

GEOMETRY AND HOPF ALGEBRA OF ONE- LOOP INTEGRALS AND THEIR CUTS

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- ▶ D -dimensional one-loop Feynman integrals with many legs and scales are complicated functions
- ▶ Is there some underlying structure?
 - first entry conditions, cuts as discontinuities
 - elegant differential equations in canonical form
- ▶ The coaction [Brown] is an important tool to probe and organise the analytic structure of Feynman integrals

- ▶ With the coaction, we will **decompose** `complicated' integrals into `simpler' ones

$$\text{ex: } \Delta(\text{Li}_2(x)) = 1 \otimes \text{Li}_2(x) + \text{Li}_2(x) \otimes 1 - \ln(1-x) \otimes \ln x$$

- ▶ We consider the coaction on one-loop Feynman integrals:
 - with completely **generic mass configurations**
 - **in dimensional regularisation** $\implies$ obtain degenerate configurations by taking masses to be equal or zero.
- ▶ We show the coaction can be written in terms of
 - **pinches**: lower point integrals
 - **cuts**: integrals with modified integration contour

$\implies$ **Simple diagrammatic construction**

OUTLINE

1. Introduction
2. Coaction of one-loop (cut) Feynman integrals
3. Diagrammatic coaction: checks and applications
4. Summary and outlook

INTRODUCTION

MULTIPLE POLYLOGARITHMS

[Goncharov, Brown, ...]

- ▶ Multiple Polylogarithms (MPLs) form a **Hopf Algebra**

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

$$G(\underbrace{0, \dots, 0}_{n \text{ times}}; z) = \frac{1}{n!} \ln^n z \qquad G(0, 1; z) = -\text{Li}_2(z) \qquad G(a; z) = \log \left(1 - \frac{1}{a} \right)$$

- ▶ The coaction **decomposes a MPL** into “simpler” pieces:

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1 \qquad \Delta(\text{Li}_n(z)) = 1 \otimes \text{Li}_n(z) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

- ▶ **Weight**: number of integrations

“simpler” ~ lower weight

COACTION ON MPL

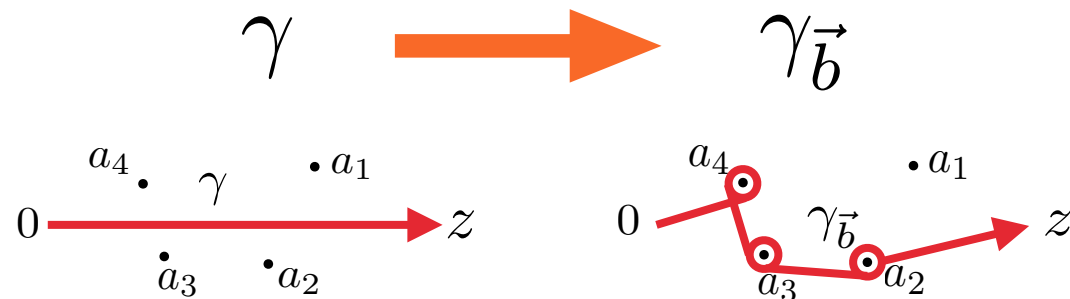
► How to get lower weight MPLs? Two natural ways:

- Remove poles in integrand, keep integration contour: $\vec{b} \subseteq \vec{a}$

$$\omega_{\vec{a}} = \frac{dt_1}{t_1 - a_1} \cdots \frac{dt_n}{t_n - a_n} \longrightarrow \omega_{\vec{b}}$$

$$G(\vec{a}; z) \longrightarrow G(\vec{b}; z)$$

- Keep same poles in integrand, modify integration contour



$$G(\vec{a}; z) \longrightarrow G(\vec{a} \setminus \vec{b}; z)$$

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$$\begin{array}{ccc} \gamma & \longrightarrow & \gamma_{\vec{b}} \\ \begin{array}{c} a_4 \cdot \quad \cdot a_1 \\ \gamma \\ 0 \longrightarrow z \\ \cdot a_3 \quad \cdot a_2 \end{array} & & \begin{array}{c} a_4 \cdot \quad \cdot a_1 \\ \gamma_{\vec{b}} \\ 0 \longrightarrow z \\ \cdot a_3 \quad \cdot a_2 \end{array} \end{array} \quad \Bigg| \quad G(\vec{a}; z) \longrightarrow G(\vec{a} \setminus \vec{b}; z)$$

$$\Delta G(\vec{a}, z) = \sum_{\vec{b} \subseteq \vec{a}} \int_{\gamma} \omega_{\vec{b}} \otimes \int_{\gamma_{\vec{b}}} \omega_{\vec{a}}$$

NOTE: integrands and contours are dual to each other: $P_{ss} \left(\int_{\gamma_{\vec{b}}} \omega_{\vec{a}} \right) = \delta_{\vec{a}, \vec{b}}$

P_{ss} : 'set to zero all but weight zero terms'

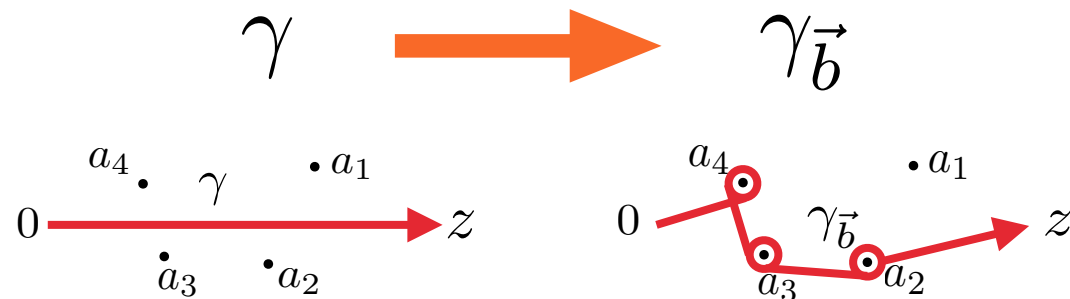
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"pinch"

"cut"

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ALGEBRAIC STRUCTURE OF MPL AND THE COACTION

- ▶ Coaction encodes discontinuities and differential equations

$$\Delta\partial = (\text{id} \otimes \partial)\Delta$$

$$\Delta\text{Disc} = (\text{Disc} \otimes \text{id})\Delta$$

- ▶ Example:

$$\Delta(\text{Li}_3(x)) = 1 \otimes \text{Li}_3(x) + \text{Li}_3(x) \otimes 1 + \text{Li}_2(x) \otimes \ln x + \text{Li}_1(x) \otimes \frac{\ln^2 x}{2}$$

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$$\begin{aligned} & (\text{id} \otimes \partial_x) (\text{Li}_2(x) \otimes \ln x) \\ &= \frac{1}{x} \text{Li}_2(x) \otimes 1 \end{aligned}$$

$$\Rightarrow \partial_x \text{Li}_3(x) = \frac{1}{x} \text{Li}_2(x)$$

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$$\begin{aligned} & (\text{Disc} \otimes \text{id}) \text{Li}_1(x) \otimes \frac{\ln^2 x}{2} \\ &= \theta(x > 1) 2\pi i \otimes \frac{\ln^2 x}{2} \\ \Rightarrow \quad & \text{Disc Li}_3(x) = 2\pi i \frac{\ln^2 x}{2} \theta(x > 1) \end{aligned}$$

Action of differential operators and discontinuities fixed
by the action on weight-one MPLs

BEYOND MULTIPLE POLYLOGARITHMS — MASTER FORMULA

- **Master formula** for coaction: [Abreu, Britto, Duhr, Gardi, 1703.05064&1704.07931]

$$\Delta \left(\int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \omega$$

- Master **integrands** ω_i
 - Master **contours** γ_i
- } **dual** to each other: $P_{ss} \left(\int_{\gamma_i} \omega_j \right) = \delta_{ij}$
- Second factor defined mod $i\pi$
 - **Discont.:** $\Delta \text{Disc} = (\text{Disc} \otimes \text{id}) \Delta$

$$\Delta \left(\text{Disc} \int_{\gamma} \omega \right) = \sum_i \text{Disc} \left(\int_{\gamma} \omega_i \right) \otimes \int_{\gamma_i} \omega$$
 - **Derivatives:** $\Delta \partial = (\text{id} \otimes \partial) \Delta$

$$\Delta \left(\partial \int_{\gamma} \omega \right) = \sum_i \int_{\gamma} \omega_i \otimes \int_{\gamma_i} \partial \omega$$

COACTION OF ONE-LOOP (CUT) FEYNMAN INTEGRALS

RELATIONS BETWEEN ONE-LOOP FEYNMAN INTEGRALS

$$I_n^{\mu_1, \dots, \mu_l; D} = \int d^D k \frac{k^{\mu_1} \dots k^{\mu_l}}{P_1^{\nu_1} P_2^{\nu_2} \dots P_n^{\nu_n}} \quad P_i = (k - q_i)^2 - m_i^2$$

► Tensor integrals

$$I_n^{\mu_1, \dots, \mu_l; D} \xrightarrow[\text{reduction}]{\text{Passarino-Veltman}} I_n^D(\nu_1, \dots, \nu_n) = \int d^D k \frac{1}{P_1^{\nu_1} P_2^{\nu_2} \dots P_n^{\nu_n}}$$

► Integrals in different dimensions

$$I_n^D(\nu_1, \dots, \nu_n) \xrightarrow[\text{dimension shift}]{\text{Tarasov/Lee}} I_n^{D \pm 2}(\nu_1, \dots, \nu_n)$$

► Integrals with propagators raised to higher powers

$$I_n^D(\nu_1, \dots, \nu_n) \xrightarrow[\text{(IBP) relations}]{\text{Integration-by-parts}} I_n^D = I_n^D(1, \dots, 1)$$

A BASIS OF ONE-LOOP FEYNMAN INTEGRALS

$$J_n = i \frac{e^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n} k \frac{1}{P_1 P_2 \dots P_n}$$

- ▶ Dimensions vary with number of propagators: $D_n = 2 \left\lceil \frac{n}{2} \right\rceil - 2\epsilon$
 - tadpole and bubbles: $2 - 2\epsilon$
 - triangles and boxes: $4 - 2\epsilon$
 - pentagon and hexagons: $6 - 2\epsilon$
- ▶ This choice of **master integrals** has nice analytic properties:
 - coefficients in Laurent expansion in ϵ have uniform weight
 - if we set the weight of ϵ to -1, can define weight of unexpanded function: $\lceil n/2 \rceil$
 - Feynman diagrams satisfy differential equations in canonical form

LANDAU CONDITIONS AND INTEGRATION CONTOURS

[Landau, Cutkosky, t'Hooft&Veltman, Analytic S-Matrix,...]

► Landau conditions characterise singularities of the integrand:

- encircle poles of propagators: Landau singularity of the 1st kind
- pinch pole at infinity: Landau singularity of the 2nd kind

[cf. talk by A. Volovich]

► We can list interesting contours of one-loop Feynman integrals:

- original contour: $\Gamma_{\emptyset}$
- encircle poles of propagators: $\Gamma_1, \Gamma_2, \Gamma_{123} \dots$
- pinch pole at infinity and encircle propagators: $\Gamma_{\infty}, \Gamma_{\infty 15} \dots$

Do these integrals form a basis of integration contours?

Homology theory

DECOMPOSITION THEOREM

[Fotiadi&Pham; Hwa&Teplitz; Federbush;
Landshof; Polkinghorne; ...]

- Relations between integration contours of one-loop integrals were explored in the 60's:

$$\Gamma_{\infty C} = -2\Gamma_C - \sum_{e \in E_J \setminus C} \Gamma_{Ce} + \dots \quad C \text{ odd}$$

$$\Gamma_{\infty C} = - \sum_{e \in E_J \setminus C} \Gamma_{Ce} - \sum_{e, f \in E_J \setminus C} \Gamma_{Cef} + \dots \quad C \text{ even}$$

E_J : all propagators in J_n ;
 C : set of propagators ;
 $\dots$: contours that vanish
 $\text{mod } i\pi$

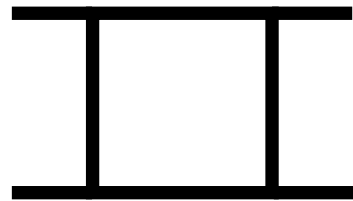
- Contours that pinch **infinity** are **not independent**
- **Basis of master contours**: label by subset of encircled propagator poles

Cut Feynman Integrals

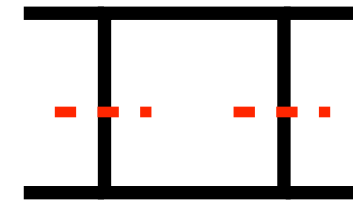
CUT ONE-LOOP INTEGRALS

[Landau, Cutkosky; t'Hooft&Veltman; Analytic S-Matrix; Fotiadi&Pham; Hwa&Teplitz; Federbush; Landshof; Polkinghorne;...]

[Abreu, Britto, Duhr, Gardi, 1702.03163]



$$\int_{\Gamma_\emptyset} \frac{d^D k}{P_1 P_2 P_3 P_4}$$



$$\int_{\Gamma_{24}} \frac{d^D k}{P_1 P_2 P_3 P_4}$$

- ▶ Contour encircle poles of propagators $\Rightarrow$ **residue calculation in D dimensions**
- ▶ **Consistent with prescription** 'replace cut propagator by δ -function'

$$\frac{1}{(k - q_i)^2 - m_i^2} \longrightarrow \delta((k - q_i)^2 - m_i^2)$$

- ▶ Residue calculation made precise with **Leray's theory of multivariate residues**
- ▶ Picture very clear in **projective space**: cuts involve projections on intersections of hyperplanes defined by cut conditions.

DUAL BASES OF MASTER INTEGRANDS AND CONTOURS

$$J_n = \int d^{D_n} k \frac{i e^{\gamma_E \epsilon}}{P_1 \dots P_n} = \int_{\Gamma_\emptyset} \omega_{E_J}$$

- ▶ Integrands with **different propagators** are **independent**
 - **Basis of master** integrands: ω_C for $C \subseteq E_J$

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► Are the bases of master contours and integrands dual?

- C **even**: $P_{ss} \int_{\Gamma_C} \omega_{\tilde{C}} = \delta_{\tilde{C}, C}$ ✓ $\gamma_C = \Gamma_C$

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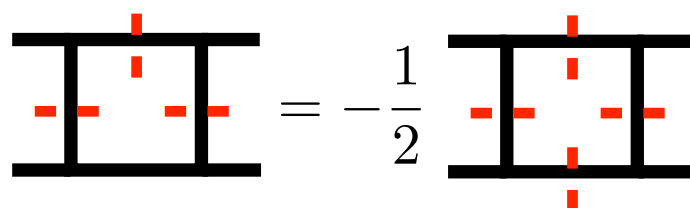
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ex: 

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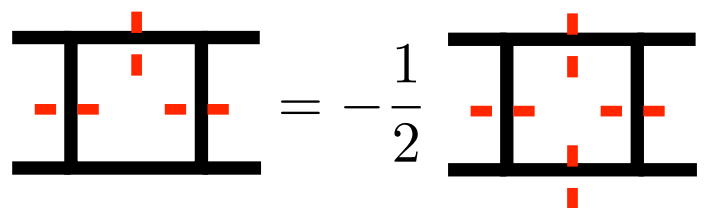
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ex: 

$$P_{ss} \left(\int_{\Gamma_C} \omega_{\tilde{C}} + \frac{1}{2} \sum_{e \in \tilde{C} \setminus C} \int_{\Gamma_{C_e}} \omega_{\tilde{C}} \right) = \delta_{\tilde{C}, C}$$

$$\gamma_C = \Gamma_C + \frac{1}{2} \sum_{e \in \tilde{C} \setminus C} \Gamma_{C_e} = -\frac{1}{2} \Gamma_{\infty C}$$

NOTE: Normalise contours or integrand by maximal cut at ϵ^0 to get dual basis

COACTION OF ONE-LOOP FEYNMAN INTEGRALS

- Coaction for J associated with Feynman graph G

$$\Delta(J_G) = \sum_{\emptyset \neq C \subseteq E_G} \int_{\gamma} \omega_C \otimes \int_{\gamma_C} \omega_{E_G}$$

- Diagrammatic representation of the coaction

$$\Delta(J_G) = \sum_{\emptyset \neq C \subseteq E_G} J_{G_C} \otimes \left(\mathcal{C}_C J_G + a_C \sum_{e \in E_G \setminus C} \mathcal{C}_{C_e} J_G \right)$$

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Sum over all subset
of propagators

Pinch integral with
propagators not in
 C pinched

Cuts of graph G :

- Cut all propagators in C
- For odd $|C|$, add 1/2 times the sum of all diagrams with one extra cut

DIAGRAMMATIC COACTION: CHECKS AND APPLICATIONS

DIAGRAMMATIC COACTION

[Abreu, Britto, Duhr, Gardi, 1703.05064&1704.07931]

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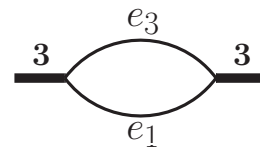
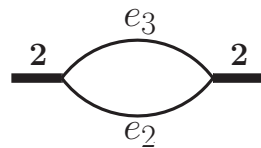
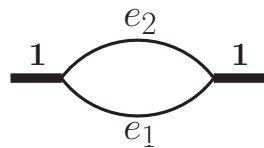
- ▶ Simple **diagrammatic rules**, valid for **any one-loop integral**
- ▶ Extends trivially to a coaction on **cut Feynman integrals**
- ▶ Valid in **dimensional regularisation** $\implies$ massless limits
- ▶ Use known coaction of MPLs to **check validity**:
 1. Evaluate integrals in dim. regularisation
 2. Expand each function in dim. regularisation parameter ε
 3. Check at each order that we **agree with the coaction of MPLs**

THREE-MASS TRIANGLE

- ▶ Massless propagators, massive external legs:

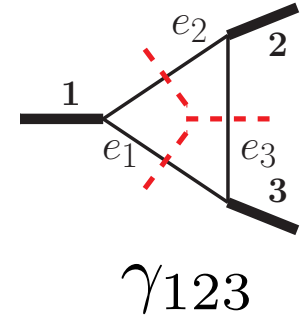
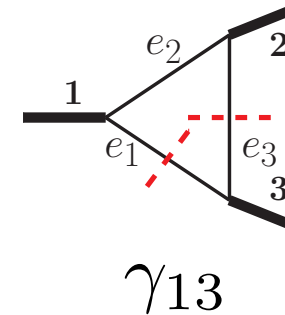
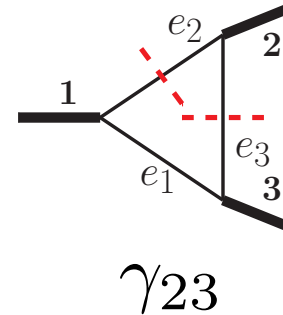
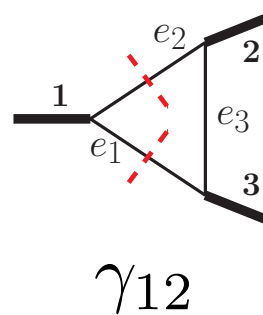
$$J_3(p_1^2, p_2^2, p_3^2) = \text{triangle diagram with external legs 1, 2, 3 and internal lines } e_1, e_2, e_3$$

- ▶ Pinches

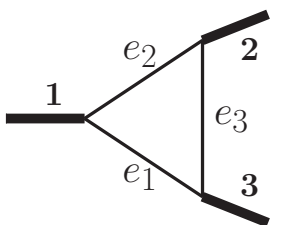


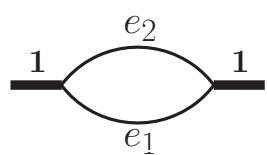
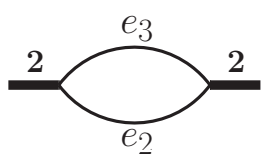
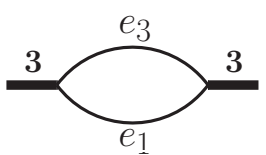
— massless tadpoles vanish

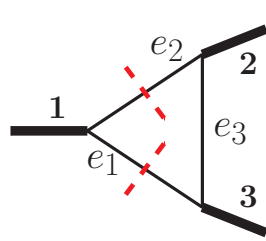
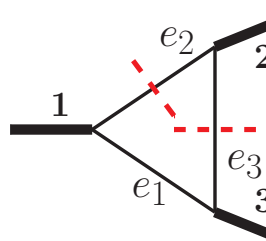
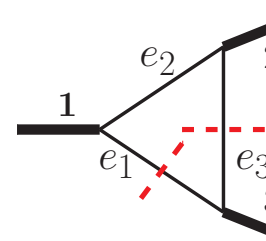
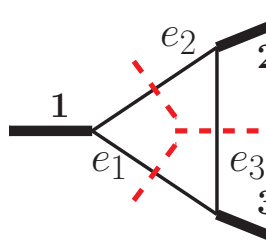
- ▶ Contours & cuts $\gamma_i \rightarrow 0$



THREE-MASS TRIANGLE

▶ Massless propagators, massive external legs: $J_3(p_1^2, p_2^2, p_3^2) =$ 

▶ Pinches    – massless tadpoles vanish

▶ Contours & cuts $\gamma_i \rightarrow 0$  γ_{12}  γ_{23}  γ_{13}  γ_{123}

▶ Coaction

$$\Delta \left[\text{triangle}(1, 2, 3) \right] = \text{bubble}(1, 1) \otimes \gamma_{12} + \text{bubble}(2, 2) \otimes \gamma_{23} + \text{bubble}(3, 3) \otimes \gamma_{13} + \text{triangle}(1, 2, 3) \otimes \gamma_{123}$$

THREE-MASS TRIANGLE

$$\Delta \left[\begin{array}{c} \text{triangle with edges } e_1, e_2, e_3 \text{ and vertices } 1, 2, 3 \end{array} \right] =$$

- ▶ Three-mass triangle is finite, **bubbles introduce poles**. Still works because:

$$\left. \begin{array}{c} \text{triangle with edges } e_1, e_2, e_3 \text{ and vertices } 1, 2, 3 \end{array} \right|_{\epsilon^n} + \left. \begin{array}{c} \text{triangle with edges } e_1, e_2, e_3 \text{ and vertices } 1, 2, 3 \end{array} \right|_{\epsilon^n} + \left. \begin{array}{c} \text{triangle with edges } e_1, e_2, e_3 \text{ and vertices } 1, 2, 3 \end{array} \right|_{\epsilon^n} = - \left. \begin{array}{c} \text{triangle with edges } e_1, e_2, e_3 \text{ and vertices } 1, 2, 3 \end{array} \right|_{\epsilon^{n-1}}$$

- Particular case of remarkable **general relation**: $\sum_{e \in E_G} \mathcal{C}_e J_G + \sum_{\substack{e, f \in E_G \\ e < f}} \mathcal{C}_{ef} J_G = -\epsilon J_G$
- ▶ Checked order-by-order in ϵ , up to weight 4.
- ▶ Requires many terms to combine in highly non-trivial way

BOX WITH ONE MASSIVE PROPAGATOR

$$\begin{aligned}
 \Delta \left[\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \right] &= \text{circle}(e_1) \otimes \left(\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} + \frac{1}{2} \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \right) \\
 &+ \text{bubble}(e_1, e_3, s) \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} + \text{bubble}(e_4, e_2, t) \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \\
 &+ \text{triangle}(e_4, e_2, t) \otimes \left(\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} + \frac{1}{2} \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \right) + \frac{1}{2} \text{triangle}(e_4, e_2, t) \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \\
 &+ \frac{1}{2} \text{triangle}(e_3, e_1, s) \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} + \frac{1}{2} \text{triangle}(e_3, e_1, s) \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \\
 &+ \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & e_1 & e_4 \\ \hline \end{array}
 \end{aligned}$$

BOX WITH ONE MASSIVE PROPAGATOR

$$\Delta \left[\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right] = \text{[Blue Box]} + \text{[Green Box]} + \text{[White Box]}$$

Blue Box: $\text{[Tadpole } e_1] \otimes \left(\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \frac{1}{2} \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right)$

Green Box: $+ \text{[Bubble } e_1 \text{ with } s \text{ cuts}] \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \text{[Bubble } e_4 \text{ with } t \text{ cuts}] \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array}$

White Box: $+ \text{[Triangle } e_4 \text{ with } t \text{ cut}] \otimes \left(\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \frac{1}{2} \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right) + \frac{1}{2} \text{[Triangle } e_4 \text{ with } t \text{ cut}] \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array}$

$+ \frac{1}{2} \text{[Triangle } e_3 \text{ with } s \text{ cut}] \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} + \frac{1}{2} \text{[Triangle } e_3 \text{ with } s \text{ cut}] \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array}$

$+ \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array}$

Annotations:

- Massless tadpoles vanish in dim.reg.
- Cuts on massless channels vanish in dim.reg.

BOX WITH ONE MASSIVE PROPAGATOR

$$\Delta \left[\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right] = \text{[Diagrammatic Expansion]}$$

The expansion consists of several terms:

- Blue Box:** A tadpole with mass e_1 multiplied by a sum of two box diagrams. The first box diagram has a vertical cut on the bottom-right propagator e_1 . The second box diagram has a vertical cut on the top-right propagator e_4 . A blue arrow points to this box with the text: "Massless tadpoles vanish in dim.reg."
- Green Box:** Two terms involving bubble diagrams. The first is a bubble with mass e_1 (labeled s) multiplied by a box diagram with a vertical cut on e_1 . The second is a bubble with mass e_4 (labeled t) multiplied by a box diagram with a vertical cut on e_4 . A green arrow points to this box with the text: "Cuts on massless channels vanish in dim.reg."
- Red Box:** Four terms involving three-propagator cuts. The first two terms are multiplied by $\frac{1}{2}$ and involve cuts on e_1 and e_4 respectively. The last two terms are multiplied by $\frac{1}{2}$ and involve cuts on e_3 and e_2 respectively. A red arrow points to this box with the text: "Three-propagator cut that isolates massless 3 point vertex vanishes"
- Bottom Term:** A box diagram with a vertical cut on the bottom-right propagator e_1 .

BOX WITH ONE MASSIVE PROPAGATOR

$$\Delta \left[\begin{array}{|c|c|c|} \hline & e_3 & \\ \hline e_2 & & e_4 \\ \hline & e_1 & \\ \hline \end{array} \right] = \text{[Diagrammatic Expansion]}$$

The expansion consists of several terms:

- Blue box:** A tadpole on line e_1 multiplied by a sum of two box diagrams. The first box diagram has a vertical cut on e_1 . The second box diagram has vertical cuts on both e_1 and e_4 . A blue arrow points to this box with the text: "Massless tadpoles vanish in dim.reg."
- Green box:** Two terms involving bubble diagrams. The first is a bubble on e_1 (with external lines s and s) multiplied by a box diagram with a vertical cut on e_1 . The second is a bubble on e_4 (with external lines t and t) multiplied by a box diagram with vertical cuts on e_1 and e_4 . A green arrow points to this box with the text: "Cuts on massless channels vanish in dim.reg."
- Red box:** Four terms involving three-propagator cuts. The first two terms are multiplied by $\frac{1}{2}$. They involve triangle diagrams (with external lines t, e_4, e_2 and s, e_3, e_1 respectively) multiplied by box diagrams with various cuts. A red arrow points to this box with the text: "Three-propagator cut that isolates massless 3 point vertex vanishes"
- Orange box:** A single term consisting of a box diagram multiplied by a box diagram with vertical cuts on e_1 and e_4 .

DISCONTINUITIES

[Abreu, Britto, Duhr, Gardi, 1703.05064&1704.07931]

$$\Delta \text{Disc} = (\text{Disc} \otimes \text{id}) \Delta$$

- ▶ Discontinuity operators act on **weight one contributions in left factor**

$$\text{Disc}_{p_1^2} \left(\text{triangle diagram with edges } e_1, e_2, e_3 \right) = \text{Disc}_{p_1^2} \left(\text{bubble diagram with edges } e_1, e_2 \right) \otimes \left(\text{triangle diagram with edges } e_1, e_2, e_3 \right) = 2\pi i \theta(p_1^2) \otimes \left(\text{triangle diagram with edges } e_1, e_2, e_3 \right)$$

- ▶ Consistent with all results on the relation between cuts and discontinuities.
- ▶ Consistent with **'first-entry condition'**: the left factor of the coaction has the same branch cuts as a Feynman integral
[Gaiotto, Maldacena, Sever, Vieira; Abreu, Britto, Grönqvist; Brown; ...]
- ▶ **Sharper version:** [cf. talk by F. Brown]

In the coaction of a (cut) Feynman integral, the **first entries are themselves Feynman integrals**, with a subset of propagators but the same set of cut propagators.

DIFFERENTIAL EQUATIONS

[Abreu, Britto, Duhr, Gardi, 1703.05064&1704.07931]

$$\Delta\partial = (\text{id} \otimes \partial)\Delta$$

- ▶ Differential operators act on **weight one contributions in right factor**:
 1. Max cut at order ε
 2. NMax cut at order ε (n even) or ε^0 (n odd)
 3. NNMax cut at order ε^0

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**WE'VE COMPUTED THEM
ALL, FOR ANY ONE-LOOP INTEGRAL!**

**FULLY DETERMINE ALPHABET
AND SYMBOL OF ANY ONE-LOOP INTEGRAL**

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[Abreu, Britto, Duhr, Gardi, 1703.05064&1704.07931]

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- Directly write down the differential equation of any J_n , no need to construct differential operators or use IBP relations

$$d \left[\text{pentagon} \right] = \left[\text{triangle} \right] d \left[\text{hexagon}_1 + \frac{1}{2} \text{hexagon}_2 \right]_{\epsilon^0} + \left[\text{box} \right] d \left[\text{hexagon}_3 \right]_{\epsilon^0} + \epsilon \left[\text{pentagon} \right] d \left[\text{hexagon}_4 \right]_{\epsilon}$$

$\ln \left(\frac{Y_n}{\text{Gram}_n} \right)$

SUMMARY & OUTLOOK

Summary

- **Conjectured** a formula for the **coaction of large class of integrals**
- When applied to one-loop integrals, **diagrammatic representation**
- Extract information (discontinuities, derivatives, ...) on complicated integrals from simpler ones

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Other results

[with James Matthew]

- Coaction on other functions defined as iterated integrals:
 1. **Hypergeometric functions** ${}_2F_1, {}_3F_2, {}_4F_3, \dots$ ✓
 2. **Appell functions** $F_1, F_2, F_3, \dots$ ✓

OUTLOOK: TWO-LOOP SUNRISE, ONE MASSIVE PROPAGATOR

► Two masters at the top level

$$\text{---} \bigcirc \text{---} = \epsilon^2 r_1 J(1, 1, 1) = \int_{\gamma_1} \omega_1 \quad \text{---} \bigcirc \bullet \text{---} = \epsilon [r_2 J(1, 1, 1) + r'_2 J(1, 2, 1)] = \int_{\gamma_2} \omega_2$$

Master contours $\gamma_1 = \Gamma_\emptyset$

$$\gamma_2 = \frac{2}{3} \Gamma_\emptyset - \frac{1}{6\epsilon} \Gamma_{123}$$

Duality contours/integrands

$$\Omega = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \int_{\gamma_2} \omega_1 \\ \int_{\gamma_1} \omega_2 & \int_{\gamma_2} \omega_2 \end{pmatrix} \xrightarrow{P_{ss}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Coaction

$$\Delta \left(\text{---} \bigcirc \text{---} \right) = \text{---} \bigcirc \text{---} \otimes \text{---} \bigcirc \text{---} + \text{---} \bigcirc \bullet \text{---} \otimes \int_{\gamma_2} \omega_1 \quad \checkmark$$

$$\Delta \left(\text{---} \bigcirc \bullet \text{---} \right) = \text{---} \bigcirc \text{---} \otimes \text{---} \bigcirc \bullet \text{---} + \text{---} \bigcirc \bullet \text{---} \otimes \int_{\gamma_2} \omega_2 \quad \checkmark$$

- Master formula predicts correct coaction!
- Larger number of contours, diagrammatic representation harder to determine.
- Many two-loop examples to study, that evaluate to MPLs or elliptic integrals.

THANK YOU!