Reducing differential systems for multiloop integrals

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Outline

- Differential equations in multiloop calculations
- Algorithm of reduction to ϵ-form
- Criterion of irreducibility[1].
- Variable change[1].
- Summary and Outlook.

[1] with A. Pomeransky, soon to be published.

Differential equation approach

Object of this talk

First-order differential systems with coefficients being the rational functions of the variable:

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Topic of this talk

Reduction of such systems to canonical form(defined below).

We will talk about the linear transformation of functions $\boldsymbol{J} \to T\boldsymbol{J}$ (T is rational in x).

We will also talk about change of variable $x \to f(x)$, where f is rational.

NB: The differential system is assumed to live on the Riemann sphere, so the definitions of the canonical forms we are considering are invariant under Moebius transformations $x \to \frac{ax+b}{cx+d}$.

Differential equation approach

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First-order differential systems with coefficients being the rational functions of the variable:

$$\partial_x \boldsymbol{J} = M(x)\,\boldsymbol{J}. \tag{DS}$$

Motivation

- Differential equations method in the multiloop calculations is one of the most powerful "general purpose" approaches, applicable to any dimensionally regularized theory irrespective of its symmetries.
- The IBP reduction approach (Chetyrkin and Tkachov, 1981) reduces the problem of multiloop calculation to that of calculationof a finite number of master integrals $\boldsymbol{J} = (J_1, \ldots J_n)^T$ and also provides the differential system (DS) for them (Kotikov, 1991; Remiddi, 1997).
  NB: The coefficients of DS depend also on ϵ.

Differential equations in ϵ-form

In 2013 J. Henn made a remarkable observation (Henn, 2013) that in many cases it is possible to choose such master integrals that the ϵ-dependence of the resulting differential system reduces to a single factor in the right-hand side:

$$\partial_x \boldsymbol{J} = \epsilon M(x)\, \boldsymbol{J}$$

Also it appeared that $M(x)$ can be represented as a sum of simple poles, $M(x) = \sum_i \frac{A_i}{x-a_i}$. It means that the ϵ-expansion of the general solution

$$P\exp\left[\epsilon \int dx M(x)\right] = \sum_n \epsilon^n \int\limits_{x>x_1>\ldots>x_n} \prod_i^n dx_i M(x_i)$$

obeys the property of uniform transcendentality, and the coefficients are expressed in terms of the Goncharov's polylogs.

Problem

Given a system

$$\partial \mathbf{J}(x) / \partial x = M(x, \epsilon) \mathbf{J}(x)$$

is it possible and how to find a change of functions reducing the system to ϵ-form

$$\partial \tilde{\mathbf{J}}(x) / \partial x = \epsilon \sum_k \frac{A_k}{x - x_k} \tilde{\mathbf{J}}(x),$$

i.e., is it possible and how to find such $T(x, \epsilon)$ that

$$T^{-1}(MT - \partial_x T) = \epsilon \sum_k \frac{A_k}{x - x_k}$$

Reduction to ϵ-form

In Ref. (RL, 2015) the algorithm of finding the ϵ-form, based on the differential system only, has been developed. It consists of 3 steps

1. Reduction to Fuchsian form
2. Normalization of the matrix residues
3. Factorization of ϵ.

In the beginning of 2017 two public implementations of this algorithm appeared:
`Fuchsia` (Gituliar and Magerya, 2017) and `epsilon` (Prausa, 2017).

Goals of this talk

- Explain the basic idea the algorithm Ref. (RL, 2015)
- Reformulate the first stage in a more simple invariant form
- Close some loopholes: criterion of irreducibility, rational variable change. New perspective: view at the problem from the point of view of vector bundles!

Fuchsian form

Regular singularities

The singular point $x = x_0$ of the differential system is regular if the general solution is bounded by power of $|x - x_0|$ when $x \to x_0$. The differential system for multiloop calculations are likely to be regular.

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Fuchsian form

The system is said to be Fuchsian in the singular point $x = x_0$ if the matrix $M(x)$ has a simple pole at $x = x_0$. In any regular singular point the system can be locally reduced to Fuchsian form by means of rational transformations of functions $\boldsymbol{J} \to T\boldsymbol{J}$ (T is rational in x).
The system is said to be globally Fuchsian if it is Fuchsian in all singular points. It means that

$$M(x) = \sum_i \frac{A_i}{x - x_i},$$

the constant matrix A_i is the *matrix residue at* $x = x_i$.

Fuchsian form

Whether the regular system can be reduced to global Fuchsian form is, in fact, equivalent to Hilbert's 21st problem, thanks to Plemelj's construction.

> Hilbert's 21st problem: To show that there always exists a linear differential system of the Fuchsian class, with given singular points and monodromic group.

Bolibrukh (1989) has proven that this problem has, in general, negative solution, so it is not always possible to reduce the regular system to global Fuchsian form.

However, Plemelj's construction allows to claim Fuchsian form in all but maybe one singular point. In particular, there is well-known algorithm by Barkatou and Pflügel (2009) of reducing the regular system in all *finite singularities, i.e. securing that*

$$M(x) = \sum_i \frac{A_i}{x - x_i} + P(x) \ ,$$

where P is some polynomial.

Normalized Fuchsian form

One might ask whether the matrix residue A at a given regular point $x = x_0$ is uniquely defined. If we are to apply transformations $T(x)$, that are regular at x_0 (i.e., $T_0 = T(x_0)$ is finite and invertible), the matrix residue is uniquely defined up to similarity $A \to T_0^{-1} A T_0$ (in particular, its spectrum are invariant).

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If we allow for general rational transformations preserving the local Fuchsian form, the eigenvalues are defined only modulo $\mathbb{Z}$.
The normalization is any rule which removes this arbitrariness in evs.
E.g., we might require that

$$\Re\lambda \in [a, a+1) \ ,$$

where a is some chosen number.
For the systems depending on ϵ with evs of the form $k + \alpha\epsilon$ (k integer) we might use $\lambda \propto \epsilon$ as a normalization condition.

Let me now review the local reduction to normalized Fuchsian form.

Balance transformation

The functions change $\boldsymbol{J} = T\tilde{\boldsymbol{J}}$ transforms the system to the form $\partial_x \tilde{\boldsymbol{J}} = \tilde{M}\tilde{\boldsymbol{J}}$, where

$$\tilde{M} = M_T \stackrel{\text{def}}{=} T^{-1}\left(MT - \partial_x T\right).$$

Our main tool is the transformation called P-balance between x_1 and x_2, defined as

$$T = B\left(P, x_1, x_2 | x\right) = \bar{P} + P\frac{x - x_2}{x - x_1},$$

where P is some projector, $\bar{P} = I - P$. As $T^{-1} = \bar{P} + P\frac{x-x_1}{x-x_2}$, the balance transformation is singular only in two points, x_1 and x_2.

NB: The balance preserves the system's properties of being regular, Fuchsian, normalized Fuchsian in any point different from x_1 and x_2.

Local reduction to Fuchsian form

Suppose

$$M = \frac{A_0}{x^{p+1}} + \frac{A_1}{x^p} + \ldots\,, \quad p > 0\,.$$

and we want to reduce the pole order[2] $p+1$.

Necessary criterion of local reducibility (Moser, 1959)
$\lim_{x\to 0} x^{\operatorname{rank} A_0} \det\left[x^p M(x) - \mu\right] = 0$ should hold identically for any μ. It is equivalent to $\dim\ker \begin{bmatrix} A_0 & A_1 - \mu \\ 0 & A_0 \end{bmatrix} > \dim\ker A_0$, i.e., in addition to null-vectors $\binom{w}{0}$ with $w \in \ker A_0$, the matrix $\begin{bmatrix} A_0 & A_1 - \mu \\ 0 & A_0 \end{bmatrix}$ has at least one null-vector of the form $\binom{u}{w}$.

[2] p is called the Poincaré rank of the differential system at $x = 0$.

Local reduction to Fuchsian form

The vector w can always be chosen as a polynomial of μ and let $\mathcal{W}$ be the linear span of its coefficients. Let P be any projector on $\mathcal{W}$. Then, applying the transformation $T = B(P, 0, x_2|x)$, we obtain

$$M_T = \frac{\tilde{A}_0}{x^{p+1}} + \ldots, \text{ where } \tilde{A}_0 = \bar{P}(A_0 + A_1 P) .$$

It can be proved[3] that $\operatorname{rank} \tilde{A}_0 < \operatorname{rank} A_0$, so in several step we secure $A_0 = 0$.

[3]**Exercise**: prove the following properties of $\mathcal{W}$:

1. $\mathcal{W} \subset \ker A_0$; 2. $A_1\mathcal{W} \subset \ker A_0 + \mathcal{W}$; 3. $\ker A_0 \cap \mathcal{W} > 0$

Using these properties, prove that

$$\dim \operatorname{Im} A_0 > \dim \operatorname{Im} \tilde{A}_0 .$$

Local normalization

Suppose

$$M = \frac{A_0}{x} + \dots .$$

and the eigenvalues of A_0 are not normalized, i.e., there is at least one e.v. λ which needs to be shifted by a positive or negative integer. Let us present two transformations which shift λ by $+1$ or by -1. Then, applying a sequence of such transformations, we may normalize all eigenvalues.

- Raising transformation: Let $A_0 u = \lambda u$. Then the transformation $B\left(uv^\dagger, 0, x_2|x\right)$ for any $v^\dagger$, provided that $v^\dagger u = 1$, replaces one eigenvalue λ by $\lambda + 1$.
- Lowering transformation: Let $v^\dagger A_0 = \lambda v^\dagger$. Then the transformation $B\left(uv^\dagger, x_2, 0|x\right)$ for any $u^\dagger$, provided that $v^\dagger u = 1$, replaces one eigenvalue λ by $\lambda - 1$.

Global reduction

A brief summary of 4 previous slides : Using a sequence of balance transformations $T = B(P, x_1, x_2|x) = \bar{P} + P\frac{x-x_2}{x-x_1}$, it is possible to reduce the system locally to normalized Fuchsian form.

Even more: Fixing the second point of each balance to be equal to some fixed point x_0, we may reduce the regular system to normalized Fuchsian form in all points except x_0.

How about the last point x_0? Are we obliged to spoil it?

Global reduction

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How about the last point x_0? Are we obliged to spoil it?

The observation of (RL, 2015) was that the construction of the suitable balances $B(P, x_1, x_2|x)$ leaves a lot of freedom, both on the stage of Fuchsification and on the stage of normalization:

- Fuchsification: $\mathcal{W} = \operatorname{Im} P$ is required to satisfy three conditions 1. $\mathcal{W} \subset \ker A_0$; 2. $A_1\mathcal{W} \subset \ker A_0 + \mathcal{W}$; 3. $\ker A_0 \cap \mathcal{W} > 0$.
- Normalization: $\mathcal{W} = \operatorname{Im} P$ is required to be eigenspace of A_0.

I.e., all conditions are imposed on the image of the projector P, while its kernel can be arbitrary.

General idea of the algorithm of Ref. (RL, 2015)

The idea of Ref. (RL, 2015) was to adjust the kernel of the projector so that the resulting balance does not spoil the behaviour of the system in the second point x_2. This reduces to the requrement that $\mathcal{K} = \ker P$ be the invariant subspace of the leading expansion coefficient in the second point. I.e., if $M(x) = \frac{B_0}{(x-x_2)^n} + \ldots$, we should require that $\mathcal{K}$ satisfy

$$B_0\mathcal{K} \subset \mathcal{K}\,.$$

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$$B_0\mathcal{K} \subset \mathcal{K}\,.$$

It is easy to understand that there is always such a subspace! But how about Bolibrukh's couterexample then?

To be the image and the kernel of the projector, subspaces $\mathcal{W}$ and $\mathcal{K}$ should intersect trivially

$$\mathcal{W} \cap \mathcal{K} = \{0\}$$

This condition may be the only obstacle for the construction of P. It is very degenerate case: since $\dim \mathcal{W} + \dim \mathcal{K} = n$, $\mathcal{W}$ and $\mathcal{K}$ usually intersect only at one point (e.g. line and plane in 3d).

General idea of the algorithm of Ref. (RL, 2015)

The third stage of the algorithm, factorization, is to find a transformation $T(\epsilon)$ (independent of x) such that for all matrix residues $A_i(\epsilon)$ holds

$$T^{-1}(\epsilon)\, A_i(\epsilon)\, T(\epsilon) = \epsilon S_i\,,$$

How can we do it without knowing S_k in r.h.s.?

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Trick

$$T^{-1}(\epsilon)\, \frac{M_k(\epsilon)}{\epsilon}\, T(\epsilon) = S_k = T^{-1}(\mu)\, \frac{M_k(\mu)}{\mu}\, T(\mu)$$

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Trick

$$T(\epsilon) \times \left(T^{-1}(\epsilon)\, \frac{M_k(\epsilon)}{\epsilon}\, T(\epsilon) = S_k = T^{-1}(\mu)\, \frac{M_k(\mu)}{\mu}\, T(\mu) \right) \times T^{-1}(\mu)$$

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$$T^{-1}(\epsilon)\,A_i(\epsilon)\,T(\epsilon) = \epsilon S_i\,,$$

Linear system for matrix elements of $T(\epsilon,\mu) = T(\epsilon)\,T^{-1}(\mu)$

$$\frac{M_k(\epsilon)}{\epsilon}\,T(\epsilon,\mu) \;=\; T(\epsilon,\mu)\,\frac{M_k(\mu)}{\mu}$$

Example

$$\mathbb{M}(\epsilon, x) = \begin{pmatrix} \frac{2\epsilon x - x - \epsilon - 1}{x(x+1)} & -\frac{\epsilon}{(x+1)(5\epsilon+1)} & -\frac{\epsilon+1}{x(x+1)(5\epsilon+1)} \\ \frac{2(4\epsilon+1)(5\epsilon+1)}{x} & -\frac{4\epsilon x+2x+3\epsilon+1}{x(x+1)} & \frac{2(\epsilon+1)}{x(x+1)} \\ \frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{x(\epsilon+1)} & -\frac{\epsilon(2\epsilon+1)}{(x+1)(\epsilon+1)} & \frac{4\epsilon x+x+5\epsilon+1}{x(x+1)} \end{pmatrix}$$

Example

$$\mathbb{M}(\epsilon,x)=\begin{pmatrix} \frac{2\epsilon x-x-\epsilon-1}{x(x+1)} & -\frac{\epsilon}{(x+1)(5\epsilon+1)} & -\frac{\epsilon+1}{x(x+1)(5\epsilon+1)} \\ \frac{2(4\epsilon+1)(5\epsilon+1)}{x} & -\frac{4\epsilon x+2x+3\epsilon+1}{x(x+1)} & \frac{2(\epsilon+1)}{x(x+1)} \\ \frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{x(\epsilon+1)} & -\frac{\epsilon(2\epsilon+1)}{(x+1)(\epsilon+1)} & \frac{4\epsilon x+x+5\epsilon+1}{x(x+1)} \end{pmatrix}$$

Reduction to ϵ-form

Stage 1 is not needed as this matrix has form $\mathbb{M}(\epsilon,x)=\frac{\mathbb{A}(\epsilon)}{x}+\frac{\mathbb{B}(\epsilon)}{x+1}$

$$\mathbb{A}(\epsilon)=\begin{pmatrix} -\epsilon-1 & 0 & -\frac{\epsilon+1}{5\epsilon+1} \\ 2(4\epsilon+1)(5\epsilon+1) & -3\epsilon-1 & 2(\epsilon+1) \\ \frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{\epsilon+1} & 0 & 5\epsilon+1 \end{pmatrix},\quad \mathbb{B}(\epsilon)=\begin{pmatrix} 3\epsilon & -\frac{\epsilon}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ 0 & -\epsilon-1 & -2(\epsilon+1) \\ 0 & -\frac{\epsilon(2\epsilon+1)}{\epsilon+1} & -\epsilon \end{pmatrix}.$$

Poles at $x=0$, $x=-1$ and $x=\infty$, the latter with matrix residue

$$\mathbb{C}(\epsilon)=-\mathbb{A}(\epsilon)-\mathbb{B}(\epsilon)=\begin{pmatrix} 1-2\epsilon & \frac{\epsilon}{5\epsilon+1} & 0 \\ -2(4\epsilon+1)(5\epsilon+1) & 2(2\epsilon+1) & 0 \\ -\frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{\epsilon+1} & \frac{\epsilon(2\epsilon+1)}{\epsilon+1} & -4\epsilon-1 \end{pmatrix}$$

$x = \infty$:

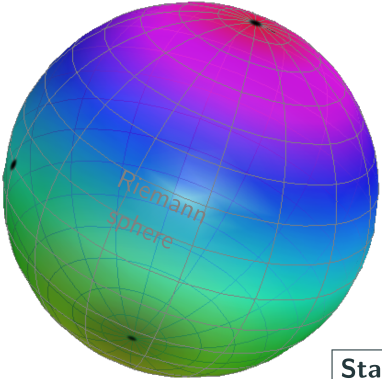
$$\begin{pmatrix} 1-2\epsilon & \frac{\epsilon}{5\epsilon+1} & 0 \\ -2(4\epsilon+1)(5\epsilon+1) & 2(2\epsilon+1) & 0 \\ -\frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{\epsilon+1} & \frac{\epsilon(2\epsilon+1)}{\epsilon+1} & -4\epsilon-1 \end{pmatrix}$$

eigenvalues: $-4\epsilon-1, 1, \underline{2\epsilon+2}$

$x = -1$:

$$\begin{pmatrix} 3\epsilon & -\frac{\epsilon}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ 0 & -\epsilon-1 & -2\epsilon-2 \\ 0 & \frac{-2\epsilon^2-\epsilon}{\epsilon+1} & -\epsilon \end{pmatrix}$$

eigenvalues: $3\epsilon, \epsilon, -3\epsilon-1$



Stage 2

$x = 0$:

$$\begin{pmatrix} -\epsilon-1 & 0 & \frac{-\epsilon-1}{5\epsilon+1} \\ \frac{40\epsilon^2+18\epsilon+2}{\frac{(2\epsilon+1)(4\epsilon+1)(5\epsilon+1)}{\epsilon+1}} & -3\epsilon-1 & 2\epsilon+2 \\ & 0 & 5\epsilon+1 \end{pmatrix}$$

Apply $B(uv^\dagger, 0, \infty | x)$

eigenvalues: $\underline{-3\epsilon-1}, \epsilon, 3\epsilon$

$x = \infty$:

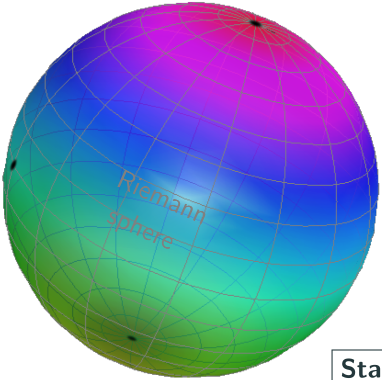
$$\begin{pmatrix} 1 & 0 & 0 \\ -2(4\epsilon+1)(5\epsilon+1) & 2\epsilon+1 & -4(\epsilon+1) \\ -\frac{(2\epsilon+1)^2(5\epsilon+1)}{\epsilon+1} & 0 & -4\epsilon-1 \end{pmatrix}$$

eigenvalues: $-4\epsilon - 1, 1, \underline{2\epsilon + 1}$

$x = -1$:

$$\begin{pmatrix} -\epsilon & \frac{\epsilon}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ 4(5\epsilon+1) & 3\epsilon-1 & 6(\epsilon+1) \\ -\frac{4\epsilon(2\epsilon+1)(5\epsilon+1)}{\epsilon+1} & \frac{\epsilon(2\epsilon+1)}{\epsilon+1} & -\epsilon \end{pmatrix}$$

eigenvalues: $3\epsilon, \epsilon, \underline{-3\epsilon - 1}$



$x = 0$:

$$\begin{pmatrix} \epsilon-1 & -\frac{\epsilon}{5\epsilon+1} & -\frac{\epsilon+1}{5\epsilon+1} \\ 2(4\epsilon-1)(5\epsilon+1) & -5\epsilon & -2(\epsilon+1) \\ \frac{(2\epsilon+1)(5\epsilon+1)(6\epsilon+1)}{\epsilon+1} & -\frac{\epsilon(2\epsilon+1)}{\epsilon+1} & 5\epsilon+1 \end{pmatrix}$$

eigenvalues: $-3\epsilon, \epsilon, 3\epsilon$

Stage 2

Apply $B(uv^\dagger, -1, \infty | x)$

$x = \infty$:

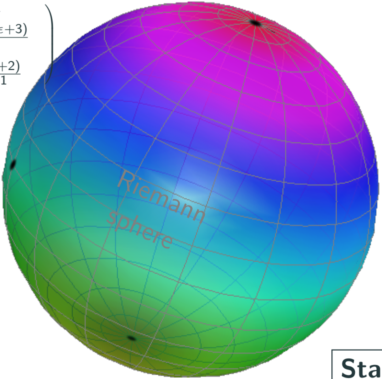
$$\begin{pmatrix} 1 & 0 & 0 \\ -2\left(4\epsilon^2 - 4\epsilon - 1\right) & \frac{\epsilon(4\epsilon+1)}{5\epsilon+1} & -\frac{(\epsilon+1)(6\epsilon+1)}{5\epsilon+1} \\ \frac{28\epsilon^3 - 4\epsilon^2 - 7\epsilon - 1}{\epsilon+1} & -\frac{\epsilon(4\epsilon+1)(6\epsilon+1)}{(\epsilon+1)(5\epsilon+1)} & -\frac{14\epsilon^2+8\epsilon+1}{5\epsilon+1} \end{pmatrix}$$

eigenvalues: $\underline{-4\epsilon - 1}$, 1, 2ϵ

$x = -1$:

$$\begin{pmatrix} -\epsilon & \frac{\epsilon}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ -2\epsilon(16\epsilon + 3) & \frac{\epsilon(21\epsilon+4)}{5\epsilon+1} & \frac{(\epsilon+1)(16\epsilon+3)}{5\epsilon+1} \\ -\frac{2\epsilon\left(44\epsilon^2+24\epsilon+3\right)}{\epsilon+1} & \frac{\epsilon\left(34\epsilon^2+17\epsilon+2\right)}{(\epsilon+1)(5\epsilon+1)} & -\frac{\epsilon(11\epsilon+2)}{5\epsilon+1} \end{pmatrix}$$

eigenvalues: $3\epsilon, \underline{\epsilon}, -3\epsilon$



$x = 0$:

$$\begin{pmatrix} \epsilon - 1 & -\frac{\epsilon}{5\epsilon+1} & -\frac{\epsilon+1}{5\epsilon+1} \\ 2(4\epsilon - 1)(5\epsilon + 1) & -5\epsilon & -2(\epsilon + 1) \\ \frac{(2\epsilon+1)(5\epsilon+1)(6\epsilon+1)}{\epsilon+1} & -\frac{\epsilon(2\epsilon+1)}{\epsilon+1} & 5\epsilon + 1 \end{pmatrix}$$

eigenvalues: $-3\epsilon, \epsilon, 3\epsilon$

Stage 2

Apply $B(uv^\dagger, \infty, -1|x)$

$x = \infty$:

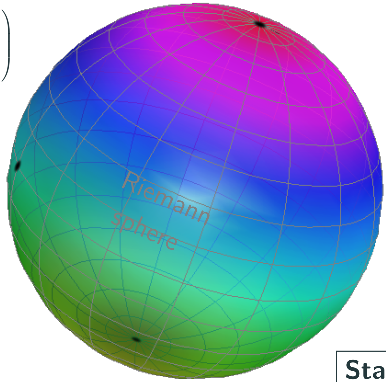
$$\begin{pmatrix} -2(8\epsilon+1) & 1 & 0 \\ -2(6\epsilon+1)(16\epsilon+3) & 3(4\epsilon+1) & 0 \\ \frac{2\epsilon\left(206\epsilon^2+91\epsilon+10\right)}{\epsilon+1} & -\frac{\epsilon(30\epsilon+7)}{\epsilon+1} & 2\epsilon \end{pmatrix}$$

eigenvalues: $-4\epsilon, \underline{1}, 2\epsilon$

$x = -1$:

$$\begin{pmatrix} 3(5\epsilon+1) & -\frac{4\epsilon+1}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ 2(4\epsilon+1)(19\epsilon+4) & -7\epsilon-3 & 2(\epsilon+1) \\ -\frac{(4\epsilon+1)\left(118\epsilon^2+29\epsilon+1\right)}{\epsilon+1} & \frac{8\epsilon(4\epsilon+1)}{\epsilon+1} & -7\epsilon-1 \end{pmatrix}$$

eigenvalues: $3\epsilon, \underline{\epsilon-1}, -3\epsilon$



$x = 0$:

$$\begin{pmatrix} \epsilon-1 & -\frac{\epsilon}{5\epsilon+1} & -\frac{\epsilon+1}{5\epsilon+1} \\ 2(4\epsilon-1)(5\epsilon+1) & -5\epsilon & -2(\epsilon+1) \\ \frac{(2\epsilon+1)(5\epsilon+1)(6\epsilon+1)}{\epsilon+1} & -\frac{\epsilon(2\epsilon+1)}{\epsilon+1} & 5\epsilon+1 \end{pmatrix}$$

eigenvalues: $-3\epsilon, \epsilon, 3\epsilon$

Stage 2

Apply $B(uv^{\dagger}, -1, \infty | x)$

$x = \infty$:

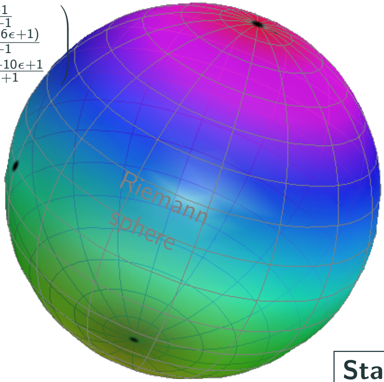
$$\begin{pmatrix} -4\epsilon & 0 & 0 \\ -2\epsilon(32\epsilon+7) & \frac{2\epsilon(4\epsilon+1)}{5\epsilon+1} & -\frac{2\epsilon(\epsilon+1)}{5\epsilon+1} \\ \frac{2\epsilon(6\epsilon+1)(7\epsilon+2)}{\epsilon+1} & -\frac{2\epsilon^2(4\epsilon+1)}{(\epsilon+1)(5\epsilon+1)} & \frac{2\epsilon^2}{5\epsilon+1} \end{pmatrix}$$

eigenvalues: $-4\epsilon, 0, 2\epsilon$

$x = -1$:

$$\begin{pmatrix} 3\epsilon+1 & \frac{\epsilon}{5\epsilon+1} & \frac{\epsilon+1}{5\epsilon+1} \\ 2(2\epsilon+1)(6\epsilon+1) & \frac{\epsilon(17\epsilon+3)}{5\epsilon+1} & \frac{2(\epsilon+1)(6\epsilon+1)}{5\epsilon+1} \\ -\frac{(3\epsilon+1)(6\epsilon+1)(8\epsilon+1)}{\epsilon+1} & \frac{\epsilon(3\epsilon+1)(6\epsilon+1)}{(\epsilon+1)(5\epsilon+1)} & -\frac{27\epsilon^2+10\epsilon+1}{5\epsilon+1} \end{pmatrix}$$

eigenvalues: $3\epsilon, \epsilon, -3\epsilon$



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eigenvalues: $-3\epsilon, \epsilon, 3\epsilon$

Stage 2

Stage 2 completed!

Example

Stage 3.

Now we solve a linear system

$$\frac{\mathbb{A}(\epsilon)}{\epsilon}\mathbb{T} = \mathbb{T}\frac{\mathbb{A}(\mu)}{\mu}, \quad \frac{\mathbb{B}(\epsilon)}{\epsilon}\mathbb{T} = \mathbb{T}\frac{\mathbb{B}(\mu)}{\mu}$$

with respect to matrix elements of $\mathbb{T}$

Example

Stage 3.

Now we solve a linear system

$$\frac{\mathbb{A}(\epsilon)}{\epsilon}\mathbb{T} = \mathbb{T}\frac{\mathbb{A}(\mu)}{\mu}, \quad \frac{\mathbb{B}(\epsilon)}{\epsilon}\mathbb{T} = \mathbb{T}\frac{\mathbb{B}(\mu)}{\mu}$$

with respect to matrix elements of $\mathbb{T}$

Solution

$$\mathbb{T} = \begin{pmatrix} (\epsilon+1)\mu(5\mu+1) & 0 & 0 \\ -2(\epsilon+1)(\epsilon-\mu)(5\mu+1) & \epsilon(\epsilon+1)(5\mu+1) & 0 \\ (7\epsilon+1)(\epsilon-\mu)(5\mu+1) & -\epsilon(\epsilon-\mu) & \epsilon(5\epsilon+1)(\mu+1) \end{pmatrix}$$

we can put μ to any number except $\mu = 0, -1, -1/5$. We choose $\mu = 1$. Finally, we obtain

$$\widetilde{\mathbb{M}}(\epsilon, x) = \epsilon \begin{pmatrix} \frac{4}{x+1} & -\frac{1}{6x(x+1)} & -\frac{1}{3x(x+1)} \\ \frac{6(13x+6)}{x(x+1)} & -\frac{5(x+3)}{3x(x+1)} & \frac{2(x-6)}{3x(x+1)} \\ -\frac{63(x-1)}{x(x+1)} & \frac{5x-9}{6x(x+1)} & -\frac{x-18}{3x(x+1)} \end{pmatrix} \xrightarrow{\text{constant tr.}} \epsilon \begin{pmatrix} -\frac{x+3}{x(x+1)} & 0 & \frac{1}{3(x+1)} \\ 0 & \frac{2x+3}{x(x+1)} & \frac{8}{3(x+1)} \\ \frac{5}{x+1} & \frac{2}{x+1} & \frac{1}{x} \end{pmatrix}$$

Loopholes of the algorithm of Ref. (RL, 2015)

The algorithm is very efficient and works for huge systems ($\sim 300 \times 300$) with big diagonal blocks ($\sim 10 \times 10$).

In particular, there is a very recent application (RL, Smirnovs, Steinhauser, 2017) to the calculation of the n_f^2 term of 4-loop FF.

Nevertheless, there are some loopholes in the algorithm which may leave some doubts about the irreducibility to ϵ-form when the algorithm fails.

1. At the third stage — factoring ϵ out — we search for x-independent transformation only — why?
2. If at first two stages it appears to be impossible to construct a suitable projector due to the condition $\mathcal{W} \cap \mathcal{K} = \{0\}$, how can we prove that ϵ-form is not possible?
3. If we fail in terms of x can we pass to a new variable y, such that there is a transformation to ϵ-form rational in terms of y?

Recent progress (in collaboration with A. Pomeransky, to be published soon): closing these loopholes essentially.

Criterion of (ir)reducibility

Proposition

First, it is easy to prove, that the transformation T being regular at the point $x = x_0$ is not only sufficient, but also necessary condition to preserve normalized Fuchsian form at $x = x_0$.

This simple observation proves that on the third stage we may restrict ourselves by considering the transformations holomorphic on the whole Riemann sphere (=x-independent). Loophole #1 is closed.

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Suppose now that we were not able to reduce the system to global normalized Fuchsian form. Still, we may reduce the system in all points, except one, chosen arbitrarily. So, suppose we have the system which is Fuchsian everywhere but at $x = \infty$. If the transformation $T(x)$ to global normalized Fuchsian form exists, by proposition it should be regular everywhere except $x = \infty$. I.e.,

both T and T^{-1} should be polynomials in x, and $\det T$ should be independent of x. How can we find this transformation?

Criterion of (ir)reducibility

Trick, again: We first find a transformation $U(x)$ which reduces the system to the normalized Fuchsian form in all singular points but one, $x = 0$. Then, using the same proposition, if T exists then the matrix

$$S = U^{-1}T$$

should be regular everywhere but at $x = 0$. I.e., both S and S^{-1} should be polynomials in x^{-1}. Threfore, we need to find the decomposition

$$U = T(x)\,S^{-1}\left(x^{-1}\right)$$

where $T(x)$, $T^{-1}(x)$, $S\left(x^{-1}\right)$, $S^{-1}\left(x^{-1}\right)$ are polynomial in their arguments, or to prove that such decomposition does not exist.

NB: This is a variant of Riemann-Hilbert problem.

Criterion of (ir)reducibility

How to solve Riemann-Hilbert problem(= find the decomposition)

1. Check that U is a Laurent polynomial in x and that $\det U$ is independent of x. If not, there is no decomposition.
2. Try to find polynomial vectors $v(x)$, such that $U^{-1}v$ is polynomial in x^{-1} (i.e., contains no positive powers of x). It reduces to finite system of linear equations because the maximal power of x in $v(x)$ is restricted by that in U.
3. If you succeed to find n independent vectors $v(x)$, put them in columns to form T. Otherwise, there is no decomposition.

So, this closes loophole #2: we have the decisive criterion of reducibility to ϵ-form.

Criterion of (ir)reducibility

How about change of variable? Let us first soppose that the eigenvalues of mtrix residues have required form $k+\alpha\epsilon$ with k being integer. Then it appears that there is no need to try any rational change $x \to f(y)$: ϵ-form either can be reached by transformation rational in x or can not be reached after an arbitrary rational variable change.

Vector bundle on Riemann sphere

It is very instructive to view U as the transition function for the holomorphic vector bundle defined by two stereographic maps of the Riemann sphere (from North and from South poles). Then the existence of the decomposition is equivalent to the triviality of the vector bundle. Classification of the holomorphic vector bundles (HVB) on the Riemann sphere is known (Birkhoff–Grothendieck theorem, check Wikipedia ;)). According to it, HVB on the Riemann sphere are extremely simple: they decouple into sum of line bundles, each charachterized by a "winding number".

Criterion of (ir)reducibility

This is equivalent to the decomposition

$$U = T(x)\, x^D S^{-1}\left(x^{-1}\right)\ ,$$

where D is some diagonal matrix whose integer eigenvalues uniquely characterize the bundle. If $D \neq 0$, the vector bundle is nontrivial [4].

The rational change $x = f(y)$ maps Riemann sphere of y onto that of x, maybe m times (m is called the degree of the mapping). This mapping simply multiplies D by m, so the vector bundle in the Riemann sphere of y is nontrivial as long as it is for x!

[4] For the two-loop massive sunrise $D = \mathrm{diag}\,(1, -1)$ btw.

Algorithm of finding ϵ-form

1. Using local reductions check whether all singular points are regular and whether the eigenvalues of matrix residues have the form $k + \alpha\epsilon,\ k \in \mathbb{Z}$.
   1.1 if there are two points with e.v. of the form $k + \frac{1}{2} + \alpha\epsilon$, map them to 0 and ∞ by Moebius transformation and make the variable change $x \to x^2$.
   1.2 if there are three points with e.v. of the form $k + \frac{1}{2} + \alpha\epsilon$, map them to 0, 1, and ∞ by Moebius transformation and make the variable change $x \to \left(\frac{1+x^2}{1-x^2}\right)^2$.
   1.3 if there are more than three points, ϵ-form can not be reached by the transformations rational in any y related to x by $x = f(y)$ with any rational function f. In particular, by the transformations rational in x.

   At this stage we are guaranteed that it makes no sense to make further rational variable change $x = f(y)$: if ϵ-form can not be reached by the transformations rational in x, it can not also be reached by transformations rational in y.

Algorithm of finding ϵ-form (contd.)

2. By means of the algorithm of Ref. (RL, 2015), reduce the system to normalized Fuchsian form in all singular points, but may be one exceptional (chosen to be $x = \infty$).
3. If the transformed system happens to be not in normalized Fuchsian form at $x = \infty$
   3.1 find the transformation $U(x)$ reducing the transformed system to normalized Fuchsian form in all singular points (including $x = \infty$) but $x = 0$.
   3.2 find the decomposition $U = TS$ with T, T^{-1} being polynomial in x, and S, S^{-1} being polynomial in x^{-1}, or prove that this decomposition does not exist.

   If the decomposition does not exist, ϵ-form can not be reached. Otherwise, apply transformation T to obtain global normalized Fuchsian form.
4. Factor ϵ out by means of x-independent transformation $T(\epsilon)$, or prove that it is impossible. The latter means that ϵ-form can not be reached.

Summary and Outlook

1. The reduction algorithm of Ref. (RL, 2015) is working very well, there are now two public codes implementing it.
2. The algorithm has now a strict termination criterion.
3. The choice of a 'proper' variable is no longer question of luck.

Outlook

- How to minimally extend the class of transformations to achieve the ϵ-form for the 'elliptic' cases? In particular, are modular forms (Adams and Weinzierl, 2017) not only sufficient but necessary (see talk by Luise Adams tomorrow)? Also, are they sufficient for general setup?
- How to develop a solid algorithm for multivariate case? See the recent progress by Meyer (2017). Also, knowing the correct alphabet (several talk yesterday) may help a lot.

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