Feynman integrals as Periods

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Amplitudes 2017, Higgs Center, Edinburgh, UK

based on [arXiv:1309.5865], [arXiv:1406.2664], [arXiv:1601.08181]

Spencer Bloch, Matt Kerr

Scattering amplitudes are essential tools to understand a variety of physical phenomena from gauge theory to classical and quantum gravity

A convenient approach is to use modern unitarity methods for expanding the amplitude on a basis of integral functions

$$A^{\mathrm{L-loop}} = \sum_{i \in \mathcal{B}(L)} \mathrm{coeff}_i \, \mathrm{Integral}_i + \mathrm{Rational}$$

What are the intrinsic properties of amplitudes of QFT in flat space?

How much can we understand about the amplitudes without having to compute them?

- Generic constraints on the integral coefficients? [cf Tourkine's talk]
- What are the elements of the basis of integral functions?

These questions are central to many talks at this meeting

-It's bigger on the inside!

-Yes, the TARDIS is dimensionally transcendental

In this talk we will present some approach to Feynman integral that makes a essential use of the algebraic geometric setup defined by the graphs

This approach is an alternative to the IBP method described in [Lee]'s and [Tancredi]'s talk

Feynman Integrals: parametric representation

Any Feynman integrals with L loops and n propagators

$$I_\Gamma = \int \frac{\prod_{i=1}^{L} d^D\ell_i}{\prod_{i=1}^{n} d_i^{\nu_i}}$$

has the parametric representation

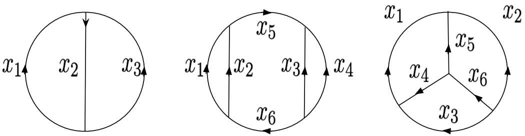
$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{x_i \geqslant 0} \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{V})^{\nu - L\frac{D}{2}}} \delta(x_n = 1) \prod_{i=1}^{n} \frac{dx_i}{x_i^{\nu_i - 1}}$$

- $\mathcal{U}$ and $\mathcal{V}$ are the Symanzik polynomials
- $\mathcal{U}$ is of degree L and $\mathcal{V}$ of degree $L+1$ in the x_i

What are the Symanzik polynomials?

$$I_\Gamma = \Gamma\left(\nu - \frac{LD}{2}\right) \int_{x_i \geqslant 0} \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{\left(\mathcal{U}\sum_i m_i^2 x_i - \mathcal{V}\right)^{\nu - L\frac{D}{2}}} \delta(x_n = 1) \prod_{i=1}^{n} \frac{dx_i}{x_i^{\nu_i - 1}}$$

$\mathcal{U} = \det \Omega$ determinant of the period matrix of the graph $\Omega_{ij} = \oint_{C_i} v_j$



(a) (b) (c)

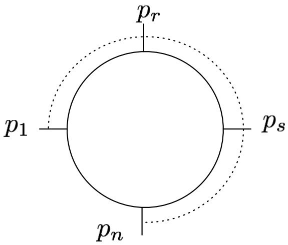
$$\Omega_{2(a)} = \begin{pmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{pmatrix}; \qquad \Omega_{3(b)} = \begin{pmatrix} x_1 + x_2 & x_2 & 0 \\ x_2 & x_2 + x_3 + x_5 + x_6 & x_3 \\ 0 & x_3 & x_3 + x_4 \end{pmatrix}$$

$$\Omega_{3(c)} = \begin{pmatrix} x_1 + x_4 + x_5 & x_5 & x_4 \\ x_5 & x_2 + x_5 + x_6 & x_6 \\ x_4 & x_6 & x_3 + x_4 + x_6 \end{pmatrix}$$

What are the Symanzik polynomials?

$$I_\Gamma \propto \int_0^\infty \frac{\delta(1-x_n)}{(\sum_i m_i^2 x_i - \mathcal{V}/\mathcal{U})^{n-L\frac{D}{2}}} \frac{\prod_{i=1}^n x_i^{\nu_i-1} dx_i}{\mathcal{U}^{\frac{D}{2}}}$$

$\mathcal{V}/\mathcal{U} = \sum_{1\leqslant r<s\leqslant n} k_r \cdot k_s G(x_r/T_r, x_s/T_s; \Omega)$ sum of Green's function



$$G^{1-loop}(\alpha_r, \alpha_s; L) = -\frac{1}{2}|\alpha_s - \alpha_r| + \frac{1}{2}\frac{(\alpha_r - \alpha_s)^2}{T}.$$

Feynman integral and periods

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{\Delta_n} \Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma(x_i)^{\nu-L\frac{D}{2}}} \prod_{i=1}^{n-1} \frac{dx_i}{x_i^{\nu_i-1}}$$

The integrand is an algebraic differential form Ω_Γ on the complement of the graph hypersurface

$$\Omega_\Gamma \in H^{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma) \qquad X_\Gamma := \{\Phi_\Gamma(\underline{x}) = 0, \underline{x} \in \mathbb{P}^{n-1}\}$$

The domain of integration is the simplex Δ_n with boundary $\partial\Delta_n \subset \mathcal{I}_n := \{x_1 \cdots x_n = 0\}$

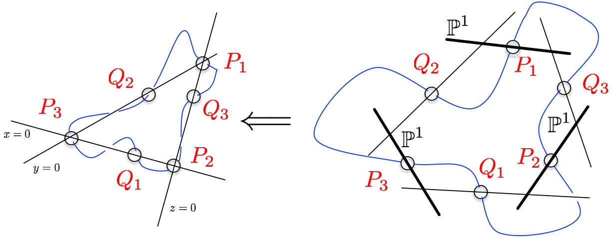
But $\partial\Delta_n \cap X_\Gamma \neq \emptyset$ therefore $\Delta_n \notin H_{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma)$

This is resolved by looking at the relative cohomology

$$H^\bullet(\mathbb{P}^{n-1} \backslash X_\Gamma; \mathcal{I}_n \backslash \mathcal{I}_n \cap X_\Gamma)$$

Feynman integral and periods

Δ_n and X_Γ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



The Feynman integral *are* periods of the relative cohomology after performing the appropriate blow-ups [Bloch, Esnault, Kreimer] and [Brown's talk]

$$H^{n-1}(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{X_F};\widetilde{\Delta_n}\backslash\widetilde{\Delta_n}\cap\widetilde{X_\Gamma})$$

Feynman integral and periods

- One can apply the construction and the discussion of the $\epsilon = (D - D_c)/2$ expansion of the Feynman integral [Belkale, Brosnan; Bogner, Weinzierl]

$$I_\Gamma = \sum_{i \geqslant -n} c_i \, \epsilon^i$$

- The periods are master integrals. The minimal number of master integral for this topology determined by the middle cohomology of the motive

$$H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\Pi_n} \backslash \widetilde{\Pi_n} \cap \widetilde{X_\Gamma})$$

Differential equation

$$\mathfrak{M}(s_{ij}, m_i) := H^\bullet(\widetilde{\mathbb{P}^{n-1}}\backslash \widetilde{X_F}; \widetilde{\Pi_n}\backslash \widetilde{\Pi_n} \cap \widetilde{X_\Gamma})$$

Since Ω_Γ varies when one changes the kinematic variables s_{ij} one needs to study a variation of (mixed) Hodge structure

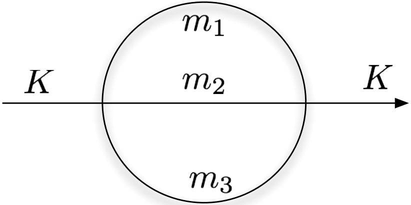
Consequently the Feynman integral will satisfy a differential equation

$$L_\Gamma \, I_\Gamma = S_\Gamma$$

The Picard-Fuchs operator will arise from the study of the variation of the differential in the cohomology when kinematic variables change

The inhomogeneous terms arises from the extension

The sunset graph



The sunset integral

We consider the sunset integral in two Euclidean dimensions

$$\mathcal{J}^2_\ominus = \int_{\Delta_3} \Omega_\ominus; \qquad \Delta_3 := \{[x:y:z] \in \mathbb{P}^2 | x \geqslant 0, y \geqslant 0, z \geqslant 0\}$$

- The sunset integral is the integration of the 2-form

$$\Omega_\ominus = \frac{z dx \wedge dy + x dy \wedge dz + y dz \wedge dx}{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz} \in H^2(\mathbb{P}^2 \backslash X_\ominus)$$

- The sunset family of open elliptic curve (modular only for all equal masses)

$$X_\ominus = \{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz = 0\}$$

The differential operator: from the period

Consider the integral which is the same integral as the sunset one with a different cycle of integration

$$\pi_0(p^2) := \int_{|x|=|y|=1} \Omega_\ominus$$

- This is the cut integral of Tancredi' talk. Changing the domain of integration realises the cut

- The other period is $\pi_1(s) = \log(s)\,\pi_0(s) + \varpi_1(s)$ with $\varpi_1(s)$ analytic. They are the two periods of the elliptic curve

- By definition they are annihilated by the Picard-Fuchs operator

$$L_{PF}\pi_0 = L_{PF}\pi_1 = 0$$

The differential operator: from the period

- The integral is the analytic period of the elliptic curve around $p^2 \sim \infty$

$$\pi_0(p^2) := -\sum_{n\geqslant 0} \frac{1}{(p^2)^{n+1}} \left(\sum_{n_1+n_2+n_3=n} \left(\frac{n!}{n_1!n_2!n_3!}\right)^2 \prod_{i=1}^{3} m_i^{2n_i} \right)$$

From the series expansion we can deduce the Picard-Fuch differential operator

$$L_{\ominus}\pi_0(p^2) = 0$$

- With this method one easily derives the PF at all loop order for the all equal mass banana and show the order(PF)=loop order
- Gives for the 3-loop banana 2 unequal masses PF of order 4, and order 6 for 3 different masses [Vanhove, to appear]
- The PF has maximal unipotent monodromy we can reconstruct all the periods using Frobenius method

The differential equation

By general consideration we know that since the integrand is a top form we have

$$L_\Gamma I_\Gamma = \int_{\Delta_n} d\beta_\Gamma = -\int_{\partial\Delta_n} \beta_\Gamma = \mathcal{S}_\Gamma \neq 0$$

Writing the differential equation as $\delta_s := s\frac{d}{ds}$ $s = 1/p^2$

$$\left(\delta_s^2 + q_1(s)\delta_s + q_0(s)\right)\left(\frac{1}{s} I_\ominus(s)\right) = \mathcal{Y}_\ominus + \sum_{i=1}^{3} \log(m_i^2) c_i(s)$$

- The right-hand side are the tadpole graphs [Tancredi's Talk]

The differential equation

The works from [del Angel,Müller-Stach] and [Doran, Kerr] on the rank of the D-module system of differential equations imply that $\mathcal{Y}_\ominus$ is the Yukawa coupling for the residue form $\omega_\ominus = \mathrm{Res}_{X_\ominus}(\Omega_\ominus) = \frac{dx}{\sqrt{P(x)}} \in \Omega^1(X_\ominus)$

$$\mathcal{Y}_\ominus(s) := \int_{X_\ominus} \omega_\ominus \wedge s\frac{d}{ds}\omega_\ominus = \frac{2s^2\prod_{i=1}^4 \mu_i - 4s\sum_i m_i^2 + 6}{\prod_{i=1}^4(\mu_i^2 s - 1)}$$

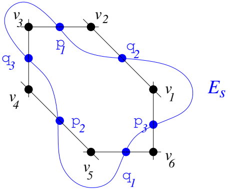
The Yukawa coupling is the Wronskian of the Picard-Fuchs operator and only depends on the form of the Picard-Fuchs operator

$$\mathcal{Y}_\ominus = \det\begin{pmatrix} \pi_0(s) & \pi_1(s) \\ s\frac{d}{ds}\pi_0(s) & s\frac{d}{ds}\pi_1(s)\end{pmatrix}$$

So far all we got can be deduced from the graph polynomial, and the associated Picard-Fuchs operator. Applies to all banana graphs

The differential equation

The logarithmic terms arise more specific to the open elliptic curve geometry

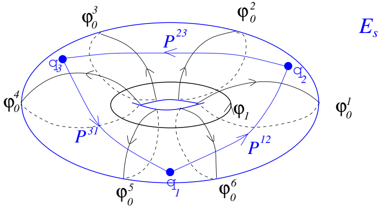


The mass dependent log-terms come from derivative of partial elliptic integrals on globally well-defined algebraic 0-cycles arising from the punctures on the elliptic curve [Bloch, Kerr,Vanhove]

$$c_i(s) = \frac{d}{ds}\int_{q_i}^{q_{i+1}} \omega_\ominus; \qquad c_1(s) + c_2(s) + c_3(s) = 0$$

They are rational function by construction.

Solving the differential equation



The integral divided by a period of the elliptic curve is a (regulator) function defined on punctured torus [Bloch, Kerr,Vanhove]

$$\mathcal{J}_\ominus \equiv \frac{i\varpi_r}{\pi}\left(\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\}+\mathcal{L}_2\left\{\frac{Z}{X},\frac{Y}{X}\right\}+\mathcal{L}_2\left\{\frac{X}{Y},\frac{Z}{Y}\right\}\right) \mod \text{period}$$

- ϖ_r is the real period on $0 < p^2 < (m_1+m_2+m_3)^2$

Solving the differential equation

Representing the ratio of the coordinates on the sunset cubic curve as functions on $\mathcal{E}_\ominus \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$

$$\frac{X}{Z}(x) = \frac{\theta_1(x/x(Q_1))\theta_1(x/x(P_3))}{\theta_1(x/x(P_1))\theta_1(x/x(Q_3))} \qquad \frac{Y}{Z}(x) = \frac{\theta_1(x/x(Q_2))\theta_1(x/x(P_3))}{\theta_1(x/x(P_2))\theta_1(x/x(Q_3))}$$

$\theta_1(x)$ is the Jacobi theta function

$$\theta_1(x) = q^{\frac{1}{8}}\frac{x^{1/2} - x^{-1/2}}{i}\prod_{n\geqslant 1}(1-q^n)(1-q^n x)(1-q^n/x)\,.$$

$$P_1 = [1,0,0]; \qquad Q_1 = [0, -m_3^2, m_2^2]; \qquad x(P_1)x(Q_1) = -1$$

$$P_2 = [0,1,0]; \qquad Q_1 = [-m_3^2, 0, m_1^2]; \qquad x(P_1)x(Q_1) = -1$$

$$P_3 = [0,0,1]; \qquad Q_1 = [-m_2^2, m_1^2, 0]; \qquad x(P_1)x(Q_1) = -1$$

Solving the differential equation

Since

$$\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\} = -\int_{x_0}^{x} \log\left(\frac{X}{Z}(y)\right)\, d\log y$$

and

$$\begin{aligned}\int \log(\theta_1(x))\, d\log x &= \sum_{n\geqslant 1}\int (\mathrm{Li}_1(q^n x) + \mathrm{Li}_1(q^n/x) + \mathrm{cste})\, d\log(x)\\ &= \sum_{n\geqslant 1}(\mathrm{Li}_2(q^n x) - \mathrm{Li}_2(q^n/x)) + \mathrm{cste}\,\log(x)\end{aligned}$$

Solving the differential equation

$$\mathcal{J}_\ominus(s) \equiv \frac{i\varpi_r}{\pi}\left(\hat{E}_2\left(\frac{x(P_1)}{x(P_2)}\right)+\hat{E}_2\left(\frac{x(P_2)}{x(P_3)}\right)+\hat{E}_2\left(\frac{x(P_3)}{x(P_1)}\right)\right) \quad \text{mod periods}$$

where

$$\hat{E}_2(x) = \sum_{n\geqslant 0}\left(\mathrm{Li}_2\left(q^n x\right)-\mathrm{Li}_2\left(-q^n x\right)\right)-\sum_{n\geqslant 1}\left(\mathrm{Li}_2\left(q^n/x\right)-\mathrm{Li}_2\left(-q^n/x\right)\right).$$

- For the all equal mass case $x(P_k) = e^{2i\pi k/6}$ are sixth root of unity
- An equivalent expression using elliptic multiple-polylogarithms has been given by [Adams, Bogner, Weinzeirl] (see as well [Brown, Levin; Adams' talk])
- The three-loop banana all equal mass is elliptic 3-logarithms because the geometry is the symmetric square of the sunset curve [Bloch, Kerr, Vanhove]

The sunset integral as an Eichler

For the all equal mass case we have family of elliptic curve with 4 singular fiber which by [Beauville] classification is a modular family $X_\ominus \simeq \mathbb{H}/\Gamma_1(6)$

- We can pullback the residue 1-form from the elliptic curve to $\mathbb{H}$ giving [Bloch,Vanhove]

$$\mathcal{I}^2_\ominus(t) = periods + \varpi_r \int_\tau^{i\infty} dx(\tau - x) \sum_{(m,n)\neq(0,0)} \frac{(-1)^{n-1}}{\sqrt{3}} \frac{\sin\frac{\pi n}{3} + \sin\frac{2\pi n}{3}}{(m+n\tau)^3}$$

$$= periods - \varpi_r \frac{12}{(2\pi i)^2} \sum_{n\in\mathbb{Z}^*} \frac{(-1)^n}{n^2} \frac{\sin\frac{\pi n}{3} + \sin\frac{2\pi n}{3}}{1-q^n}$$

Similar result at 3 loops the elliptic 3-log is the Eichler integral of a weight 4 Eisenstein series [Bloch, Kerr, Vanhove]

Mirror Symmetry

The sunset Gromov-Witten invariants

Around $1/s = p^2 = \infty$ the sunset Feynman integral has the expansion

$$\mathcal{J}_\ominus(s) = -\pi_0 \left(3R_0^3 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0\\(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\backslash(0,0,0)}} \ell(1-\ell R_0) N_{\ell_1,\ell_2,\ell_3} \prod_{i=1}^3 m_i^{\ell_i} Q^\ell \right) .$$

N_{ℓ_1,ℓ_2,ℓ_3} are local genus zero Gromov-Witten numbers

$$N_{\ell_1,\ell_2,\ell_3} = \sum_{d|\ell_1,\ell_2,\ell_3} \frac{1}{d^3} n_{\frac{\ell_1}{d},\frac{\ell_2}{d},\frac{\ell_3}{d}} .$$

the Kaehler parameter is the logarithmic Mahler measure

$$\log Q = i\pi - \int_{|x|=|y|=1} \log(\Phi_\ominus(x,y)/(xy)) \, \frac{d\log x d\log y}{(2\pi i)^2} .$$

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For the all equal masses case $m_1 = m_2 = m_3 = 1$, the mirror map is

$$Q = -q \prod_{n\geqslant 1} (1-q^n)^{n\delta(n)}; \qquad \delta(n) := (-1)^{n-1} \left(\frac{-3}{n}\right)$$

The sunset mirror symmetry

- The sunset elliptic curve is embedded into a singular compactification X_0 of the local Hori-Vafa 3-fold

$$Y := \{1 - s(m_1^2 x + m_2^2 y + m_3^2)(1 + x^{-1} + y^{-1}) + uv = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2 ,$$

- We have an isomorphism of A- and B-model $\mathbb{Z}$-variation of Hodge structure

$$H^3(\mathrm{X}_{z_0}) \cong H^{even}(\mathrm{X}^\circ_{Q_0}) ,$$

and taking (the invariant part of) limiting mixed Hodge structure on both sides yields

- the number of independent periods given by the surviving periods
- One can map the computation to the one of [Huang, Klemm, Poretschkin] who studied elliptically fibered CY 3-fold with a based given by a toric del Pezzo surface and get the expansion at infinity

Outlook

- We have described a systematic way of deriving the differential operators for the general multiloop sunset graphs
- The inhomogeneous terms arise from total derivative going back and forth because parametric and Feynman representation we hope this can help with the integration by part method
- They are many interesting *useful* connection to mathematics
  - **i)** The connection to the work on mirror symmetry gives a natural way of understanding the basis of master integral
  - **ii)** [Broadhurst] shows that banana graph at $p^2 = 1$ gives L-functions values in the critical band. They are realisation of [Deligne]'s conjectures on periods [Bloch, Kerr, Vanhove]
  - **iii)** The graph polynomials and the differential equation are sometime known to mathematician cf. [v. Straten et al.] classification of Fano varieties
  - **iv)** Interesting connection to Gamma class conjecture of [Golyshev, Zagier]