First Two-Loop Amplitudes with the Numerical Unitarity Method

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Based on work with S. Abreu, F. Febres-Cordero, H. Ita, M. Jaquier and M. Zeng.

[arXiv:1703.05273] and [arXiv:1703.05255]

LHC Era Phenomenology

- ▶ High-luminisoty run of LHC will substantially improve experimental precision.

- ▶ Need for NNLO predictions for many processes.

- ▶ ⇒ Two-loop multiparton QCD amplitudes.

ATLAS Simulation

$\sqrt{s}$ = 14 TeV: $\int$Ldt=300 fb^{-1} ; $\int$Ldt=3000 fb^{-1}

$\int$Ldt=300 fb^{-1} extrapolated from 7+8 TeV

H→μμ
ttH,H→μμ
VBF,H→ττ
H→ ZZ
VBF,H→ WW
H→ WW
VH,H→γγ
ttH,H→γγ
VBF,H→γγ
H→γγ (+j)
H→γγ

0   0.2   0.4   0.6   0.8

$\frac{\Delta\mu}{\mu}$
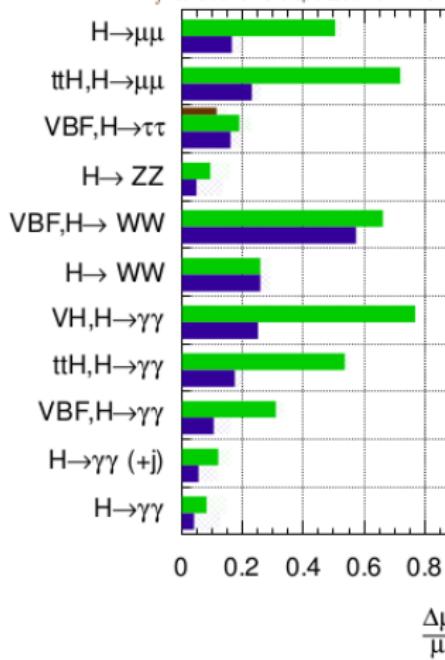
The Canonical Approach to Two-loop Amplitudes

Feynman diagrams

|
Tensor reduction
[Tarasov '96; Anastasiou, Glover, Oleari '99]
IBPs
[Tkachov, Chetyrkin '81]
↓

Sum of master integrals

$$A = \sum_{\Gamma \in \Delta} \sum_{i \, \in \, M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}.$$

|
Differential equations
[Gehrmann, Remiddi '01; Henn '13]
↓

Integrated form

- ▶ Standard procedure.

- ▶ Process specific.

- ▶ Large intermediate expressions.

- ▶ Can we repeat the leap made at NLO? [OPP '07]

The Numerical Unitarity Approach

▶ Take an ansatz for loop-amplitude integrand, decomposing into master (M_Γ) and surface (S_Γ) integrands.

$$\mathcal{A}(\ell_I) = \sum_{\text{Topologies } \Gamma} \ \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}(D) \, m_{\Gamma,i}(\ell_I, D)}{\prod_{j \in P_\Gamma} \rho_j}.$$

▶ Numerically determine $c_{\Gamma,i}(D)$ from on-shell information.

$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_I^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(D) \, m_{\Gamma',i}(\ell_I^\Gamma, D)}{\prod_{j \in (P_{\Gamma'} \setminus P_\Gamma)} \rho_j(\ell_I^\Gamma)}.$$

[BDDK '94, '95]

▶ Insert master integrals, expand $\Rightarrow$ integrated amplitude.

# Setting Up Numerical Unitarity @ 2-Loops

$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(D)\, m_{\Gamma',i}(\ell_l^\Gamma, D)}{\prod_{j \in (P_{\Gamma'} \setminus P_\Gamma)} \rho_j(\ell_l^\Gamma)} \, .$$

New challenges:

- ▶ Master/surface decomposition.
- ▶ Sub-leading poles.

Old friends:

- ▶ Fast products of trees, T_Γ.
- ▶ Colour decomposition.
- ▶ Coefficient determination, $c_{\Gamma,i}$.
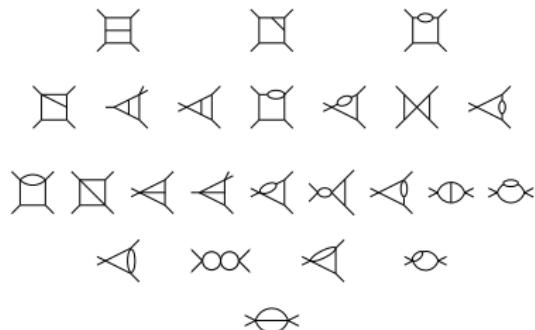
4-gluon Amplitude Topologies Γ

Master/Surface Decomposition [Abreu, Febres-Cordero, Ita, Jaquier, B.P., Zeng '17]

- Surface terms naturally produced by IBPs.

$$0 = \int \prod_{l=1,2} d^D \ell_l \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right].$$

- Generate complete set for a topology Γ by controlling powers of ρ_j.

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j.$$

[Gluza et al '11]

- Write loop momenta in propagator and ISP coordinates, controlling extra DoFs with constraint equations.

$$\ell_l^\mu \to (\rho_i, \alpha_j),$$

$$\mu_{ll} = (\mu_l)^2 = \rho_{l0} - \sum_{\nu=0}^{3} \ell_l^\nu \ell_{l\nu}.$$

[Ita '15]

Master/Surface Decomposition [Abreu, Febres-Cordero, Ita, Jaquier, B.P., Zeng '17]

- Vectors can be expressed in the new coordinates.

- They satisfy a compatability condition - a syzygy relation solvable with SINGULAR.

related [Larsen, Zhang '16; Georgoudis, Larsen, Zhang '16]

- Total derivatives using vectors → surface terms.

- Master integrands, M_Γ, fill remaining space.

$$u = f_i \rho_i \frac{\partial}{\partial \rho_i} + u_j \frac{\partial}{\partial \alpha_j} + \sum_{l,l'=1,2} f^l_{l'} \mu^k_l \frac{\partial}{\partial \mu^k_{l'}},$$

$$= \bar{u} + u_\epsilon.$$

$$\bar{u}(\mu_{11}) = 2\mu_{11} f^1_1 + 2\mu_{12} f^2_1,$$
$$\bar{u}(\mu_{22}) = 2\mu_{22} f^2_2 + 2\mu_{12} f^1_2,$$
$$\bar{u}(\mu_{12}) = \mu_{12}(f^1_1 + f^2_2) + \mu_{11} f^1_2 + \mu_{22} f^2_1.$$

$$m_{\Gamma,u}(\ell_l) = \left[-(\nu_i - 1)f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha^j} + \left(D - \frac{n_\alpha + 1}{2} \right)(f^1_1 + f^2_2) \right].$$

Sub-leading Poles

[Abreu, Febres-Cordero, Ita, Jaquier, B.P. '17]

- Beyond one loop multiple poles can be associated to a given factorization limit.

- Only leading poles given by product of trees.

- Some numerators lack associated cut equation.

- Numerators determined from descendant cut equations.
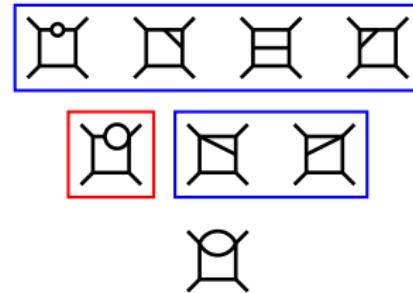
$$\boxed{} - \sum_{\substack{\Gamma \in \Delta \setminus \tilde{\Delta} \\ \Gamma > \Gamma'}} \frac{N(\Gamma, \ell_I^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_I^{\Gamma'})} \cdot$$

$$= N\left(\boxed{}\right) + \frac{1}{\rho} N\left(\boxed{}\right)$$

A Proof of Principle - Gluon Gluon Scattering

▶ Aim: To test the Numerical Unitarity approach @ 2 loops for phenomenologically relevant amplitudes.

▶ We recompute the known 4-gluon scattering amplitude in an arbitrary helicity configuration [Anastasiou et al '01; Bern et al '02].

▶ A gym with all necessary ingredients and an analytic target.

Tree Amplitudes and Colour

- Off-shell recursion. [Berends, Giele '87]

- Flexible spin DoF - "D_s".

- Extensive caching across different helicities and orderings.
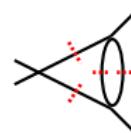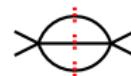
- Leading colour produced by projecting colour decomposition of product of trees to integrand.

  [B.P., Ochirov '16]

$$\tilde{N}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\!\!\prec\!\!\lhd\right) = \ C\left(\begin{smallmatrix}1\\2\end{smallmatrix}\!\!\prec\!\!\lhd\right)N\left(\begin{smallmatrix}1\\2\end{smallmatrix}\!\!\prec\!\!\lhd\right)$$
$$+ \ C\left(\begin{smallmatrix}2\\1\end{smallmatrix}\!\!\prec\!\!\lhd\right)N\left(\begin{smallmatrix}2\\1\end{smallmatrix}\!\!\prec\!\!\lhd\right)$$

Solving for Master Integral Coefficients

$$\sum_{\substack{\Gamma \in \widetilde{\Delta} \\ \Gamma \geq \Gamma' \\ i \in M_\Gamma}} \frac{c_{\Gamma,i}(D)\, m_{\Gamma,i}(\ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})} = \sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^{\Gamma}) \; - \sum_{\substack{\Gamma \in \Delta \setminus \widetilde{\Delta} \\ \Gamma > \Gamma'}} \frac{N(\Gamma, \ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})} .$$

- ▶ Sample randomly on-shell phase space ℓ_l^Γ to constrain $c_{\Gamma,i}$.
- ▶ PLU factorization for $n \times n$ systems.
- ▶ QR factorization for $n \times m$ overdetermined systems.
- ▶ LAPACK for double precision. [Anderson et al '99]
- ▶ MPACK for high precision. [Nakata '10]

Can determine master integral coefficients for fixed D and D_s.

Ds-Dependence of Coefficients

- Amplitude depends on $(D_s - 2)$, the number of spin states.

- D_s dependence of amplitude is at most quadratic.

- One power of D_s from each separable component e.g.
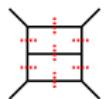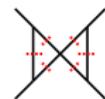
   is linear in D_s is quadratic in D_s

- Interpolate from products of trees with different D_s. [Giele et al '08]

- Then use t'Hooft Veltman scheme, $D_s = D$.

D Dependence of Coefficients

- Coefficients inherit D-dependence from surface terms recall

$$m_{\Gamma,u}(\ell_l) = \left[-(\nu_i - 1)f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha^j} + \left(D - \frac{n_\alpha + 1}{2} \right)(f_1^1 + f_2^2) \right].$$

- Resulting coefficients are rational functions of D, i.e.

$$c(D) = \frac{P(D, s, t)}{Q(D)} = \frac{p_0(s,t) + p_1(s,t)D + ... + p_i(s,t)D^i}{q_0 + q_1 D + ... + q_{j-1}D^{j-1} + D^j}.$$

- Can reconstruct $c(D)$ by sampling various D.

  related [Peraro '16], [Schabinger, von Manteuffel '14]

- Only $P(D, s, t)$ is kinematically dependent.

- We determine $Q(D)$ from a single high precision evaluation.

The Finished Product

- We thus determine $A(D)$ for a given numerical (s, t).

- $A(D)$ is exact in D and we verify it against analytics.

- We insert the integrals and expand around $D = 4$.

$$A(D) = c_0 \, (\boxminus) \, I_0 \, (\boxminus) + c_1 \, (\boxminus) \, I_1 \, (\boxminus)$$
$$+ c \, (\boxtimes) \, I \, (\boxtimes) + c \, (\triangleleft) \, I \, (\triangleleft)$$
$$+ c \, (\boxtimes) \, I \, (\boxtimes) + c \, (\infty) \, I \, (\infty)$$
$$+ c \, (\infty) \, I \, (\infty) + (s \leftrightarrow t).$$

With, $g_s = 1$, $\mu = 1$, $s = -\frac{1}{4}$ and $t = -\frac{3}{4}$ we find:

$\mathcal{A}/(\mathcal{A}_0 N_c^2)(4\pi)^4$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^0
$(1_g^-, 2_g^+, 3_g^-, 4_g^+)$	8.00000	55.6527	176.009	332.296	486.502
$(1_g^-, 2_g^-, 3_g^+, 4_g^+)$	8.00000	55.6527	164.642	222.327	-8.39044

Numerical Stability of MMPP Amplitude
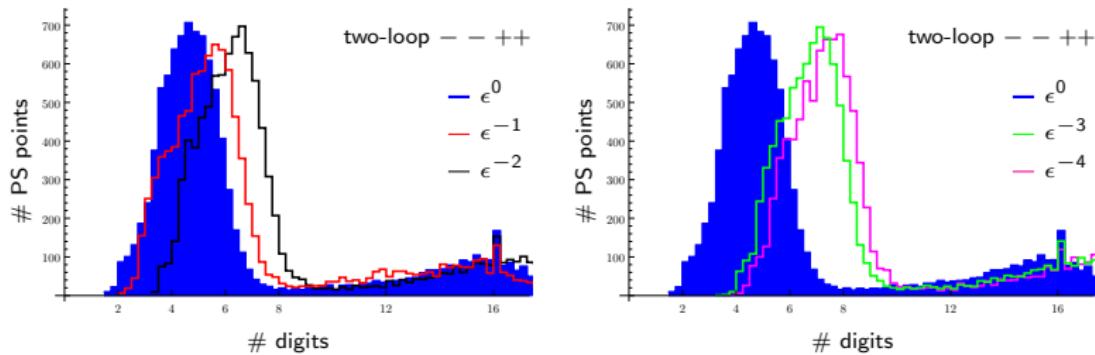
- ▶ Coefficient precision often good to ~ 8 digits.
- ▶ ~ 4 digit precision loss when inserting integrals.
- ▶ There might exist a basis which better controls the precision.

Analytics from Numerics

- Rescaled integral coefficients are rational functions of $x = \frac{t}{s}$.

- Reconstruct using same techniques as for the regulator.

  related [Peraro '16], [Schabinger, von Manteuffel '14]

- Exact analytic results from numerics, e.g (mmpp):

$$c_0 \left(\boxed{}\right) = \frac{9x + \frac{\epsilon\left(-x^3 - \frac{32x^2}{11} - \frac{97x}{44} - \frac{5}{22}\right)}{\frac{x^2}{33} + \frac{2x}{33} + \frac{1}{33}} + \frac{\epsilon^2\left(-x^3 - \frac{385x^2}{51} - \frac{937x}{102} - \frac{77}{34}\right)}{-\frac{2x^2}{51} - \frac{4x}{51} - \frac{2}{51}} + \cdots}{-9 + 66\epsilon - 184\epsilon^2 + 240\epsilon^3 - 144\epsilon^4 + 32\epsilon^5}$$

- Requires 15 evaluations of the amplitude in high precision.

Conclusions

- We set up a numerical algorithm for two-loop amplitudes.

- Analytic results can be reconstructed from numerical samples.

- A proof of principle calculation of the 4-gluon amplitude.

- The method shows promise for phenomenological calculations.

Univariate Rational Function Reconstruction

- Thiele's formula gives $c(D)$ as a continued fraction [Peraro '16]

$$c(D) = a_0 + \cfrac{D - D_0}{a_1 + \cfrac{D - D_1}{a_2 + \cfrac{D - D_2}{\cdots + \cfrac{D - D_{N-1}}{a_N}}}}.$$

- Can fix coefficients a from values $c(D_i)$.
- Easy to convert back to canonical $\frac{P(D)}{Q(D)}$.
- Only $P(D)$ is kinematically dependent.
- We fix the rational coefficients of $Q(D)$ once in high precision, recovering the exact form with continued fractions.

Numerical Stability for MPMP
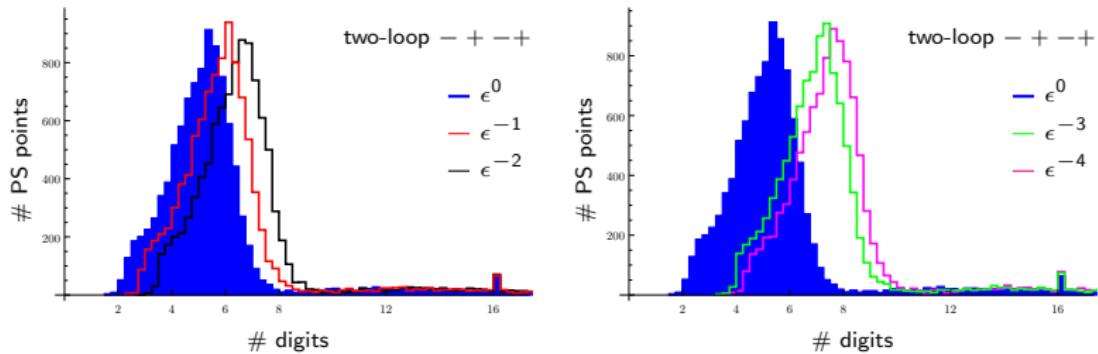
- Comparison to analytics over 10000 phase space points.
- Rescue system based on accuracy of universal $\frac{1}{\epsilon}$ pole.