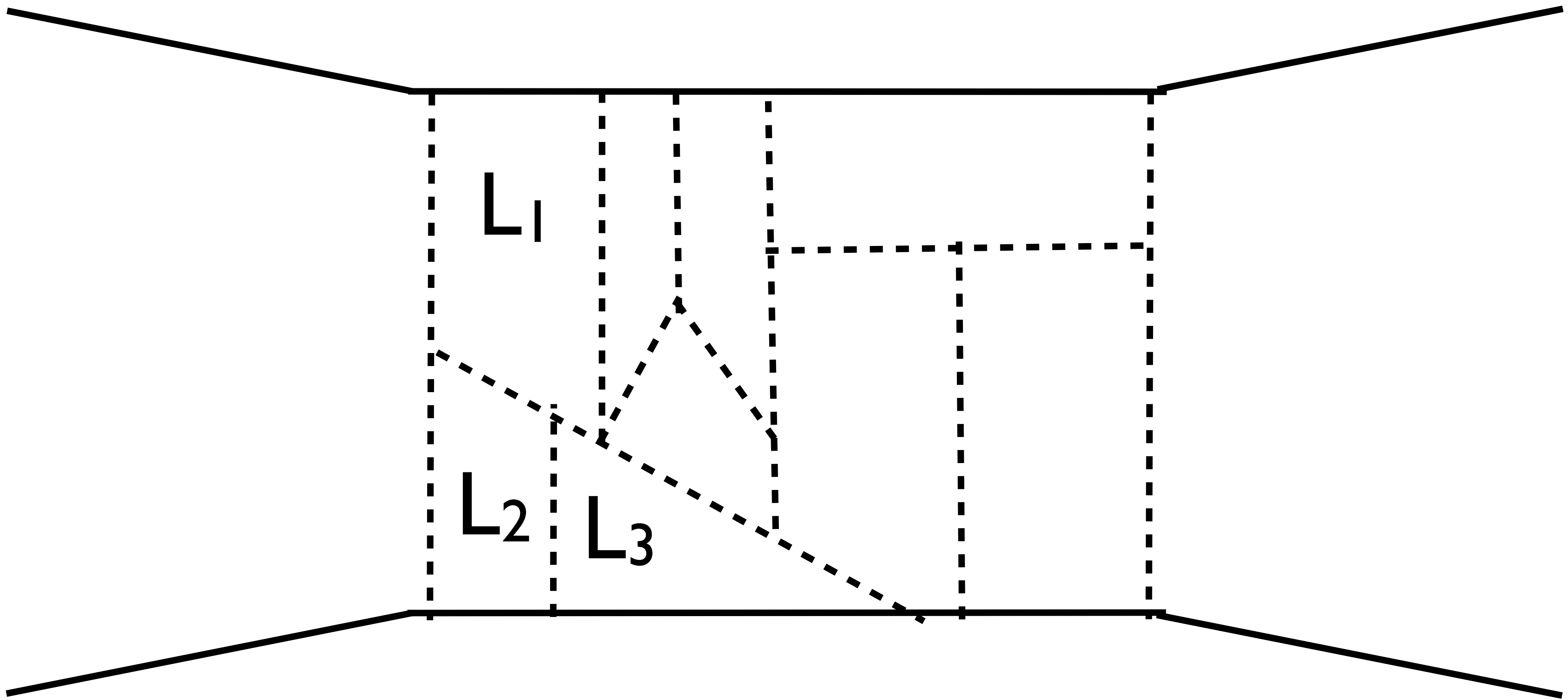
- The integration measure is also important, but let me skip it for now.

$$\cdots \int \text{``}d^4L_2\text{''} \frac{1}{(L_1\cdot L_2)(L_2\cdot Y_1)(L_2\cdot Y_3)(L_2\cdot L_3)} \cdots$$

- Since everything (incl. measure) depends only on 6-dimensional dot products, there is a natural SO(6) (really SO(4,2)) symmetry
- The two vectors Y_1, Y_3 obviously break it to SO(4).
- This SO(4) contains the usual SO(3) $\vec{J}$ as a subgroup.
- What are the remaining three generators? The Runge-Lenz vector!

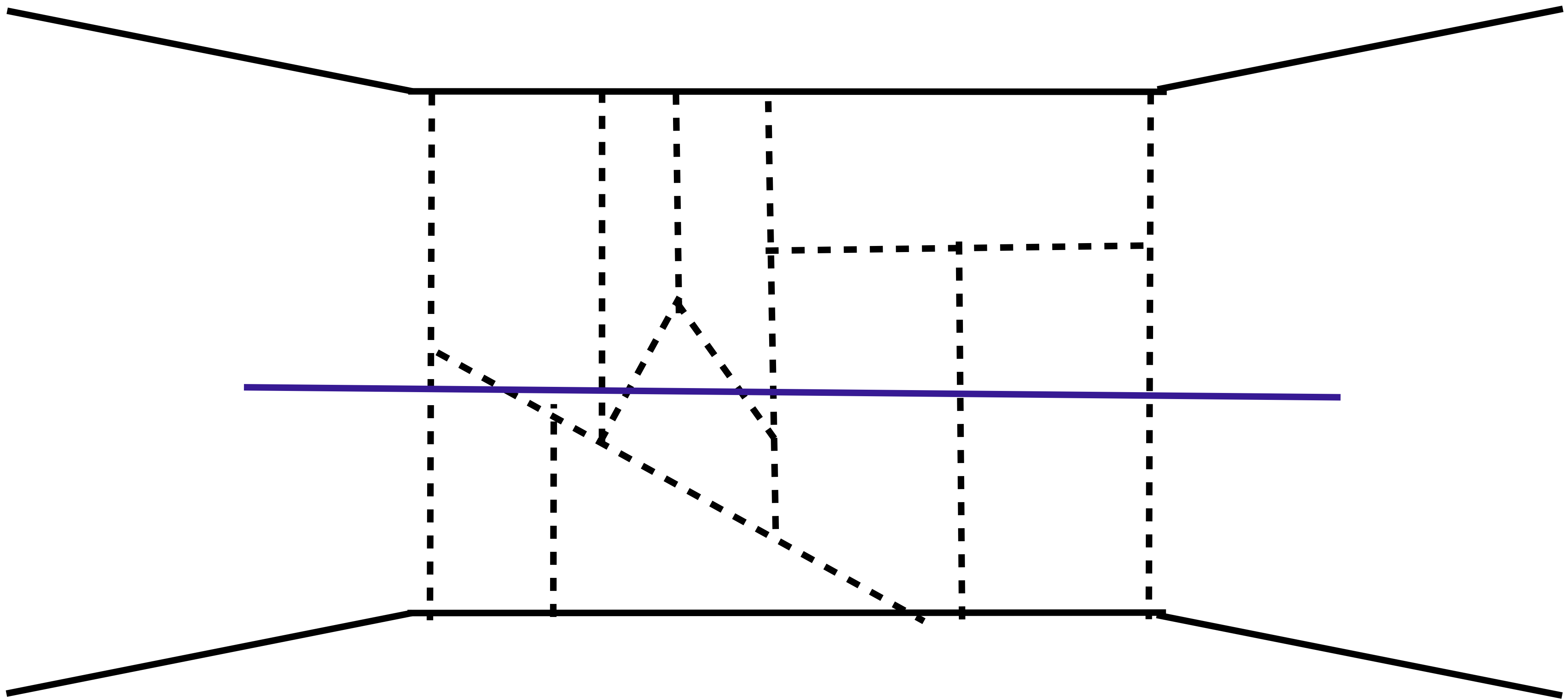
- Unfortunately, the ladder approximation is not consistent relativistically.
- (It lacks multi-particle channels and so has problems with unitarity)
- For this reason this symmetry appears to have been mostly abandoned, like a curiosity
- Wick and Cutkowski's study nonetheless left us the ``Wick rotation''

- The simplest way to imagine a consistent QFT with this symmetry requires a planar limit:



- The Feynman rules would have to respect the SO(6) symmetry, which acts in *momentum space*

- Can such a thing exist?



If such a theory exists, by unitarity surely it must contain massless particles.
Their self-interactions will have to respect the dual conformal symmetry.

N=4 super Yang-Mills

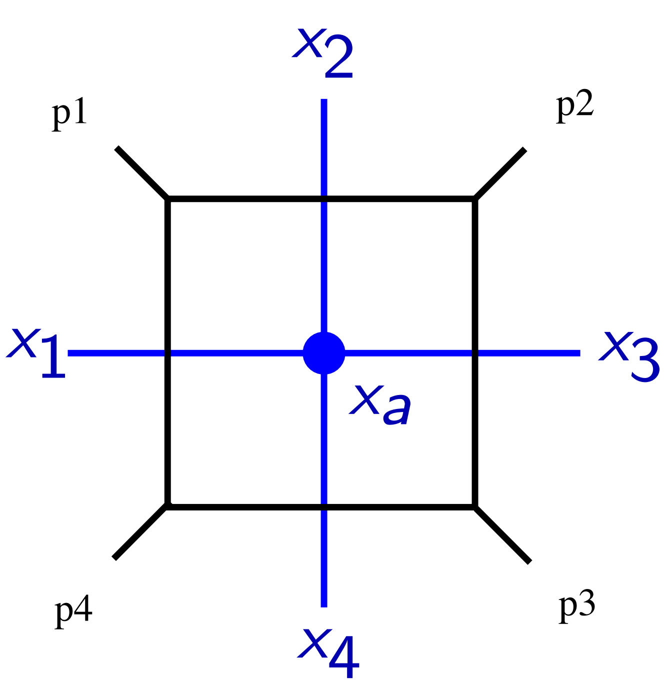
fast-forward from 1950's to 2000's

N=4 sYM has dual conformal symmetry

[Drummond, JMH, Smirnov, Sokatchev; Alday, Maldacena; Drummond, JMH, Korchemsky, Sokatchev; ...]

in massless sector:

$$\boxed{p_i^\mu = x_i^\mu - x_{i+1}^\mu}$$



$$= x_{13}^2 x_{24}^2 \int \frac{d^D x_a}{x_{1a}^2 x_{2a}^2 x_{3a}^2 x_{4a}^2}$$

invariant under SO(4,2) in dual space $x^\mu \to x^\mu / x^2$

- this symmetry is at the heart of many developments
  - duality Wilson loops/scattering amplitudes
  - integrability of N=4 SYM theory
  - and other recent developments
- we have just argued that it is a natural generalization of the hydrogen atom's SO(4), itself inherited from the Kepler problem
- but where are the massive particles?

introducing massive particles

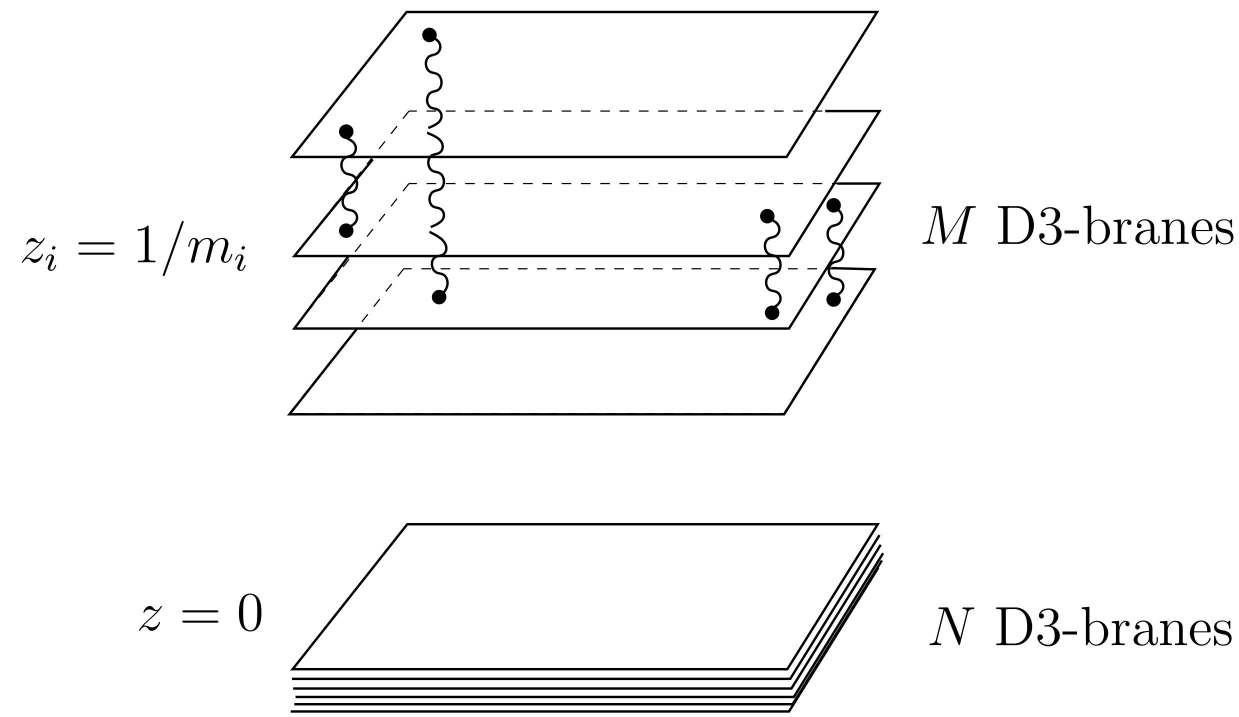
gauge theory

Higgs mechanism

$$\Phi \longrightarrow \langle\Phi\rangle + \varphi$$

$$U(N+M) \longrightarrow U(N) \times U(M)$$

string theory



(a)

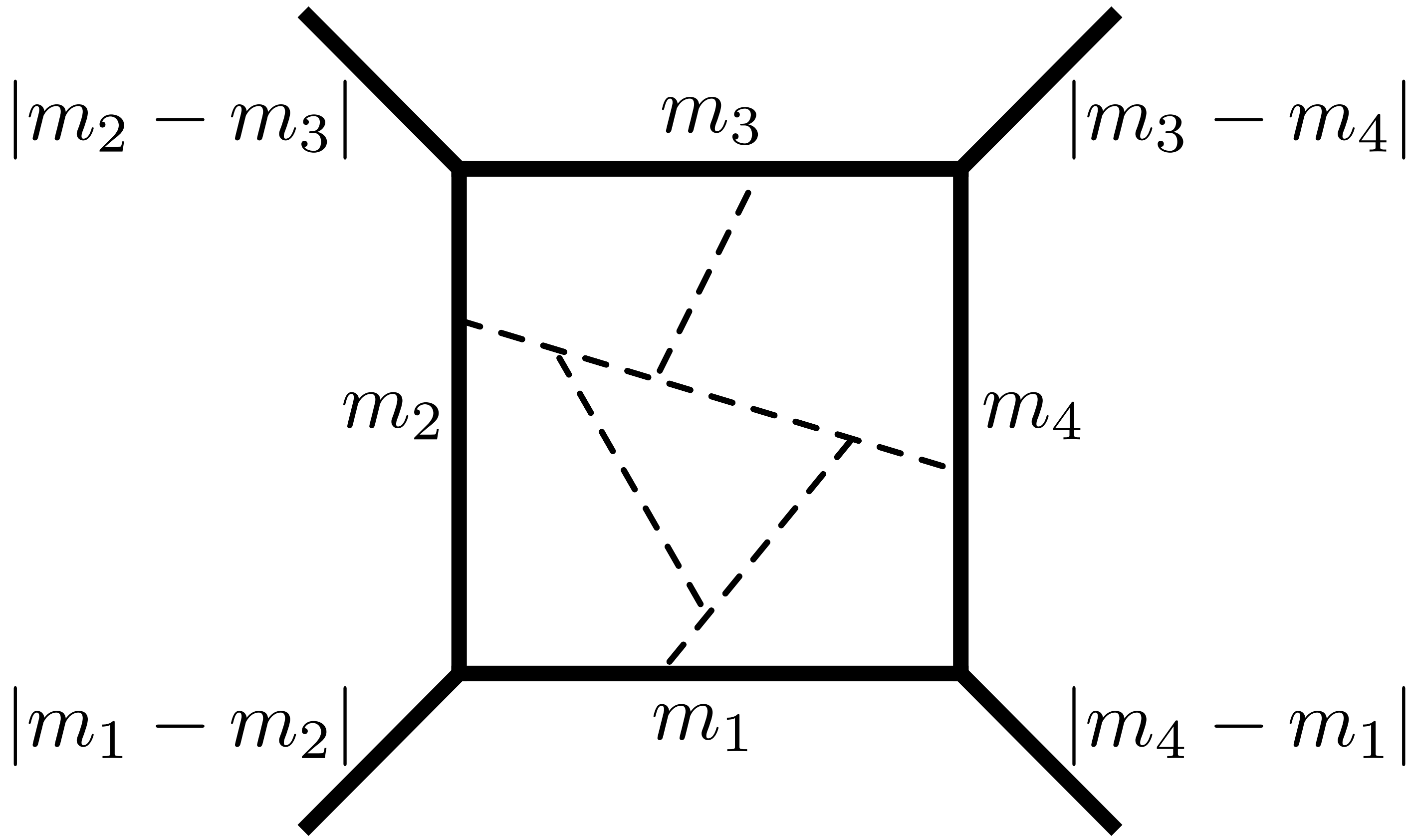
- e.g. four-particle scattering

$$U(N+4) \longrightarrow U(N) \times U(4)$$

consider scattering of SU(4) fields in large N limit

- infrared finite
- preserves dual conformal symmetry

- four-particle scattering (planar)



- dual conformal symmetry (planar) [Alday, JMH, Plefka,Schuster]

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \qquad p_i^2 = -(m_i - m_{i+1})^2$$

$$y_i^A \to \frac{y_i^A}{y_i^2} \qquad y_i^A = (x_i^\mu, m_i)$$

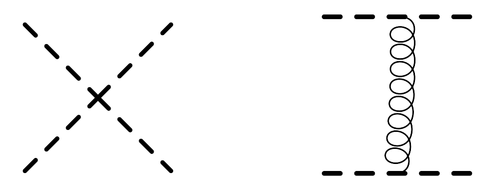
isometries of AdS_5 space
Poincare coordinates

[proof of DCS for loop integrands: Dennen, Huang; Caron-Huot, O`Connell]

Massive 4-particle amplitudes in N=4 sYM

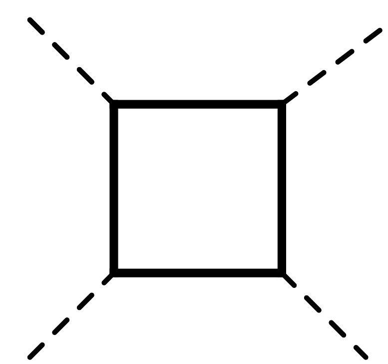
- scatter scalars from unbroken SU(4) part

$$A^{\text{tree}}_{Y\bar{Y}\to Y\bar{Y}} = -2g^2_{\text{YM}}\frac{s}{t}$$



- loops: SU(4) particle interact via massive W bosons

$$A_{Y\bar{Y}\to Y\bar{Y}} = A^{\text{tree}}_{Y\bar{Y}\to Y\bar{Y}}\ M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right)$$



- we take Nc large

- mostly studied with the mass serving as a (dual-conformal-symmetry-preserving) regulator

[Alday, JMH, Plefka, Schuster 09][JMH, Naculich, Schnitzer, Spradlin 10]

- we started a systematic investigation of massive four-particle amplitudes in N=4 SYM

- analytic result to 3 loops [Caron-Huot & JMH, 2014]

- e.g. one loop $M = 1 + g^2 M^{(1)} + \mathcal{O}(g^4)$

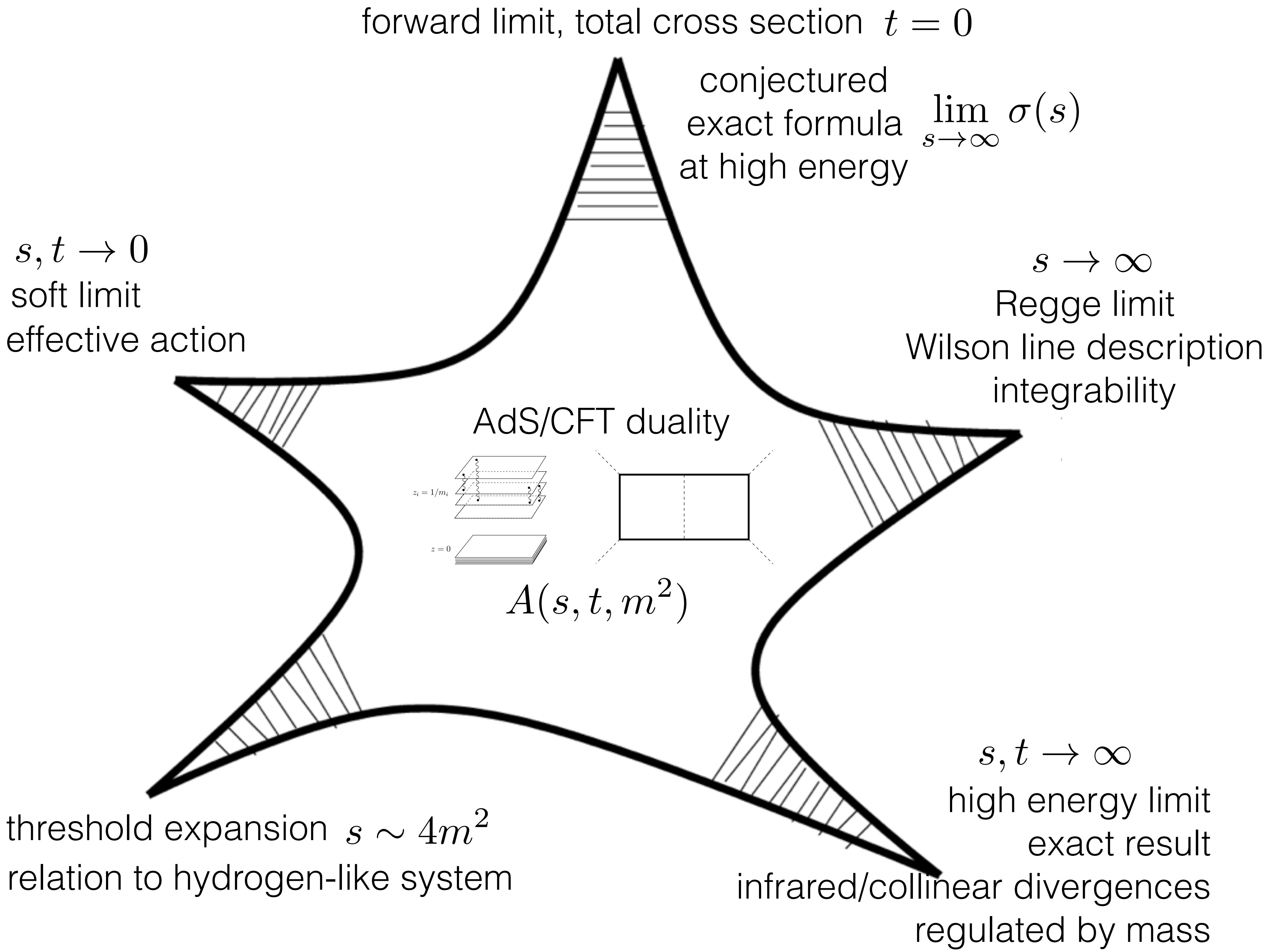
$$M^{(1)} = -\frac{2}{\beta_{uv}}\Big\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2} + \sum_{i=1,2}\Big[2\,\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\Big]\Big\}.$$

with $u = \frac{4m^2}{-s}\,, \qquad v = \frac{4m^2}{-t}\,,$

$$\beta_u = \sqrt{1+u}\,, \qquad \beta_v = \sqrt{1+v}\,, \qquad \beta_{uv} = \sqrt{1+u+v}\,.$$

- compact formula containing a lot of physics

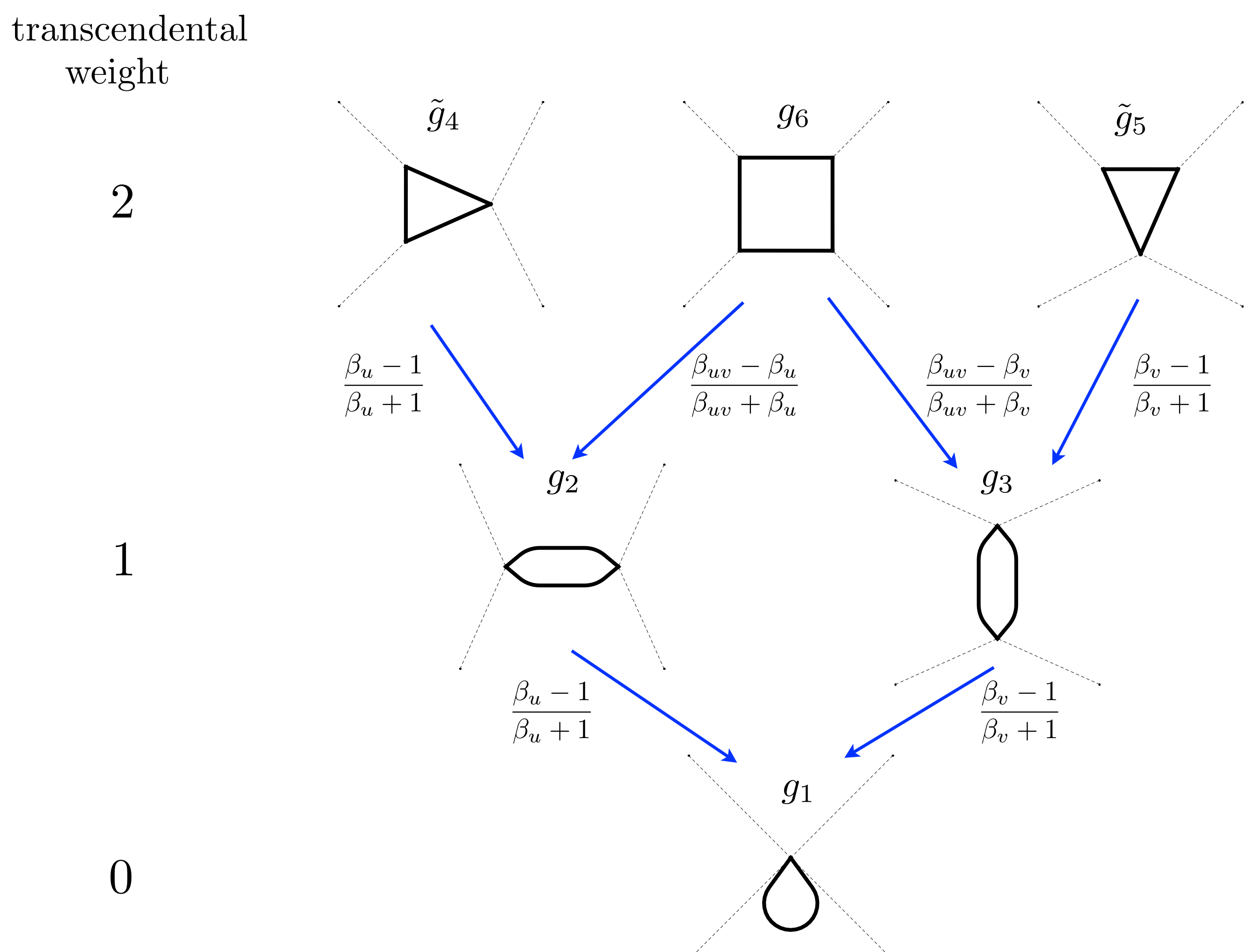
Overview of interesting limits



Integrals from differential equations

- canonical form of differential equations for finite integrals

[Caron-Huot & JMH, 2014]



- makes symbol and weight structure manifest

$$g_6 = \int_\gamma d\,\log \frac{\beta_u - 1}{\beta_u + 1}\, d\log \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u} + \int_\gamma d\,\log \frac{\beta_v - 1}{\beta_v + 1}\, d\,\log \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}\,.$$

$$u = \frac{4m^2}{-s}\,, \qquad v = \frac{4m^2}{-t}\,, \qquad \beta_u = \sqrt{1+u}\,, \qquad \beta_v = \sqrt{1+v}\,, \qquad \beta_{uv} = \sqrt{1+u+v}\,.$$

Function alphabets

- massive four-point alphabet at 2 loops in D=4

[Caron-Huot & JMH, 2014]

$$u, 1+u, v, 1+v, u+v,$$

$$\frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}$$

- additional letters at 2 loops for arbitrary D

$$\{1+u+v\,, \frac{4-v+\beta}{4-v-\beta}\,, \frac{4+v+\beta}{4+v-\beta}\,, \frac{(4\beta_u+\beta)(4\beta_u+\beta_u v+\beta)}{(4\beta_u-\beta)(4\beta_u+\beta_u v-\beta)}\,, \frac{(4\beta_{uv}+\beta)(4\beta_{uv}-\beta_{uv} v+\beta)}{(4\beta_{uv}-\beta)(4\beta_{uv}-\beta_{uv} v-\beta)}\,,$$

$$\beta = \sqrt{16+16u+8v+v^2}\,.$$

- additional letters at 3 loops in D=4

$$u^2 - 4v, v^2 - 4u, \frac{2 - 2\beta_{uv} + u}{2 + 2\beta_{uv} + u}, \frac{2 - 2\beta_{uv} + v}{2 + 2\beta_{uv} + v}$$

- D=4 alphabets can be made rational by changing variables

Obtaining the expansions

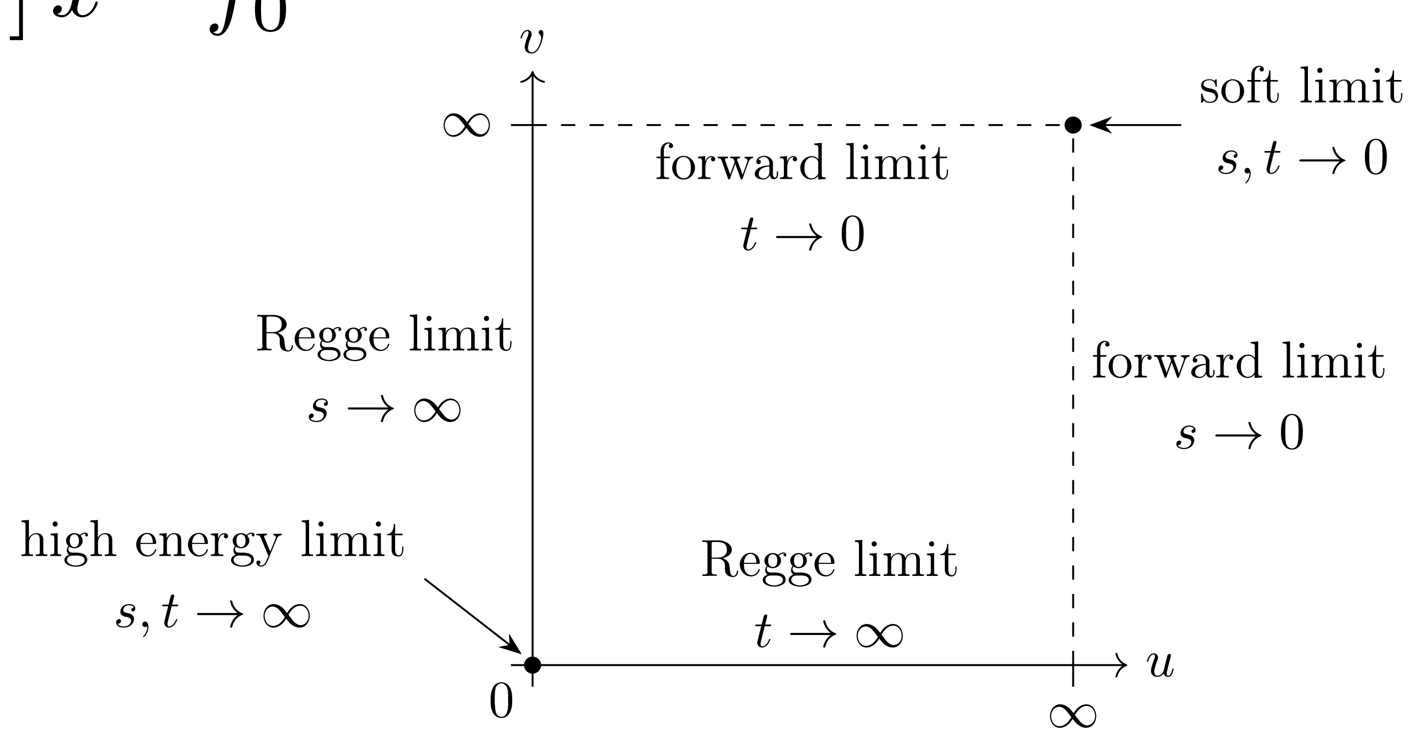
- derive them from differential equations obtained in

$$d\, f(s,t,m^2) = d\left[\tilde{A}(s,t,m^2)\right] f(s,t,m^2)$$ [Caron-Huot, JMH, 2014]

- expansion in small parameter [Wasow 1965]

$$x f'(x) = \bar{A}(x) f(x)$$
$$\bar{A}(x) = \bar{A}_0 + \bar{A}_1 x + \ldots$$
$$f(x) = [1 + P_1 x + \ldots]\, x^{\bar{A}_0} f_0$$

- boundary value f_0: start from soft limit, transport it along the boundary of this square



Soft limit $|s|, |t| \ll m^2$

- massive W bosons can be integrated out
- effective field theory description
- 1/m^4 term one-loop exact

$$\frac{1}{st} M \left(\frac{4m^2}{-s}, \frac{4m^2}{-t} \right) = \frac{1}{st} - \frac{g^2}{6m^4} + \mathcal{O}(1/m^6).$$

in agreement with non-renormalization threorems, see e.g. [Buchbinder, Petrov, Tseytlin 01], and references therein

- we derive the expansion up to three loops, e.g.

$$+\frac{st}{m^6}\left(-\frac{g^2}{60} - \frac{g^4}{12} + \frac{g^6}{3}\right) + \frac{st}{m^8}\left(-\frac{g^2}{840} - \frac{g^4}{180}\right)$$

$$+\frac{s^2+t^2}{m^8}\left(-\frac{g^2}{420} - \frac{g^4}{45} + \frac{g^6}{24}\right) + \mathcal{O}\left(\frac{1}{m^{10}}, g^4\right)$$

Note: Soft limit for U(1) external states was discussed in [Bianchi, Morales, Wen 15]

High-energy limit $m^2/s \to 0, m^2/t \to 0, \text{ with } s/t \text{ fixed}$

- mass serves as a regulator for infrared/collinear divergences

$$M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = 1 + g^2\left[-2\log\left(\frac{m^2}{-s}\right)\log\left(\frac{m^2}{-t}\right) + \pi^2\right] + \mathcal{O}(g^4)\,.$$

- structure of IR-divergent terms known

- finite part fixed by dual conformal Ward identity for Wilson loops [Drummond, Korchemsky, JMH, Sokatchev, 07] [Alday, JMH, Plefka, Schuster 09]

$$\log M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = -\frac{1}{8}\gamma(g^2)\left[\log^2\left(\frac{-s}{m^2}\right) + \log^2\left(\frac{-t}{m^2}\right)\right] - \tilde{\mathcal{G}}_0\left[\log\left(\frac{-s}{m^2}\right) + \log\left(\frac{-t}{m^2}\right)\right]$$
$$+\frac{1}{8}\gamma(g^2)\left[\log^2\left(\frac{-s}{-t}\right) + \pi^2\right] + \tilde{c}(g^2) + \mathcal{O}(m^2)\,. \qquad (2.15)$$

$\gamma(g)$ light-like cusp [Beisert, Eden, Staudacher, 07]

- confirms a previous conjecture [Bern, Dixon, Smirnov, 05]
- note: formula is Regge exact:

$$\log M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = \frac{\gamma(g)}{8}\left[-2\log\left(\frac{m^2}{-s}\right)\log\left(\frac{m^2}{-t}\right) + \pi^2\right] + \tilde{\mathcal{G}}_0(g)\left[\log\left(\frac{m^2}{-s}\right) + \log\left(\frac{m^2}{-t}\right)\right] + \tilde{c}(g) + \mathcal{O}(m^2)\,.$$

[Drummond, Korchemsky, Sokatchev, 07] [Naculich, Schnitzer, 07]

Regge limit and cusp anomalous dimension

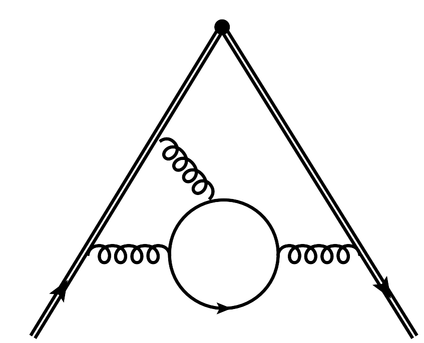
- expected Regge behavior

$$\lim_{s\to\infty} M\left(\frac{4m^2}{-s},\frac{4m^2}{-t}\right) = r_0(t)\left(\frac{-s}{m^2}\right)^{j_0(t)+1} + \mathcal{O}(1/s)$$

$$j_0(t)+1 = \frac{2g^2}{\beta_v}\log\frac{\beta_v-1}{\beta_v+1} + \mathcal{O}(g^4)$$

$$r_0(s) = 1 + \mathcal{O}(g^2)$$

- in planar N=4 sYM, Regge trajectory is related to cusp anomalous dimension



- subleading powers 1/s in limit poorly studied

An implication of dual conformal symmetry

$$M_4(s,t;m_1,m_2,m_3,m_4) = M(u,v)$$

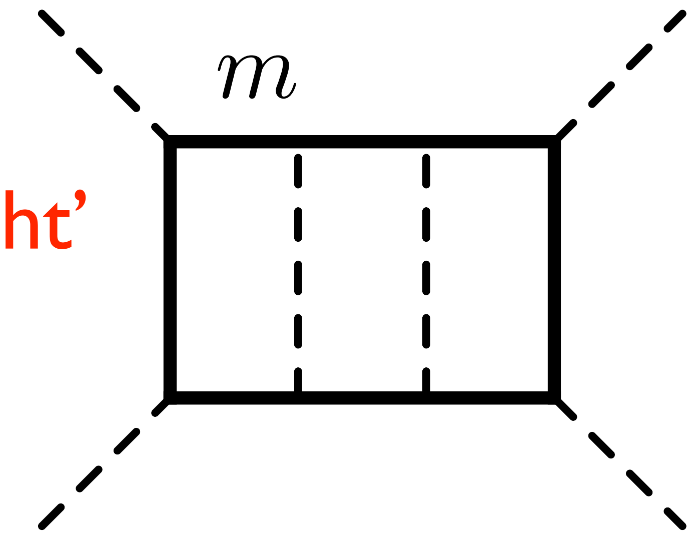
[Alday, JMH, Plefka,Schuster 09]

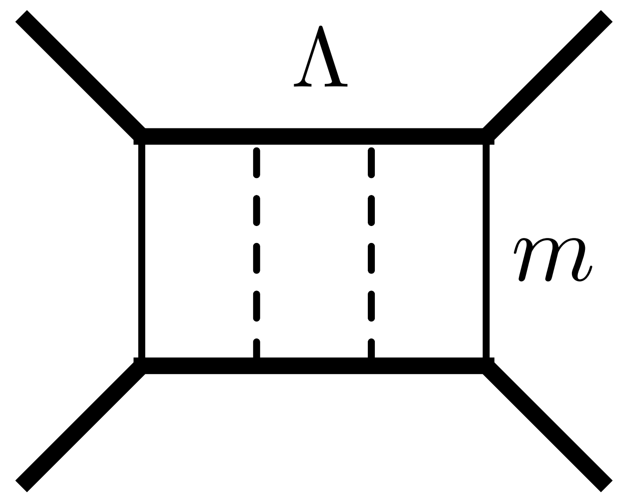
$$u = \frac{4m_1 m_3}{-s+(m_1-m_3)^2}\,, \quad v = \frac{4m_2 m_4}{-t+(m_2-m_4)^2}$$

- implies equivalence [JMH, Naculich, Schnitzer, Spradlin 10]

$m_i = m$ $\qquad$ $m_2 = m_4 = m \quad m_1 = m_3 = \Lambda \gg m$

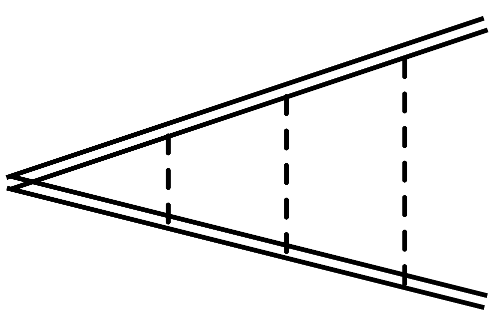
'light-by-light' scattering





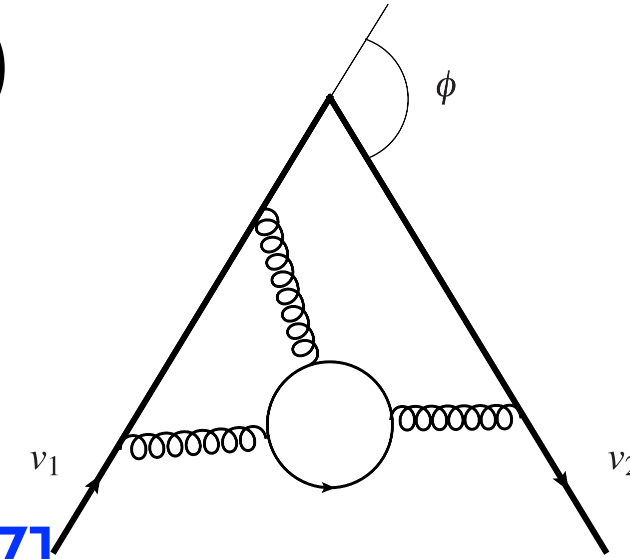
'Bhabha' scattering

$$M \sim_{t\to\infty} t^{(j(s)+1)} \qquad M \sim_{m\to 0} m^{\Gamma_{\rm cusp}(\phi)}$$



$$\boxed{j(s)+1 = -\Gamma_{\rm cusp}(\phi) \quad \text{where} \quad s = 4m^2 \sin^2\frac{\phi}{2}\,.}$$

Anomalous dimension $\Gamma_{\rm cusp}(\phi)$ of a Wilson loop with cusp

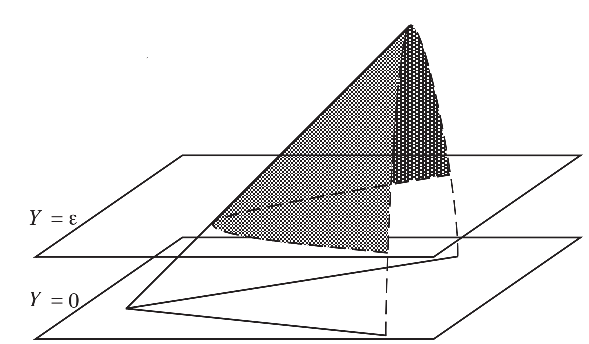


- known in QCD up to 3 loops
  [Polyakov 1980] [Korchemsky, Radyushkin, 1987]
  [Grozin, JMH,Korchemsky, Marquard, 2016]
- known in planar N=4 sYM up to 4 loops
  [Drukker, Forini 06] [Correa, JMH, Maldacena, Sever 12]
  [JMH, Huber 13]
- exact result [Correa, JMH, Maldacena, Sever 12]

$$\Gamma_{\rm cusp}(\phi) = -B\,\phi^2 + \mathcal{O}(\phi^4)$$

$$B = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \approx g^2 - \frac{2}{3}\pi^2 g^4 + \frac{2}{3}\pi^4 g^6 + \dots$$

- planar case governed by integrability
  [Drukker 12] [Correa, Maldacena, Sever 12]
- strong coupling computation from a minimal surface [Drukker, Gross,Ooguri, 1999]



power suppressed terms in Regge limit

- we find only one 'daughter' trajectory

$$\lim_{s\to\infty} M(\frac{4m^2}{-s}, \frac{4m^2}{-t}) = \left(\frac{-s}{m^2}\right)^{j_0(t)+1} \left(1 + \frac{c_1(t)}{s}\right) + \frac{c_2(t)}{s} \left(\frac{-s}{m^2}\right)^{g^2 c_3(t)} + \mathcal{O}(1/s^2)$$

- tested up to 3 loops

- dual conformal symmetry suggests O(4) partial wave expansion

The first two terms in the Regge limit are pure powers, *when using O(4) variables*!

$$\lim_{s\to\infty} \frac{1+e^{-\rho}}{1-e^{-\rho}} M(\frac{4m^2}{-s}, \frac{4m^2}{-t}) = r_0(t) e^{(j_0(t)+1)\rho} + r_1(t) e^{(j_1(t)+1)\rho} + \mathcal{O}(e^{-2\rho}),$$

$$\left(\cosh\rho = 1 + \frac{2s}{t} - \frac{s}{2m^2}\right)$$

O(4) partial wave expansion

$$\boxed{\lim_{s\to\infty} \frac{1+e^{-\rho}}{1-e^{-\rho}} M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = r_0(t) e^{(j_0(t)+1)\rho} + r_1(t) e^{(j_1(t)+1)\rho} + \mathcal{O}(e^{-2\rho})}$$

- 3-loop result agrees with this form!

$$\cosh\rho = 1 + \frac{2s}{t} - \frac{s}{2m^2}$$

- We find ($\frac{\beta_v - 1}{\beta_v + 1} = e^{-\varphi}, \qquad \xi = \frac{1}{\beta_v},$)

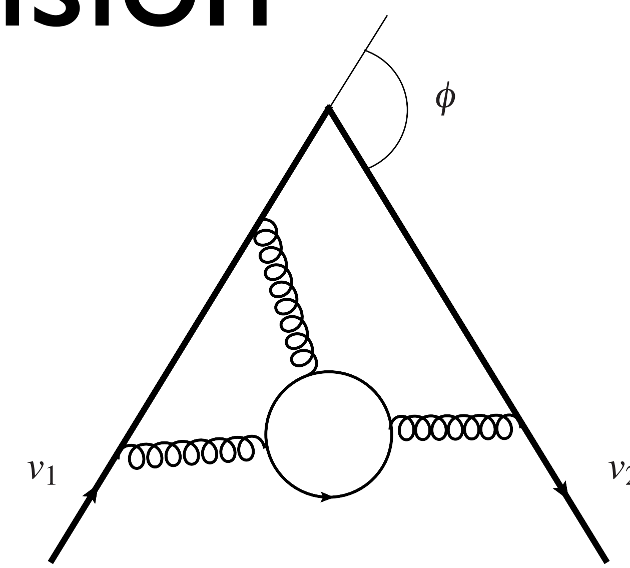
sub-leading trajectory:

$$\begin{aligned} j_1 = &-2 - 4g^2 + g^4\left(16 - \frac{4}{3\xi}\varphi^3 + 8(\varphi - 2\xi)(\varphi - \frac{1}{\xi}\zeta_2)\right) \\ &+ g^6\Bigg[\frac{24}{\xi}\mathrm{Li}_4(e^{-2\varphi}) + \left(64 + \frac{16\varphi}{\xi}\right)\mathrm{Li}_3(e^{-2\varphi}) + 64(\varphi+\xi)\mathrm{Li}_2(e^{-2\varphi}) - 128\varphi\xi\log(1-e^{-2\varphi}) \\ &\quad + \frac{8}{5\xi}\varphi^5 - \frac{8}{3}\varphi^4\left(5 + \frac{1}{\xi}\right) + \frac{16}{3}\varphi^3\left(4 + 7\xi + \frac{1+4\zeta_2}{\xi}\right) - 16\varphi^2\left(3 + 6\zeta_2 + 4\xi + 2\xi^2 + \frac{\zeta_2}{\xi}\right) \\ &\quad + 16\varphi\left(4\zeta_2 + 6\xi(2+\zeta_2) + \frac{11\zeta_4 - \zeta_3 + 2\zeta_2}{\xi}\right) - 24\zeta_4\left(10 + \frac{1}{\xi}\right) + 32\zeta_3 - 64\zeta_2(1+\xi) - 128\Bigg]\,. \end{aligned}$$

residue:

$$r_1 = 2 + 8g^2\left(2\log\frac{1+e^{-\phi}}{1-e^{-\phi}} - 1\right) + \mathcal{O}(g^4)$$

Subleading Regge trajectory from an anomalous dimension



- leading Regge trajectory is cusp anomalous dimension j_0

- we conjecture that the first subleading trajectory is computed from the anomalous dimension of a Wilson loop with a scalar insertion at the cusp j_1

- two-loop test underway [Brueser, Caron-Huot, JMH]

Conclusion

- studied massive amplitudes on the Coulomb branch of N=4 sYM
- many limits governed by integrability, or exact results available, at leading order in expansion

New results:

- we found a simple structure in the Regge limit
- only one daughter trajectory at 1/s
- trajectory computable from Wilson loop

Outlook

- confirm conjecture for subleading Regge trajectory?
- Wilson loop with scalar insertion from integrability?

  [Gromov, Kazakov, Leurent, Volin, 13; Gromov, Levkovich-Maslyuk 15]

- for massless amplitudes, expansion derived around collinear limit using integrability; can the same be done for the Regge limit?

  [Alday, Gaiotto, Maldacena, Sever, Vieira 06; Basso, Sever, Vieira 13]

- amplitudes at strong coupling: so far, computed only for small mass; it would be interesting to extend this to finite mass, at least in Regge limit

  [Drukker, Gross, Ooguri 1999; Alday, Maldacena 07]

Extra slides

Forward limit and cross section

- optical theorem relates $A_{Y\bar{Y}\to Y\bar{Y}}$ to

$$\sigma_{\rm tot} = \sigma_{Y\bar{Y}\to WW+{\rm gluons}}$$

$$\sigma_{\rm tot} = \frac{1}{2E_{\rm cm}p_{\rm cm}} \lim_{t\to 0} {\rm Im}(A) = \frac{1}{s} \lim_{t\to 0} {\rm Im}(A)$$

- one-loop example:

$$\sigma_{\rm tot} = \frac{32\pi^3 g^2}{N_c m^2}\sqrt{1-4m^2/s} + \mathcal{O}(g^4)$$

- cross section goes to a constant at high energies

$$\lim_{s\to\infty} \sigma_{\rm tot} = \frac{32\pi^3 g^2}{N_c m^2} + \mathcal{O}(g^4)$$

- we will make a conjecture for this limit at any coupling!

Exact cross section at high energies

- leading Regge behavior

$$\lim_{s\to\infty} M\left(\frac{4m^2}{-s}, \frac{4m^2}{-s}\right) = r_0(t)(-s-i0)^{1+j_0(t)} + \mathcal{O}(1/s)$$

- analytically continue and take imaginary part,

take forward limit

$$\lim_{t\to 0}\lim_{s\to\infty} \frac{1}{-t}\,\mathrm{Im}\, M(s,t) = \pi \frac{d}{dt} j_0(t)|_{t=0}.$$

- the slope is given by the ‘Bremsstrahlung function’

$$\frac{d}{dt} j_0(t)|_{t=0} = \frac{B}{m^2}\,, \qquad B = \frac{1}{4\pi^2}\frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \approx g^2 - \frac{2}{3}\pi^2 g^4 + \frac{2}{3}\pi^4 g^6 + \ldots.$$

- *Assuming the above limits commute*, we obtain

$$\boxed{\lim_{s\to\infty} \sigma_{Y\bar{Y}\to WW+X} = \frac{2\pi^2 g^2}{m^2} B}$$

- confirmed by explicit calculation up to 3 loops

Perturbative check to 3 loops

- we find $\sigma_{\mathrm{tot}} = \dfrac{32\pi^3 g^2}{N_c m^2}\left[g^2 X_1 + g^4 X_2 + g^6 X_3 + \mathcal{O}(g^8)\right]$

$$
\begin{aligned}
X_1 =& \frac{1+x}{1-x}\,,\\
X_2 =& 16\mathrm{Li}_2(-x) + 8\log(-x)\log(x+1) - \frac{2\pi^2}{3}\,,\\
X_3 =& -48H_{-3,0}(-x) + 64H_{3,0}(-x) + 48H_{-2,0,0}(-x) - 64H_{2,0,0}(-x)\\
&- 48\zeta_2 H_{-2}(-x) + 64\zeta_2 H_2(-x) + 32\zeta_4\\
&+ \frac{1+x}{1-x}\Big[16H_{-3,0}(-x) + 96H_{-2,2}(-x) - 32H_{2,2}(-x) + 128H_{3,1}(-x)\\
&+ 64H_{-2,0,0}(-x) + 32H_{-2,1,0}(-x) + 32H_{2,-1,0}(-x) - 80H_{2,0,0}(-x)\\
&- 96H_{2,1,0}(-x) - 112\zeta_2 H_{-2}(-x) + 96\zeta_2 H_2(-x) + 28\zeta_4\Big]\,.
\end{aligned}
$$

- here $x = \dfrac{\sqrt{1-4m^2/s} - 1)}{\sqrt{1-4m^2/s}+1}$ and $-1 < x < 0$; H are harmonic polylogarithms
- high-energy limit $x \to 0$

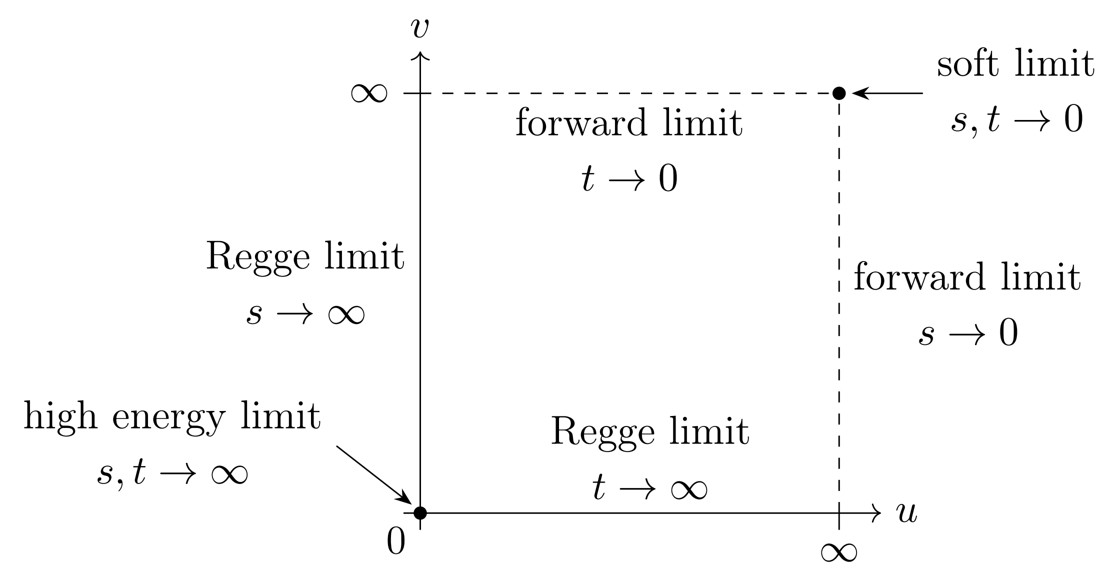
$$X_1 \to 1\,, \qquad X_2 \to -\frac{2\pi^2}{3}\,, \qquad X_3 \to \frac{2\pi^4}{3}\,.$$

- perfectly agrees with conjectured formula!

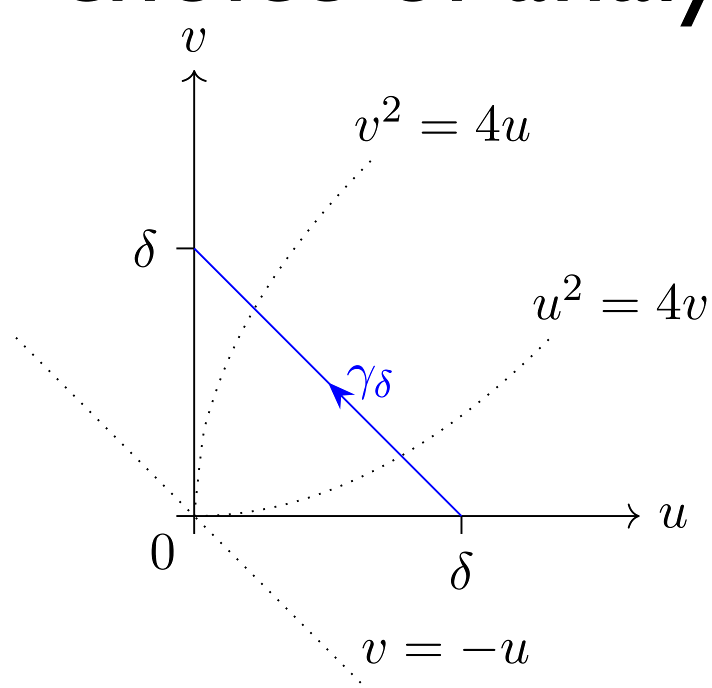
$$\boxed{\lim_{s\to\infty} \sigma_{Y\bar{Y}\to WW+X} = \frac{2\pi^2 g^2}{m^2} B}$$

Transporting the boundary value

- boundary value f_0: start from soft limit, transport it along the boundary of this square



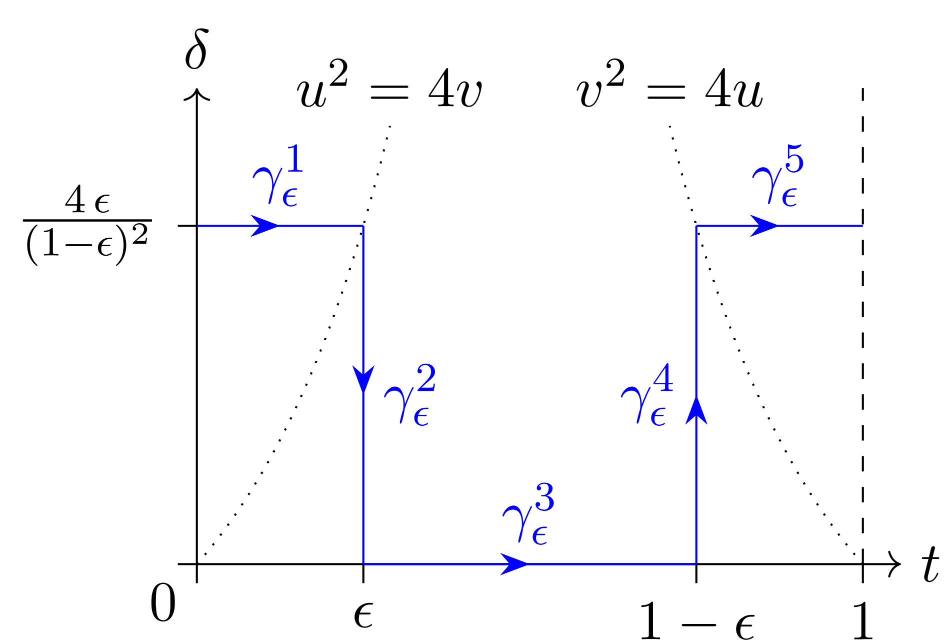
- choice of analytic continuation path



$$\gamma_\delta(t) = (u,v) = (\delta(1-t), \delta t)\,, \quad t \in [0,1]$$

$$\{\log(u), \log(v), \log(u+v)\} \longrightarrow \{\log(\delta), \log(t), \log(1-t)\}\,.$$

- extra step at three loops:



bound state energy of pair of W bosons

- Regge theory: extract spectrum from

$$\Gamma_{\text{cusp}}(\phi_n) = -n \qquad\qquad E_n = 2m \sin\frac{\phi_n}{2}$$

n integer

- obtain bound state energy from

cusp anomalous dimension

$$\Gamma_{\text{cusp}}(\phi) = -\frac{\lambda}{8\pi^2}\,\phi\,\tan\frac{\phi}{2} + \mathcal{O}(\lambda^2)\,.$$

$$\Gamma_{\text{cusp}}(\pi - \delta) \approx -\frac{\lambda}{4\pi\delta}$$

we find $$E_n - 2m = -\frac{\lambda^2 m}{64\pi^2 n^2} + O(\lambda^3)\,.$$

as expected from $$H = \frac{p^2}{2\mu} - \frac{\lambda}{4\pi}\frac{1}{|x|}$$

higher orders

- resummation required (ultrasoft effects) [systematic EFT, e.g. Pineda 2007]

$$\Gamma_{\text{cusp}}(\pi-\delta) = \frac{-\lambda}{4\pi\delta}\left(1-\frac{\delta}{\pi}\right) + \frac{\lambda^2}{8\pi^3\delta}\log\frac{\epsilon_{\text{uv}}}{2\delta}$$
$$-\frac{\lambda}{4\pi^2}\int_{\epsilon_{\text{uv}}}^{\infty}\frac{d\tau}{\cosh(\tau)-1}\left(e^{-\tau\frac{\lambda}{4\pi\delta}}-1\right) + \mathcal{O}(\lambda^3)\,.$$

[Correa, JMH, Maldacena, Sever, 2012]

- result for energy

$$(E_n - 2m)\,|_{\lambda^3} = \frac{-\lambda^3 m}{64\pi^4 n^2}\left[S_1(n) + \log\frac{\lambda}{2\pi n} - 1 - \frac{1}{2n}\right] \qquad S_1(n) = \sum_{k=1}^{n}\frac{1}{k}$$

- checks
  - [...] bounded for any n
  - n large correctly gives quark-antiquark potential

    [Ericksson, Semenoff, Szabo Zarembo, 1999; Pineda 2007]
  - confirmed by standard 'Coulomb resummation'

    [Caron-Huot, JMH]

    [see Beneke, Kiyo & Schuller 1312.4791]

Strong coupling check

- cusp anomalous dimension $\Gamma_{\rm cusp}(\phi)$ at strong coupling was computed from minimal surface

[Drukker, Gross, Ooguri, 1999]

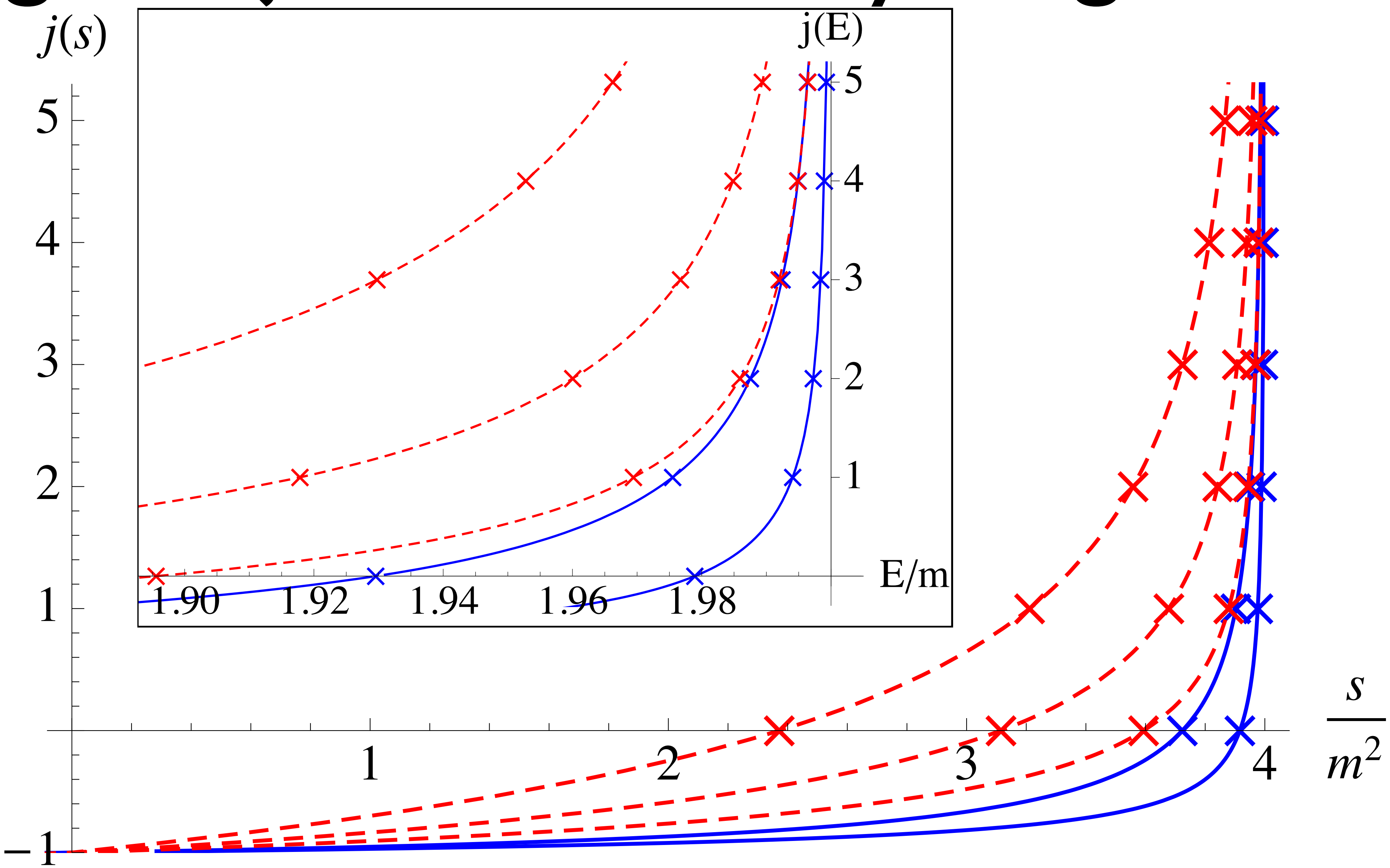
- spectrum of 'mesons' was computed at strong coupling in 2003

[Kruczensky, Mateos, Myers, Winters, 2003]

- the two curves agree perfectly, once one uses the correct dictionary!

$$E_n = 2m \sin \frac{\phi_n}{2}$$

Regge trajectories of Hydrogen-like states



$\lambda = 5, 10, 10, 30, 100$ (bottom to top)

solid/blue: based on weak-coupling formulas

dashed/red: based on strong-coupling formulas

- exact spectrum should be computable from TBA for $\Gamma_{\text{cusp}}(\phi)$

[Correa, Maldacena, Sever; Drukker]