The S-matrix Bootstrap

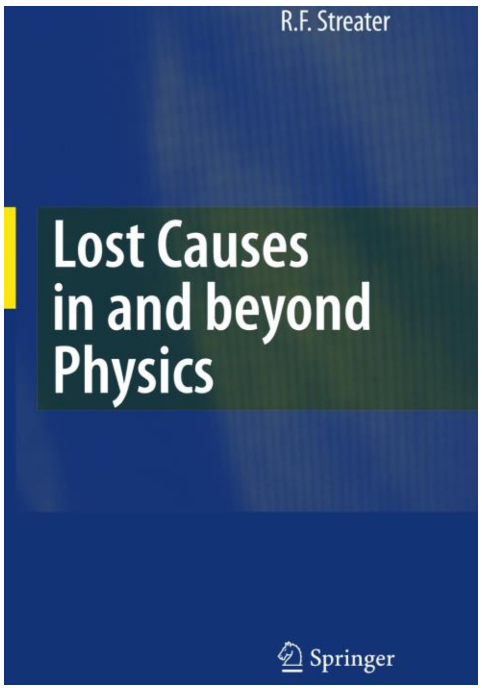
Miguel F. Paulos
CERN

Based on: arXiv:1607.06109, 1607.06110, *17xx…*

w/ J. Penedones, J. Toledo, B. van Rees and P. Vieira.

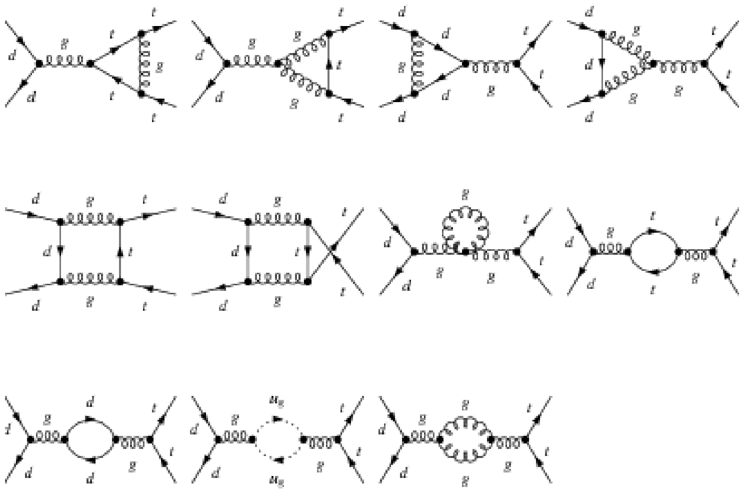
Amplitudes 2017

Why do this?

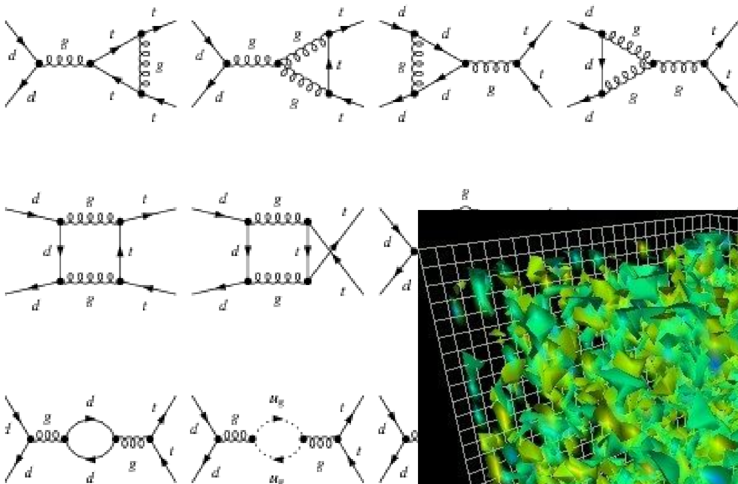


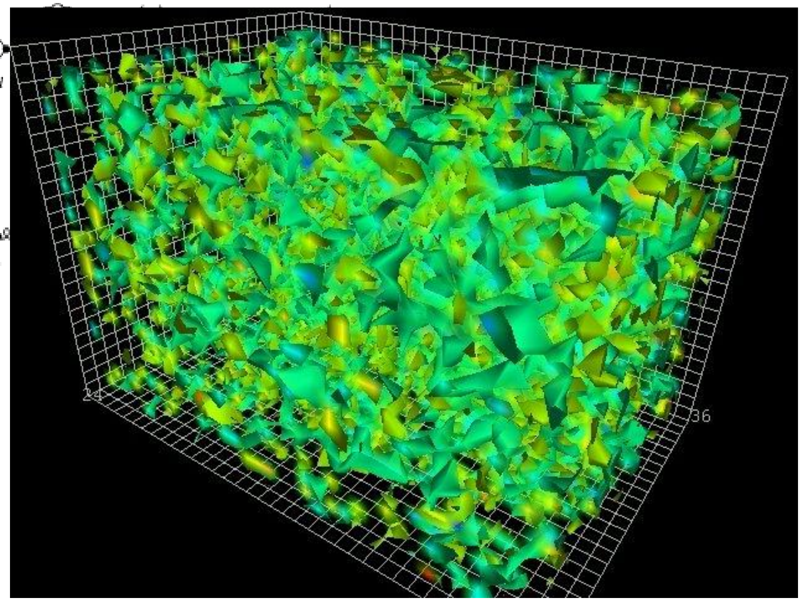
"It may be flogging a dead horse, but I will mention this subject, and why it failed."

Quantum Field Theory is hard!

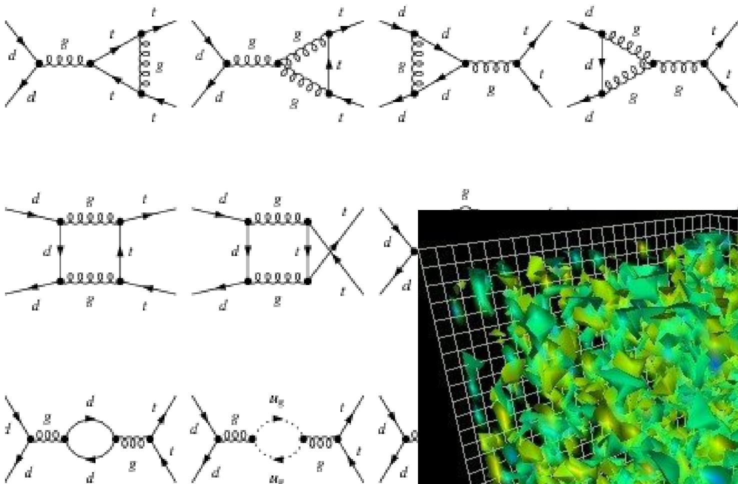


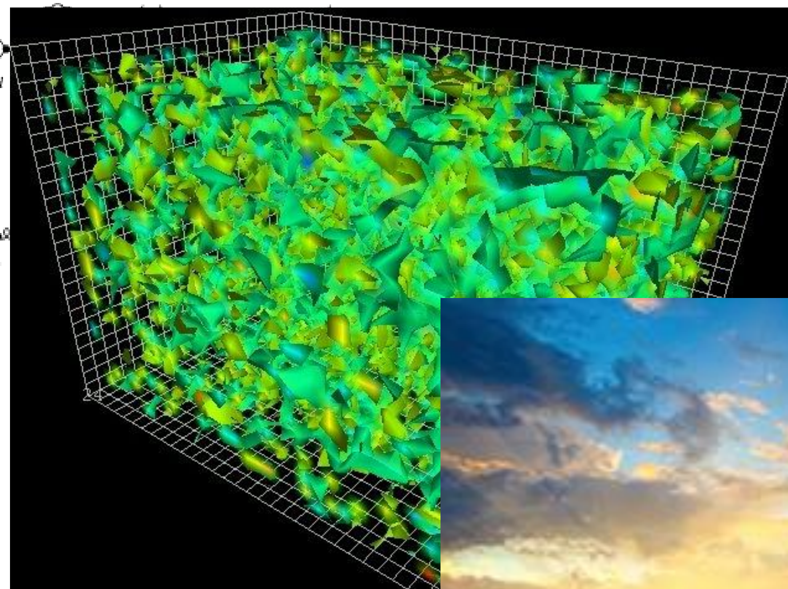
Quantum Field Theory is hard!





Quantum Field Theory is hard!





The bootstrap manifesto:

To constrain and determine
quantum field theories
from basic principles

Unitarity

Locality

Crossing

The Analytic S-matrix

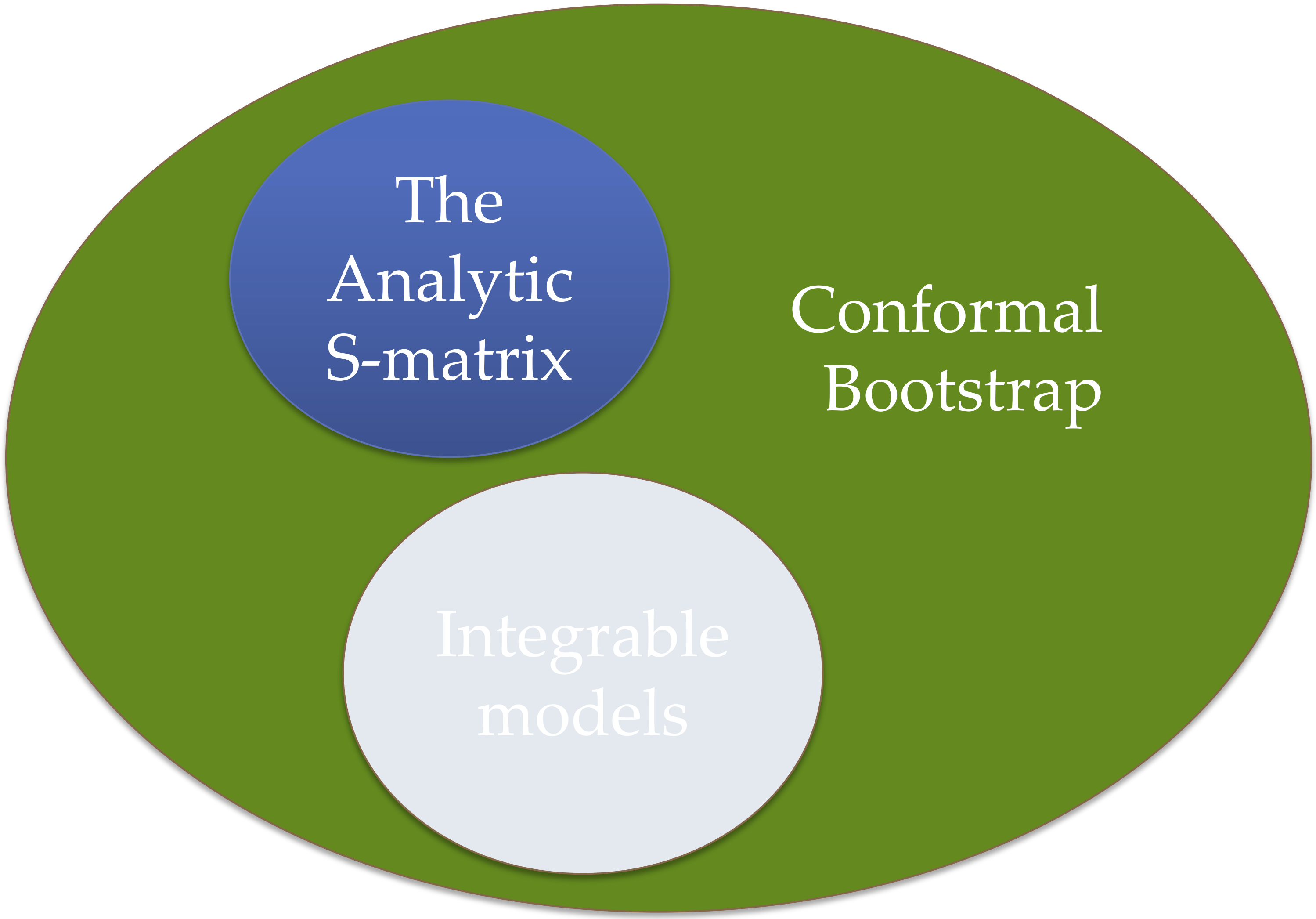
Lehmann, Symanzik, Zimmerman Froissart, Gribov, Martin, Chew, Frautschi, Mandelstam…

Conformal Bootstrap

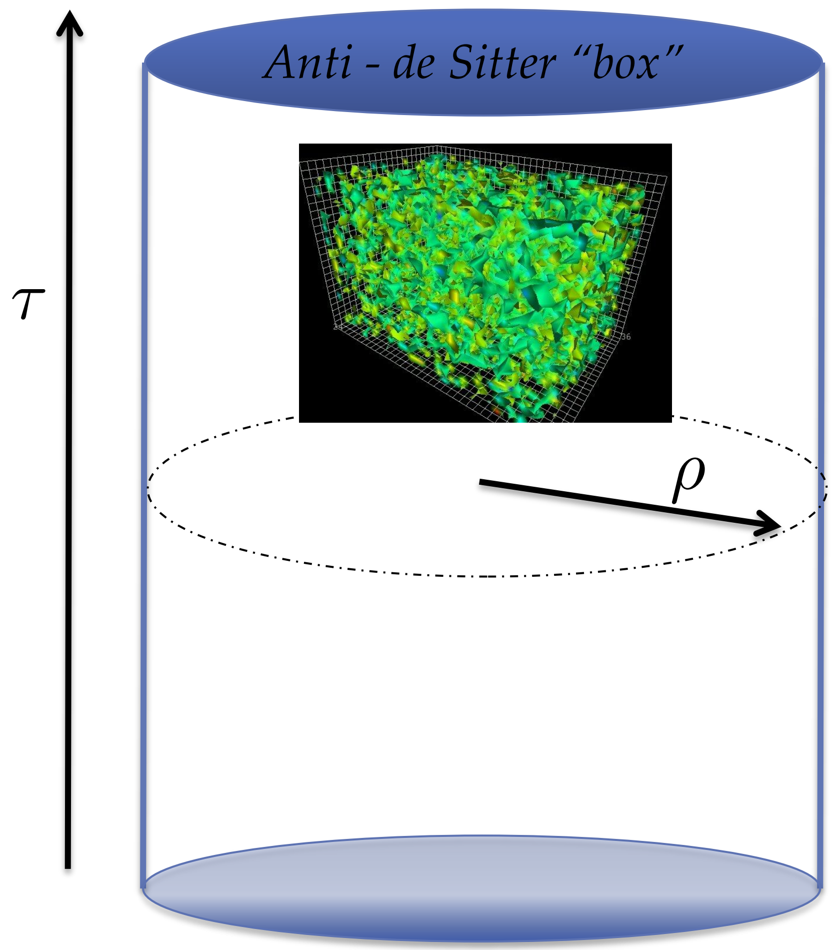
Ferrara, Gatto, Grillo, Parisi, Belavin, Polyakov, Al. Zamolodchikov, Rattazzi, Rychkov, Tonni, Vichi…

Integrable models

Bethe, Yang, Baxter, Faddeev, Onsager, Zamolodchikov^2,…


The Analytic S-matrix
Conformal Bootstrap
Integrable models

QFT in a box



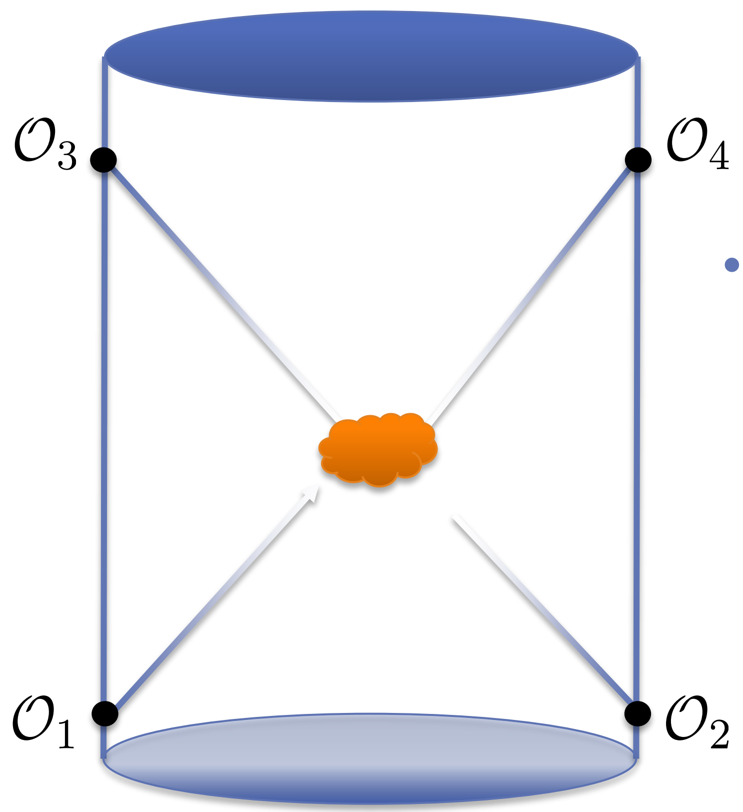
- Put a D-dimensional QFT in anti-de Sitter space.
- AdS acts as a nice box preserving as many symmetries as flat space.
- On the boundary of AdS lives a CFT in D-1 dimensions which should capture all the bulk physics.
- ((Somewhat) boring) example of holography.

CFT to S-matrix

- Large AdS radius recovers flat space scattering.

$$\Delta_{\mathcal{O}} = mR_{\text{AdS}}$$

- CFT operators with large scaling dimension!

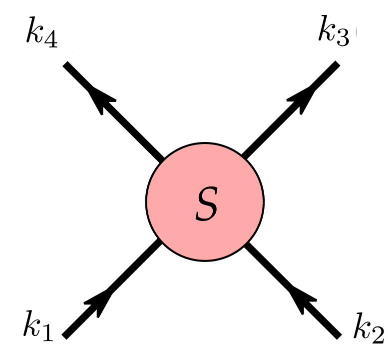


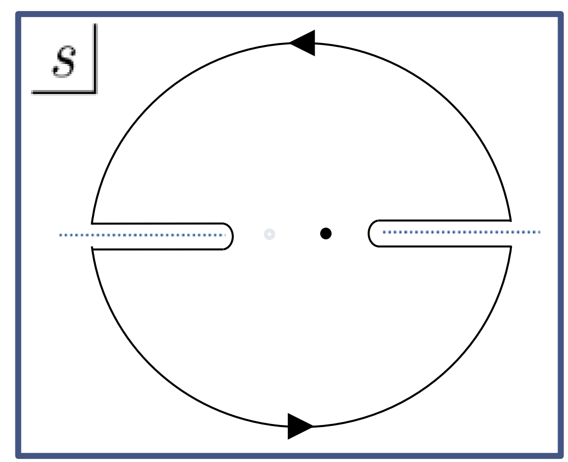
- Rest of the dictionary:

$$\langle \mathcal{O}(x_4)\mathcal{O}(x_3)\mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$$
$$\downarrow$$
$$\langle k_3, k_4 | 1 + i\mathcal{T} | k_1, k_2 \rangle$$

The S-matrix bootstrap

- We will parametrize the S-matrix directly.
- Using known analyticity and crossing properties, write down dispersion relation.
- Use unitarity to place constraints on parameters.
- Later, parametrize directly without using dispersion relation.





Bootstrapping the S-matrix

The S-matrix in 1+1 QFT

- Real analytic.

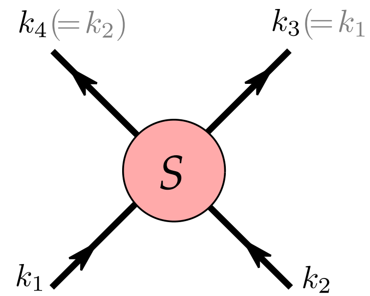
- Crossing symmetric:

$$S(s) = S(4m^2 - s)$$

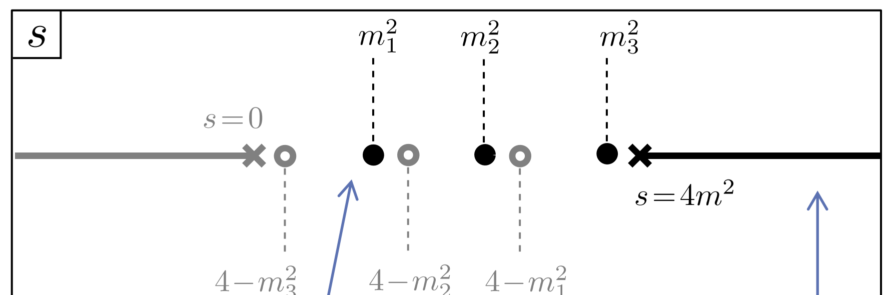
- Poles at position of stable particles.

- Cuts corresponding to multiparticle exchange.

- Polynomially bounded.



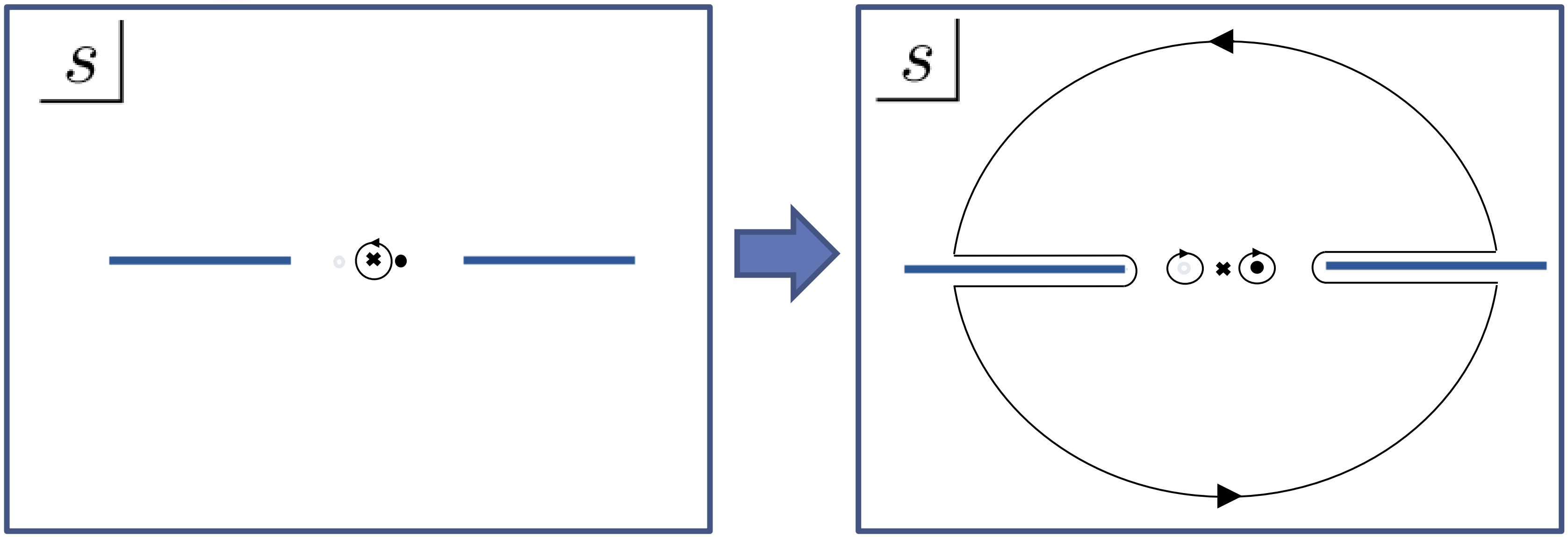
$$s \equiv (k_1 + k_2)^2$$
$$t \equiv (k_2 - k_3)^2 = 4m^2 - s$$
$$u \equiv (k_3 - k_1)^2 = 0$$



Multiparticle s-channel cut

s-channel pole

Dispersion relation



$$S(s) = S_\infty - \sum_j \mathcal{J}_j \left(\frac{g_j^2}{s - m_j^2} + \frac{g_j^2}{4m^2 - s - m_j^2} \right) + \int_{4m^2}^{\infty} dx\, \rho(x) \left(\frac{1}{x - s} + \frac{1}{x - 4m^2 + s} \right)$$

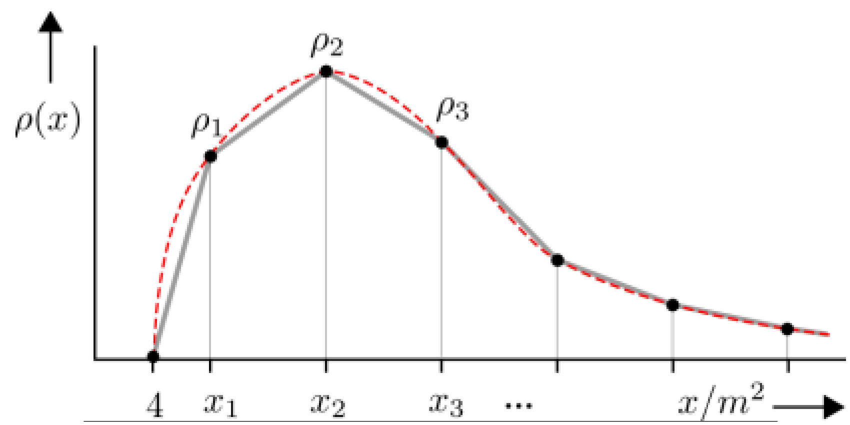
Simple Jacobian from working with S not T.

Discontinuity across cut

Unitarity constraints

- For a physical scattering process, unitarity has to be obeyed.

$$|S(s)|^2 \leq 1, \qquad s \geq 4m^2$$

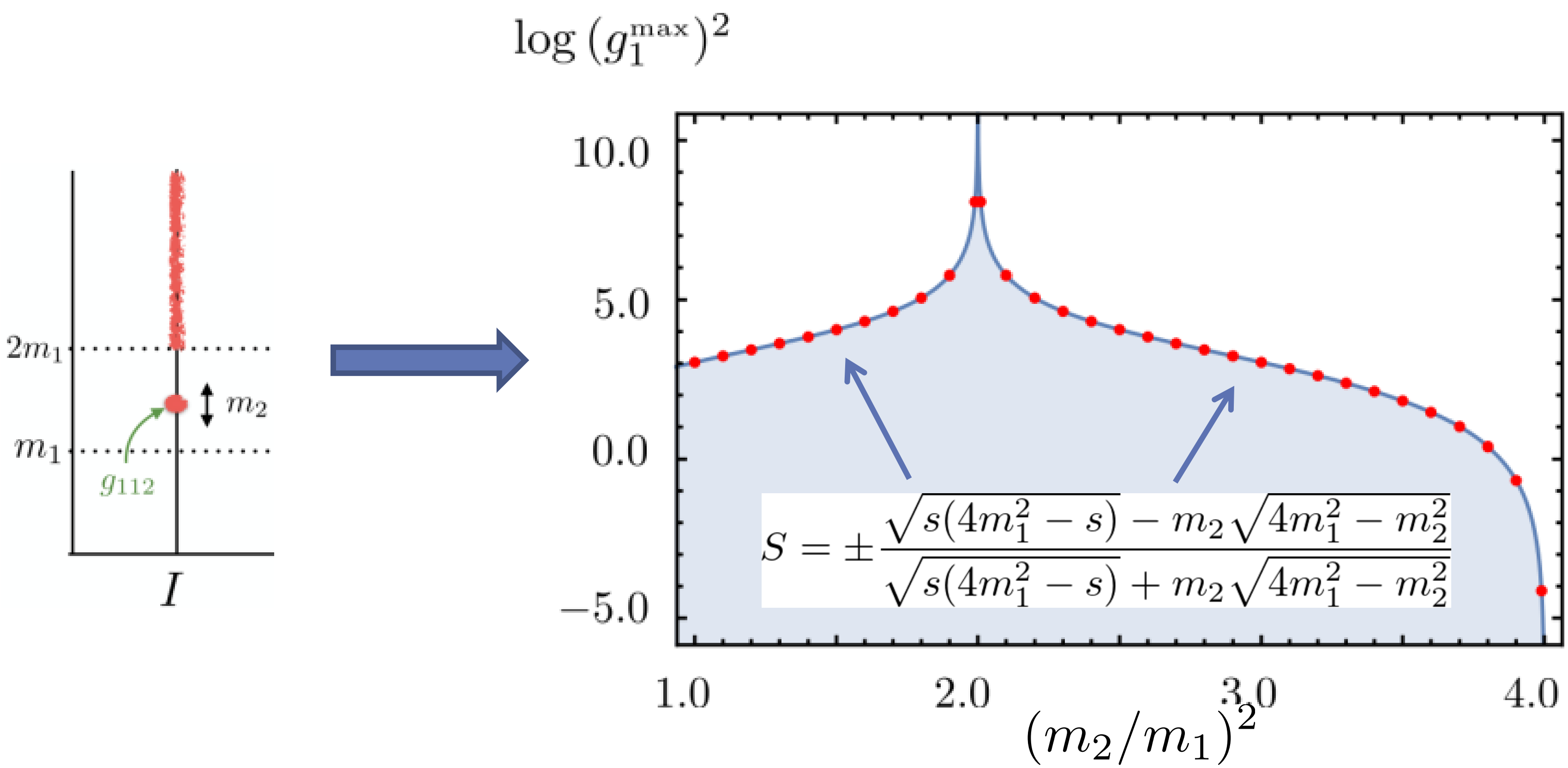


- This implies constraints on couplings and discontinuity. We parametrize discontinuity in a simple way.

$$\boxed{S(s) \sim S_\infty - \sum_j \mathcal{J}_j \left(\frac{g_j^2}{s - m_j^2} + \frac{g_j^2}{4m^2 - s - m_j^2} \right) + \sum_{a=1}^{M} \rho_a K_a(s)}$$

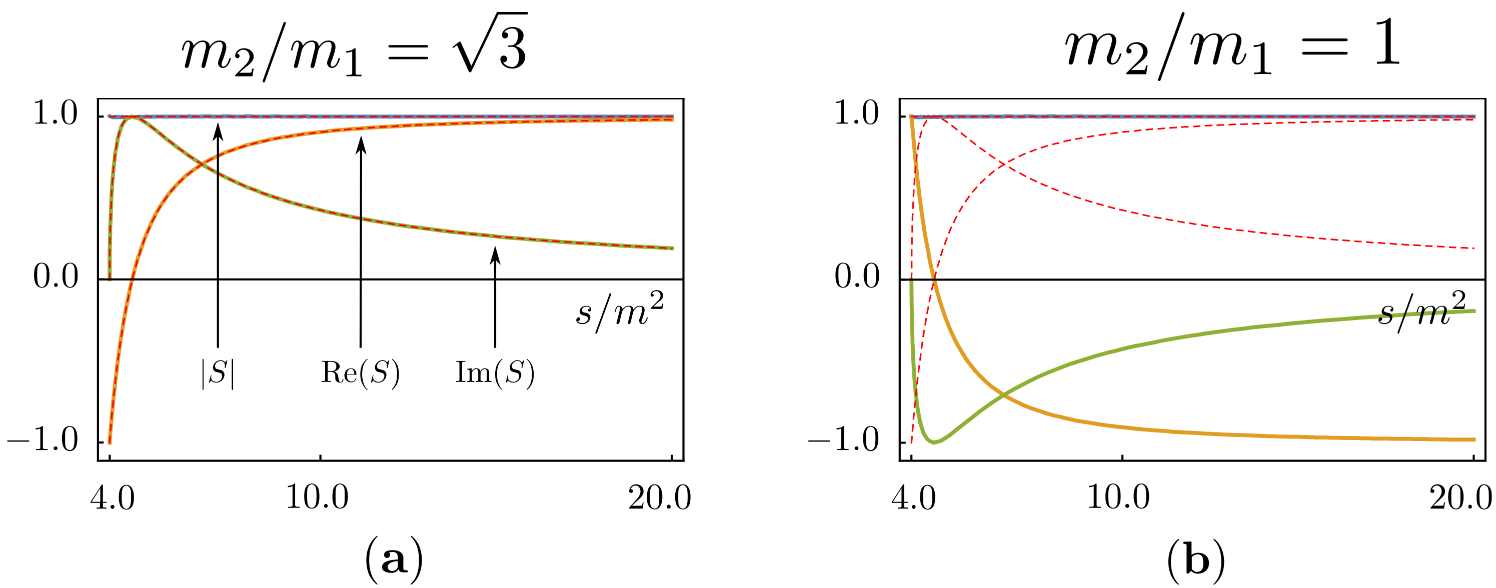
- We maximize coupling subject to unitarity constraints to obtain a bound.

Single particle exchange



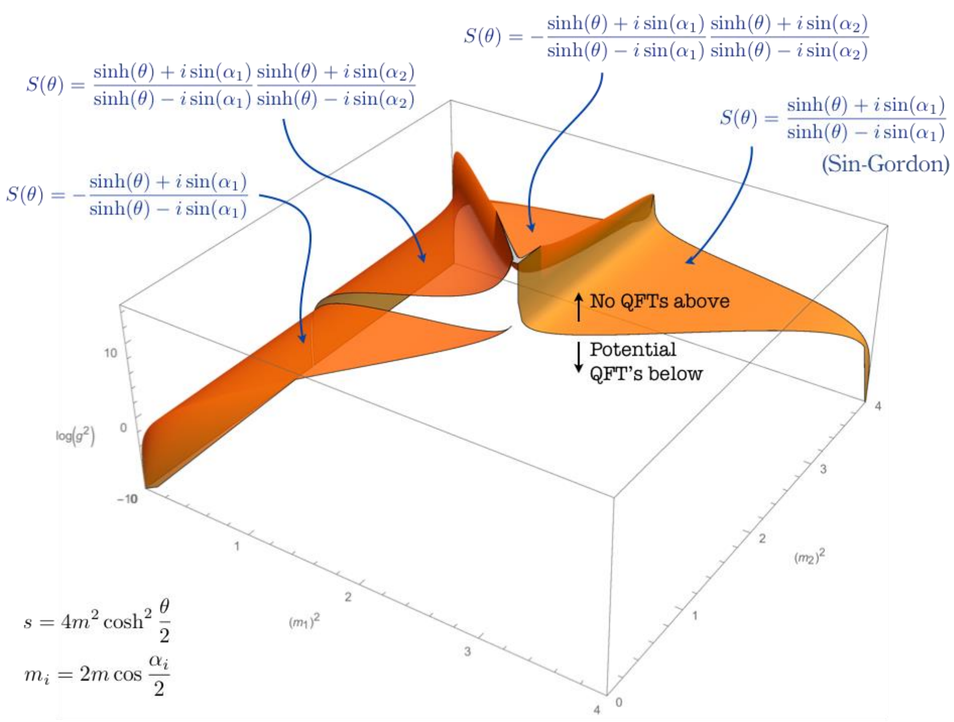
- Universal bounds in 1+1d QFTs

Single particle exchange



- Red dashed curves: S-matrix of Sin-Gordon model, scattering of two lightest breathers. Integrable model, explains absence of particle production.

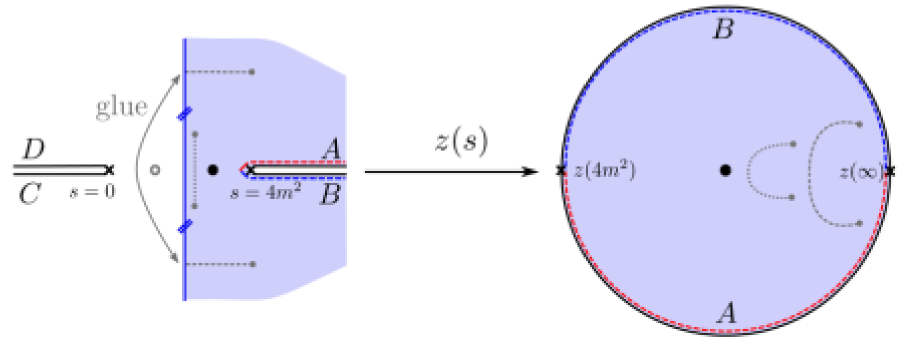
S-matrix with two bound states



- We can generalize analysis to more complicated scenarios.
- In all cases, S-matrix that saturates bounds is given by a product of Castillejo-Dalitz-Dyson (CDD) factors, i.e. it can be written down explicitly.

Analytic bootstrap

- Bounds are exact and can be obtained analytically (Creutz '72 (!))

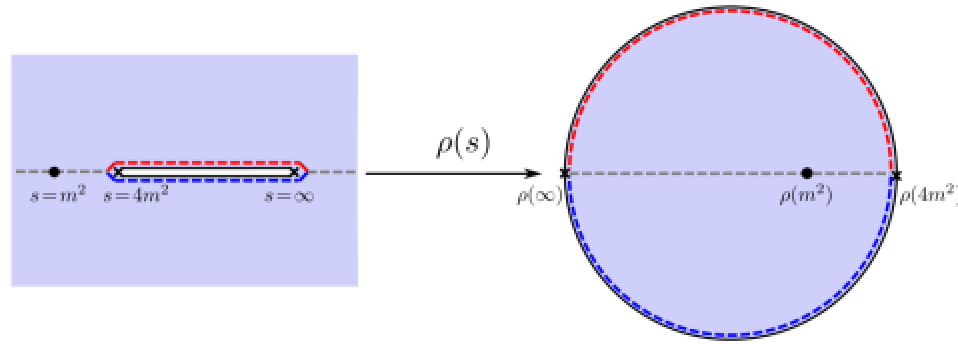


$$z = \frac{\sqrt{s(4m^2 - s)} - \sqrt{s_0(4m^2 - s_0)}}{\sqrt{s(4m^2 - s)} + \sqrt{s_0(4m^2 - s_0)}}$$

$$g^2 \propto \left| \oint \frac{dz}{2\pi i} S(z) \right| \leq \frac{1}{2\pi} \oint \frac{dz}{iz} |S(z)| \leq 1$$

A better ansatz

- Map again to the disk, but separating out the s- and t-channel cuts.



$$s \to \rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

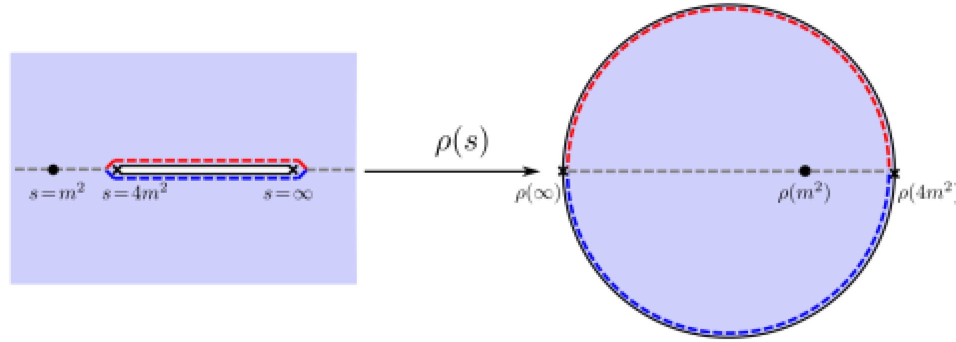
- Make the new ansatz, independent of discretizations:

$$S(s,t) = \left[-\frac{\hat{g}^2}{s - m_b^2} - \frac{\hat{g}^2}{t - m_b^2} + \sum_{a,b=0}^{+\infty} c_{ab} \rho_s^a \rho_t^b \right]_{s+t=4m^2}$$

- Recover previous results but more efficiently.

A better ansatz

- Map again to the disk, but separating out the s- and t-channel cuts.



$$s \to \rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

- Suggests higher dimensional generalization:

$$M(s,t,u) = \left[-\frac{g^2}{s - m_b^2} - \frac{g^2}{t - m_b^2} - \frac{g^2}{u - m_b^2} + \sum_{a,b,c=0} \alpha_{abc} \rho_s^a \rho_t^b \rho_u^c \right]_{s+t+u=4m^2}$$

- This ansatz is consistent , i.e. analytic extension is well defined because maths. (*cf*. Cartan's theorem B for Stein manifolds).

General strategy

$$M(s,t,u) = -\frac{g^2}{s-m_b^2} - \frac{g^2}{t-m_b^2} - \frac{g^2}{u-m_b^2} + \sum_{a,b,c=0}^{Nmax} \alpha_{abc}\rho_s^a \rho_t^b \rho_u^c$$

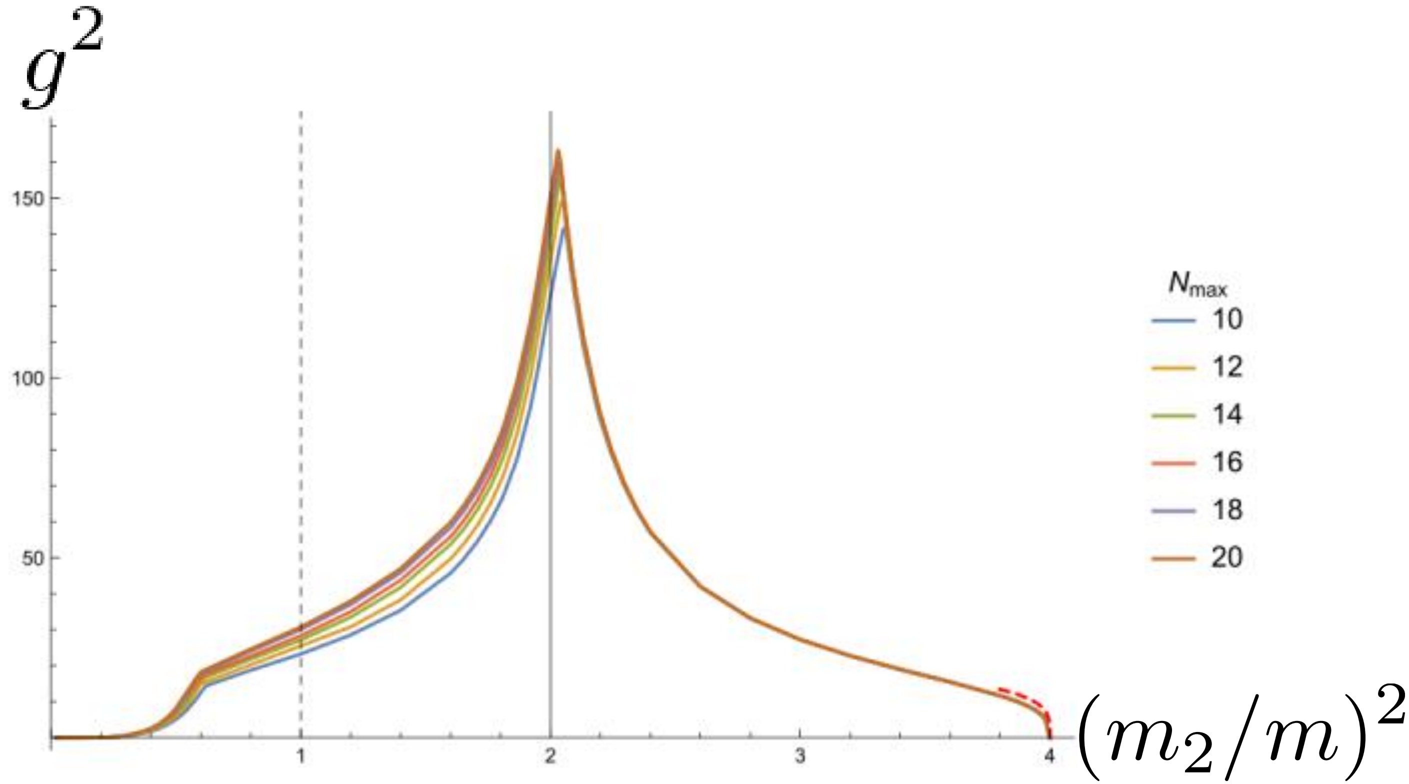
- Given this parametrization, we proceed as before: maximize coupling subject to unitarity.
- The latter is imposed by decomposing the amplitude in terms of partial waves,

$$S_l(s) = 1 + i\,\frac{(s-4)^{\frac{d-2}{2}}}{\sqrt{s}} \int_{-1}^{1} dx (1-x^2)^{\frac{d-3}{2}}\, P_l^{(d)}(x)\, M(s,t)|_{t\to\frac{1}{2}(s-4)(x-1)}$$

- and imposing |S_L|<=1 in the physical region up to some fixed maximum spin which we can vary.

Results: one bound state

- Universal bound on D=3+1 Quantum Field Theories



- Puzzle: as numerics improve, bounds seem to be saturated by S-matrices with almost no particle production.

Maximizing the pion coupling

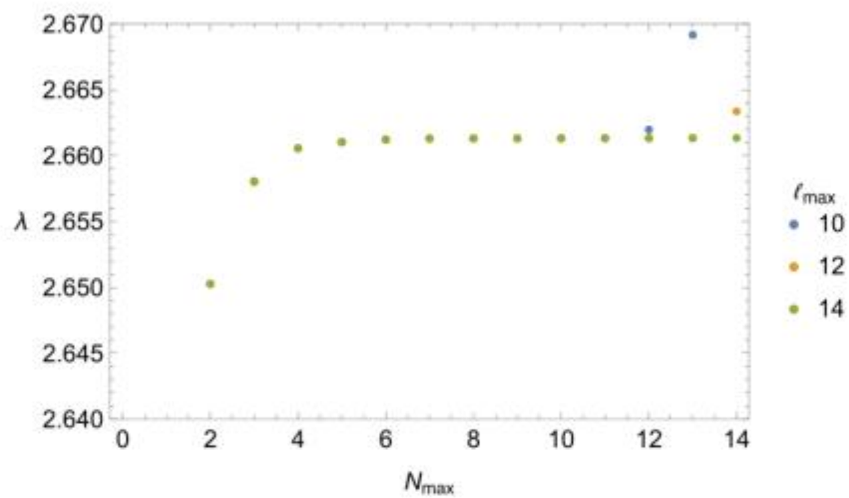


Figure 6: Maximal value of the quartic coupling $\lambda \equiv \frac{1}{32\pi} M(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ achieved with the ansatz (13) (with $g = 0$) plus the term (18). With this improved ansatz, the maximal quartic coupling effectively saturates for $N_{max} \gtrsim 6$.

$$M(s,t,u) \to M(s,t,u) + c_0 \left(\frac{1}{\sqrt{4m^2 - s}} + \frac{1}{\sqrt{4m^2 - t}} + \frac{1}{\sqrt{4m^2 - u}} \right) + \dots$$

Explicitly constructed (approximate) amplitude $\longrightarrow$ $$2.62 < 2.661 < 2.75$$ $\longleftarrow$ *Exact (old) analytic result*

Minimizing the pion coupling

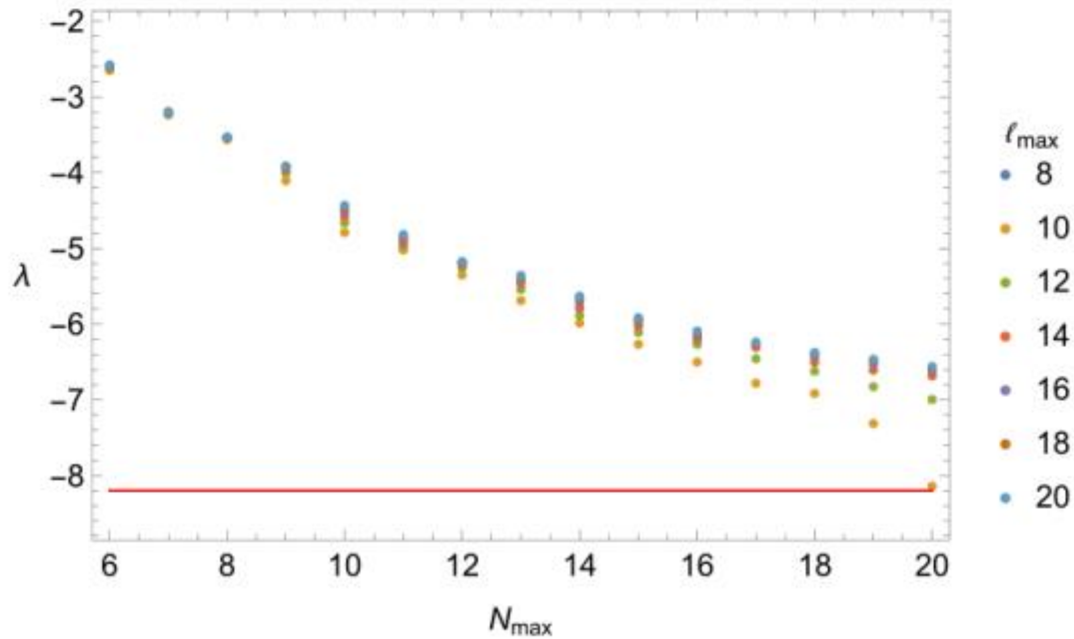


Figure 7: Minimal value of the quartic coupling $\lambda \equiv \frac{1}{32\pi} M(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ achieved with the ansatz (13) (with $g = 0$). With this ansatz, the minimal quartic coupling continues to decrease significantly with N_{max} even for $N_{max} = 20$.

Summary and Outlook

- The S-matrix can be bootstrapped directly. Clever choice of parametrization + computer power (<<< M.C.).
- In 1+1, bounds saturated by integrable or integrable-like solutions which can be written down explicitly. In higher dimensions no exact results, but low/trivial particle production?
- Once multiple scattering processes are included, expect particle production to appear – not all S-matrices obtained corresponded to consistent theories.
- Other generalizations: global symmetries (isospin,…), vector particles, fermions, SUSY…
- What can we say analytically in higher dimensions? Take inspiration from CFT bootstrap..