Finite Integrals, Finite Fields, Finite Masses

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Amplitudes 2017

Higgs Centre
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Status of NNLO calculations for Large Hadron Collider

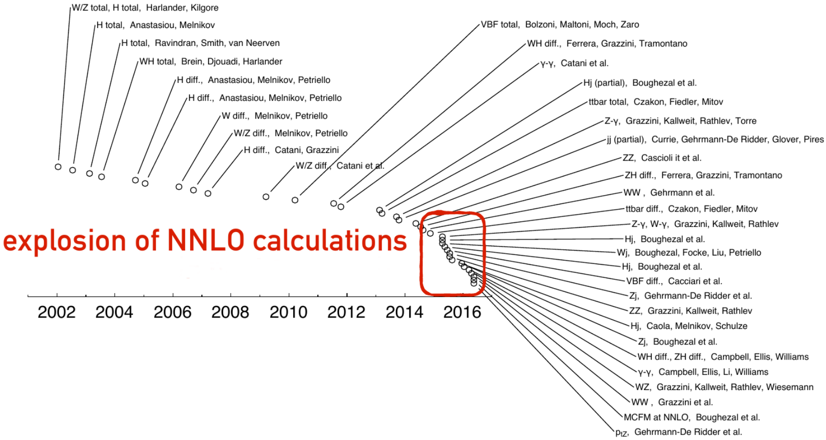
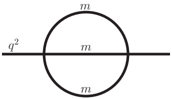


figure by Gavin P. Salam, CMS meeting 2017 this talk:

- finite integrals: *singularity resolution for direct integration*
  [with Erik Panzer, Robert Schabinger]

- finite fields: *fast reduction techniques*
  [with Robert Schabinger]

- finite masses: *beyond multiple polylogarithms*
  [with Lorenzo Tancredi]



Part I: Finite Integrals

An improved basis for Feynman parameters

consider Feynman parameter representation of multi-loop integral

$$I = N\left[\prod_{j=1}^{N}\int_0^{\infty} \mathrm{d}x_j x_j^{\nu_k-1}\right]\delta(1-x_N)\,\mathcal{U}^{\nu-(L+1)\frac{d}{2}}\,\mathcal{F}^{-\nu+L\frac{d}{2}}$$

where

- $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity
- $\mathcal{U}$ and $\mathcal{F}$ are Symanzik polynomials in x_i

problem:

- can't directly expand in $\epsilon = (4-d)/2$: divergencies from x_i integrations
- no straight-forward analytical integration [Brown '08]
- no straight-forward numerical integration

generic approaches to singularity resolution:

1. sector decomposition [Hepp '66, Binoth, Heinrich '00]
2. polynomial exponent raising [Bernstein '72, Tkachov '96, Passarino '00]
3. analytic regularisation [Panzer '14]
4. basis of finite Feynman integrals ("dims & dots") [AvM, Schabinger, Panzer '14]

Multi-Loop Feynman integrals

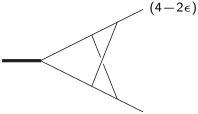
$$I = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \qquad a_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

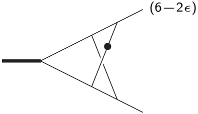
- fulfill linear relations: integration-by-parts identities
- systematic reduction to master integrals possible
- think of it as linear vector space with some finite basis
- specific basis choices:
  - canonical basis for method of differential equations [Henn '13]
  - basis of finite integrals for direct integration (analyt., numeric.): this talk

New proposal for singularity resolution [AvM, Panzer, Schabinger '14]

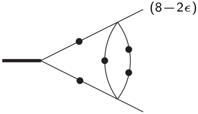
observation: always possible to decompose wrt basis of finite integrals



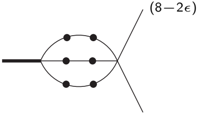
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$- \frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



$$+ \frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2 q^2}$$



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basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)

Practical algorithm for basis construction

Algorithm: construction of finite basis

- systematic scan for finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

Practical algorithm for basis construction

Algorithm: construction of finite basis

- systematic scan for finite integrals with dim-shifts and dots
- IBP + dimensional recurrence for actual basis change

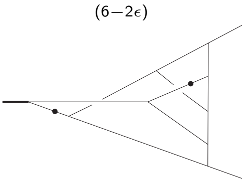
remarks:

- implementation available in `Reduze 2.1`
- computationally expensive part shifted to IBP solver
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon
- finite integrals heavily employed in $N = 4$ calculations since long time

Analytical integration @ 4-loops

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3$$

$$- 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

- only shallow ϵ expansion needed
- numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation
- starts at weight 7, not expected to contribute to cusp anomalous dimension

Numerical evaluations

advantages of (quasi-)finite basis:

- straight-forward to integrate numerically (in principle)
- no cancellation of spurious singularities
- no blow up in number of sectors
- very simple integrands also at high orders in ϵ

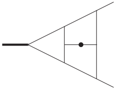
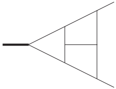
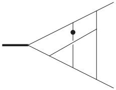
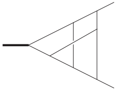
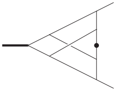
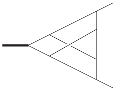
experiments with numerical evaluations:

- naive straight-forward implementation possible but not ideal
- better: employ existing sector decomposition programs
  - `Fiesta` [A. Smirnov]
  - `SecDec` [Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke]
  - `sector_decomposition` [Bogner, Weinzierl]
- finite integrals: faster & more reliable
- used for *HH* @ NLO [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke '16]

Numerical performance

[AvM, Schabinger '17]

improvement wrt conventional basis:

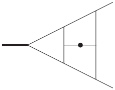
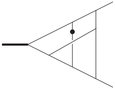
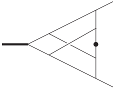
finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	$(4-2\epsilon)$ 	39094 s	9.91×10^{-4}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	$(4-2\epsilon)$ 	19025 s	9.38×10^{-5}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	$(4-2\epsilon)$ 	19586 s	1.07×10^{-4}

timings with Fiesta 4, ϵ expansion through to weight 6

Numerical performance

[AvM, Schabinger '17]

ϵ expansions to high weights feasible:

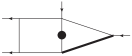
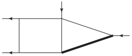
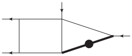
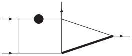
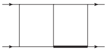
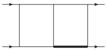
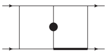
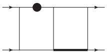
	weight 6		weight 8	
	time	rel. err.	time	rel. err.
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	491 s	2.22×10^{-5}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	761 s	5.84×10^{-6}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	485 s	8.45×10^{-6}

timings with Fiesta 4

Numerical performance

[AvM, Schabinger '17]

basis of finite integrals renders problematic double boxes numerically accessible

finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$  (s,t)	201 s	2.34×10^{-4}	$(4-2\epsilon)$  (s,t)	384 s	8.12×10^{-4}
$(6-2\epsilon)$  (s,t)	150 s	4.83×10^{-4}	$(4-2\epsilon)$  (s,t)	56538 s	1.67×10^{-2}
$(6-2\epsilon)$  (s,t)	280 s	1.00×10^{-3}	$(4-2\epsilon)$  (s,t)	214135 s	8.29×10^{-3}
$(6-2\epsilon)$  (s,t)	294 s	1.21×10^{-3}	$(4-2\epsilon)$  (s,t)	3484378 s	30.9

timings with SecDec 3 in physical region

Part II: Finite Fields

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right)$$

where p_j are external momenta, $a_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

integral reduction:

- express arbitrary integral for given problem via few basis integrals
- integration-by-parts (IBP) reductions [Chetyrkin, Tkachov '81]
- public codes: `Air` [Anastasiou], `Fire` [Smirnov], `Reduze 1` [Studerus], `Reduze 2` [AvM, Studerus], `LiteRed` [Lee], `Kira` [Maierhoefer, Usovitsch, Uwer]
- possible: exploit structure at algebra level
- here: Laporta's approach

IBP reductions from finite field samples

A novel approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

IBP reductions from finite field samples

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1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

finite field techniques:

- no intermediate expression swell by construction
- early discard of redundant and auxiliary quantities
- big potential for parallelisation

established in math literature, becomes popular in physics:

- rational reconstruction proposal: [Wang '81; Wang, Guy, Davenport '82]
- dense solver: [Kauers]
- filtering: `Ice` [Kant '13]
- tensor reduction: [Heller]
- supersymmetric integrand construction: [Bern, Carrasco, Johansson, Roiban]
- QCD integrand construction: [Peraro '16]
- symbol algebra: [Dixon, Drummond, Harrington, McLeod, Papathanasiou, Spradlin '16]

A fast univariate solver

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

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\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

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I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots[x]}} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2[x]}} & \longrightarrow & O_{\mathbb{Z}_{p_2[x]}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3[x]}} & \longrightarrow & O_{\mathbb{Z}_{p_3[x]}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p[x]} & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p[x]} & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_p,x_2} & \longrightarrow & O_{\mathbb{Z}_p,x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p,x3} & \longrightarrow & O_{\mathbb{Z}_p,x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

note: massively parallisable

First QCD form factor @ 4-loops for gluons [AvM, Schabinger '16]

$$
\begin{aligned}
\mathcal{F}_4^g|_{N_f^3} = C_F \Bigg[& -\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{32}{3}\zeta_3 - \frac{145}{9}\right) + \frac{1}{\epsilon}\left(\frac{352}{45}\zeta_2^2 + \frac{1040}{9}\zeta_3 + \frac{68}{9}\zeta_2 - \frac{10003}{54}\right) \\
& + \frac{4288}{27}\zeta_5 - 64\zeta_3\zeta_2 + \frac{2288}{27}\zeta_2^2 + \frac{24812}{27}\zeta_3 + \frac{3074}{27}\zeta_2 - \frac{508069}{324} + \mathcal{O}(\epsilon)\Bigg] \\
+ C_A \Bigg[& \frac{1}{27\epsilon^5} + \frac{5}{27\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{14}{27}\zeta_2 - \frac{55}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{586}{81}\zeta_3 - \frac{70}{27}\zeta_2 - \frac{24167}{1458}\right) \\
& + \frac{1}{\epsilon}\left(-\frac{802}{135}\zeta_2^2 - \frac{5450}{81}\zeta_3 - \frac{262}{81}\zeta_2 - \frac{465631}{2916}\right) - \frac{14474}{135}\zeta_5 + \frac{4556}{81}\zeta_3\zeta_2 \\
& - \frac{1418}{27}\zeta_2^2 - \frac{99890}{243}\zeta_3 + \frac{38489}{729}\zeta_2 - \frac{20832641}{17496} + \mathcal{O}(\epsilon)\Bigg]
\end{aligned}
$$

- gluon cusp anomalous dimension: $\Gamma_4^g|_{N_f^3} = C_A\left[\frac{64}{27}\zeta_3 - \frac{32}{81}\right]$
- respects Casimir scaling
- non-planar C_F pieces do not contribute to $\Gamma_4^g|_{N_f^3}$

First QCD form factor @ 4-loops for gluons [AvM, Schabinger '16]

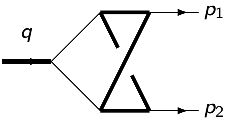
$$\mathcal{F}_4^g|_{N_f^3} = C_F \left[-\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{32}{3}\zeta_3 - \frac{145}{9}\right) + \frac{1}{\epsilon}\left(\frac{352}{45}\zeta_2^2 + \frac{1040}{9}\zeta_3 + \frac{68}{9}\zeta_2 - \frac{10003}{54}\right) + \frac{4288}{27}\zeta_5 - 64\zeta_3\zeta_2 + \frac{2288}{27}\zeta_2^2 + \frac{24812}{27}\zeta_3 + \frac{3074}{27}\zeta_2 - \frac{508069}{324} + \mathcal{O}(\epsilon) \right]$$
$$+ C_A \left[\frac{1}{27\epsilon^5} + \frac{5}{27\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{14}{27}\zeta_2 - \frac{55}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{586}{81}\zeta_3 - \frac{70}{27}\zeta_2 - \frac{24167}{1458}\right) + \frac{1}{\epsilon}\left(-\frac{802}{135}\zeta_2^2 - \frac{5450}{81}\zeta_3 - \frac{262}{81}\zeta_2 - \frac{465631}{2916}\right) - \frac{14474}{135}\zeta_5 + \frac{4556}{81}\zeta_3\zeta_2 - \frac{1418}{27}\zeta_2^2 - \frac{99890}{243}\zeta_3 + \frac{38489}{729}\zeta_2 - \frac{20832641}{17496} + \mathcal{O}(\epsilon) \right]$$

- gluon cusp anomalous dimension: $\Gamma_4^g|_{N_f^3} = C_A \left[\frac{64}{27}\zeta_3 - \frac{32}{81}\right]$
- respects Casimir scaling
- non-planar C_F pieces do not contribute to $\Gamma_4^g|_{N_f^3}$
- more four-loop results:
  - $N = 4$ form factor [Boels, Kniehl, Tarasov, Yang '12, '15; Boels, Huber, Yang '17],
  - quarks form factor in QCD: leading N_c [Henn, Smirnov, Smirnov, Steinhauser '16 + Lee '16], N_f^2 [Lee, Smirnov, Smirnov, Steinhauser '17],
  - further partial results for cusp anomalous dimensions: [Grozin, Henn, Korchemsky, Marquard '15; Ruijl, Ueda, Vermaseren, Davies, Vogt '16]

Part III: Finite Masses

A triangle beyond multiple polylogarithms

[Tancredi, AvM '17]



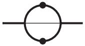
- $q^2 = s$, $p_1^2 = p_2^2 = 0$, four massive internal lines
- relevant for $t\bar{t}$, $\gamma\gamma$ production
- no massive sunrise subsector
- 2 master integrals in d dimensions at top level
- beyond multiple polylogarithms
- closely related recent work: [Tancredi, Remiddi; Adams, Bogner, Chaubey, Schweitzer, Weinzierl; Aglietti, Bonciani, Grassi, Remiddi; Bonciani, Del Duca, Frellesvig, Henn, Moriello, Smirnov; Ablinger, Blümlein, De Freitas, van Hoeij, Imamoglu, Raab, Radu, Schneider]
- see talks by [Adams, Broadhurst, Vanhove]

 :

$$m_1 = \epsilon^2\, I^{2,10}_{0,2,0,2,0,0,0}\,,$$

 :

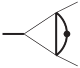
$$m_2 = \epsilon^2\, s\, I^{3,7}_{2,2,1,0,0,0,0}\,,$$

 :

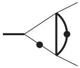
$$m_3 = \epsilon^2\, s\, I^{3,22}_{0,2,1,0,2,0,0}\,,$$

 :

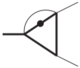
$$m_4 = \epsilon^2 \sqrt{s(s-4\,m^2)}\left[I^{3,22}_{0,2,2,0,1,0,0} + \frac{1}{2} I^{3,22}_{0,2,1,0,2,0,0}\right]$$

 :

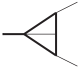
$$m_5 = \epsilon^3\, s\, I^{4,15}_{1,2,1,1,0,0,0}\,,$$

 :

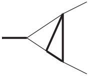
$$m_6 = \epsilon^2 \sqrt{s(s+4\,m^2)}\left[s\, I^{4,15}_{2,2,1,1,0,0,0} - \frac{\epsilon}{2\,m^2(1+2\epsilon)}\, I^{2,10}_{0,2,0,2,0,0,0}\right]$$

 :

$$m_7 = \epsilon^3\, s\, I^{4,30}_{0,2,1,1,1,0,0}\,,$$

 :

$$m_8 = \epsilon^4\, s\, I^{5,59}_{1,1,0,1,1,1,0}\,,$$

 :

$$m_9 = \epsilon^4\, s\, I^{5,31}_{1,1,1,1,1,0,0}$$

Differential equation for subsectors

$$\mathrm{d}\vec{m} = \epsilon \sum_{i=1}^{5} \mathrm{d} \ln(l_i(x))\, A_i\, \vec{m}$$

with the root-valued letters

$$l_1 = \sqrt{x}, \quad l_2 = \tfrac{1}{2}(\sqrt{x} + \sqrt{x+4}), \quad l_3 = \sqrt{x+4}, \quad l_4 = \tfrac{1}{2}(\sqrt{x} + \sqrt{x-4}), \quad l_5 = \sqrt{x-4}$$

where

$$x = -q^2/m^2$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

INTEGRATION OF SUBSECTORS

construct ansatz in [Duhr, Gangl, Rhodes '11] basis:

$$\ln, \quad \mathrm{Li}_2, \quad \mathrm{Li}_3, \quad \mathrm{Li}_4, \quad \mathrm{Li}_{2,2}$$

constrain arguments of polylogs by considering symbol

$$\mathrm{sym}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{(n-1) \text{ times}} .$$

and requiring absence of spurious letters

$$1 - z = c^{a_0} l_1^{a_1} \cdots l_5^{a_5}$$

extend to root-valued alphabet

numerical sample and heuristic integer relation search [Lenstra, Lenstra, Lovász] for

$$\ln(1-z) - a_0 \ln(c) - a_1 \ln(l_1) - \ldots - a_5 \ln(l_5) = 0 .$$

non-obvious factorizations like e.g.:

$$1 - \frac{1}{l_2 l_3} = \frac{l_2}{l_3}$$

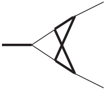
$$1 - \frac{1}{\frac{1}{2}(\sqrt{x} + \sqrt{x+4})\sqrt{x+4}} = \frac{\frac{1}{2}(\sqrt{x} + \sqrt{x+4})}{\sqrt{x+4}} .$$

- match ansatz to derivative (symbol)
- solutions for all subsector integrals in terms of

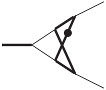
$$\ln(l_1), \ln(l_2), \ln(l_3), \ln(l_4), \ln(l_5), \mathrm{Li}_2(l_2^{-2}), \mathrm{Li}_2(1/(l_2 l_3))), \mathrm{Li}_2(-l_4^{-2}), \mathrm{Li}_2(-1/(l_4 l_5)), \mathrm{Li}_3(l_2^{-2}),$$
$$\mathrm{Li}_3(l_1/l_2), \mathrm{Li}_3(-1/(l_1 l_2^2 l_3)), \mathrm{Li}_3(1/(l_2 l_3))), \mathrm{Li}_3(l_2/l_3)), \mathrm{Li}_3(-l_4^{-2}), \mathrm{Li}_3(l_4^{-2}), \mathrm{Li}_3(1/(l_1 l_4)),$$
$$\mathrm{Li}_3(-1/(l_4 l_5))), \mathrm{Li}_3(l_1 l_5)/l_4^2), \mathrm{Li}_4(-l_2^{-2}), \mathrm{Li}_4(l_2^{-2}), \mathrm{Li}_4(-1/(l_1 l_2)), \mathrm{Li}_4(l_1/l_2), \mathrm{Li}_4(-1/(l_1 l_2^2 l_3)),$$
$$\mathrm{Li}_4(1/(l_2 l_3)), \mathrm{Li}_4(l_2/l_3), \mathrm{Li}_4(l_1 l_3)/l_2^2), \mathrm{Li}_4(-l_4^{-2}), \mathrm{Li}_4(l_4^{-2}), \mathrm{Li}_4(1/(l_1 l_4)), \mathrm{Li}_4(l_4/l_1), \mathrm{Li}_4(-1/(l_1 l_4^2 l_5)),$$
$$\mathrm{Li}_4(-1/(l_4 l_5)), \mathrm{Li}_4(l_1 l_5/l_4^2), \mathrm{Li}_4(l_5/l_4), \mathrm{Li}_{2,2}(-1, -l_2^{-2}), \mathrm{Li}_{2,2}(-1, -l_4^{-2}), \mathrm{Li}_{2,2}(-1/(l_1 l_2), l_1/l_2),$$
$$\mathrm{Li}_{2,2}(1/(l_1 l_4), l_1/l_4)$$

- non-trivial boundary constants fixed by regularity conditions

Integration of top level sector



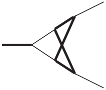
: $m_{10} = \epsilon^4 s^2 I^{6,63}_{1,1,1,1,1,1,0}\,,$



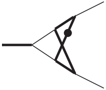
: $m_{11} = \epsilon^4 \frac{s^2(s+16m^2)}{2(1+2\epsilon)} I^{6,63}_{1,2,1,1,1,1,0}\,.$

$$\frac{d}{dx}\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} = B(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \epsilon\, D(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \begin{pmatrix} N_{10}(\epsilon;x) \\ N_{11}(\epsilon;x) \end{pmatrix}$$

Integration of top level sector



$$m_{10} = \epsilon^4 s^2 I^{6,63}_{1,1,1,1,1,1,0}\,,$$



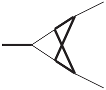
$$m_{11} = \epsilon^4 \frac{s^2(s+16m^2)}{2(1+2\epsilon)} I^{6,63}_{1,2,1,1,1,1,0}\,.$$

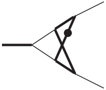
$$\frac{d}{dx}\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} = B(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \epsilon\, D(x)\begin{pmatrix} m_{10} \\ m_{11} \end{pmatrix} + \begin{pmatrix} N_{10}(\epsilon; x) \\ N_{11}(\epsilon; x) \end{pmatrix}$$

homogeneous solution (for $0 < x < 16$) e.g. from second order deq or maximal cuts

$$G(x) = \sqrt{x}\begin{pmatrix} \mathrm{K}(x/16) & \mathrm{K}(1-x/16) \\ -\mathrm{E}(x/16) & \mathrm{E}(1-x/16)-\mathrm{K}(1-x/16) \end{pmatrix}, \qquad \frac{d}{dx}G(x) = B(x)G(x)$$

Integration of top level sector

 : $$m_{10} = \epsilon^4 s^2 I^{6,63}_{1,1,1,1,1,1,0}\,,$$

 : $$m_{11} = \epsilon^4 \frac{s^2(s+16m^2)}{2(1+2\epsilon)} I^{6,63}_{1,2,1,1,1,1,0}\,.$$

$$\frac{d}{dx}\begin{pmatrix} m_{10} \\ m_{11}\end{pmatrix} = B(x)\begin{pmatrix} m_{10} \\ m_{11}\end{pmatrix} + \epsilon\, D(x)\begin{pmatrix} m_{10} \\ m_{11}\end{pmatrix} + \begin{pmatrix} N_{10}(\epsilon;x) \\ N_{11}(\epsilon;x)\end{pmatrix}$$

homogoneous solution (for $0 < x < 16$) e.g. from second order deq or maximal cuts

$$G(x) = \sqrt{x}\begin{pmatrix} \mathrm{K}(x/16) & \mathrm{K}(1-x/16) \\ -\mathrm{E}(x/16) & \mathrm{E}(1-x/16)-\mathrm{K}(1-x/16)\end{pmatrix}, \qquad \frac{d}{dx}G(x) = B(x)G(x)$$

basis change to eliminate homog. term:

$$\begin{pmatrix} \tilde{m}_{10} \\ \tilde{m}_{11}\end{pmatrix} = G^{-1}(x)\begin{pmatrix} m_{10} \\ m_{11}\end{pmatrix} \quad \Rightarrow \quad \frac{d}{dx}\begin{pmatrix} \tilde{m}_{10} \\ \tilde{m}_{11}\end{pmatrix} = \epsilon G^{-1}(x)D(x)G(x)\begin{pmatrix} \tilde{m}_{10} \\ \tilde{m}_{11}\end{pmatrix} + G^{-1}(x)\begin{pmatrix} N_{10}(\epsilon;x) \\ N_{11}(\epsilon;x)\end{pmatrix}$$

for more recent work on cuts see also [Tancredi, Primo '16,'17; Frellesvig, Papadopoulos '17; Zeng '17; Abreu, Britto, Duhr, Gardi '17; Bosma, Sogaard, Zhang '17; Harley, Moriello, Schabinger '17] expansion in ϵ gives for finite parts

$$\frac{d}{dx}\begin{pmatrix}\tilde{m}_{10}^{(4)}\\ \tilde{m}_{11}^{(4)}\end{pmatrix} = \frac{2}{\pi}\sqrt{x}\begin{pmatrix}\mathrm{E}(1-x/16)-\mathrm{K}(1-x/16) & -\mathrm{K}(1-x/16)\\ \mathrm{E}(x/16) & \mathrm{K}(x/16)\end{pmatrix}\begin{pmatrix}0\\ \mathcal{R}^{(4,16)}(x)+i\,\mathcal{Q}^{(4,16)}(x)\end{pmatrix}$$

with

$$\mathcal{R}^{(4,16)}(x) = 5\ln^2(l_2) - 3\,l_1\,\frac{\zeta_2/2+\ln^2(l_4)+\mathrm{Li}_2(-1/l_4^2)}{l_5}\,,$$
$$\mathcal{Q}^{(4,16)}(x) = 0\,.$$

⇒ one-fold integral representation for solution

Analytic continuation to physical kinematics

$$l_i' = \{\sqrt{-x}, \tfrac{1}{2}(\sqrt{-x}+\sqrt{-x-4}), \sqrt{-x-4}, \tfrac{1}{2}(\sqrt{-x}+\sqrt{-x+4}), \sqrt{-x+4}\}$$

region $4 < x < 16$:

$$\mathcal{R}^{(4,16)}(x) = 5\ln^2(l_2) - 3\,l_1\,\frac{\zeta_2/2 + \ln^2(l_4) + \mathrm{Li}_2(-1/l_4^2)}{l_5}\,,$$

$$\mathcal{Q}^{(4,16)}(x) = 0\,.$$

region $0 < x < 4$:

$$\mathcal{R}^{(0,4)}(x) = 5\ln^2(l_2) + 3\,l_1\,\frac{\mathrm{Cl}_2(-2\,\mathrm{arccsc}(2/l_1))}{l_5'}\,,$$

$$\mathcal{Q}^{(0,4)}(x) = 0\,,$$

region $-4 < x < 0$:

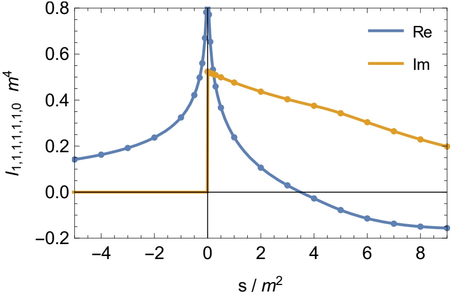
$$\mathcal{R}^{(-4,0)}(x) = -5\,\mathrm{arcsec}^2\,(2/l_3) + 3\,l_1'\,\frac{(\zeta_2 - \ln^2\,(l_4') - \mathrm{Li}_2(1/l_4'^2))}{l_5'}\,,$$

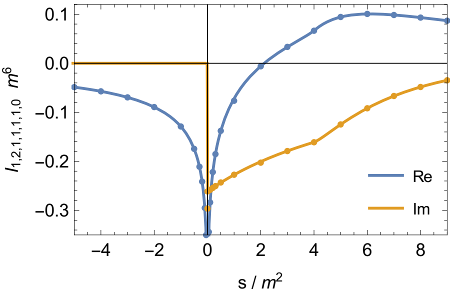
$$\mathcal{Q}^{(-4,0)}(x) = \pi\,\frac{3\,l_1'}{l_5'}\,\ln\,(l_4')\,,$$

region $-\infty < x < -4$:

$$\mathcal{R}^{(-\infty,-4)}(x) = 5\ln^2\,(l_2') - \frac{15}{2}\zeta_2 + 3\,l_1'\,\frac{(\zeta_2 - \ln^2\,(l_4') - \mathrm{Li}_2(1/l_4'^2))}{l_5'}\,,$$

$$\mathcal{Q}^{(-\infty,-4)}(x) = \pi\,\left(\frac{3\,l_1'}{l_5'}\,\ln\,(l_4') - 5\,\ln\,(l_2')\right)\,,$$





- fast evaluation: 15ms for generic phase space point
- also high precision no problem:

$$m^4 I_{1,1,1,1,1,1,0}\big|_{s=5m^2,d=4} \approx -0.07776462028160023644086669458011467822536257409024 + i\ 0.34306740464518688969054397597465622650767181505054$$

Conclusions

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations

reductions via finite field sampling:

- speeds up integration-by-parts reductions
- useful also in other contexts

analytical solutions for integrals with masses:

- Li functions preferable also for root-valued letters
- more to be explored beyond multiple polylogarithms