TWO-PARTON SCATTERING IN THE HIGH-ENERGY LIMIT

Leonardo Vernazza

Higgs centre for theoretical physics, School of Physics and astronomy

THE UNIVERSITY of EDINBURGH

Amplitudes 2017, Edinburgh, 11/07/2017

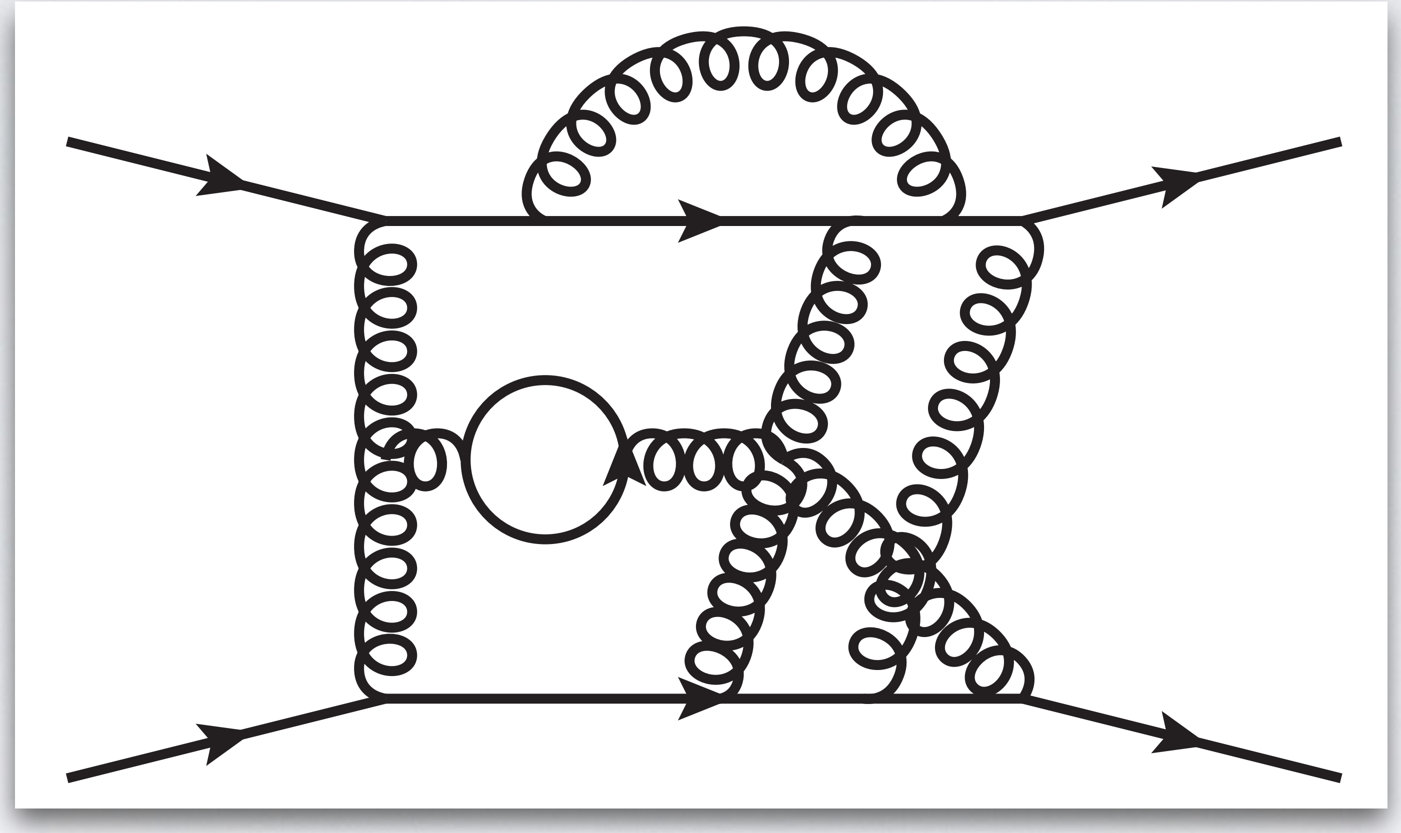
OUTLINE

- **Aspects of 2 → 2 scattering amplitudes in the high-energy limit**
- **High-energy rapidity evolution and the Balitsky-JIMWLK equation**
- **The three-Reggeon cut**
- **The two-Reggeon cut**

In collaboration with Simon Caron-Huot, Einan Gardi and Joscha Reichel,

Based on arXiv:1701.05241 and work in progress

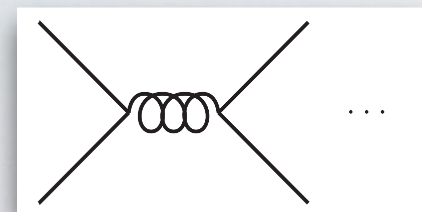
ASPECTS OF 2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

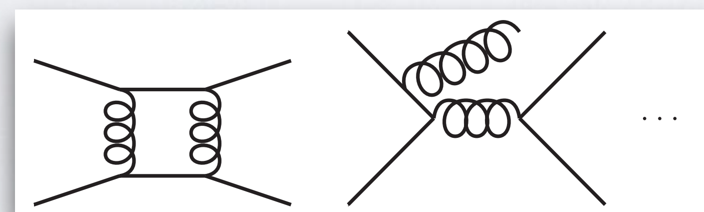


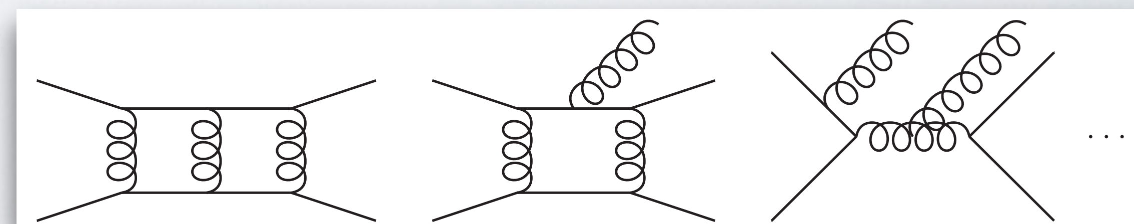
2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Calculation of scattering amplitudes at high order in perturbation theory is one of the main ingredients for the program of precision physics at the LHC

$$\mathcal{M} = 4\pi\alpha_s\left[\mathcal{M}^{(0)} + \frac{\alpha_s}{4\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\mathcal{M}^{(2)} + \ldots\right]$$



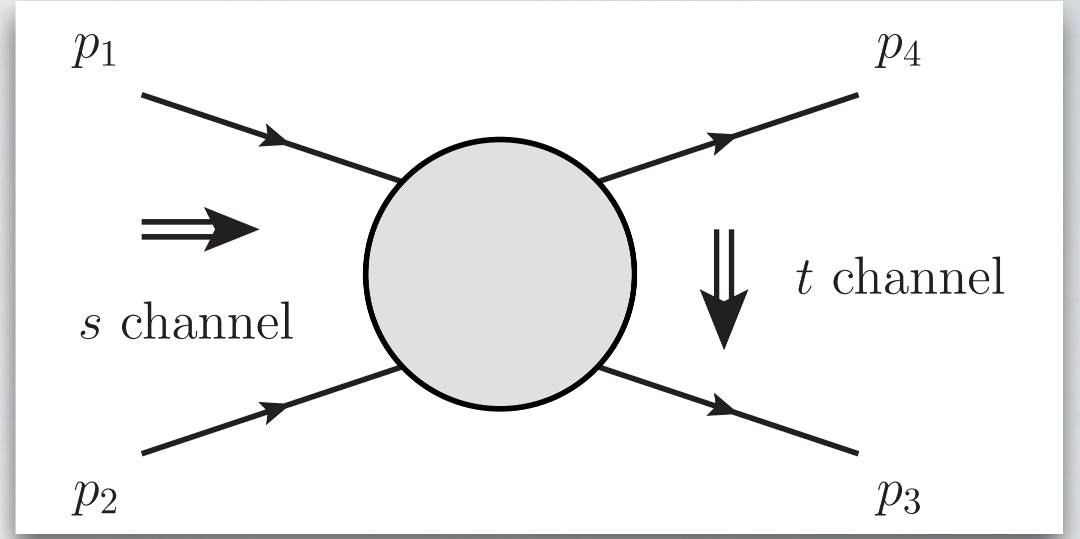




- Amplitudes are complicated functions of the kinematical invariants, their calculation is non-trivial, and it is subject of intense study.
  - Express Feynman integrals in terms of known functions (harmonic polylogarithms, elliptic integrals, etc)
  - Amplitudes contains infrared divergences, which must cancel when summing virtual and real corrections.

2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Information and constraints can be obtained by considering kinematical limits:
  - it reduces the number of invariants;
  - it helps identifying factorisation properties and iterative structures of the amplitude;
  - it may be relevant for phenomenology: because of soft and collinear enhancement, amplitudes in specific kinematic limit develops large logarithms, which may spoil the convergence of the perturbative expansion in that region of the parameter space.



- Consider 2 → 2 scattering amplitudes in the high-energy limit:

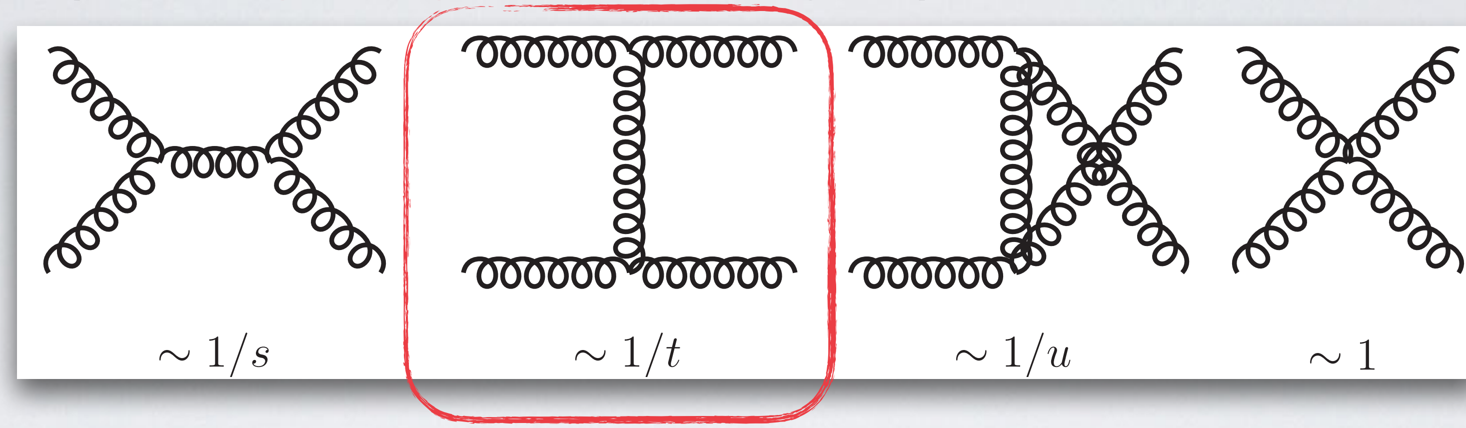
$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0$$

- The amplitude becomes a function of the ratio |s/t|; here we consider the leading power term in this expansion

$$\mathcal{M}(s,t,\mu) = \mathcal{M}_{LP}\left(\frac{s}{-t}, \frac{-t}{\mu^2}\right)\left[1 + \mathcal{O}\left(\frac{-t}{s}\right)\right].$$

2 ➔ 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

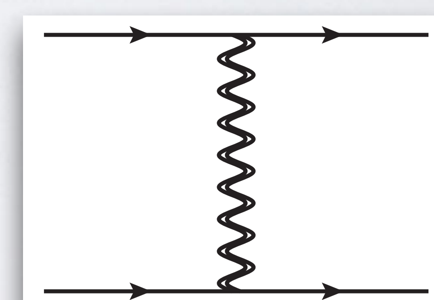
- Consider, as an example, the gluon-gluon scattering amplitude at tree level:



- In the high-energy limit only the second diagram contributes at leading power. The amplitude is simply

$$\mathcal{M}(s,t) = 4\pi\alpha_s \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \mathcal{M}^{(n)}(s,t), \qquad \mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t} (T_i^b)_{a_1 a_4} (T_j^b)_{a_2 a_3}\, \delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}.$$

- The amplitude at higher orders contains logarithms of the ratio |s/t|. In the sixties the dominant behaviour in the high-energy limit was characterised in terms of Regge poles and cuts. These can now be studied in the context of QCD. One has



Regge, Gribov

$$\mathcal{M}_{ij\to ij}|_{\mathrm{LL}} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi} C_A\, \alpha_g^{(1)}(t)} 4\pi\alpha_s\, \mathcal{M}^{(0)}_{ij\to ij},$$

- where the function $\alpha_g(t)$ is known as the Regge trajectory:

$$\alpha_g(t) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \alpha_g^{(n)}(t), \qquad \alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \overset{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon},$$

- and r_Γ is a ubiquitous 1-loop factor:

$$r_\Gamma = e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1 - \frac{1}{2}\zeta_2\,\epsilon^2 - \frac{7}{3}\zeta_3\,\epsilon^3 + \dots$$

2 → 2 SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

- Determining the amplitude beyond LL requires to understand the structure of Regge cuts. At this purpose, the following considerations hold:
  - The amplitudes which develop definite factorisation properties in the high-energy limit are the so called even and odd amplitudes, i.e. the projection onto eigenstates of signature, (crossing symmetry s ↔ u):

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

  - $M^{(+)}$ and $M^{(-)}$ are respectively imaginary and real, when expressed in terms of the natural signature-even combination of logs:

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

  - Beyond tree level the amplitude has a non-trivial color structure:

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\,\mathcal{M}^{[i]}(s,t).$$

  - Decompose the amplitude in a color orthonormal basis in the t-channel: for gluon scattering one has

$$8\otimes 8 = 1\oplus 8_s\oplus 8_a\oplus 10\oplus\overline{10}\oplus 27\oplus 0$$

  - Invoking Bose symmetry we deduce that $M^{(+)}$, which is symmetric under permutation of the kinematic variables s and u, picks out the colour components which are symmetric under permutation of the indices of particles 2 and 3, and $M^{(-)}$, which is antisymmetric upon swapping s and u, picks out the colour-antisymmetric part:

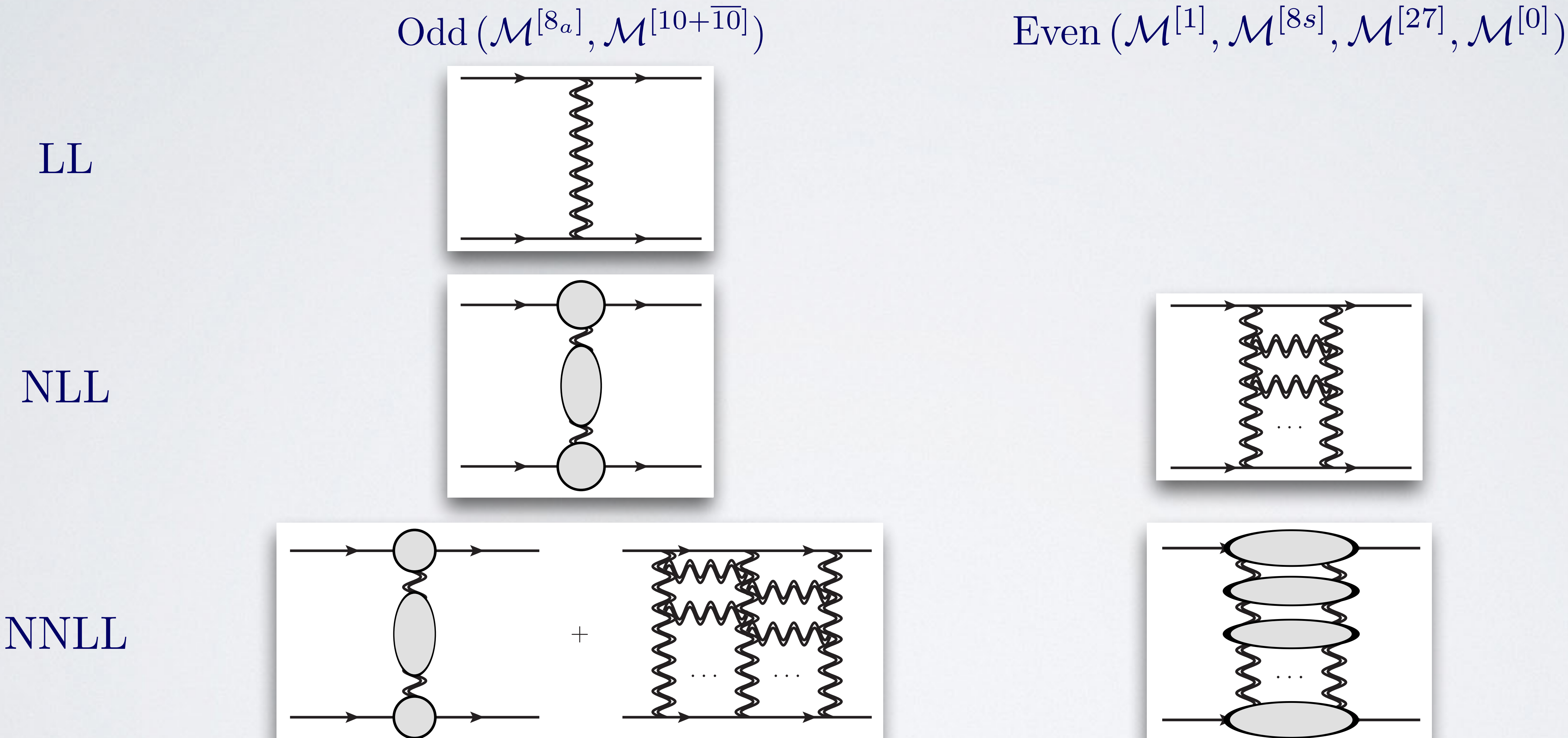
$$\text{odd: } \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even: } \mathcal{M}^{[1]}, \mathcal{M}^{[8_s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]} \qquad (gg \text{ scattering}).$$

FACTORISATION STRUCTURE

- Write the amplitude as the sum of odd and even component, with expansion in the strong coupling constant

$$\mathcal{M}(s,t) = \mathcal{M}^{(-)}(s,t) + \mathcal{M}^{(+)}(s,t), \qquad \mathcal{M}^{(\pm)}(s,t) = 4\pi\alpha_s \sum_{l,m} \left(\frac{\alpha_s}{\pi}\right)^l L^m \, \mathcal{M}^{(\pm,l,m)} \, .$$

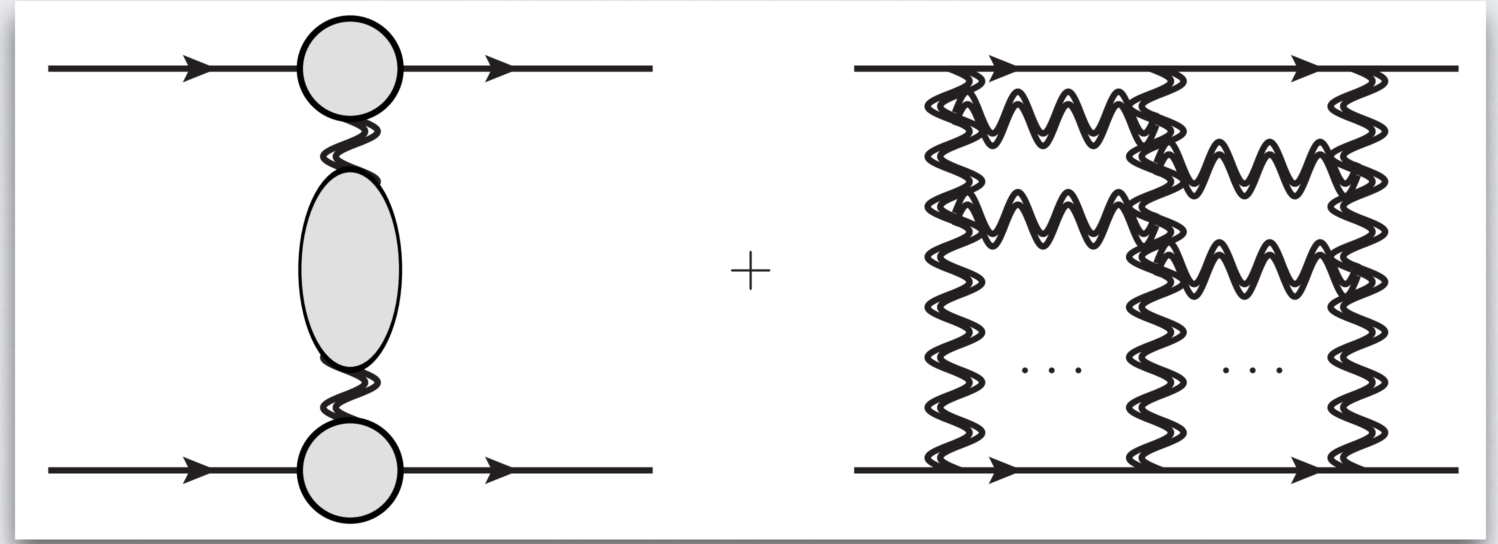
- Up to NNLL, the amplitude in the high-energy limit has the following factorisation structure:



- aIn order to display the Regge-cut contributions in the most transparent way, it proves useful to define a "reduced" amplitude by removing from it the Reggeized gluon and collinear divergences as follows:

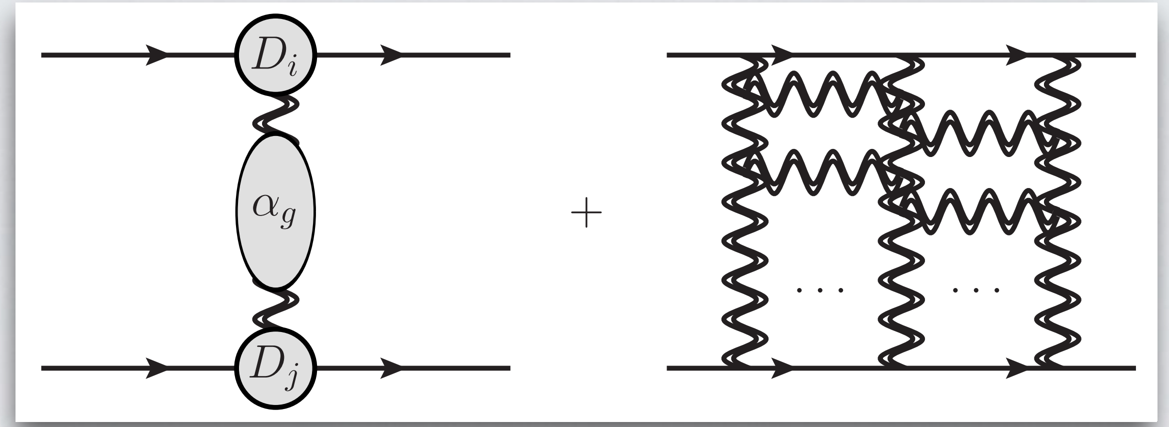
$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\mathbf{T}_t^2 \, \alpha_g(t) \, L} \, \mathcal{M}_{ij\to ij} \, ,$$

THE BALITSKY-JIMWLK EQUATION AND THE THREE REGGEON CUT



THE ODD AMPLITUDE AT NNLL

- Starting at NNLL, one has mixing between one- and three-Reggeons exchange:

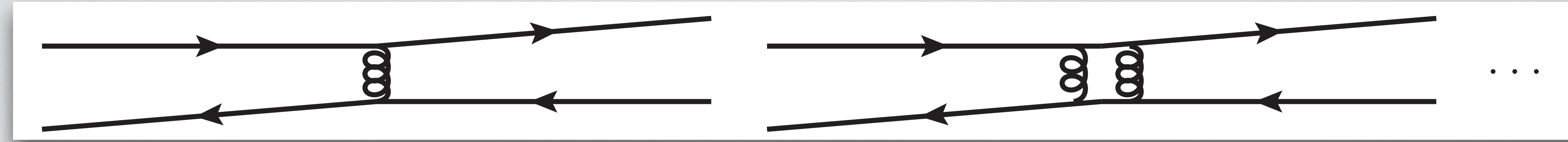


Del Duca, Glover, 2001; Del Duca, Falcioni, Magnea, LV, 2013

- The mixing between one- and three-Reggeons exchange has significant consequences:
  - It is at the origin of the breaking of the simple power law one has up to NLL accuracy. Such breaking appears for the first time at two loops.
  - It implies that, starting at three loops, there will be a single-logarithmic contribution originating from the three-Reggeon exchange, and from the interference of the one- and three-Reggeon exchange: the interpretation of the Regge trajectory at three loops needs to be clarified.
- Schematically, the whole amplitude at NNLL is composed of

$$\hat{\mathcal{M}}_{ij\to ij}|_{\rm NNLL} = \hat{\mathcal{M}}^{(-)}_{ij\to ij}|_{\text{1-Reggeon + 3-Reggeon}} + \hat{\mathcal{M}}^{(+)}_{ij\to ij}|_{\text{2-Reggeon}}.$$

BFKL THEORY ABRIDGED



- The high-energy limit correspond to a configuration of forward scattering:

$$t = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -\frac{s}{2}(1 - \cos\theta),$$
$$u = (p_1 - p_3)^2 = (p_2 - p_4)^2 = -\frac{s}{2}(1 + \cos\theta),$$

$$s \gg -t \quad \Rightarrow \theta \to 0.$$

- The high-energy logarithm correspond to the rapidity difference between the target and the projectile:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

- Such kinematical configuration is described conveniently in terms of Wilson lines stretching from $-\infty$ to $+\infty$. The Wilson lines follow the paths of color charges inside the projectile, and are thus null and labelled by transverse coordinates **z**:

Korchemskaya, Korchemsky, 1994, 1996

$$U(z_\perp) = \mathcal{P}\exp\left[ig_s\int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+T^a\right].$$

- The idea is to approximate, to leading power, the fast projectile and target by Wilson lines and then compute the scattering amplitude between Wilson lines.

Babansky, Balitsky, 2002

- The full transverse structure needs to be retained. A projectile necessarily contains multiple color charges at different transverse positions: the number of Wilson lines cannot be held fixed.

Caron-Huot, 2013

BFKL THEORY ABRIDGED

- However, in perturbation theory, the unitary matrices U(z) will be close to identity and so can be usefully parametrised by a field W:

Caron-Huot, 2013

$$U(z) = e^{ig_s\, T^a W^a(z)}\,.$$

- The color-adjoint field W sources a BFKL Reggeised gluon. A generic projectile, created with four-momentum p_1 and absorbed with p_4, can thus be expanded at weak coupling as

$$|\psi_i\rangle \;\equiv\; \frac{Z_i^{-1}}{2p_1^+} a_i(p_4) a_i^\dagger(p_1)|0\rangle \;\sim\; g_s\, D_{i,1}(t)\,|W\rangle + g_s^2\, D_{i,2}(t)\,|WW\rangle + g_s^3\, D_{i,3}(t)\,|WWW\rangle + \ldots$$
$$\equiv |\psi_{i,1}\rangle + |\psi_{i,2}\rangle + |\psi_{i,3}\rangle + \ldots.$$

- The factors $D_{i,j}$ depend on the transverse coordinates of the W fields, but not on the center of mass energy. They correspond to the impact factors for the exchange of one-, two- and three-Reggeons.
- The energy dependence enters from the fact that the Wilson lines have rapidity divergences which must be regulated, which leads to a rapidity evolution equation (Balitsky-JIMWLK):

$$-\frac{d}{d\eta}\,|\psi_i\rangle = H\,|\psi_i\rangle.$$

BFKL THEORY ABRIDGED

- The inner product is by definition the scattering amplitude of Wilson lines renormalized to equal rapidity.
- For our purposes, it suffices to know that it is Gaussian to leading-order:

$$G_{11'} \equiv \langle W_1 | W_{1'} \rangle = i \, \frac{\delta^{a_1 a_1'}}{p_1^2} \, \delta^{(2-2\epsilon)}(p_1 - p_1') + \mathcal{O}(g_s^2).$$

- Multi-Reggeon correlators are obtained by Wick contractions:

Caron-Huot, 2013

$$\langle W_1 W_2 | W_{1'} W_{2'} \rangle = G_{11'} G_{22'} + G_{12'} G_{21'} + \mathcal{O}(g_s^2),$$
$$\langle W_1 W_2 W_3 | W_{1'} W_{2'} W_{3'} \rangle = G_{11'} G_{22'} G_{33'} + (5 \text{ permutations}) + \mathcal{O}(g_s^2),$$
$$\cdots$$

- There are also off-diagonal elements, which can be defined to have zero overlap (at equal rapidity):

$$\langle W_1 W_2 W_3 | W_4 \rangle = \langle W_4 | W_1 W_2 W_3 \rangle = 0.$$

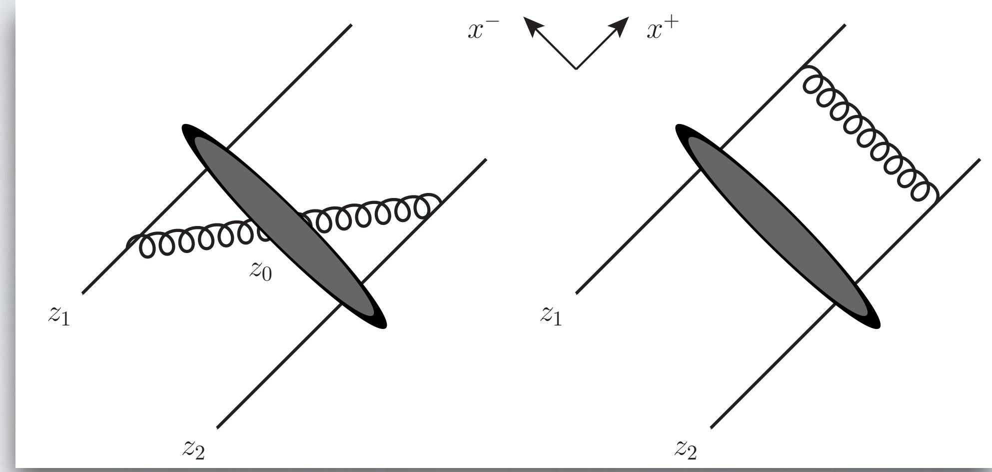
- Choosing the 1-W and 3-W states to be orthogonal, combined with symmetry of the Hamiltonian, (boost invariance):

$$\frac{d}{d\eta} \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = 0 \quad \Leftrightarrow \quad \langle H \mathcal{O}_1 | \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 | H \mathcal{O}_2 \rangle \equiv \langle \mathcal{O}_1 | H | \mathcal{O}_2 \rangle,$$

- implies that in this scheme $H_{k \to k+2} = H_{k+2 \to k}$. This relation is known as projectile-target duality.

THE BALITSKY-JIMWLK EQUATION

- The Balitsky-JIMWLK equation for an arbitrary number of Wilson lines $U(z_i)$ can be written in the form



$$-\frac{d}{d\eta}\Big[U(z_1)\dots U(z_n)\Big] = \sum_{i,j=1}^{n} H_{ij}\cdot\Big[U(z_1)\dots U(z_n)\Big],$$

- with

Caron-Huot, 2013

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int [dz_i][dz_j][dz_0]K_{ij;0}\Big[T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\Big] + \mathcal{O}(\alpha_s^2).$$

- We work now in dimensional regularisation with $2\text{-}2\varepsilon$ dimensions, and $dz = d^{2-2\varepsilon}z$, and $T_{L/R}$'s are generators for left and right color rotations:

Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016

$$T^a_{i,L} = [T^a U(z_i)]\,\frac{\delta}{\delta U(z_i)}, \qquad T^a_{i,R}(z) = [U(z_i)T^a]\,\frac{\delta}{\delta U(z_i)}.$$

- In our analysis we need only the leading-order conformal invariant kernel K_{ij}, which has a very simple dimension-independent expression in momentum space:

$$K_{ij;0} \equiv S_\epsilon(\mu^2)\int [\text{đ}q][\text{đ}p]\, e^{iq\cdot(z_i-z_0)}e^{ip\cdot(z_j-z_0)}(-2\pi^2)\frac{(q+p)^2}{q^2p^2} = S_\epsilon(\mu^2)\frac{\Gamma(1-\epsilon)^2}{\pi^{-2\epsilon}}\frac{z_{0i}\cdot z_{0j}}{(z^2_{0i}z^2_{0j})^{1-\epsilon}},$$

- The corrections to the Balitsky-JIMWLK Hamiltonian are suppressed by α_s in a power-counting where the Wilson lines are generic, $U \sim 1$. This is more general than the perturbative counting where $1 - U \sim g_s W \sim g_s$, implying that the equation resums infinite towers of Reggeon iterations.

THE BALITSKY-JIMWLK EQUATION

- To see this, expand U in powers of W:

$$U = e^{ig_s\, W^a T^a} = 1 + ig_s\, W^a\, T^a - \frac{g_s^2}{2} W^a W^b\, T^a T^b - i\frac{g_s^3}{6} W^a W^b W^c\, T^a T^b T^c$$
$$+ \frac{g_s^4}{24} W^a W^b W^c W^d\, T^a T^b T^c T^d + \mathcal{O}(g_s^5\, W^5).$$

- The expansion of the color generators follows by using the Backer-Campbell-Hausdorff formula. Then, it is possible to expand the leading Hamiltonian H_{ij} in powers of g_s:

$$H = H_{k\to k} + H_{k\to k+2} + \dots$$

- We get

$$H_{k\to k} = \frac{\alpha_s C_A}{2\pi^2} \int [dz_i][dz_0] K_{ii;0}\, (W_i - W_0)^a \frac{\delta}{\delta W_i^a}$$
$$- \frac{\alpha_s}{2\pi^2} \int [dz_i][dz_j][dz_0] K_{ij;0} (W_i - W_0)^x (W_j - W_0)^y (F^x F^y)^{ab} \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

- The first non-linear correction is new:

$$H_{k\to k+2} = \frac{\alpha_s^2}{3\pi} \int [dz_i][dz_0]\, K_{ii;0}\, (W_i - W_0)^x W_0^y (W_i - W_0)^z\, \mathrm{Tr}\big[F^x F^y F^z F^a\big] \frac{\delta}{\delta W_i^a}$$
$$+ \frac{\alpha_s^2}{6\pi} \int [dz_i][dz_j][dz_0]\, K_{ij;0}\, (F^x F^y F^z F^t)^{ab} \Big[(W_i - W_0)^x W_0^y W_0^z (W_j - W_0)^t$$
$$- W_i^x (W_i - W_0)^y W_0^z (W_j - W_0)^t - (W_i - W_0)^x W_0^y (W_j - W_0)^z W_j^t\Big] \frac{\delta^2}{\delta W_i^a \delta W_j^b}.$$

Caron-Huot, Gardi, LV, 2017

THE BALITSKY-JIMWLK EQUATION

- More on the Balitsky-JIMWLK power counting ($U \sim 1$) vs the BFKL power-counting ($W \sim 1$):
- Inserting the expansion of U in terms of W in the leading-order Balitsky-JIMWLK equation, one finds that an $m \to m+k$ transition is proportional to g_s^{2l+k}. Thus for $k \geq 0$, all the leading interactions can be extracted from the leading-order equation.

$$H \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} .$$

LO BFKL kernel

From LO B-JIMWLK

Terms in NNLO B-JIMWLK - predicted by symmetry $H = H^T$

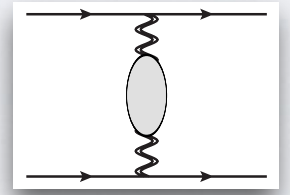
- On the other hand, interactions with $k < 0$ are suppressed by at least $g_s^{2l+|k|}$, which means that they can first appear in the ($|k|+1$)-loop Balitsky-JIMWLK Hamiltonian.
- Thus to obtain the $m \to m-2$ transition by direct calculation of the Hamiltonian would require three- loop non-planar computation.
- For our purposes this is unnecessary, since the symmetry of H predicts the result.

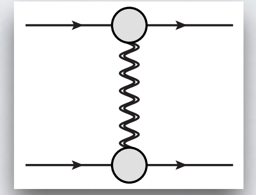
THE ODD AMPLITUDE UP TO THREE LOOPS

- We can now list the ingredients which build up the amplitude up to three loops. Since the odd and even sectors are orthogonal and closed under the action of Ĥ (signature symmetry), we have

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij}\right) \equiv \langle\psi_j^{(+)}|e^{-\hat{H}L}|\psi_i^{(+)}\rangle + \langle\psi_j^{(-)}|e^{-\hat{H}L}|\psi_i^{(-)}\rangle.$$

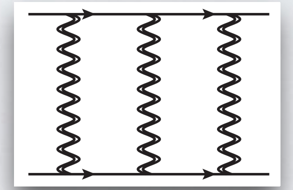
- Using that multi-Reggeon impact factors are coupling-suppressed, $|\psi_{ik}\rangle \sim g_s^k$, and using the suppression by powers of α_s of off-diagonal elements in H, the signature odd amplitude becomes to three loops:



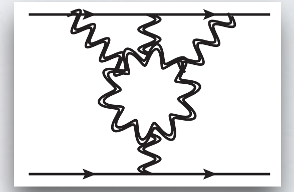


$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{tree}}_{ij\to ij} = \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{LO})},$$

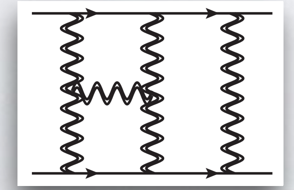
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{1-loop}}_{ij\to ij} = -L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NLO})},$$



$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{2-loops}}_{ij\to ij} = +\frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{LO})} - L\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NLO})} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{LO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{NNLO})},$$



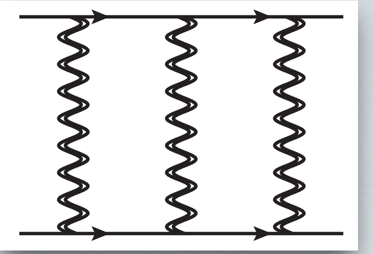
$$\frac{i}{2s}\hat{\mathcal{M}}^{(-)\,\text{3-loops}}_{ij\to ij} = -\frac{1}{6}L^3\langle\psi_{j,1}|(\hat{H}_{1\to1})^3|\psi_{i,1}\rangle^{(\text{LO})} + \frac{1}{2}L^2\langle\psi_{j,1}|(\hat{H}_{1\to1})^2|\psi_{i,1}\rangle^{(\text{NLO})} - L\Big\{\langle\psi_{j,1}|\hat{H}_{1\to1}|\psi_{i,1}\rangle^{(\text{NNLO})} + \Big[\langle\psi_{j,3}|\hat{H}_{3\to3}|\psi_{i,3}\rangle + \langle\psi_{j,3}|\hat{H}_{1\to3}|\psi_{i,1}\rangle + \langle\psi_{j,1}|\hat{H}_{3\to1}|\psi_{i,3}\rangle\Big]^{(\text{LO})}\Big\} + \langle\psi_{j,3}|\psi_{i,3}\rangle^{(\text{NLO})} + \langle\psi_{j,1}|\psi_{i,1}\rangle^{(\text{N}^3\text{LO})}.$$



RESULT: THE ODD AMPLITUDE AT NNLL TO THREE LOOPS

- Up to two loops the amplitude reads

Three-Reggeon cut



$$\hat{\mathcal{M}}^{(-,1)}_{ij\to ij} = \left(D_i^{(1)} + D_j^{(1)}\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

$$\hat{\mathcal{M}}^{(-,2)}_{ij\to ij} = \left[D_i^{(2)} + D_j^{(2)} + D_i^{(1)}D_j^{(1)} + \pi^2 R^{(2)}\left((\mathbf{T}^2_{s-u})^2 - \tfrac{1}{12}(C_A)^2\right)\right]\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

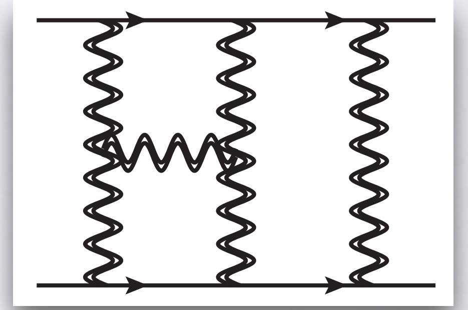
- with

$$R^{(2)} \equiv -\frac{1}{24}(r_\Gamma)^2\,\mathcal{I}[1] = -\frac{(r_\Gamma)^2}{6\epsilon^2}\frac{B_{1,1+\epsilon}(\epsilon)}{B_{1,1}(\epsilon)} = (r_\Gamma)^2\left(-\frac{1}{8\epsilon^2} + \frac{3}{4}\epsilon\zeta_3 + \frac{9}{8}\epsilon^2\zeta_4 + \ldots\right),$$

- At three loops we find the following amplitude:

$$\hat{\mathcal{M}}^{(-,3,1)}_{ij\to ij} = \pi^2\left(R_A^{(3)}\,\mathbf{T}^2_{s-u}[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}] + R_B^{(3)}\,[\mathbf{T}^2_t, \mathbf{T}^2_{s-u}]\mathbf{T}^2_{s-u} + R_C^{(3)}\,(C_A)^3\right)\hat{\mathcal{M}}^{(0)}_{ij\to ij},$$

- where the loop functions $R_{A,B,C}$ are



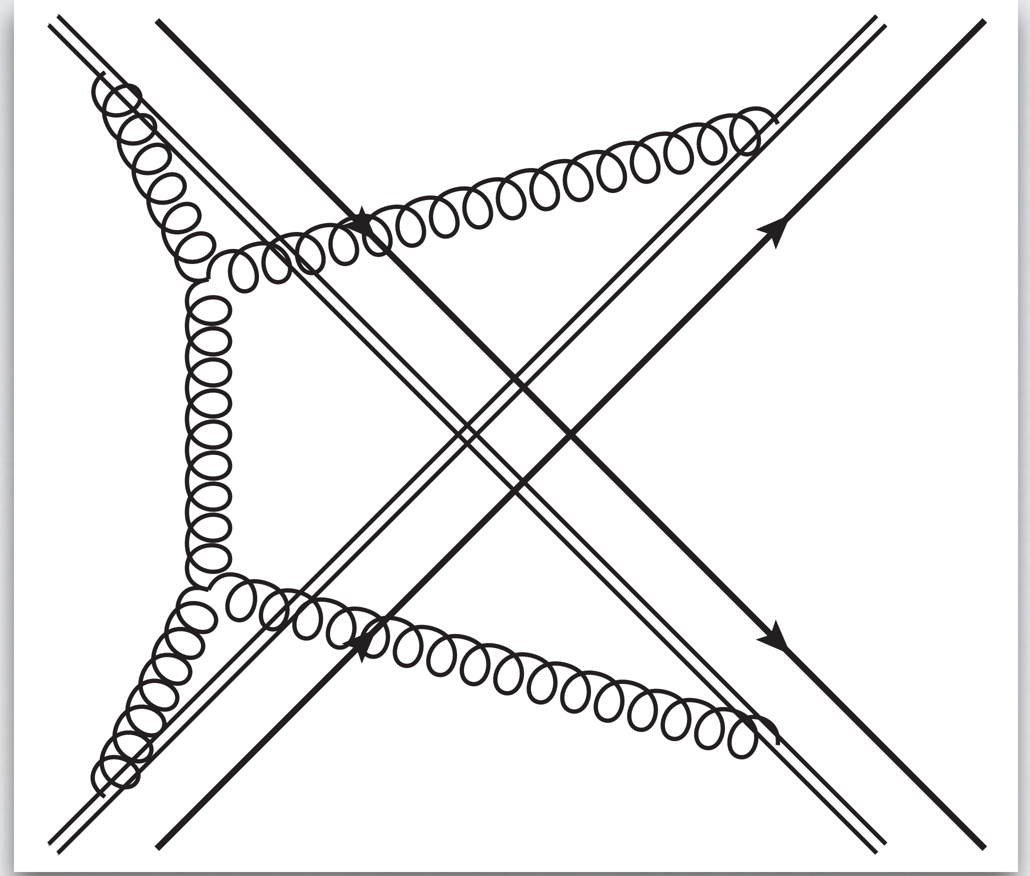
$$R_A^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_a - \mathcal{I}_c) = (r_\Gamma)^3\left(\frac{1}{48\epsilon^3} + \frac{37}{24}\zeta_3 + \ldots\right),$$

$$R_B^{(3)} = \frac{1}{16}(r_\Gamma)^3(\mathcal{I}_c - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{24\epsilon^3} + \frac{1}{12}\zeta_3 + \ldots\right),$$

$$R_C^{(3)} = \frac{1}{288}(r_\Gamma)^3(2\mathcal{I}_c - \mathcal{I}_a - \mathcal{I}_b) = (r_\Gamma)^3\left(\frac{1}{864\epsilon^3} - \frac{35}{432}\zeta_3 + \ldots\right).$$

Caron-Huot, Gardi, LV, 2017

COMPARISON BETWEEN REGGE AND INFRARED FACTORIZATION



BFKL VS INFRARED FACTORISATION

- The calculation of the amplitude so far is based solely on evolution equations of the Regge limit, and has taken no input from the theory of infrared divergences.
- This gives a highly nontrivial consistency test: the prediction must be consistent with the known exponentiation pattern and the anomalous dimensions governing infrared divergences.
- Conversely, the prediction for the reduced amplitude gives a constraint on the soft anomalous dimension, which helps in determining it beyond three loops. → Almelid, Duhr, Gardi, McLeod, White, 2017
- The infrared divergences of scattering amplitudes are controlled by a renormalization group equation, whose integrated version takes the form Becher, Neubert, 2009; Gardi, Magnea, 2009

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

- where $\mathbf{Z}$ is given as a path-ordered exponential of the soft-anomalous dimension:

$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\rm dip.}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012, …

BFKL VS INFRARED FACTORISATION

- Γ^{dip}_n involves only only pairwise interactions amongst the hard partons, and is therefore referred to as the "dipole formula":

$$\mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = -\frac{\gamma_K(\alpha_s)}{2}\sum_{i<j}\log\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_i \gamma_i(\alpha_s)\,.$$

- The term $\Delta_n(\rho_{ijkl})$ involves interactions of up to four partons, and is called the "quadrupole correction":

$$\mathbf{\Delta}_n(\{\rho_{ijkl}\}) = \sum_{i=3}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^i \mathbf{\Delta}_n^{(i)}(\{\rho_{ijkl}\}).$$

- The three loop correction has been calculated recently, and reads

$$\begin{aligned}\mathbf{\Delta}_n^{(3)}(\{\rho_{ijkl}\}) = \frac{1}{4}f^{abe}f^{cde}\sum_{1\le i<j<k<l\le n}\Big[&\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ikjl},\rho_{iljk})\\ &+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj})\Big]\\ &-\frac{C}{4}f^{abe}f^{cde}\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n,\\ j,k\neq i}}\{\mathbf{T}_i^a,\mathbf{T}_i^d\}\mathbf{T}_j^b\mathbf{T}_k^c,\end{aligned}$$

Almelid, Duhr, Gardi, 2015, 2016

- where $\mathcal{F}$ is a function of cross ratios: $\rho_{ijkl} = (-s_{ij})(-s_{kl})/(-s_{ik})(-s_{jl})$. Explicitly, one has

$$\mathcal{F}(\rho_{ikjl},\rho_{ilkj}) = F(1-z_{ijkl}) - F(z_{ijkl}),\quad \text{with}\quad F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\Big(\mathcal{L}_{001}(z)+\mathcal{L}_{100}(z)\Big),$$

- where the $\mathscr{L}$ are Brown's single-valued harmonic polylogarithms, and the constant term reads

$$C = \zeta_5 + 2\zeta_2\zeta_3.$$

BFKL VS INFRARED FACTORISATION

- In the high-energy limit the dipole formula reduces to

Del Duca, Duhr, Gardi, Magnea, White, 2011

$$\Gamma^{\rm dip.}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) \xrightarrow{\text{Regge}} \frac{\gamma_K(\alpha_s)}{2}\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2 + \frac{C_{\rm tot}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$

- and the quadrupole correction reads:

$$\mathbf{\Delta}^{(3)} = i\pi\,[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\frac{1}{4}\left[\zeta_3 L + 11\zeta_4\right] + \frac{1}{4}[\mathbf{T}_{s-u}^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\left[\zeta_5 - 4\zeta_2\zeta_3\right]$$
$$-\frac{\zeta_5+2\zeta_2\zeta_3}{8}\Big\{f^{abe}f^{cde}\Big[\{\mathbf{T}_t^a,\mathbf{T}_t^d\}\Big(\{\mathbf{T}_{s-u}^b,\mathbf{T}_{s-u}^c\}+\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\Big)$$
$$+\{\mathbf{T}_{s-u}^a,\mathbf{T}_{s-u}^d\}\{\mathbf{T}_{s+u}^b,\mathbf{T}_{s+u}^c\}\Big] - \frac{5}{8}C_A^2\mathbf{T}_t^2\Big\},$$

Only NNLL term

- where

$$\mathbf{T}_{s-u}^a \equiv \frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a - \mathbf{T}_u^a\right), \qquad \mathbf{T}_{s+u}^a \equiv \frac{1}{\sqrt{2}}\left(\mathbf{T}_s^a + \mathbf{T}_u^a\right).$$

Caron-Huot, Gardi, LV, 2017

- Because of the form of $\boldsymbol{\Gamma}^{\rm dip}$ and $\boldsymbol{\Delta}(\rho_{ijkl})$ in the High-energy limit, the **Z** factor factorises

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) Z_i\left(t,\mu,\alpha_s(\mu^2)\right) Z_j\left(t,\mu,\alpha_s(\mu^2)\right),$$

- where the relevant bit for us is

$$\tilde{\mathbf{Z}}\left(\frac{s}{t},\mu,\alpha_s(\mu^2)\right) = \exp\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\mathbf{\Delta}}^{(3)}\right\}$$

- The factors K and Q_{Δ} involve integrals over the scale:

$$K = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\alpha_s(\lambda^2)\right) = \frac{1}{2\epsilon}\frac{\alpha_s(\mu^2)}{\pi} + \ldots, \quad Q_{\mathbf{\Delta}}^{(3)} = -\frac{\mathbf{\Delta}^{(3)}}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^3 = \frac{\mathbf{\Delta}^{(3)}}{6\epsilon}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^3.$$

BFKL VS INFRARED FACTORISATION

- The scalar factors $Z_{i,j}$ are the same as those we removed from the reduced amplitude in the BFKL context, and at LL accuracy the exponent in $\tilde{\mathbf{Z}}$ is also very similar to the gluon Regge trajectory subtracted in the reduced amplitude. This makes the relation between the "infrared-renormalized" amplitude (hard function) H and reduced matrix element particularly simple:

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \exp^{-1}\left\{K\left(\alpha_s(\mu^2)\right)\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2\right] + Q_{\mathbf{\Delta}}^{(3)}\right\}$$
$$\cdot\,\exp\left\{\alpha_g(t) L\,\mathbf{T}_t^2\right\}\hat{\mathcal{M}}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right).$$

- This equation allows us to pass from directly from the reduced amplitude predicted using BFKL theory, to the hard function.
- In particular, the statement that the left-hand-side H is finite, which is equivalent to the exponentiation of infrared divergences, is a highly nontrivial constraint on our result.
- By using Baker-Campbell-Hausdorff formula one gets

$$\mathcal{H}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \Bigg(1 + \tfrac{K^3(\alpha_s)}{3!}\Big(2\pi^2\,L\,[\mathbf{T}_{s-u}^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]] - i\,\pi L^2\,[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]\Big)$$
$$+\,i\pi\tfrac{K^2(\alpha_s)}{2}L\,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2] - Q_{\mathbf{\Delta}}^{(3)}\Bigg)\cdot\exp\left\{-\,i\pi\,K(\alpha_s)\,\mathbf{T}_{s-u}^2\right\}$$
$$\cdot\,\exp\left\{\Big(\alpha_g(t) - K(\alpha_s)\Big)\,L\,\mathbf{T}_t^2\right\}\hat{\mathcal{M}}_{ij\to ij}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right).$$

BFKL VS INFRARED FACTORISATION

- Some coefficients, like the impact factors, are not predicted explicitly from Regge theory: in that case, we can use these equations in the reverse direction.
- The BFLK approach we have developed allows us to extract these quantities consistently, and use them to predict higher orders. Consider for instance the impact factors at two loops:

$$\mathrm{Re}[\mathcal{H}^{(2,0)}] = \Big[D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)} - \pi^2 R^{(2)} \tfrac{1}{12} (C_A)^2 + \pi^2 \Big(R^{(2)} + \tfrac{1}{2} (K^{(1)})^2 + K^{(1)} \hat{\alpha}_g^{(1)} \Big) (\mathbf{T}_{s-u}^2)^2 \Big] \hat{\mathcal{M}}^{(0)}.$$

Del Duca, Glover, 2001; Del Duca, Falcioni, Magnea, LV, 2013

- In our framework the impact factors at two loops can be extracted consistently by taking the projection of the amplitude onto the antisymmetric octet component:

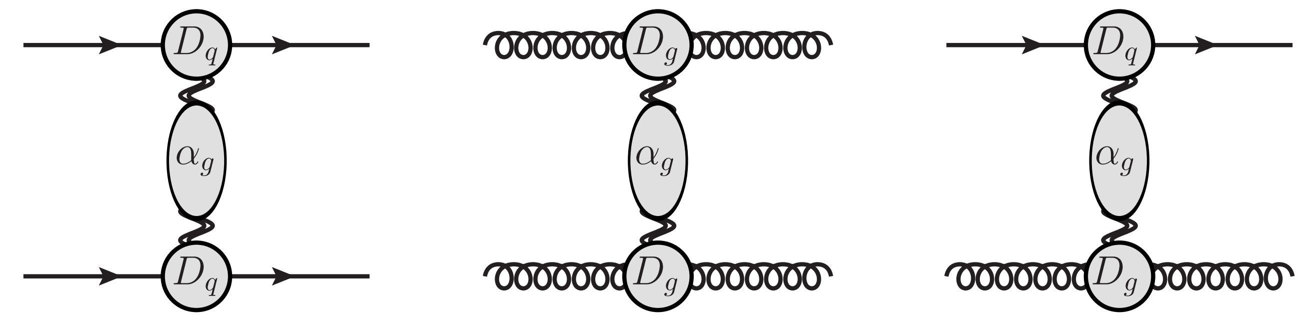
$$2D_g^{(2)} = \frac{\mathcal{H}_{gg\to gg}^{(2,0)[8_a]}}{\mathcal{H}_{gg\to gg}^{(0)[8_a]}} - (D_g^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2 + 24}{4},$$

$$D_q^{(2)} + D_g^{(2)} = \frac{\mathcal{H}_{qg\to qg}^{(2,0)[8_a]}}{\mathcal{H}_{qg\to qg}^{(0)[8_a]}} - D_q^{(1)} D_g^{(1)} + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^2 + 4}{4},$$

$$2D_q^{(2)} = \frac{\mathrm{Re}[\mathcal{H}_{qq\to qq}^{(2,0)[8_a]}]}{\mathcal{H}_{qq\to qq}^{(0)[8_a]}} - (D_q^{(1)})^2 + \pi^2 R^{(2)} \frac{N_c^2}{12} - \pi^2 \hat{R}^{(2)} \frac{N_c^4 - 4N_c^2 + 12}{4N_c^2}.$$

Caron-Huot, Gardi, LV, 2017

- The effect of the three-Reggeon cut is evident from the color-dependent term in the equations above. Once again, consistency requires the three equations above to be satisfied simultaneously.



BFKL VS INFRARED FACTORISATION

- At three loops, at NNLL, the calculation of the odd sector within Regge theory gives

$$\begin{aligned}\mathrm{Re}[\mathcal{H}^{(3,1)}] = &\Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)}\Big)\Big] \mathbf{T}_t^2\, \hat{\mathcal{M}}^{(0)} \\ &+ \pi^2 \Big[R_C^{(3)} - \tfrac{1}{12}\hat{\alpha}_g^{(1)} R^{(2)}\Big] (\mathbf{T}_t^2)^3\, \hat{\mathcal{M}}^{(0)} + \pi^2\, \hat{\alpha}_g^{(1)}\, \hat{R}^{(2)}\, \mathbf{T}_t^2 (\mathbf{T}_{s-u}^2)^2\, \hat{\mathcal{M}}^{(0)} \\ &+ \pi^2 \Big[R_A^{(3)} + \tfrac{1}{6} K^{(1)} \Big(2(K^{(1)})^2 + 3\hat{\alpha}_g^{(1)} K^{(1)} + 3\mathrm{d}_2\Big)\Big] \mathbf{T}_{s-u}^2 [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\, \hat{\mathcal{M}}^{(0)} \\ &+ \pi^2 \Big[R_B^{(3)} - \tfrac{1}{3} K^{(1)} \Big((K^{(1)})^2 + 3\hat{\alpha}_g^{(1)} K^{(1)} + 3(\hat{\alpha}_g^{(1)})^2\Big)\Big] [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \mathbf{T}_{s-u}^2\, \hat{\mathcal{M}}^{(0)}.\end{aligned}$$

must be finite

- Which is consistent with infrared factorisation. This is a rather non-trivial check, given that the two calculations are done in two completely different ways.
- We get also some parts of the finite amplitude. In the orthonormal basis in the t-channel we have

$$\begin{aligned}\mathrm{Re}[\mathcal{H}^{(3,1),[8_a]}] = \Big\{ &C_A \Big[\hat{\alpha}_g^{(3)} + \hat{\alpha}_g^{(2)}\Big(D_i^{(1)} + D_j^{(1)}\Big) + \hat{\alpha}_g^{(1)}\Big(D_i^{(2)} + D_j^{(2)} + D_i^{(1)} D_j^{(1)}\Big)\Big] \\ &+ C_A^3 \frac{\pi^2}{864}\Big(\frac{1}{\epsilon^3} - \frac{15\zeta_2}{4\epsilon} - \frac{175\zeta_3}{2}\Big) - C_A \pi^2 \frac{2\zeta_3}{3} + \mathcal{O}(\epsilon)\Big\} \hat{\mathcal{M}}^{(0),[8_a]},\end{aligned}$$

$$\mathrm{Re}[\mathcal{H}^{(3,1),[10+\overline{10}]}] = \sqrt{2} C_A \sqrt{C_A^2 - 4} \Big\{ \frac{11\pi^2\zeta_3}{24} + \mathcal{O}(\epsilon)\Big\} \hat{\mathcal{M}}^{(0),[8_a]}.$$

Caron-Huot, Gardi, LV, 2017

- The antisymmetric octet amplitude cannot be predicted entirely, given the unknown Regge trajectory at three loops; The $10 + \overline{10}$ component, however, can be predicted exactly, and it agrees with a recent calculation of the gluon-gluon scattering amplitude at three loops in N=4 SYM. Henn, Mistlberger, 2016

THE REGGE TRAJECTORY AT THREE LOOPS IN N=4 SYM

- Consider the relation between the three-loop "gluon Regge trajectory" and the single logarithmic term.
- Starting from three loops the "gluon Regge trajectory" is scheme-dependent. Here we defined it to be the $1 \to 1$ matrix element of the Hamiltonian, $\alpha_g(t) = -H_{1\to1}/C_A$, in the scheme where states corresponding to a different number of Reggeon are orthogonal:

$$\log \frac{\mathcal{M}^{[8_a]}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}} = L\left\{ -H_{1\to1}(t) + \left(\frac{\alpha_s}{\pi}\right)^3 \pi^2 \left[N_c\left(-2R_A^{(3)} + 2R_B^{(3)}\right) + N_c^3 R_C^{(3)}\right]\right\} + \mathcal{O}(L^0, \alpha_s^4),$$

- Thanks to a recent calculation of the gluon-gluon amplitude in N=4 SYM, in this theory one has

$$\log \frac{\mathcal{M}^{[8_a],\mathcal{N}=4}_{gg\to gg}}{\mathcal{M}^{(0)[8_a]}_{gg\to gg}}\Bigg|_L = N_c\left[\frac{\alpha_s}{\pi}k_1 + \left(\frac{\alpha_s}{\pi}\right)^2 k_2 + \left(\frac{\alpha_s}{\pi}\right)^3 k_3 + \cdots\right],$$

Henn, Mistlberger, 2016

- Define the Regge trajectory as

$$-H_{1\to1}^{\mathcal{N}=4\,\mathrm{SYM}} = N_c\left[\frac{\alpha_s}{\pi}\alpha_g^{(1)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \left(\frac{\alpha_s}{\pi}\right)^2 \alpha_g^{(2)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \left(\frac{\alpha_s}{\pi}\right)^3 \alpha_g^{(3)}|_{\mathcal{N}=4\,\mathrm{SYM}} + \cdots\right],$$

- Then, matching these two results we get

$$\alpha_g^{(1)}|_{\mathcal{N}=4} = k_1 = \frac{1}{2\epsilon} - \epsilon\frac{\zeta_2}{4} - \epsilon^2\frac{7}{6}\zeta_3 - \epsilon^3\frac{47}{32}\zeta_4 + \epsilon^4\left(\frac{7}{12}\zeta_2\zeta_3 - \frac{31}{10}\zeta_5\right) + \mathcal{O}(\epsilon^5),$$

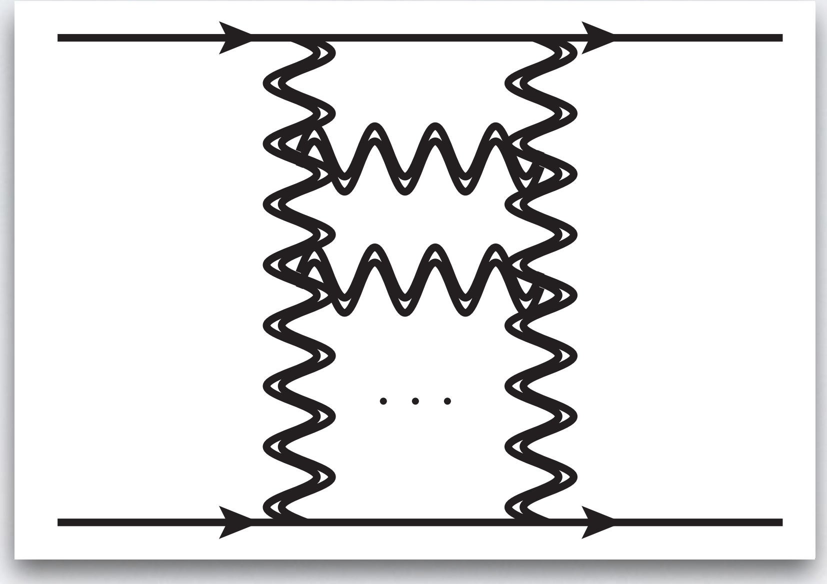
$$\alpha_g^{(2)}|_{\mathcal{N}=4} = k_2 = N_c\left[-\frac{\zeta_2}{8}\frac{1}{\epsilon} - \frac{\zeta_3}{8} - \epsilon\frac{3}{16}\zeta_4 + \epsilon^2\left(\frac{71}{24}\zeta_2\zeta_3 + \frac{41}{8}\zeta_5\right) + \mathcal{O}(\epsilon^3)\right],$$

$$\begin{aligned}\alpha_g^{(3)}|_{\mathcal{N}=4} &= k_3 - \pi^2\left[N_c\left(-2R_A^{(3)} + 2R_B^{(3)}\right) + N_c^3 R_C^{(3)}\right] \\ &= N_c^2\left[-\frac{\zeta_2}{144}\frac{1}{\epsilon^3} + \frac{49\zeta_4}{192}\frac{1}{\epsilon} + \frac{107}{144}\zeta_2\zeta_3 + \frac{\zeta_5}{4} + \mathcal{O}(\epsilon)\right] + N_c^0\left[0 + \mathcal{O}(\epsilon)\right].\end{aligned}$$

Caron-Huot, Gardi, LV, 2017

- The amplitude is really a sum of multiple powers. Simply exponentiating the log of the full amplitude at three loops predicts an incorrect four-loop amplitude. The correct, predictive, procedure is to exponentiate the BFKL Hamiltonian. With the "trajectory" fixed as above, this procedure does not require any new parameter for the odd amplitude at NNLL to all loop orders.

THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

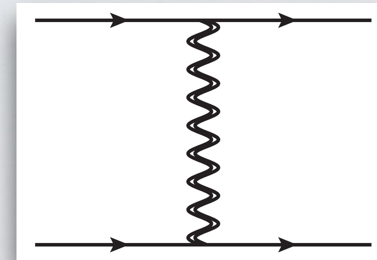


THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

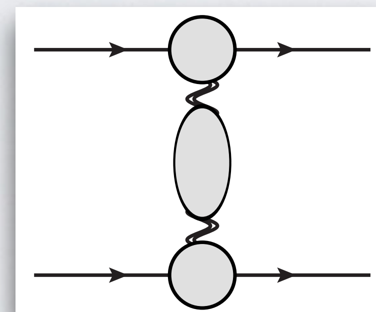
- Consider now the two Reggeon cut at NLL

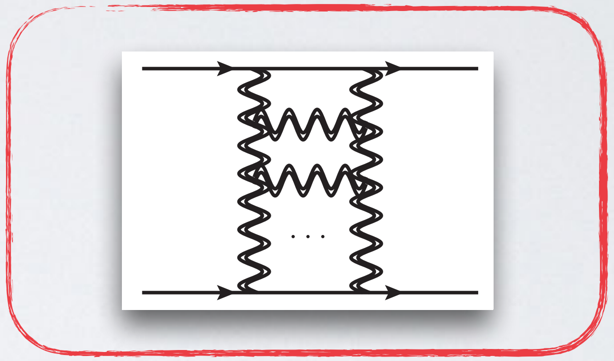
Odd $(\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]})$ | Even $(\mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]})$

LL

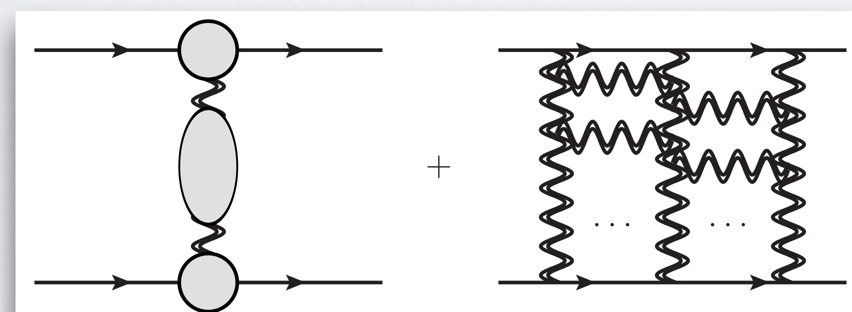


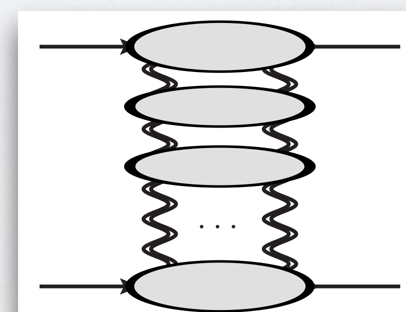
NLL





NNLL





- The amplitude reads

$$\frac{i}{2s}\hat{\mathcal{M}}^{\rm NLL}_{ij\to ij} \xrightarrow{\rm Regge} \frac{i}{2s}\left(\hat{\mathcal{M}}^{(+),\rm NLL}_{ij\to ij} + \hat{\mathcal{M}}^{(-),\rm NLL}_{ij\to ij}\right) \equiv \langle\psi^{(+)}_{j,2}|e^{-\hat{H}L}|\psi^{(+)}_{i,2}\rangle^{\rm (LO)} + \langle\psi^{(-)}_{j,1}|e^{-\hat{H}L}|\psi^{(-)}_{i,1}\rangle^{\rm (NLO)},$$

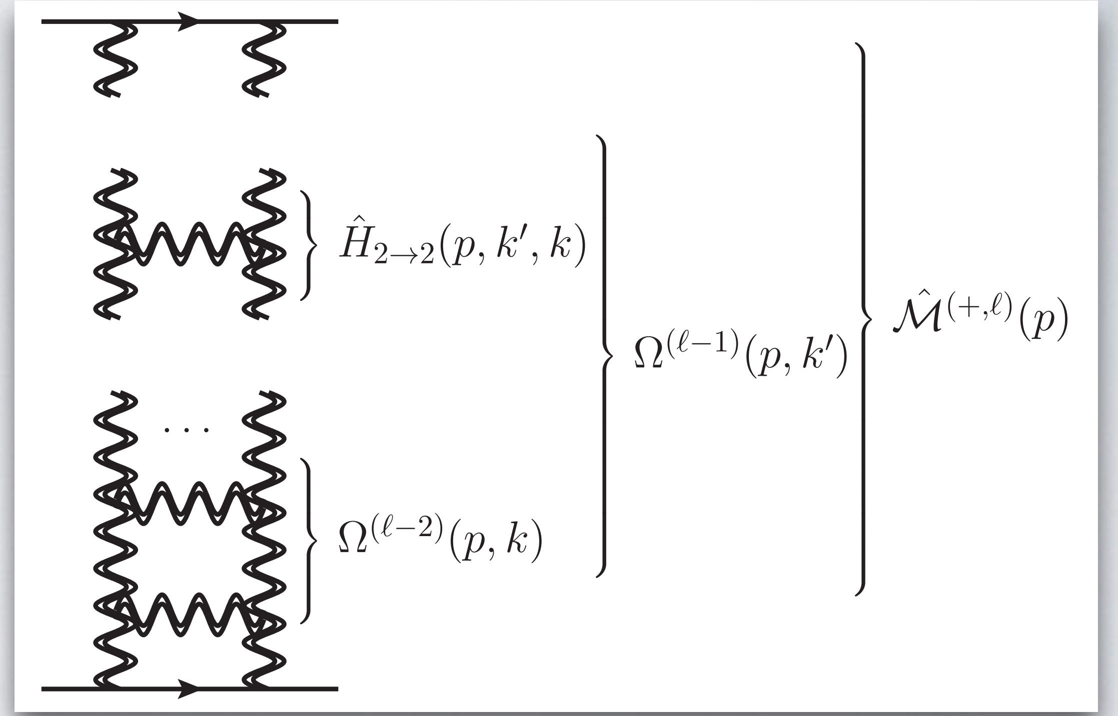
- and the even amplitude at NLL is given by

$$\frac{i}{2s}\hat{\mathcal{M}}^{(+,\ell,\ell-1)} \equiv \frac{i}{2s}\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = \frac{L^{\ell-1}}{(\ell-1)!}\langle\psi_2^{(+)}|\left(-\hat{H}_{2\to 2}\right)^{\ell-1}|\psi_2^{(+)}\rangle^{\rm (LO)}.$$

THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

- The even amplitude reads

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k] \, \frac{p^2}{k^2(k-p)^2} \, \Omega^{(\ell-1)}(p,k) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$



- with

$$[{\rm D}k] \equiv \frac{\pi}{B_0} \left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^\epsilon \frac{{\rm d}^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}},$$

- and

$$B_0 = r_\Gamma = e^{\epsilon\gamma_{\rm E}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- and the "target averaged wave function" reads

$$\Omega^{(\ell-1)}(p,k) = (2C_A - \mathbf{T}_t^2)\, \Psi^{(\ell-1)}(p,k) + (C_A - \mathbf{T}_t^2)\, \Phi^{(\ell-1)}(p,k),$$

- with

$$\Psi^{(\ell-1)}(p,k) = \int [{\rm D}k'] \, f(p,k,k') \left[\Omega^{(\ell-2)}(p,k') - \Omega^{(\ell-2)}(p,k)\right], \qquad \Phi^{(\ell-1)}(p,k) = \frac{1-J(p,k)}{2\epsilon} \Omega^{(\ell-2)}(p,k),$$

- and the initial condition is fixed to

$$\Omega^{(0)}(p,k) = 1.$$

- Furthermore, the function f is given by the BFKL kernel

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2}, \qquad J(p,k) = -2\epsilon \int [{\rm D}k'] \, f(p,k,k').$$

THE BALITSKY-JIMWLK EQUATION AND THE TWO REGGEON CUT

- Up to four loops one gets

$$\hat{\mathcal{M}}^{(+,1)}_{\rm NLL} = -i\pi \, \frac{B_0}{2\epsilon} \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

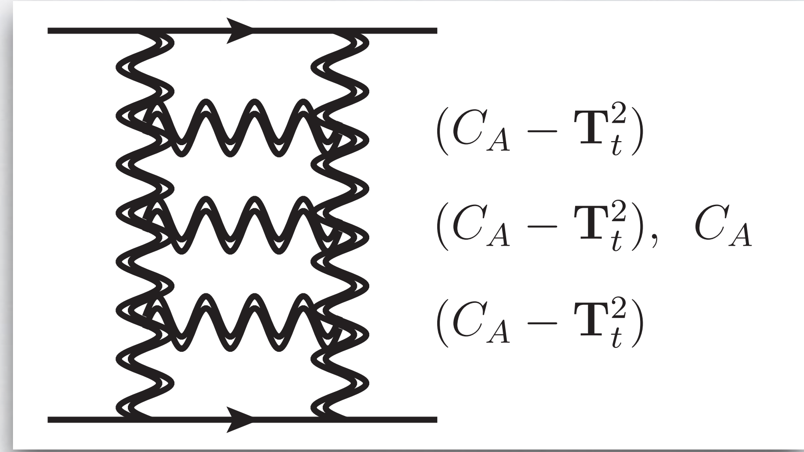
$$\hat{\mathcal{M}}^{(+,2)}_{\rm NLL} = i\pi \, \frac{(B_0)^2}{2} \left[\frac{1}{(2\epsilon)^2} + \frac{9\zeta_3}{2}\epsilon + \frac{27\zeta_4}{4}\epsilon^2 + \frac{63\zeta_5}{2}\epsilon^3 + \mathcal{O}(\epsilon^4) \right] (C_A - \mathbf{T}^2_t) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,3)}_{\rm NLL} = i\pi \, \frac{(B_0)^3}{3!} \left[\frac{1}{(2\epsilon)^3} - \frac{11\zeta_3}{4} - \frac{33\zeta_4}{8}\epsilon - \frac{357\zeta_5}{4}\epsilon^2 + \mathcal{O}(\epsilon^3) \right] (C_A - \mathbf{T}^2_t)^2 \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \, \frac{(B_0)^4}{4!} \Big\{ (C_A - \mathbf{T}^2_t)^3 \left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2) \right) + C_A (C_A - \mathbf{T}^2_t)^2 \left(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2) \right) \Big\} \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)}.$$

Caron-Huot, 2013

- At four loop a new color structure appear, with a single pole not predicted by the dipole formula of infrared divergences!
- The fact that it arises only at four loops is a consequence of the "top-bottom" symmetry of the ladder. The new color structure appears in the target-averaged wave function already at three loops, but it cancels out due to this symmetry.



TWO REGGEON CUT: SOFT APPROXIMATION

- It would be possible to calculate few order higher in perturbation theory; the problem becomes rapidly quite involved.
- However, this is not necessary, if we are interested to know only the infrared singularities. Reconsider the wave function:

$$\Omega^{(\ell-1)}(p,k) = (2C_A - \mathbf{T}_t^2)\,\Psi^{(\ell-1)}(p,k) + (C_A - \mathbf{T}_t^2)\,\Phi^{(\ell-1)}(p,k),$$

- with

$$\Psi^{(\ell-1)}(p,k) = \int [\mathrm{D}k']\, f(p,k,k') \left[\Omega^{(\ell-2)}(p,k') - \Omega^{(\ell-2)}(p,k)\right], \qquad \Phi^{(\ell-1)}(p,k) = \frac{1 - J(p,k)}{2\epsilon}\Omega^{(\ell-2)}(p,k),$$

- where

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2},$$

$$J(p,k) = \left(\frac{p^2}{k^2}\right)^{\epsilon} + \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon} - 1.$$

finite!

- The wave function is actually finite. All divergences must arise from the last integration!

$$\hat{\mathcal{M}}^{(+,\ell)}_{\mathrm{NLL}} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [\mathrm{D}k]\, \frac{p^2}{k^2(k-p)^2}\, \Omega^{(\ell-1)}(p,k)\, \mathbf{T}^2_{s-u}\, \mathcal{M}^{(0)},$$

- We see that divergences arises only from the limit $k \to p$ or $k \to 0$ limit. Consider one of the two regions, and multiply the result by two.

TWO REGGEON CUT: SOFT APPROXIMATION

- In the soft limit the integrations becomes trivial ("bubble" type integrals), and we are able to obtain an all-order solution for the target-averaged wave function:

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p^2}{k^2}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right\},$$

- where

$$\hat{B}_n(\epsilon) = \frac{B_n(\epsilon)}{B_0(\epsilon)} - 1, \quad \text{and} \quad B_n(\epsilon) = e^{\epsilon\gamma_{\rm E}} \frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)} \frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}.$$

- It is immediate to get a result for the reduced amplitude:

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} \left(1 + \hat{B}_{-1}\right) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n} \times \prod_{m=0}^{n-2} \left[1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right] \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

- This result is valid only up to the single poles. Taking this into account, it is possible to achieve a tremendous simplification:

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} \left(1 - R(\epsilon) \frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} (C_A - \mathbf{T}_t^2)^{\ell-1} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

- where

Caron-Huot, Gardi, Reichel, LV, preliminar

$$R(\epsilon) \equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)} - 1 = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\epsilon^3 - 3\zeta_4\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO REGGEON CUT: SOFT APPROXIMATION

- Expand for a few orders in the strong coupling constant:

$$\hat{\mathcal{M}}^{(+,\ell=1,2,3)}_{\rm NLL}|_s = i\pi \, \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} (C_A - \mathbf{T}_t^2)^{\ell-1} \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}^{(+,\ell=4,5,6)}_{\rm NLL}|_s = i\pi \, \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon) C_A (C_A - \mathbf{T}_t^2)^{\ell-2} \Big\} \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}^{(+,\ell=7,8,9)}_{\rm NLL}|_s = i\pi \, \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon) C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon) C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3} \Big\} \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

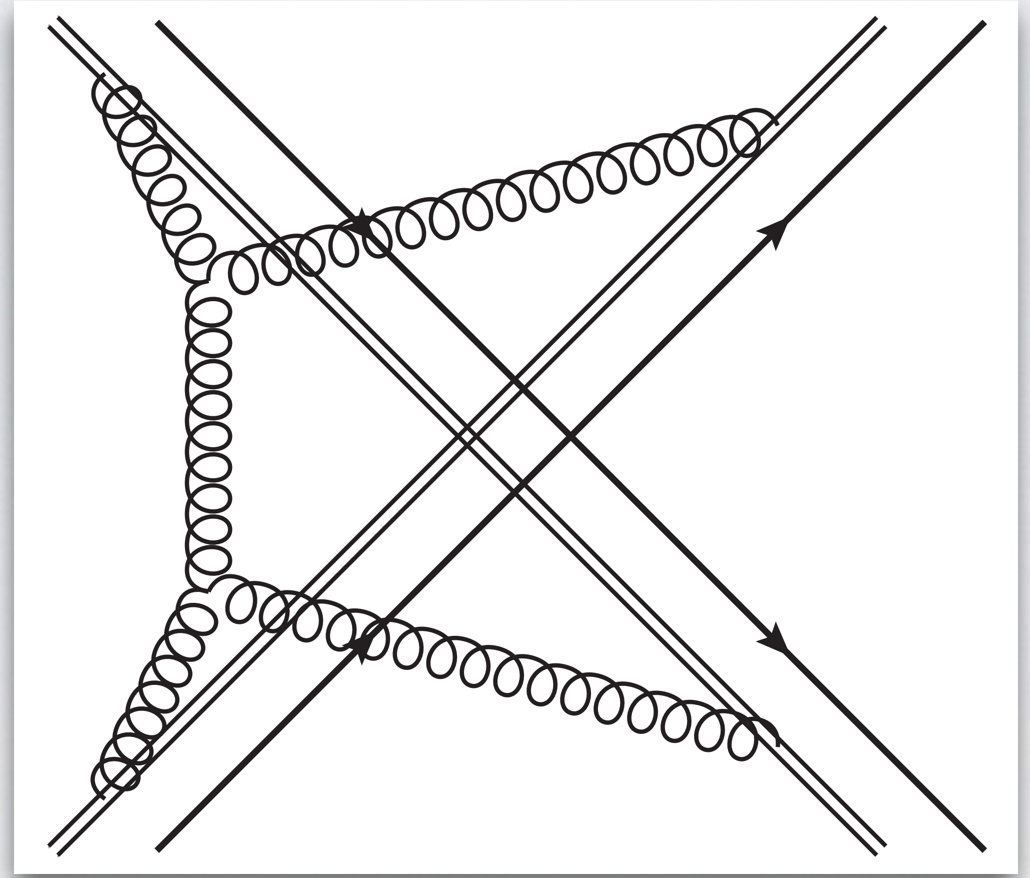
$$\hat{\mathcal{M}}^{(+,\ell=10,11,12)}_{\rm NLL}|_s = i\pi \, \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \Big\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon) C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon) C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3} + R^3(\epsilon) C_A^3 (C_A - \mathbf{T}_t^2)^{\ell-4} \Big\} \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

- A new color structure appears every three loops!
- Resumming the amplitude to all loops we get

Caron-Huot, Gardi, Reichel, LV, preliminar

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}|_s = 4\pi\alpha_s \, \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \left(1 - R(\epsilon) \frac{C_A}{C_A - \mathbf{T}_t^2} \right)^{-1} \left[\exp\left\{ \frac{B_0(\epsilon)}{2\epsilon} \frac{\alpha_s}{\pi} L (C_A - \mathbf{T}_t^2) \right\} - 1 \right] \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

COMPARISON BETWEEN REGGE AND INFRARED FACTORIZATION



TWO REGGEON CUT: BFKL VS INFRARED FACTORISATION

- Consider the soft anomalous dimension

$$\mathbf{\Gamma}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)=\tilde{\mathbf{\Gamma}}\left(\frac{s}{t},\lambda,\alpha_s(\lambda^2)\right)+\sum_{i=1}^{4}\mathbf{\Gamma}_i\left(t,\lambda,\alpha_s(\lambda^2)\right)+\mathcal{O}\left(\frac{t}{s}\right),$$

- with

$$\tilde{\mathbf{\Gamma}}\left(\alpha_s(\lambda^2)\right)=\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NNLL}\left(\alpha_s(\lambda^2)\right)+\ldots$$

- Parameterise the soft anomalous dimension at NLL according to

$$\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)=\sum_{\ell=1}^{\infty}\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}=\sum_{\ell=1}^{\infty}\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL}\left(\frac{\alpha_s(p^2)}{\pi}\right)^{\ell}\left(\frac{p^2}{\lambda^2}\right)^{\ell\epsilon}.$$

- Within the dipole formula one has

$$\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)=\frac{\gamma_K\left(\alpha_s(\lambda^2)\right)}{2}\,L\,\mathbf{T}_t^2,\qquad\qquad \tilde{\mathbf{\Gamma}}^{(1)}_{\rm NLL}=i\pi\,\mathbf{T}_{s-u}^2,$$

- Recall now the infrared factorisation formula

$$\mathcal{M}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)=\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

- with

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)=\mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\}.$$

TWO REGGEON CUT: BFKL VS INFRARED FACTORISATION

- We get the infrared-factorised representation of the reduced amplitude:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = 4\pi\alpha_s \, \exp\left\{\frac{(B_0-1)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t^2)\right\}\exp\left\{-\frac{1}{2\epsilon}\frac{\alpha_s}{\pi}L\mathbf{T}_t^2\right\}$$
$$\times\,\mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{p^2}\frac{{\rm d}\lambda^2}{\lambda^2}\left[\tilde{\mathbf{\Gamma}}_{\rm LL}\left(\alpha_s(\lambda^2)\right)+\tilde{\mathbf{\Gamma}}_{\rm NLL}\left(\alpha_s(\lambda^2)\right)\right]\right\}\mathcal{M}^{(0)}+\mathcal{O}(\epsilon^0),$$

- and comparing with the result from the Regge theory allows us to obtain

$$\tilde{\mathbf{\Gamma}}^{(\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!}\left[\frac{\alpha_s}{\pi}\left(1-R\left(\frac{\alpha_s}{2\pi}L(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\right]_{\alpha_s^\ell}\mathbf{T}^2_{s-u}.$$

- Explicitly, for the first few orders we have:

$$\tilde{\mathbf{\Gamma}}^{(1)}_{\rm NLL} = i\pi\,\mathbf{T}^2_{s-u}, \qquad \tilde{\mathbf{\Gamma}}^{(2)}_{\rm NLL} = 0, \qquad \tilde{\mathbf{\Gamma}}^{(3)}_{\rm NLL} = 0,$$
$$\tilde{\mathbf{\Gamma}}^{(4)}_{\rm NLL} = -i\pi\,L^3\,\frac{\zeta_3}{24}\,C_A(C_A-\mathbf{T}_t^2)^2\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(5)}_{\rm NLL} = -i\pi\,L^4\,\frac{\zeta_4}{128}\,C_A(C_A-\mathbf{T}_t^2)^3\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(6)}_{\rm NLL} = -i\pi\,L^5\,\frac{\zeta_5}{640}\,C_A(C_A-\mathbf{T}_t^2)^4\,\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(7)}_{\rm NLL} = i\pi\frac{L^6}{720}\left[\frac{\zeta_3^2}{16}\,C_A^2(C_A-\mathbf{T}_t^2)^4+\frac{1}{32}\left(\zeta_3^2-5\zeta_6\right)\,C_A(C_A-\mathbf{T}_t^2)^5\right]\mathbf{T}^2_{s-u},$$
$$\tilde{\mathbf{\Gamma}}^{(8)}_{\rm NLL} = i\pi\frac{L^7}{5040}\left[\frac{3\zeta_3\zeta_4}{32}\,C_A^2(C_A-\mathbf{T}_t^2)^5+\frac{3}{64}\left(\zeta_3\zeta_4-3\zeta_7\right)\,C_A(C_A-\mathbf{T}_t^2)^6\right]\mathbf{T}^2_{s-u}.$$

Caron-Huot, Gardi, Reichel, LV, preliminar

Almelid, Duhr, Gardi, McLeod, White, 2017

- The result can be used as constraint in a bootstrap approach to the soft anomalous dimension.

CONCLUSION

- Using the non-linear Balitsky-JIMWLK rapidity evolution equation we have computed the three-Reggeon cut to three loops, at NNLL in the signature-odd sector, and the IR singular part of the two-Reggeon cut to all orders, at NLL in the signature-even sector, for $2 \to 2$ scattering amplitudes.
- Concerning the three-Reggeon cut, we have shown how to take systematically into account the effect of mixing between states with k and k+2 Reggeized gluons, due non-diagonal terms in the Balitsky-JIMWLK Hamiltonian, which contribute first at NNLL.
- Our results are consistent with a recent determination of the infrared structure of scattering amplitudes at three loops, as well as a computation of $2 \to 2$ gluon scattering in $N = 4$ super Yang-Mills theory. Combining the latter with our Regge-cut calculation we extract the three-loop Regge trajectory in this theory.
- The calculation of the infrared singular part of the two-Reggeon cut allows us to extract the soft anomalous dimension to all orders in perturbation theory, in this kinematical limit.
- The information obtained concerning infrared singularities has been/will be used to constrain the structure of the soft anomalous dimension in general kinematics. (See Almelid, Duhr, Gardi, McLeod, White, 2017).