Monodromy relations at higher loops

Piotr Tourkine

Amplitudes 2017, July 14th, Edinburgh

[Tourkine, Vanhove, **PRL** *'16], [Ochirov, Tourkine, Vanhove, work in progress]*

Outline

- Motivations
- Tree-level amplitude relations : Veneziano
- Results : all-order amplitude relations

Amplitudes

- Whole week discussing discoveries in amplitudes
- Observation : we actually do not understand some of the important basic principles underlying quantum field theories
- I'm actually not going to explain the origin of these miracles, just give a string theory take on colour/kinematics in loops *[Bern, Carrasco, Johansson]*

Results

- I'm going to present to you a **new set of relations for open string & gauge theory higher-loop amplitudes**,

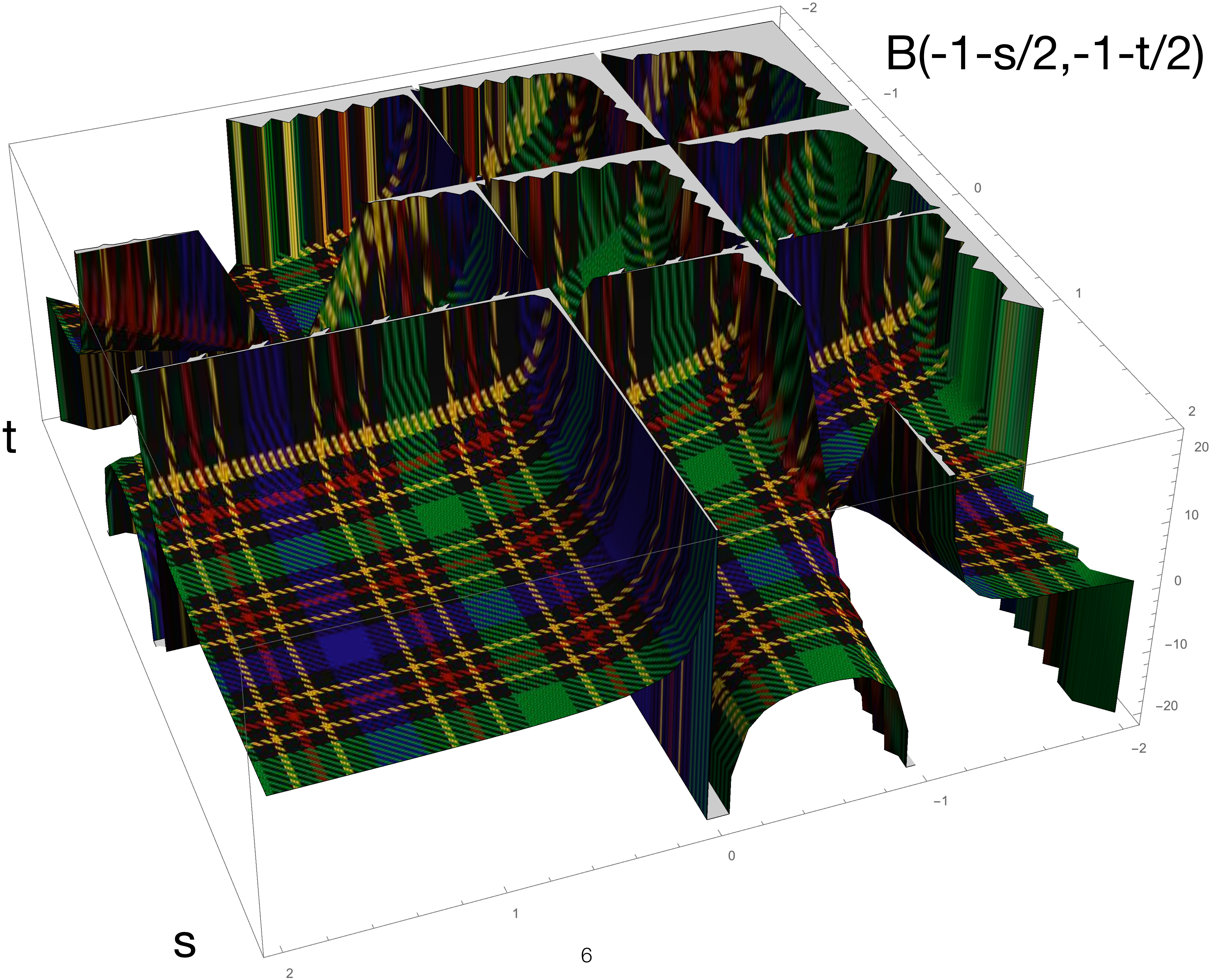
- Loops in field theory : relations between integrands that will generalise

$$\sum_{i=1}^{n-1} p_1 \cdot (\ell + \sum_{j=1}^{i} p_j)\, \mathcal{I}(p_2, \ldots, \underbrace{p_1}_{\text{position } i}, \ldots, p_n) \approx 0\,.$$

[Boels, Isermann]

Derivation

- The derivation relies only on:
  - Cauchy's theorem
  - Gaussian un-integration
- At loops : the string integrand no more holomorphic,
  - because of zero modes which can be reverse engineered to loop momenta *[D'Hoker, Phong, 1989]*



B(-1-s/2,-1-t/2)
t
s
-2
-1
0
1
2
20
10
0
−10
−20
−2
−1
0
1
2

Veneziano amplitude

$$A_V = g^2 \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' s/4)}{\Gamma(2+\alpha' s/4+\alpha' t/4)}$$

Veneziano amplitude

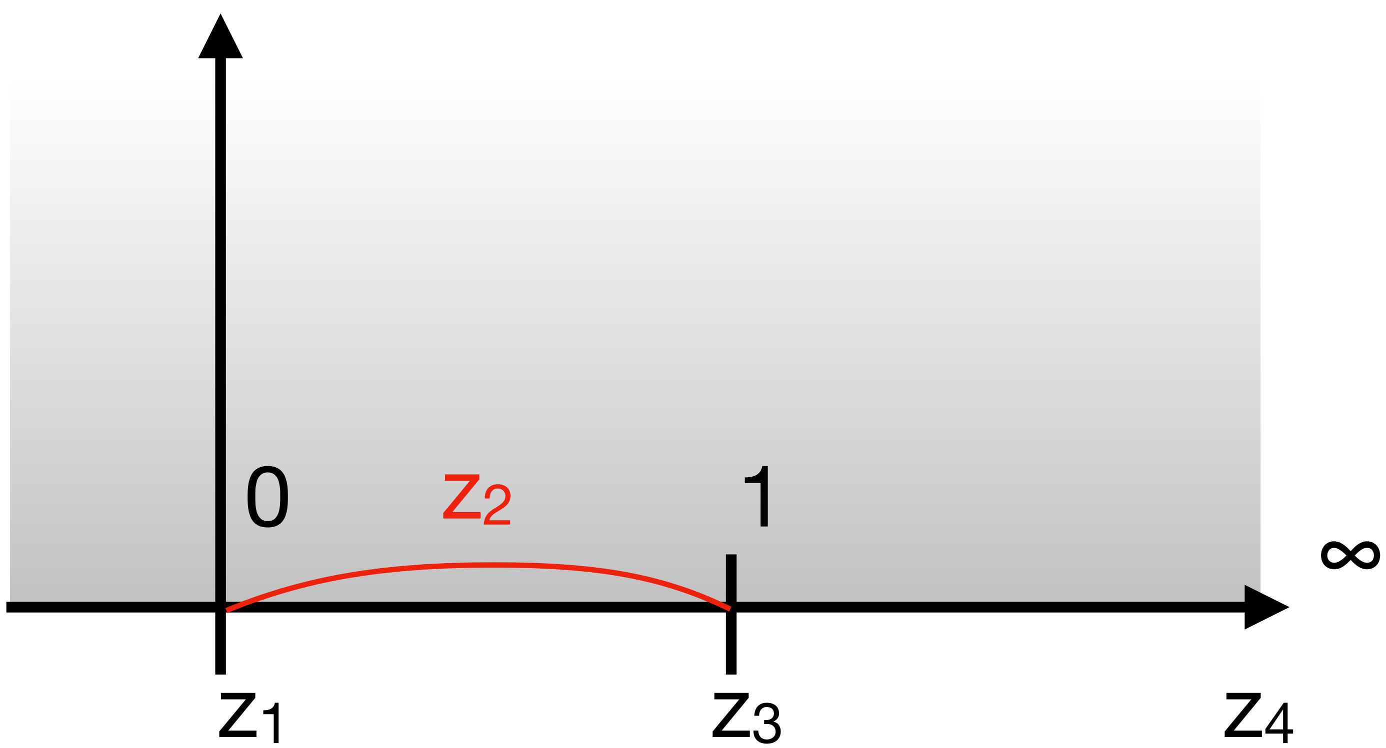
$$A_V = g^2 \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' s/4)}{\Gamma(2+\alpha' s/4+\alpha' t/4)}$$

$$e^{i\pi\alpha' k_1.k_2} A_V(s,u) + A_V(s,t) + e^{-i\pi\alpha' k_1.k_4} A_V(t,u) = 0$$

Plahte, 1972

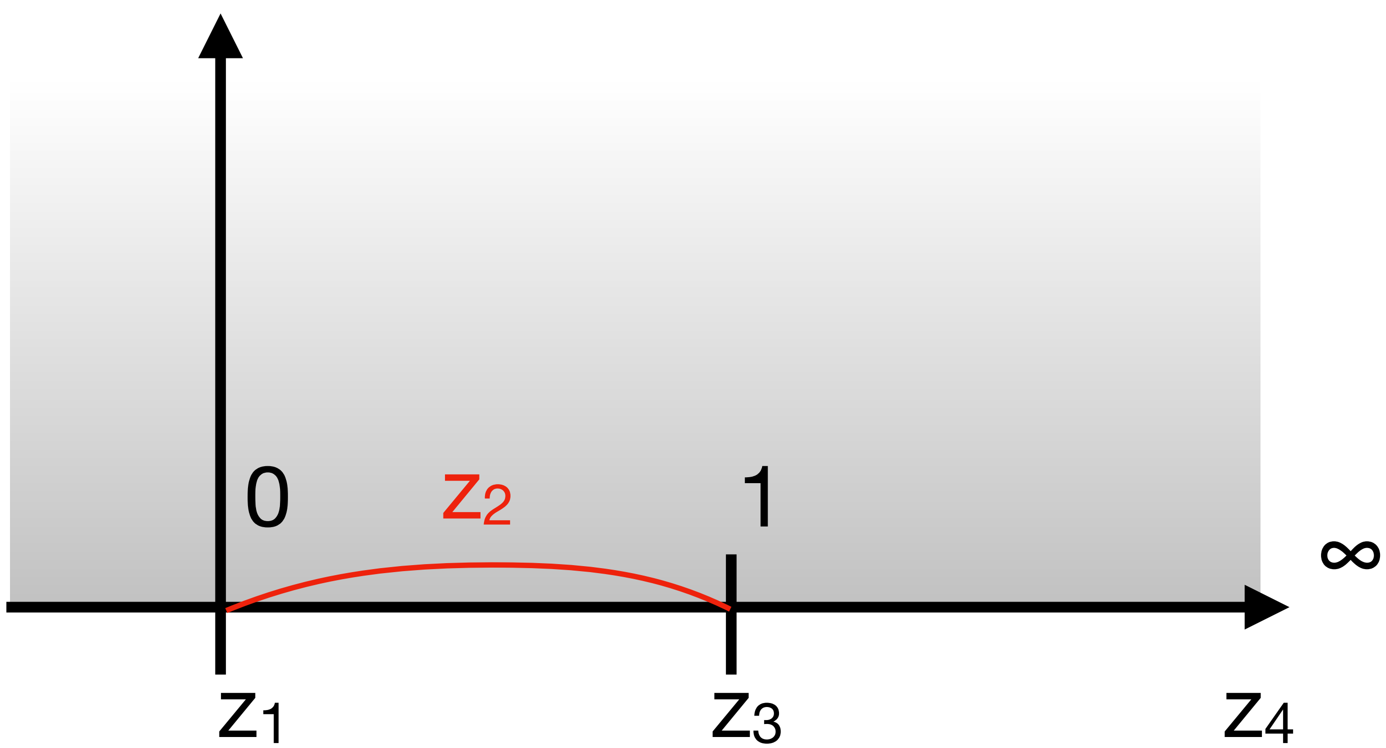
Veneziano amplitude

$$A_V = \int_0^1 dz_2 \, \exp\left(\alpha' k_1.k_2 \log|z_2| + \alpha' k_2.k_3 \log|1 - z_2|\right)$$



Veneziano amplitude

$$A_V = \int_0^1 dz_2 \, \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right)$$

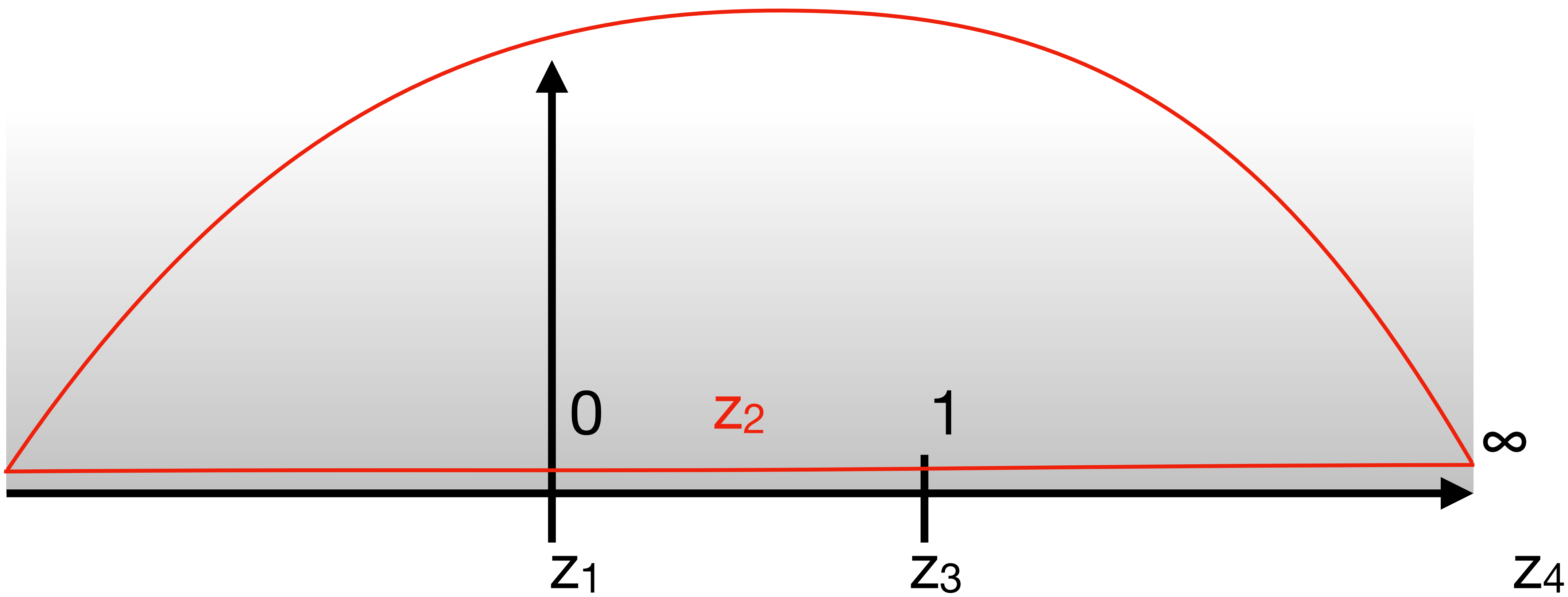


$\log|\,.\,| \to \log(\,.\,)$

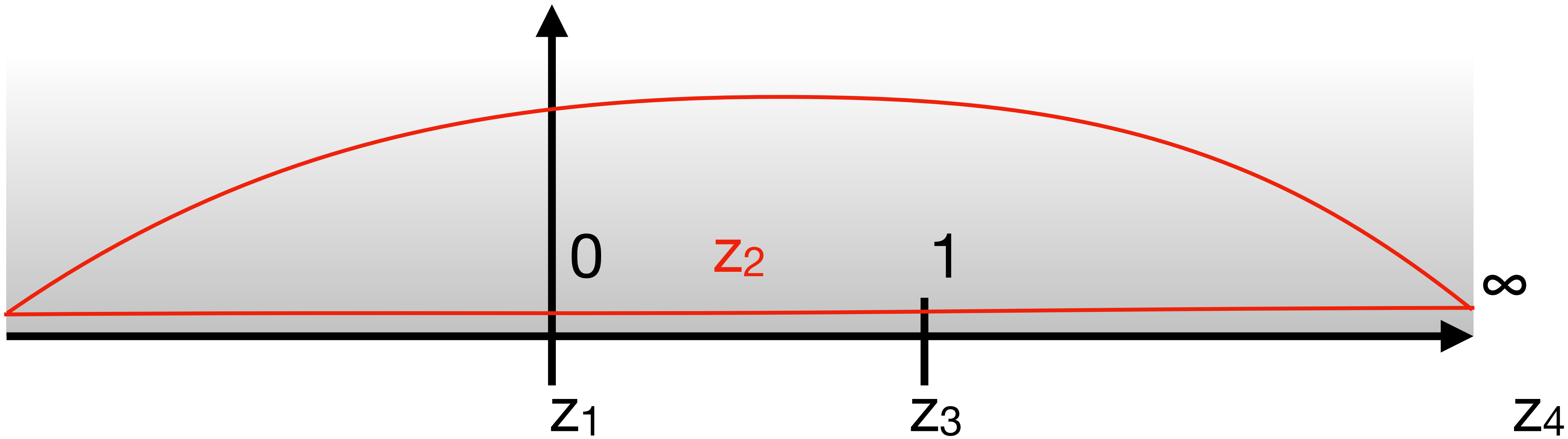
⇨ holomorphic integrand

Veneziano amplitude

$$\oint dz_2 \,\exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right) = 0$$

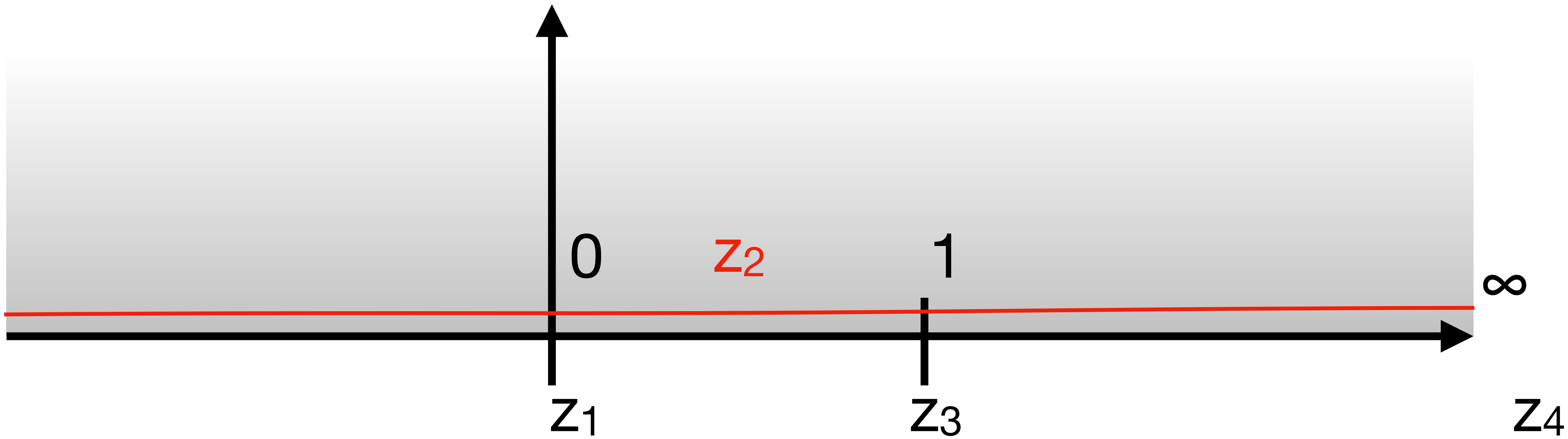


Veneziano amplitude

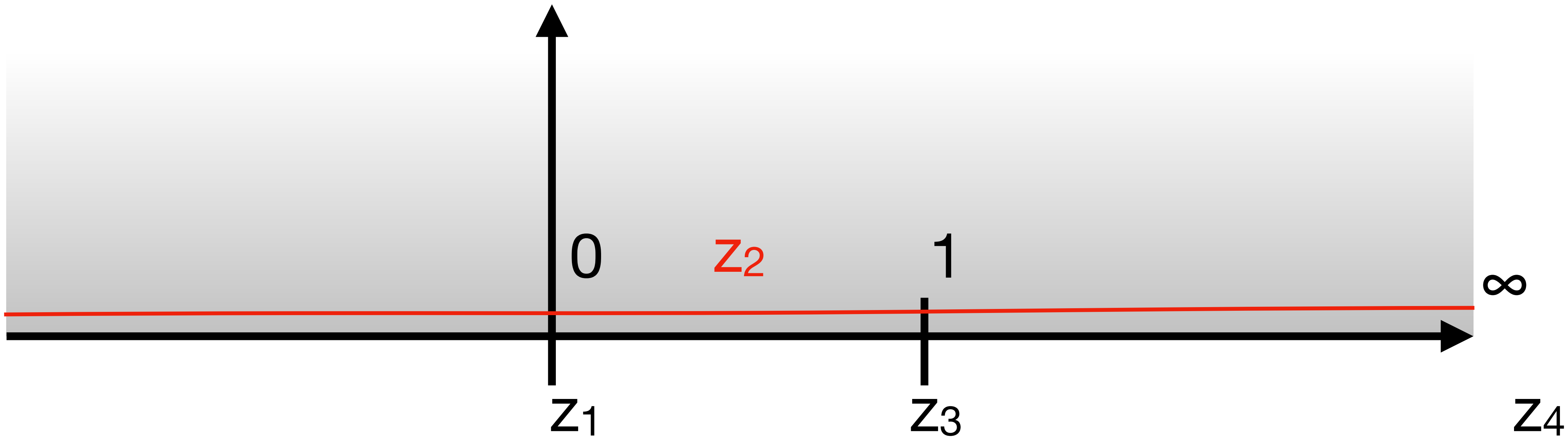


Upper contour integral vanishes
by momentum conservation
and on-shellness

Veneziano amplitude

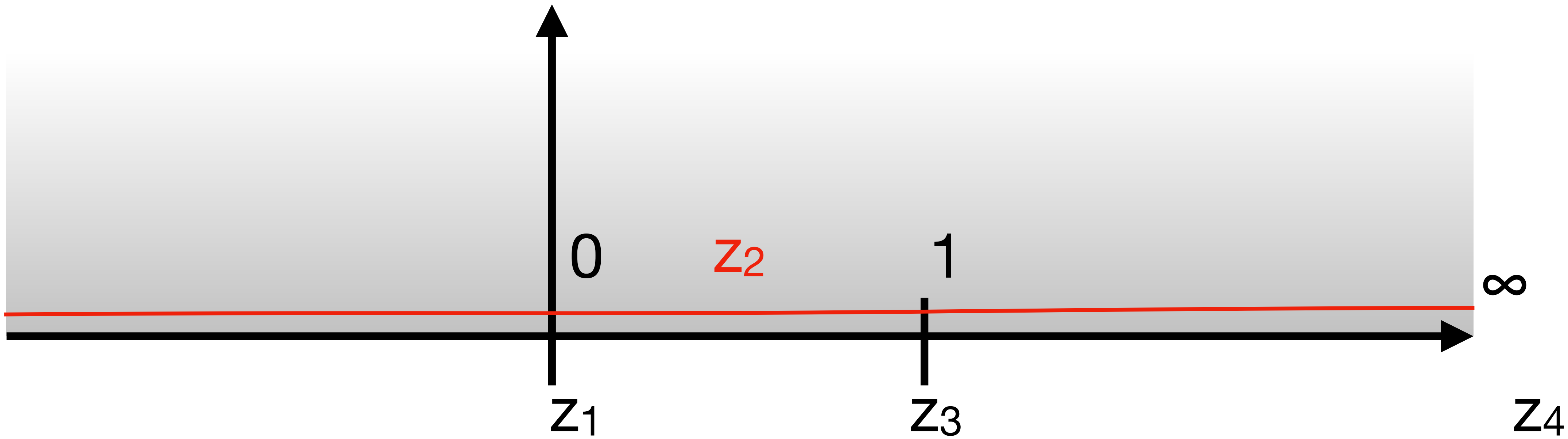


Veneziano amplitude



$$\int_{-\infty}^{0} dz_2 \; \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right)$$
$$+ \int_{0}^{1} dz_2 \; \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right)$$
$$+ \int_{1}^{\infty} dz_2 \; \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right) = 0$$

Veneziano amplitude



$$\int_{-\infty}^{0} dz_2 \, \exp\left(\alpha' k_1 . k_2 \log(z_2) + \alpha' k_2 . k_3 \log(1 - z_2)\right)$$

$$\neq A(2, 1, 3, 4)$$

Veneziano amplitude

$$A(2,1,3,4) =$$

$$\int_{-\infty}^{0} dz_2 \exp\left(\alpha' k_1.k_2 \log|z_2| + \alpha' k_2.k_3 \log|1-z_2|\right)$$

$$\neq$$

$$\int_{-\infty}^{0} dz_2 \ \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1-z_2)\right)$$

Veneziano amplitude

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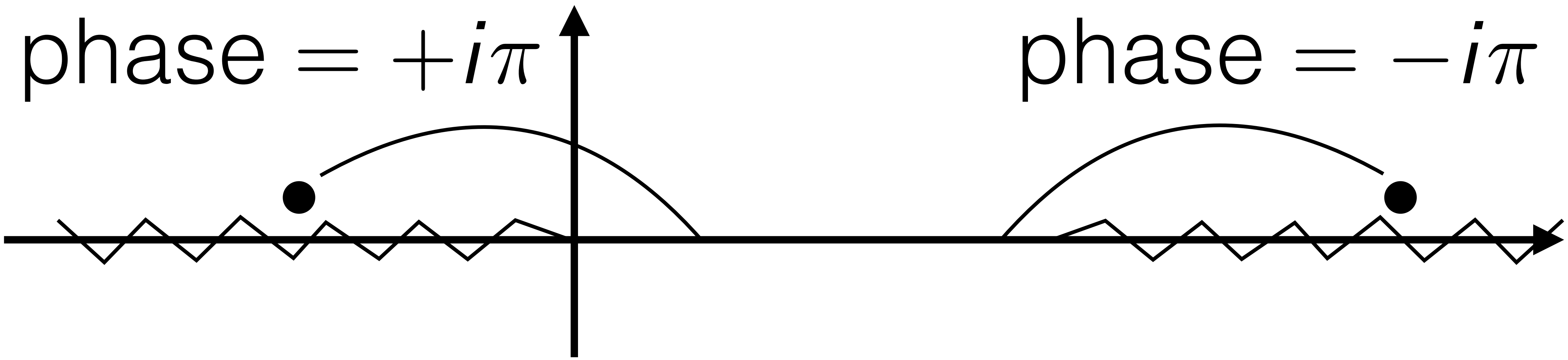
$$\neq$$

$$\int_{-\infty}^{0} dz_2 \, \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1-z_2)\right)$$

$$\ln(z) = \ln(-|z|) = \ln(|z|) + i\pi$$

Veneziano amplitude

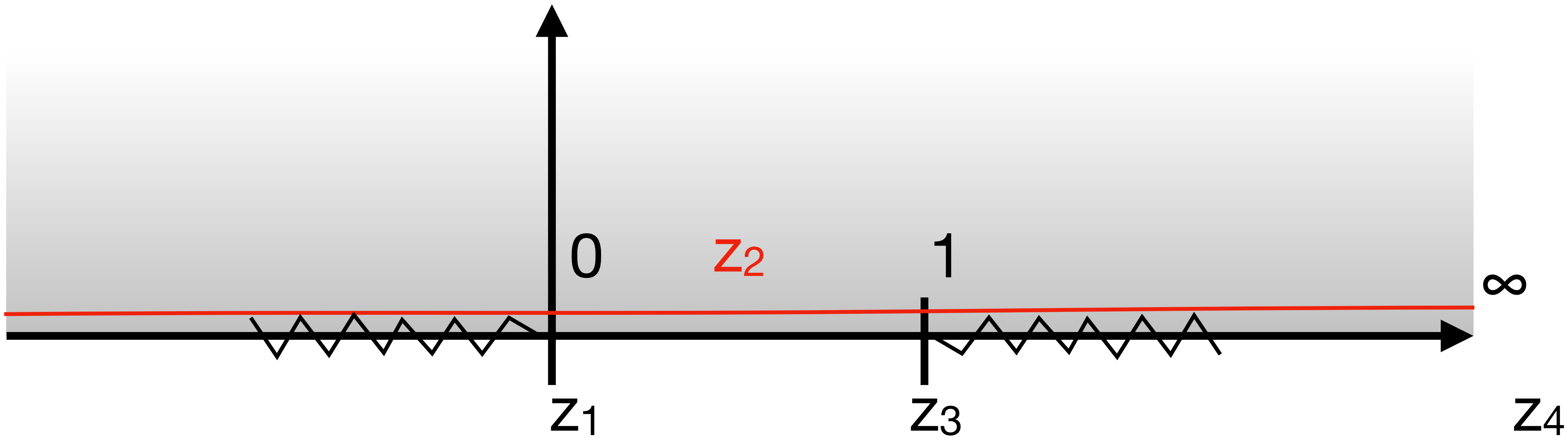


Therefore :

$$\int_{-\infty}^{0} dz_2 \, \exp\left(\alpha' k_1.k_2 \log(z_2) + \alpha' k_2.k_3 \log(1 - z_2)\right)$$

$$= \exp(i\pi\alpha' k_1.k_2) A(2, 1, 3, 4)$$

Veneziano amplitude



$$e^{i\pi\alpha' k_1.k_2} A_V(s,u) + A_V(s,t) + e^{-i\pi\alpha' k_1.k_4} A_V(t,u) = 0$$

Veneziano amplitude

$$e^{i\pi\alpha' k_1.k_2} A_V(s,u) + A_V(s,t) + e^{-i\pi\alpha' k_1.k_4} A_V(t,u) = 0$$

field theory limit : α’ ⟶ 0

Real part: KK relations

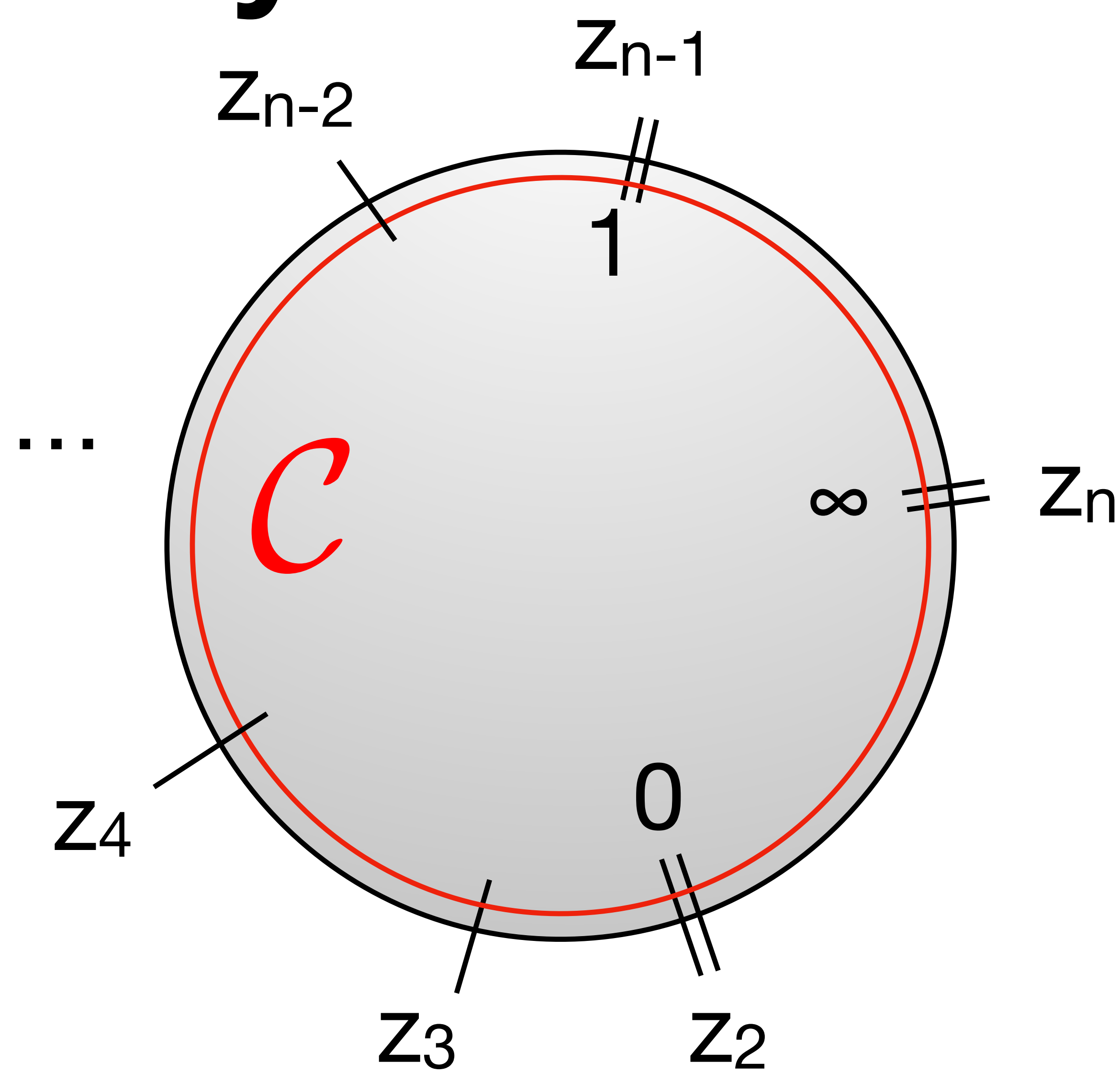
$$A(s,u) + A(s,t) + A(t,u) = 0$$

Imaginary part: fundamental BCJ relations

$$s\,A(s,u) - t\,A(t,u) = 0$$

Monodromies in string theory

$$\oint_{\mathcal{C}} dz_1 (\dots) = 0$$



Monodromies in string theory

$$\oint_{\mathcal{C}} dz_1 (\ldots) = 0$$

$$A(k_1, k_2, \ldots, k_n) + \sum_{i=2}^{n} e^{i\alpha' k_1 \cdot k_{2\ldots i}} A(k_2, \ldots, \underbrace{k_1}_{\text{position } i}, \ldots, k_n) = 0$$

Plahte, 1972

Give Kleiss-Kuijf relations at O(1)

and Bern-Carrasco-Johansson relations at O(α')

n-point relations

- Obtain (n-3)! independent amplitudes

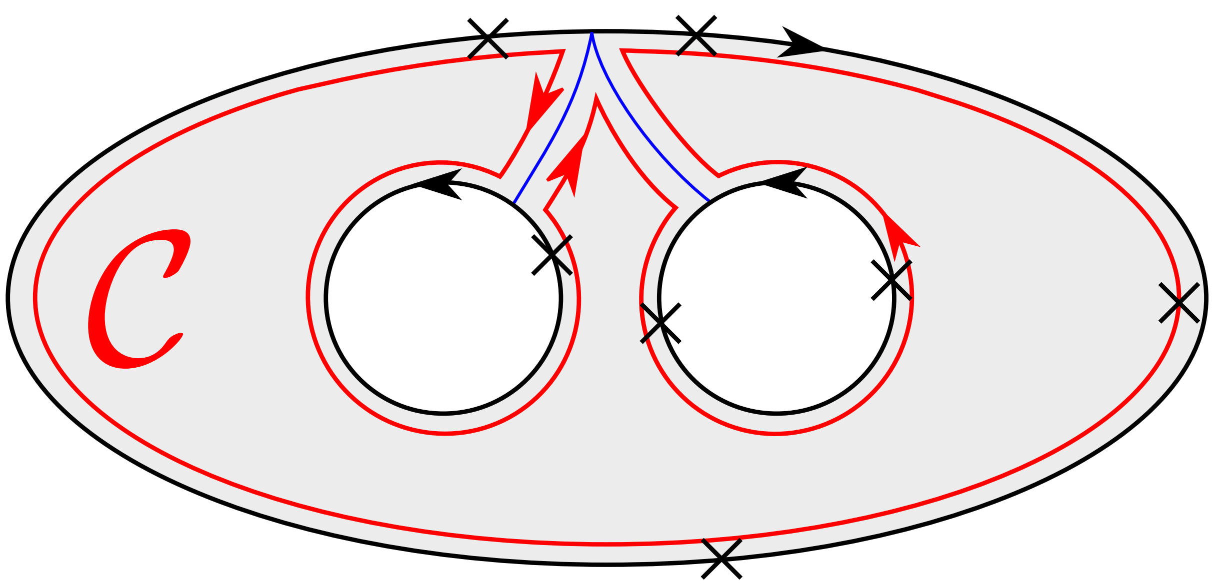
[N. Bjerrum-Bohr, P. H. Damgaard, P. Vanhove]

- Theory-independent statement

Result: higher loop monodromies

Tourkine, Vanhove, 2016

$$\oint_{\mathcal{C}} dz_1(\ldots) = 0$$



Outline of the loop story

1. Holomorphy

2. One loop : detailed analysis

3. Higher loops & definition of the loop momenta

1. Non-holomorphy and zero modes

- Higher loop : multi-periodicity forbids holomorphic functions

  (related theorem ; $\nexists$ elliptic function with single poles)

- Connected to presence of new zero-modes for $\partial X^\mu / \partial z$

- Zero-modes = neither left nor right

1. Non-holomorphy and zero modes

$$A_{\text{1-loop}} = \int_0^\infty \frac{dt}{t^{D/2}} \prod_{1\leq r<s\leq n} e^{-\alpha' k_r\cdot k_s G^{NH}(\nu_r,\nu_s)} \times f(\nu_r,\nu_s)$$

1. Non-holomorphy and zero modes

$$A_{\text{1-loop}} = \int_0^\infty \frac{dt}{t^{D/2}} \prod_{1\leq r<s\leq n} e^{-\alpha' k_r \cdot k_s G^{NH}(\nu_r,\nu_s)} \times f(\nu_r, \nu_s)$$

$$G^{NH} = G^{Hol} + \frac{\text{Im}(\nu_r - \nu_s)^2}{t}$$

1. Non-holomorphy and zero modes

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$$\propto \int d^D\ell \int_0^\infty dt(d^{n-1}\nu) \boxed{e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i}}$$

$$\times \prod_{1\le r<s\le n} e^{-\alpha' k_r\cdot k_s\, G^{Hol}(\nu_r,\nu_s)} \times f(\nu_r,\nu_s)$$

$$G^{NH} = G^{Hol} + \frac{\text{Im}(\nu_r - \nu_s)^2}{t}$$

1. Non-holomorphy and zero modes

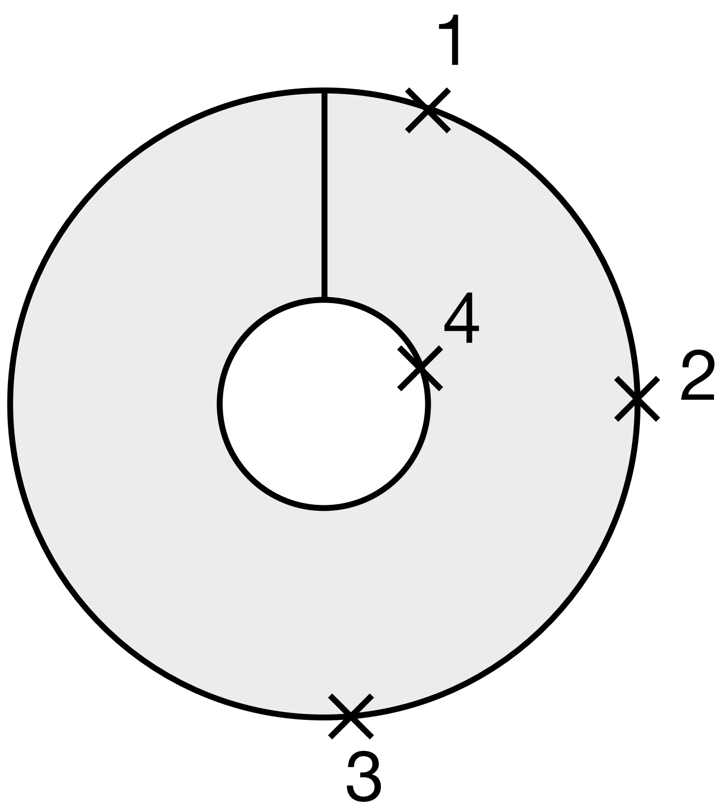
Similar expression at all loops [D'Hoker Phong '89] ('Chiral Splitting')

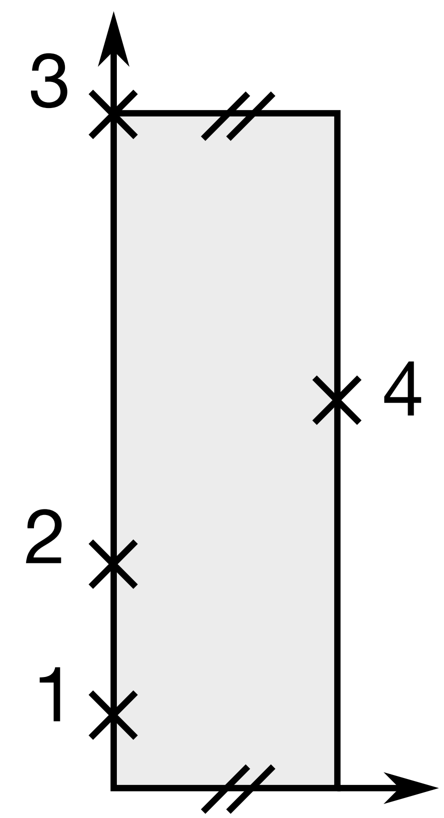
Morale:

for holomorphy, introduce loop-momentum

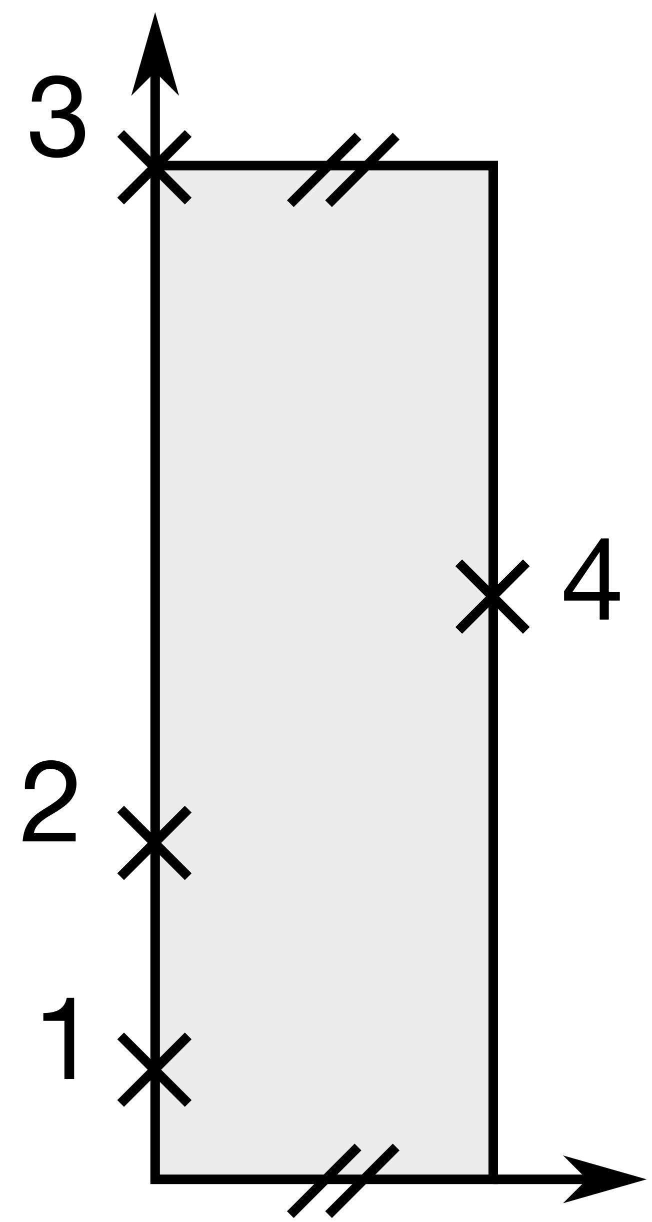
Consequence ; a lot of what follows is for integrands

2. One loop



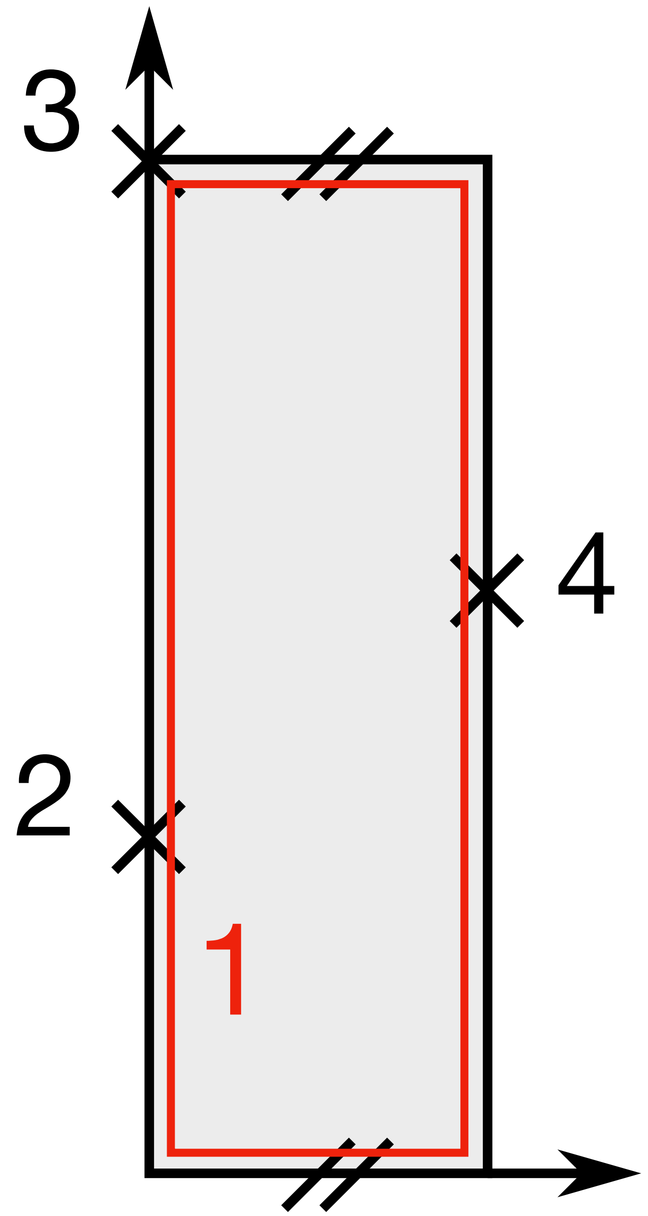


Loop monodromies

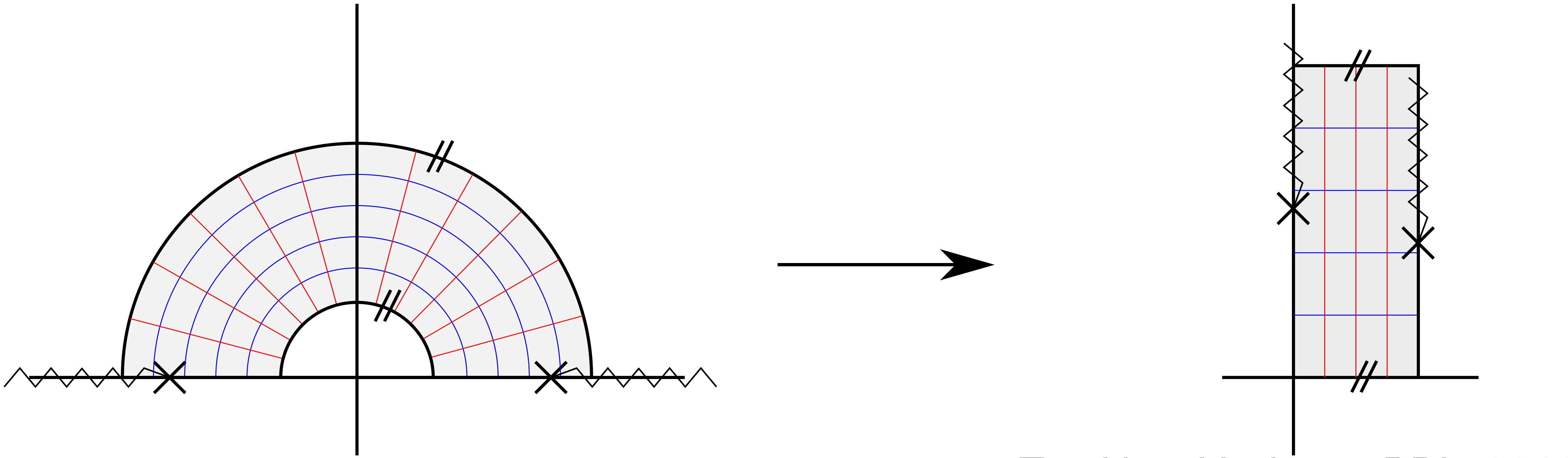


Loop monodromies

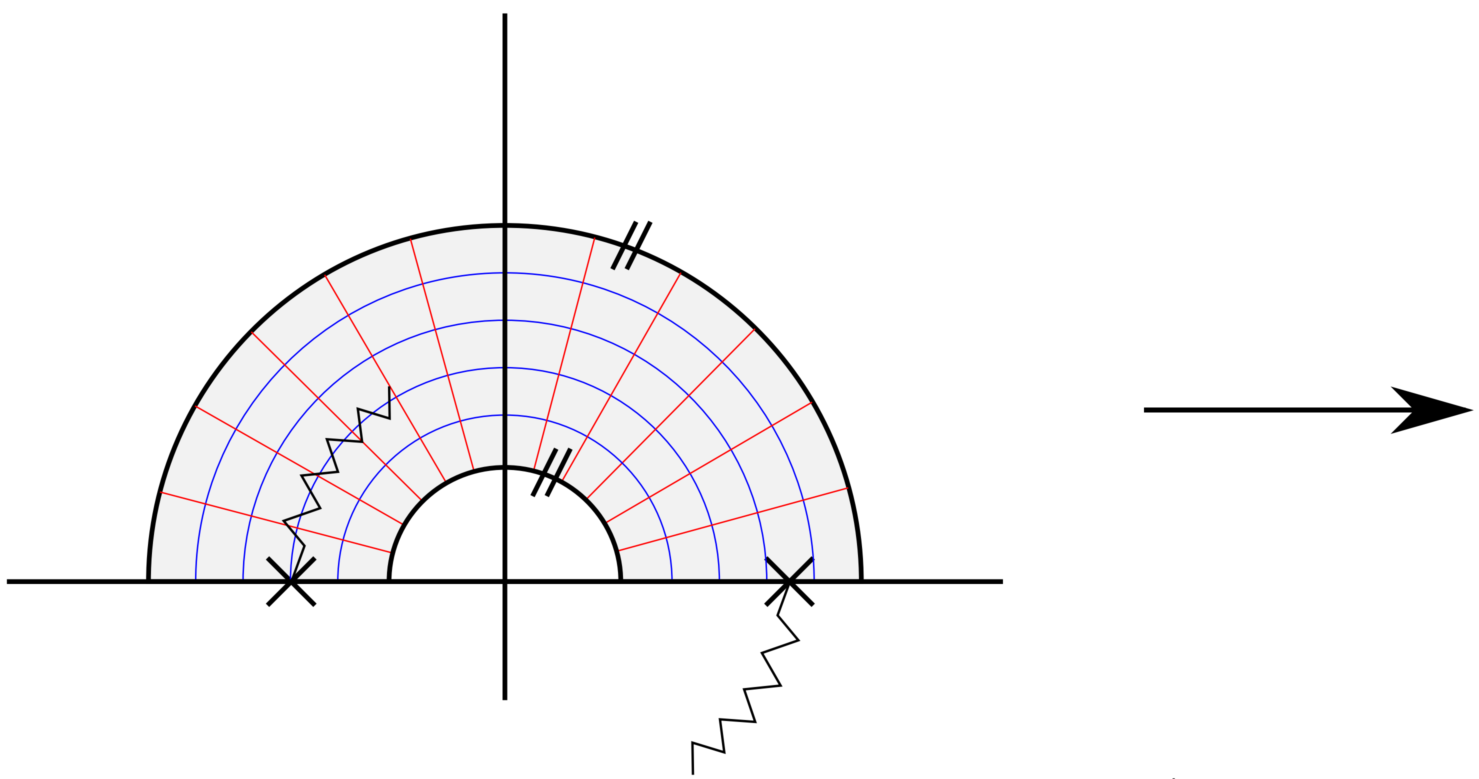
$$\oint A(z_1, \ldots)dz_1 = 0$$

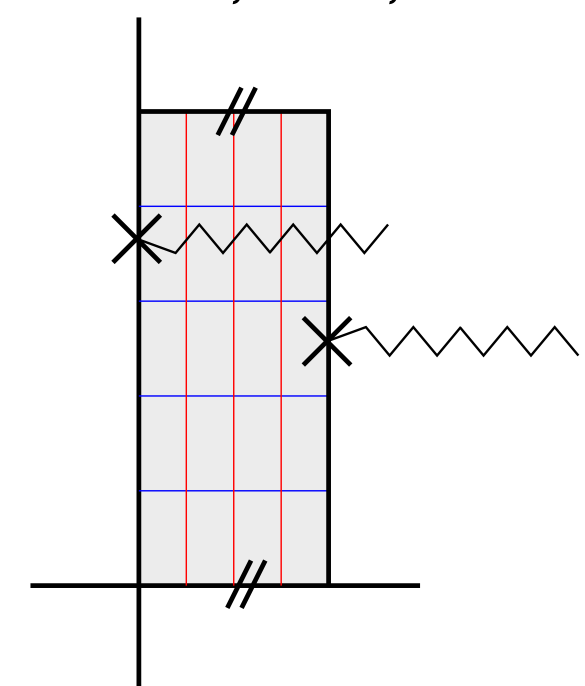


Comment on cuts



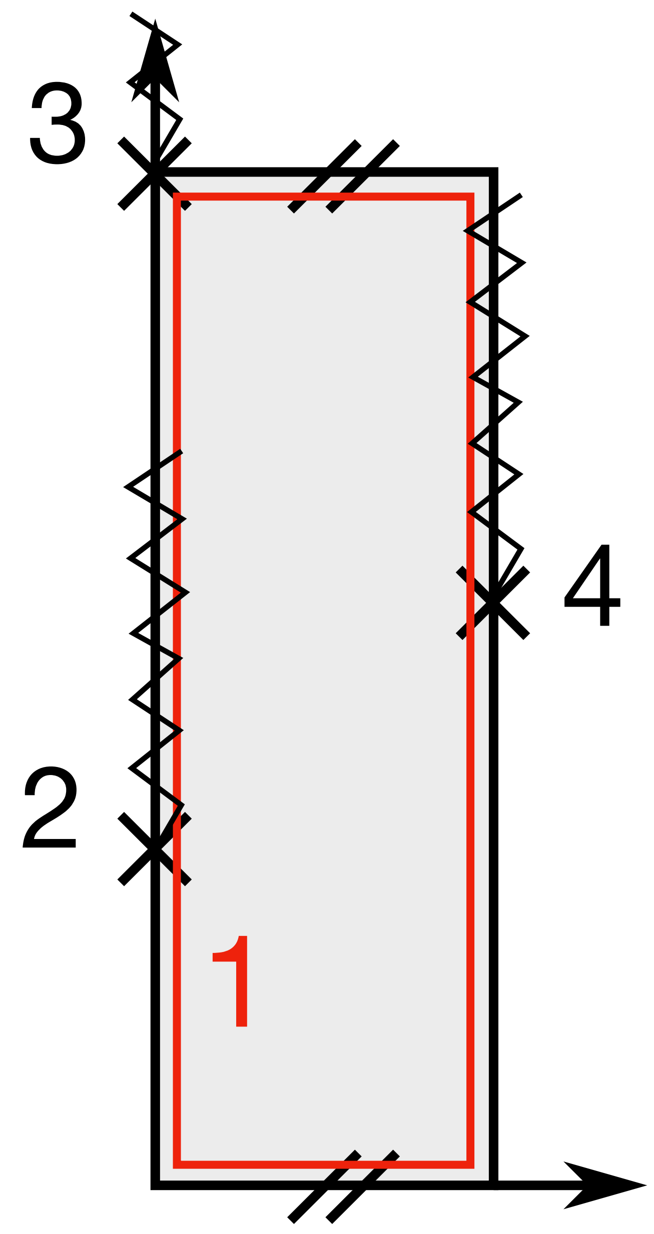
Tourkine, Vanhove, PRL, 2016



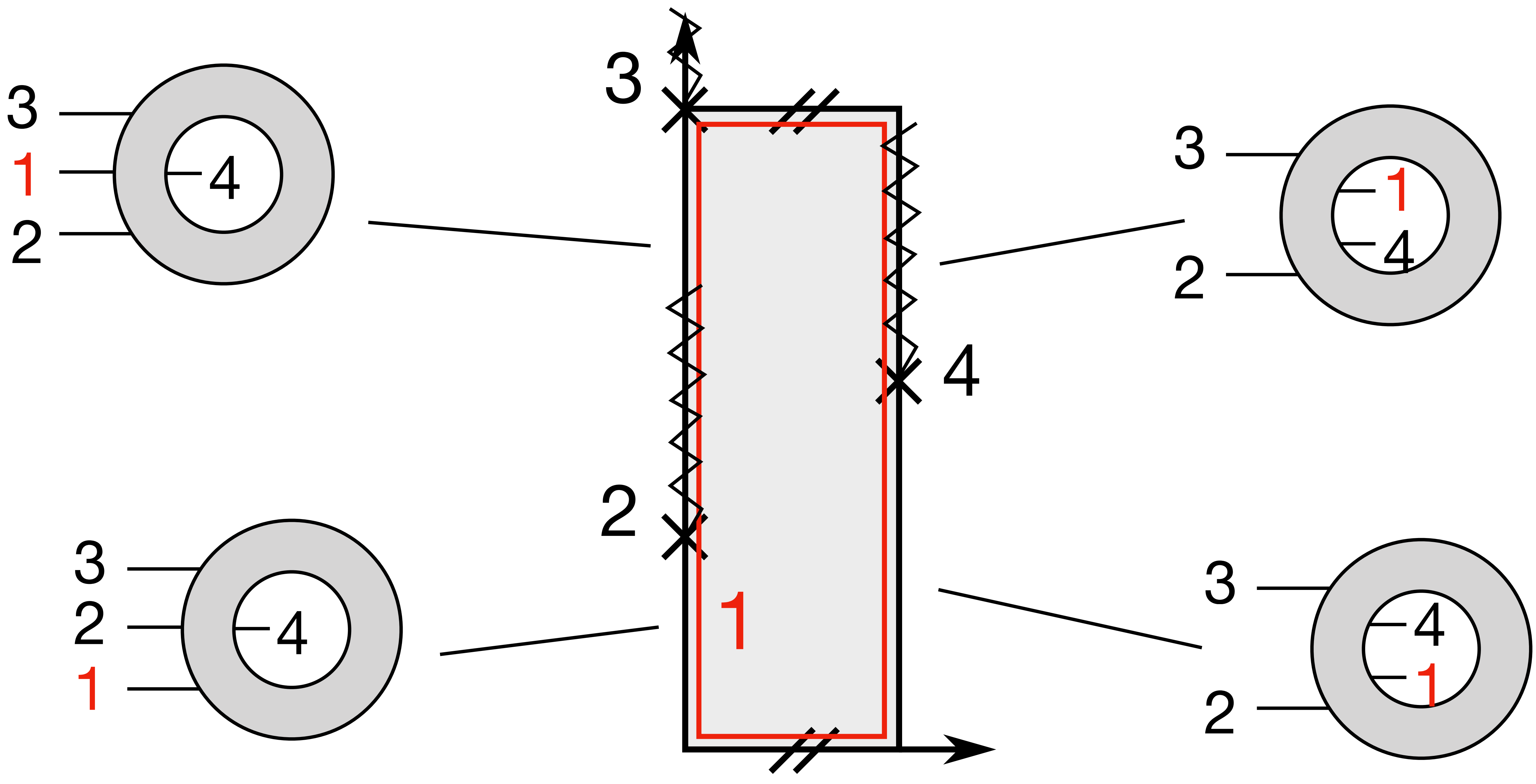


Hohenneger, Stieberger 2017

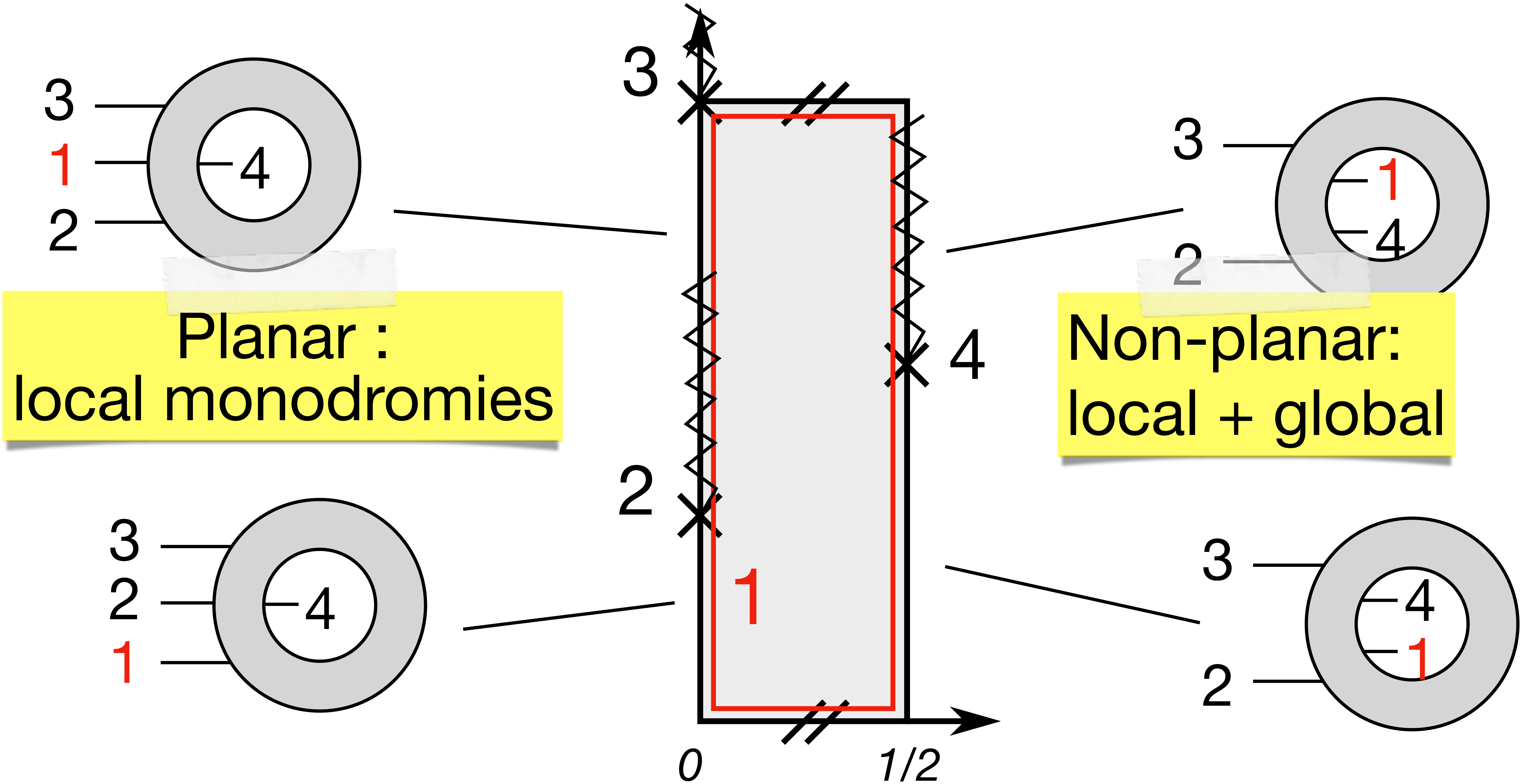
Loop monodromies



Loop monodromies



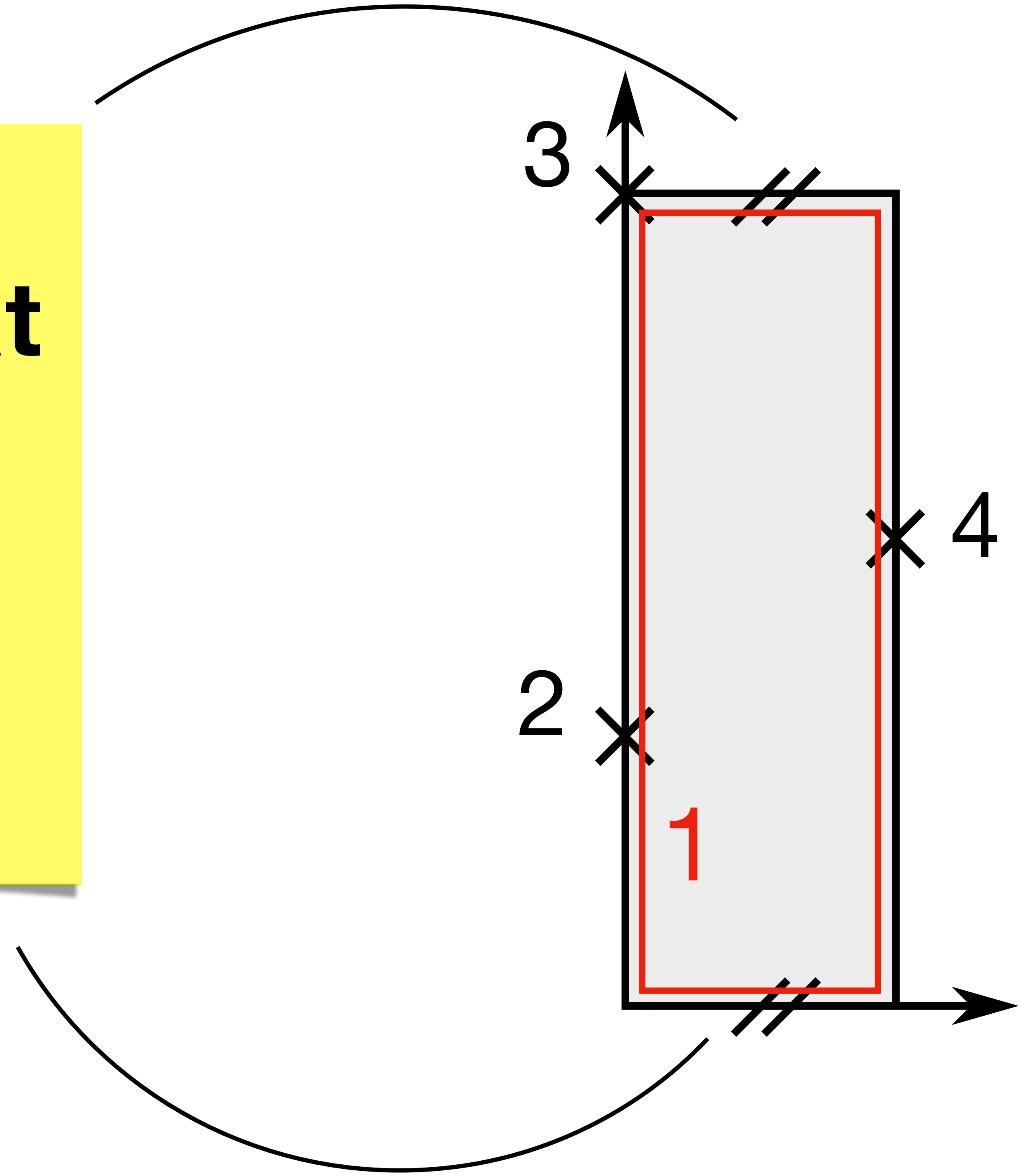
Loop monodromies



Loop monodromies

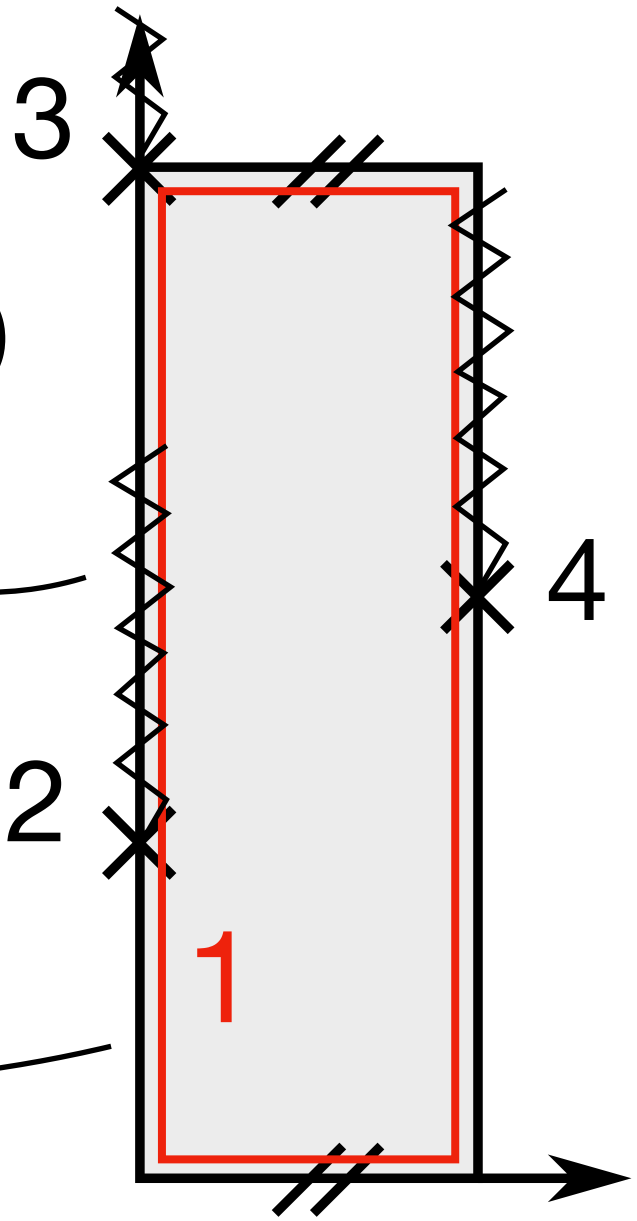
Boundary terms:
contributions that integrate to zero

both in string and field theory



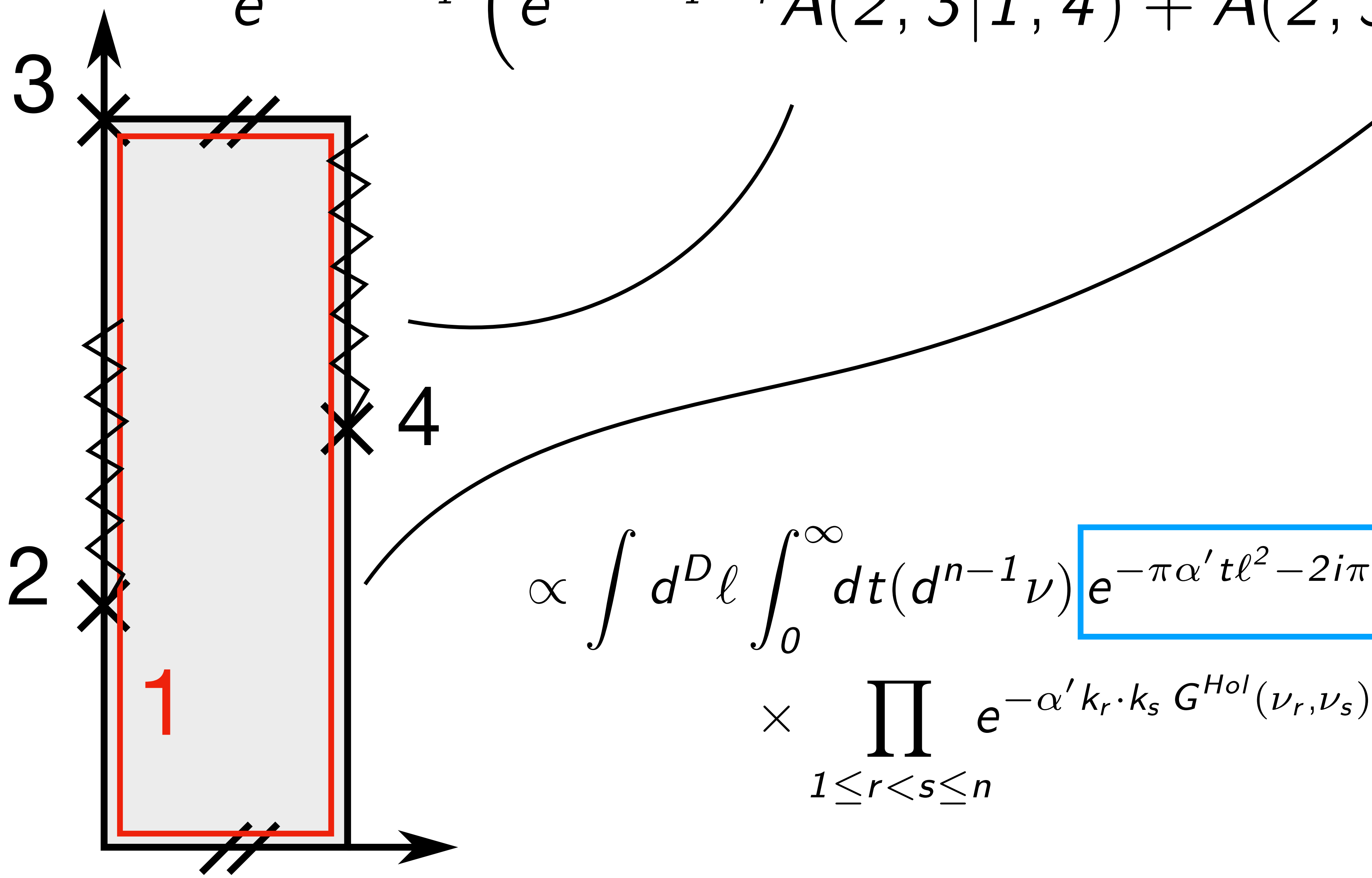
Planar : local phases

$$A(1,2,3|4) + \exp(i\pi\alpha' k_1 \cdot k_2) A(2,1,3|4)$$



Global phases

$$e^{i\pi\alpha'\ell\cdot k_1}\left(e^{i\pi\alpha' k_1\cdot k_4}A(2,3|1,4)+A(2,3|4,1)\right)$$



$$\propto \int d^D\ell \int_0^\infty dt (d^{n-1}\nu)\, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i}$$

$$\times \prod_{1\leq r<s\leq n} e^{-\alpha' k_r\cdot k_s\, G^{Hol}(\nu_r,\nu_s)} \times f(\nu_r,\nu_s)$$

Global phases

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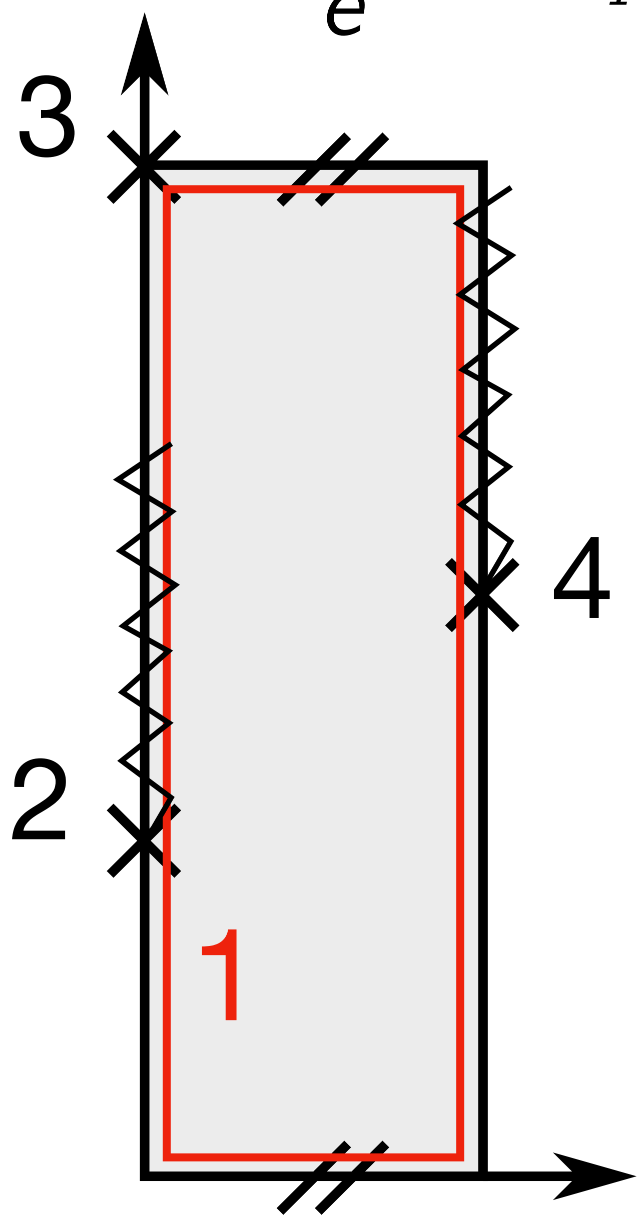


again :

$A(..|..) =$ integrand !

Also : consequences for graph interpretation

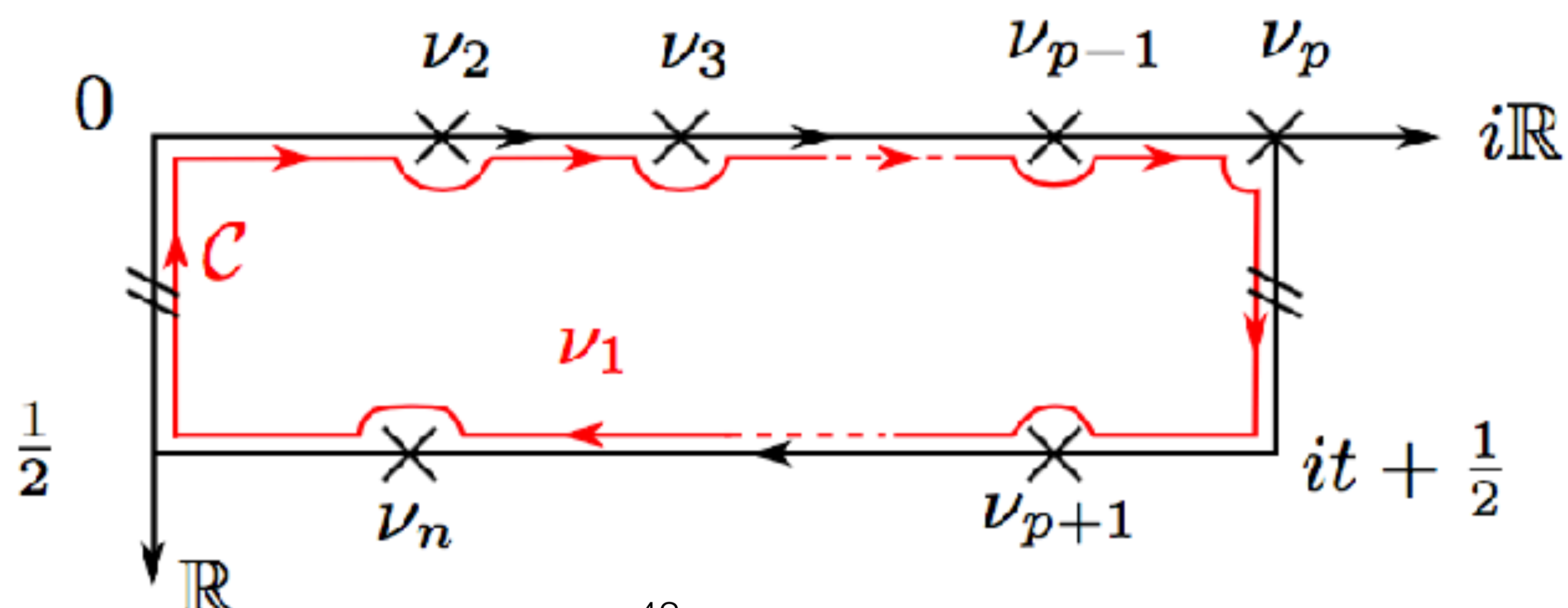
[Ochirov, Tourkine, Vanhove], to appear

Monodromy relations

$$A(1,2,\ldots,p|p+1,\ldots,n)+$$

$$\sum_{i=2}^{p-1} e^{i\alpha'\pi k_1\cdot k_{2\ldots i}} A(2,\ldots,i,1,i+1,\ldots,p|p+1,\ldots,n) =$$

$$-\sum_{i=p}^{n}\left(e^{-i\alpha'\pi k_1\cdot k_{i+1\cdots n}} \times A(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n)[e^{-i\pi\alpha'\ell\cdot k_1}]\right)$$



Monodromy relations

$O(1)$:

$$A(\alpha|\beta^T) = (-1)^{|\beta|} \sum_{\gamma \in (\alpha \text{ shuffle } \beta)} A(\gamma)\,.$$

[Bern Dixon Dunbar Kosower,'94]

[Feng, Huang, Jia, '10,'11]

$O(\alpha')$:

$$\sum_{i=2}^{p-1} k_1 \cdot k_{2\ldots i}\, \mathcal{I}(2,\ldots,i,1,i+1,\ldots,p|p+1,\ldots,n) +$$

$$\sum_{i=p}^{n} k_1 \cdot (\ell + k_{i+1\ldots n})\, \mathcal{I}(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n) = 0$$

Monodromy relations

Particular case (planar start):

$$\sum_{i=1}^{n-1} p_1 \cdot (\ell + \sum_{j=1}^{i} p_j)\, \mathcal{I}(p_2, \ldots, \underbrace{p_1}_{\text{position } i}, \ldots, p_n) \approx 0\,.$$

[Boels, Iserman '11], [Brown, Naculich '16]

Monodromy relations

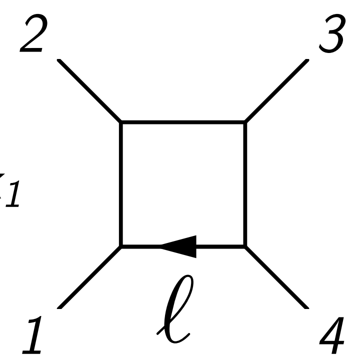
- Relations are **generic**
- Just need to be able to build the theory in open string theory

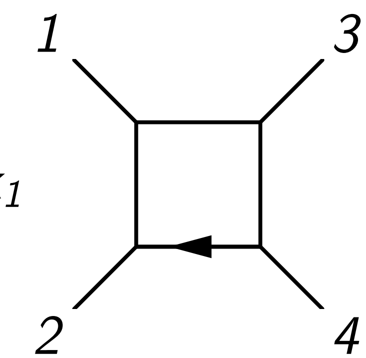
Remark : ∃ equivalent in terms of Schwinger proper-time parametrization

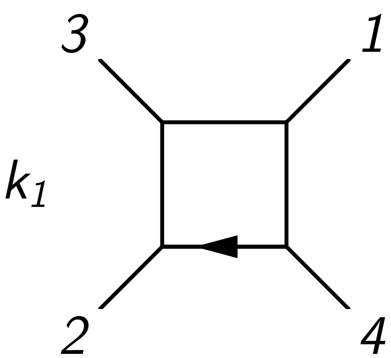
- Used in *[Hohenneger, Stieberger ’17]*
- Relates different Feynman graphs
- α’ expansion : also infinite number of relations between elliptic MZV’s

[Hohenneger, Stieberger ’17],
[J. Broedel, N. Matthes, G. Richter, O. Schlotterer ’17]

Field Theory : BCJ satisfies one-loop relations

$\ell \cdot k_1$ 

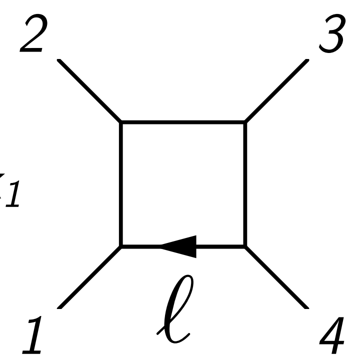
$+ (\ell + k_2) \cdot k_1$ 

$+ (\ell + k_{23}) \cdot k_1$ 

[Boels, Iserman; Y. Du, H. Luo; Brown, Naculich]

Field Theory : BCJ satisfies one-loop relations

$\ell \cdot k_1$



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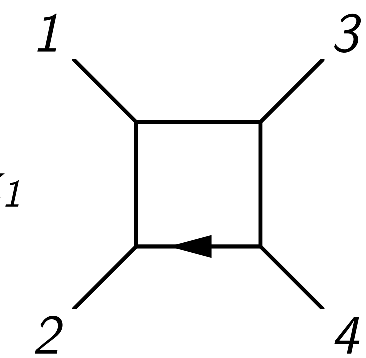
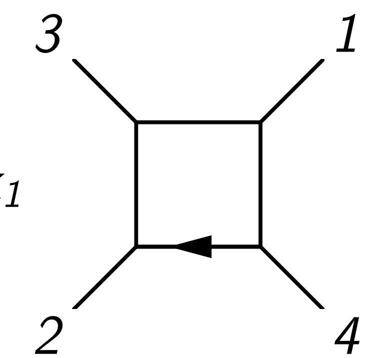


$+ (\ell + k_{23}) \cdot k_1$

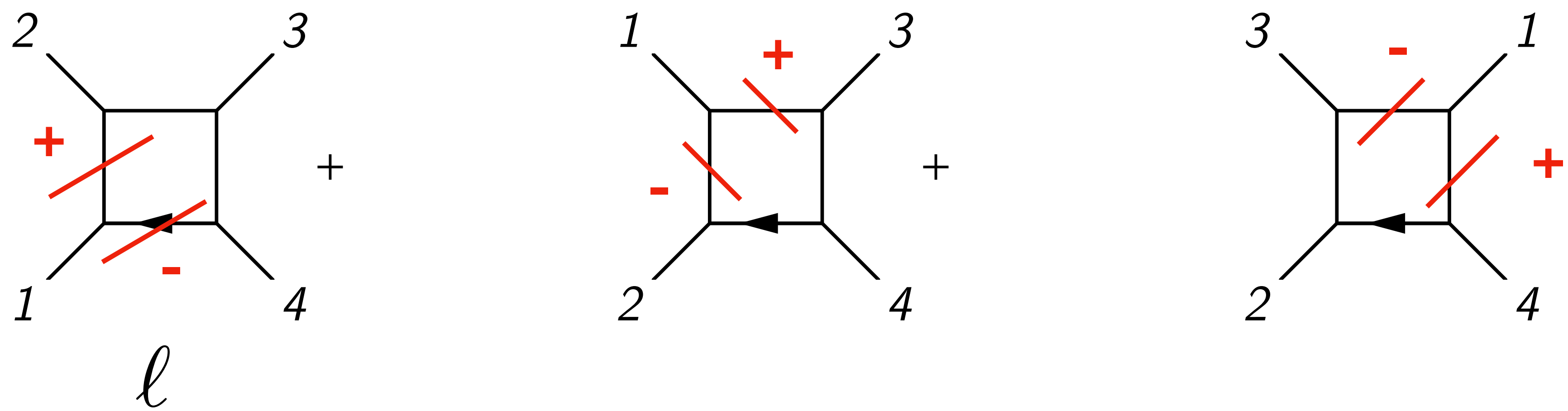


$$2\ell \cdot k_1 = (\ell + k_1)^2 - \ell^2$$

$$2(\ell + k_2) \cdot k_1 = (\ell + k_1 + k_2)^2 - (\ell + k_2)^2$$

$$2(\ell + k_2 + k_3) \cdot k_1 = (\ell + k_1 + k_2 + k_3)^2 - (\ell + k_2 + k_3)^2$$

Field Theory : BCJ satisfies one-loop relations



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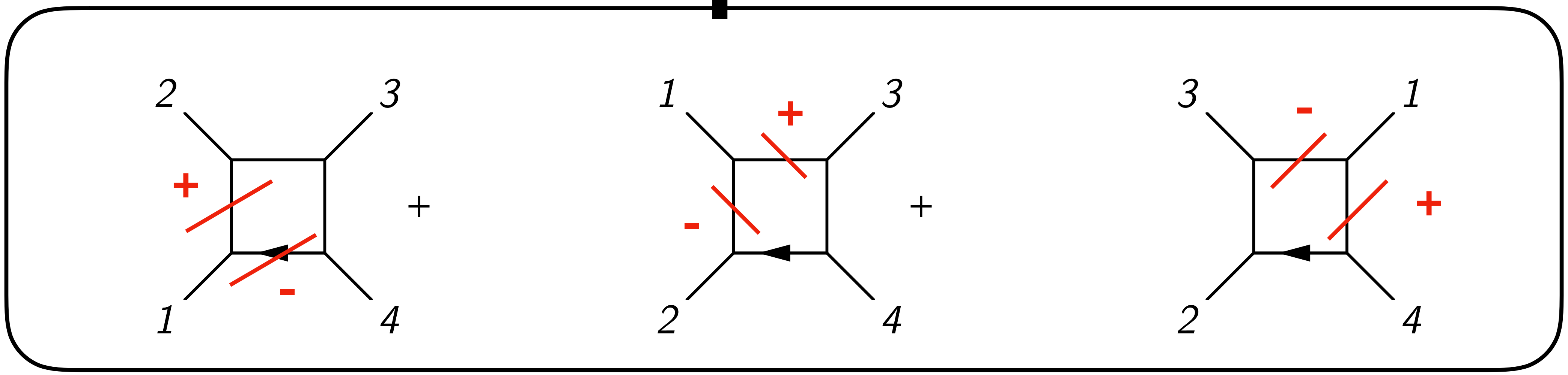
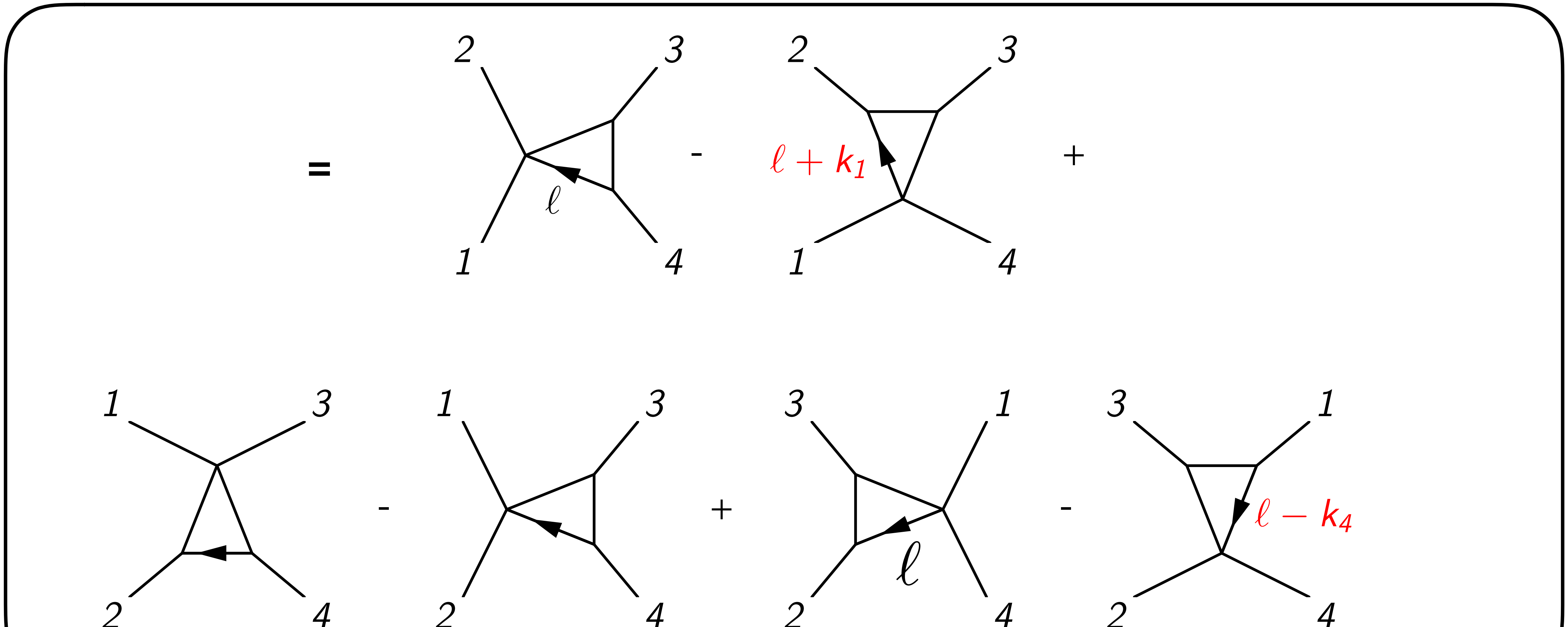
Field Theory : BCJ satisfies one-loop relations





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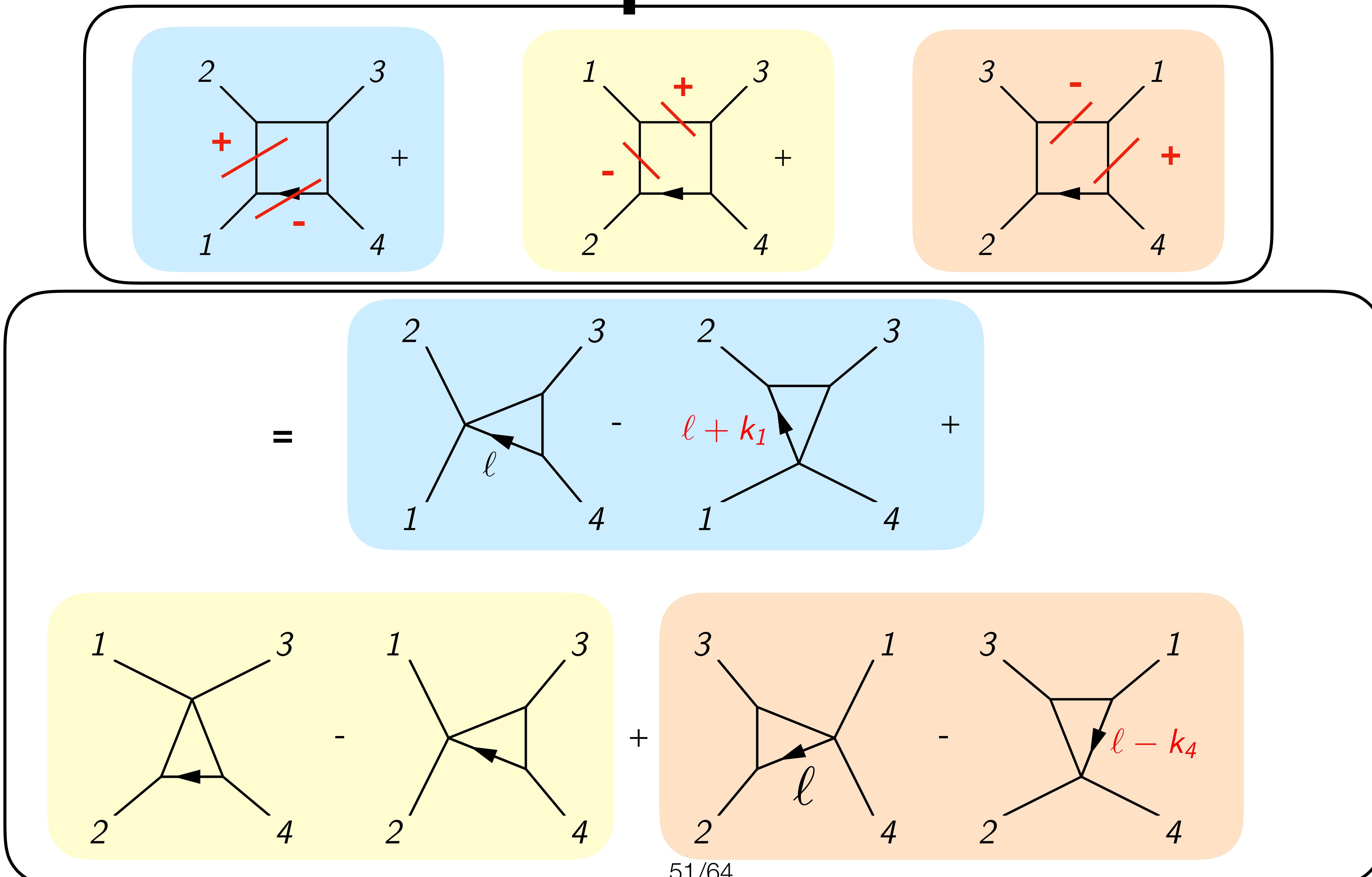

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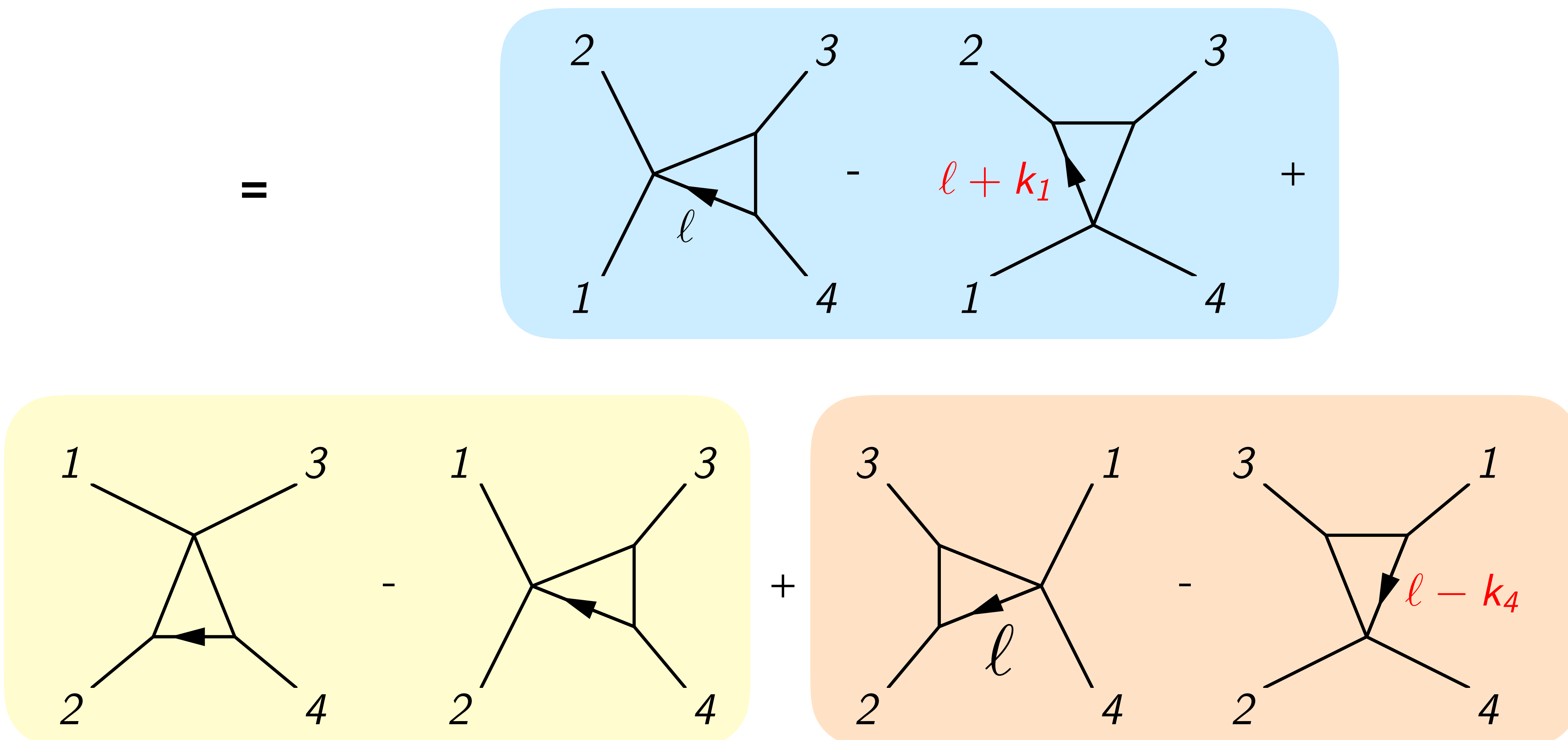
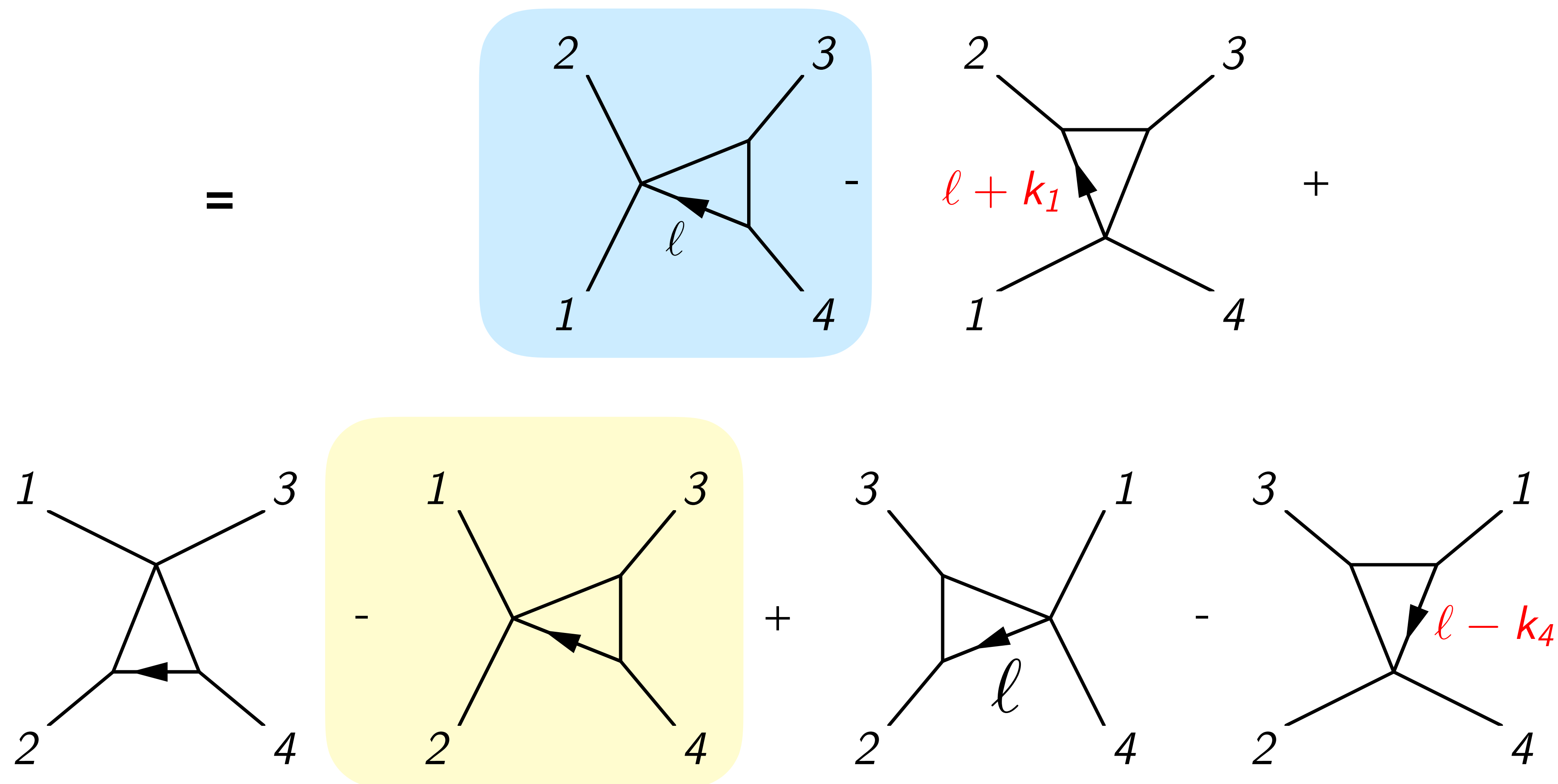
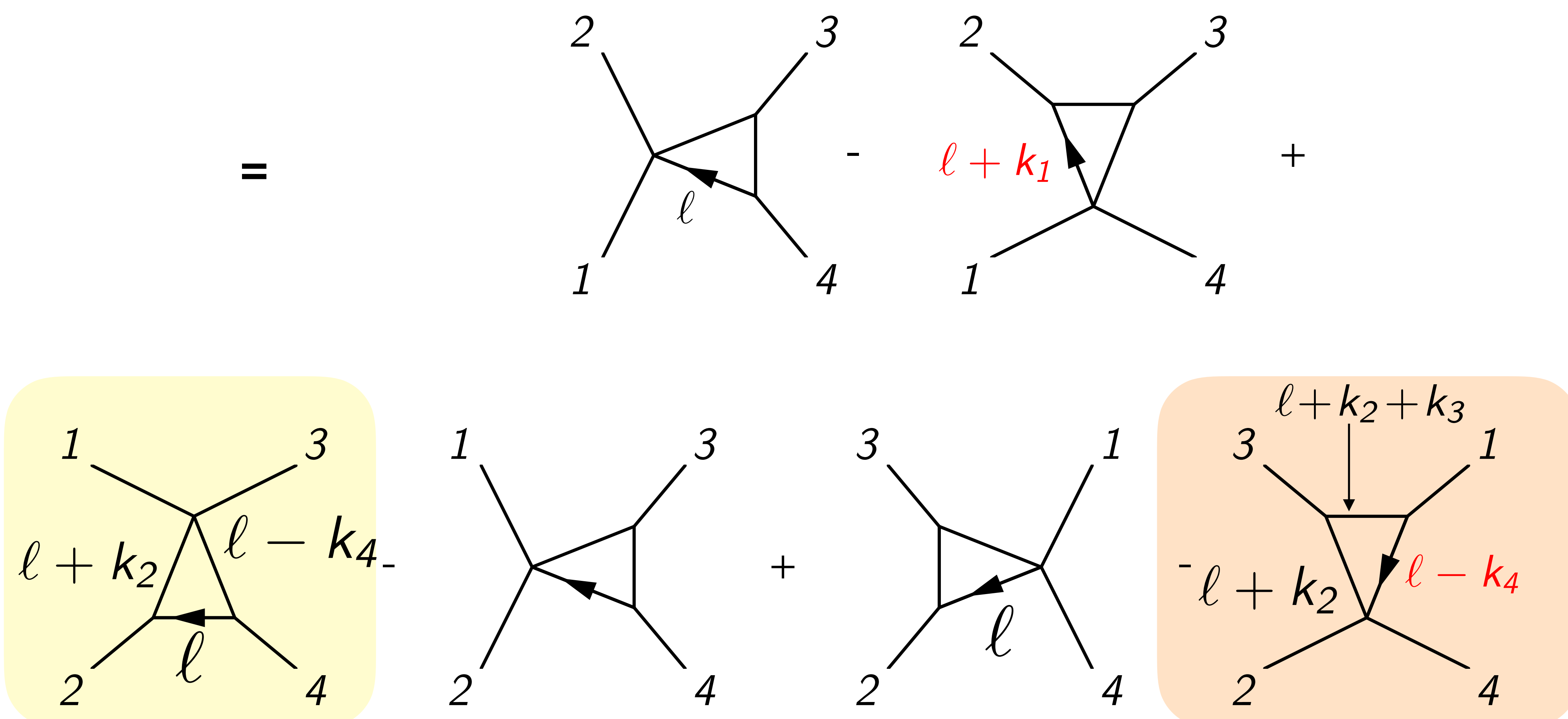


Field Theory : BCJ satisfies one-loop relations

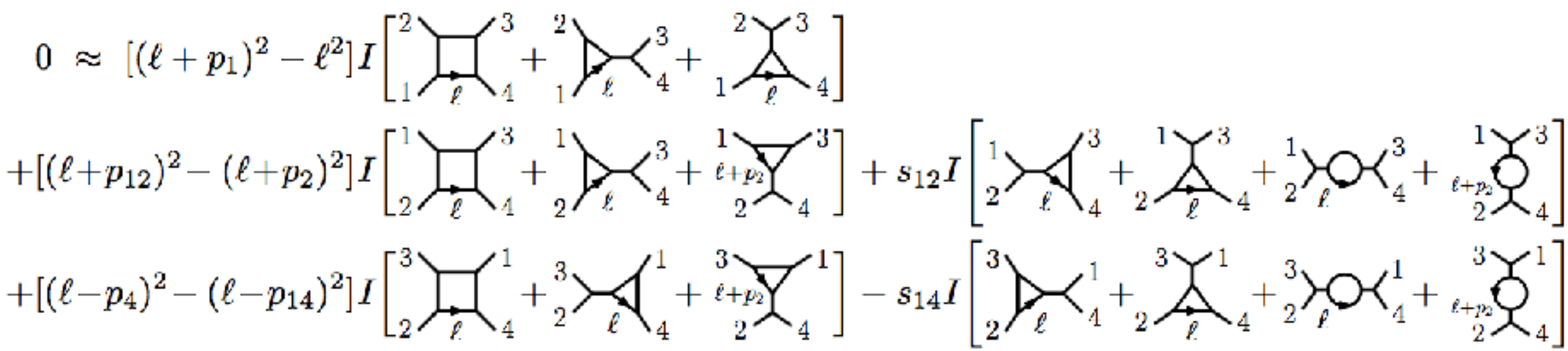


Field Theory : BCJ satisfies one-loop relations



Field Theory : BCJ satisfies one-loop relations

[Ochirov, Tourkine, Vanhove], to appear



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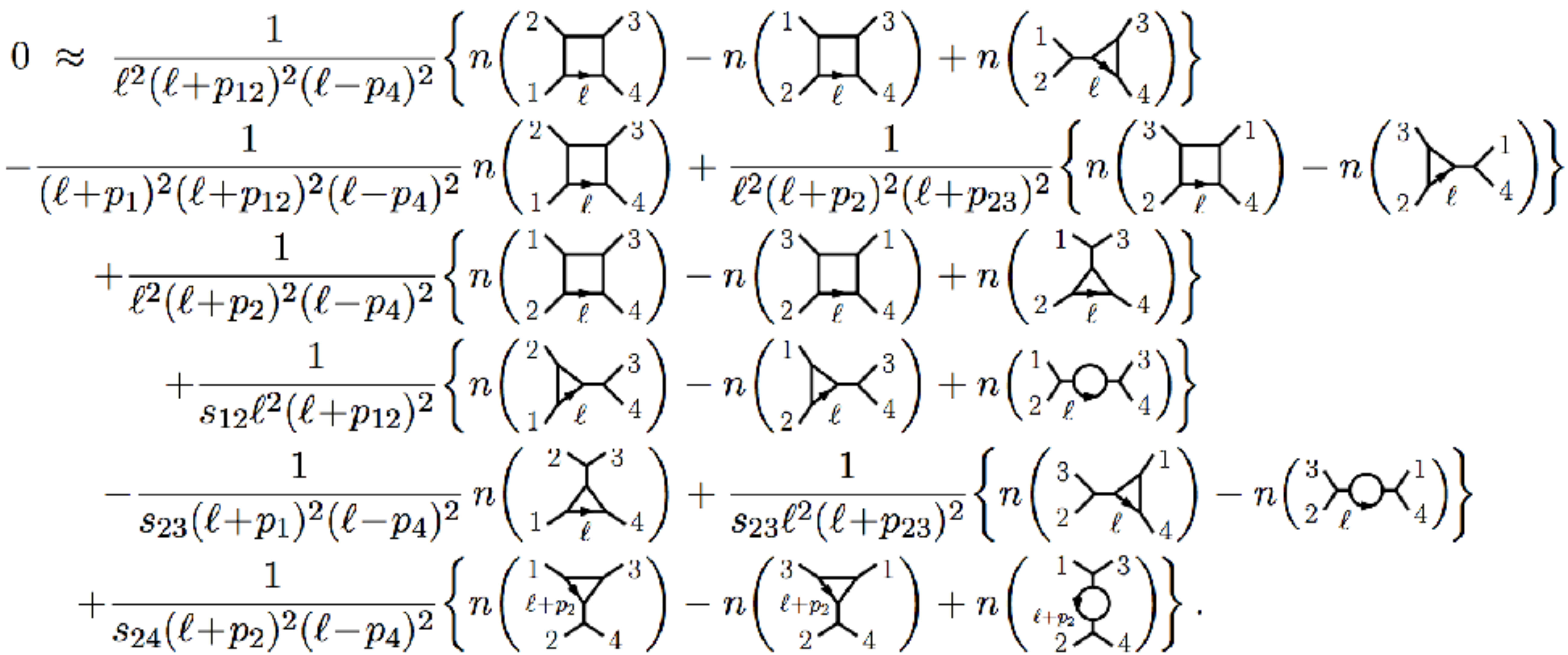
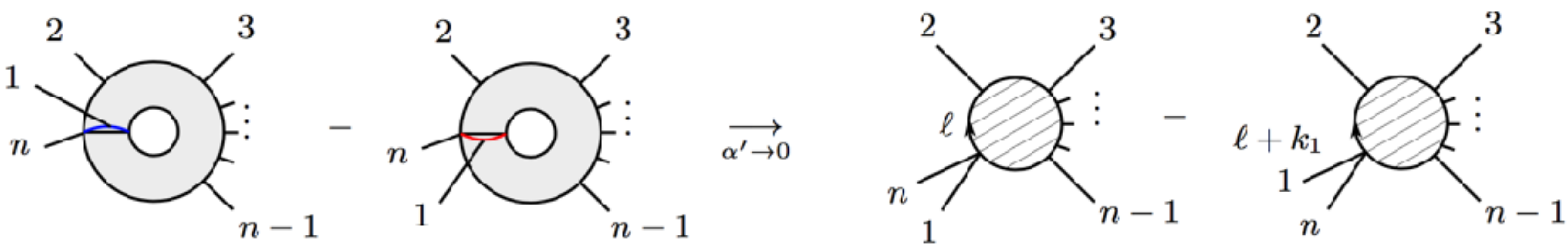


Boundary terms

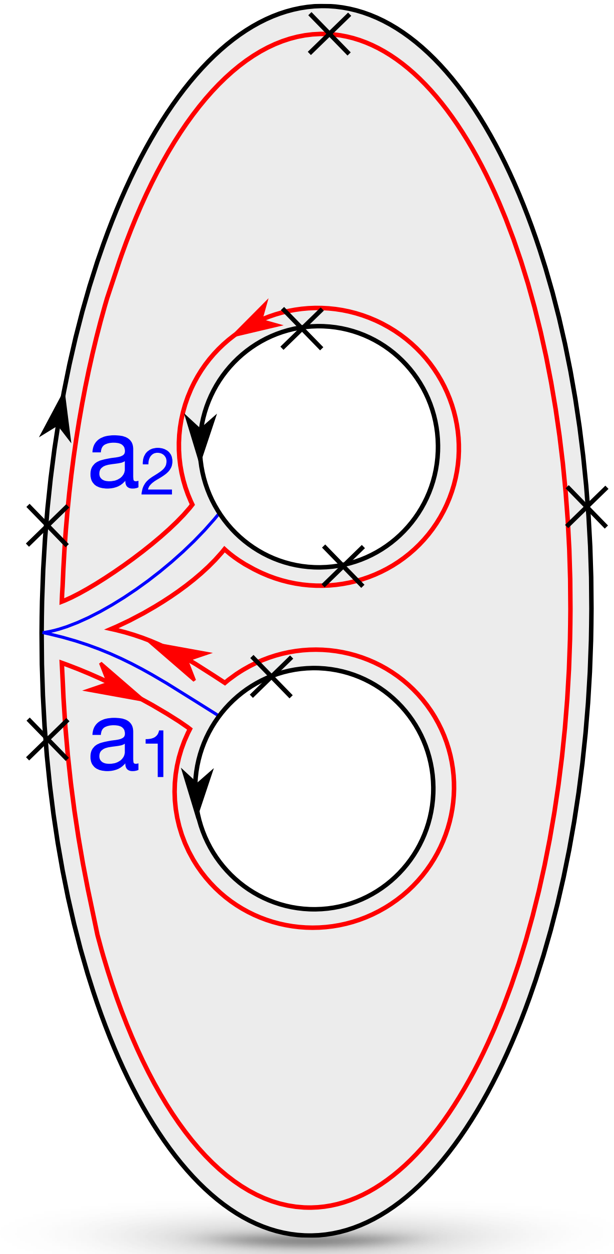
Constrain the form of terms that integrate to zero



[Tourkine, Vanhove '16]

Higher loop relations

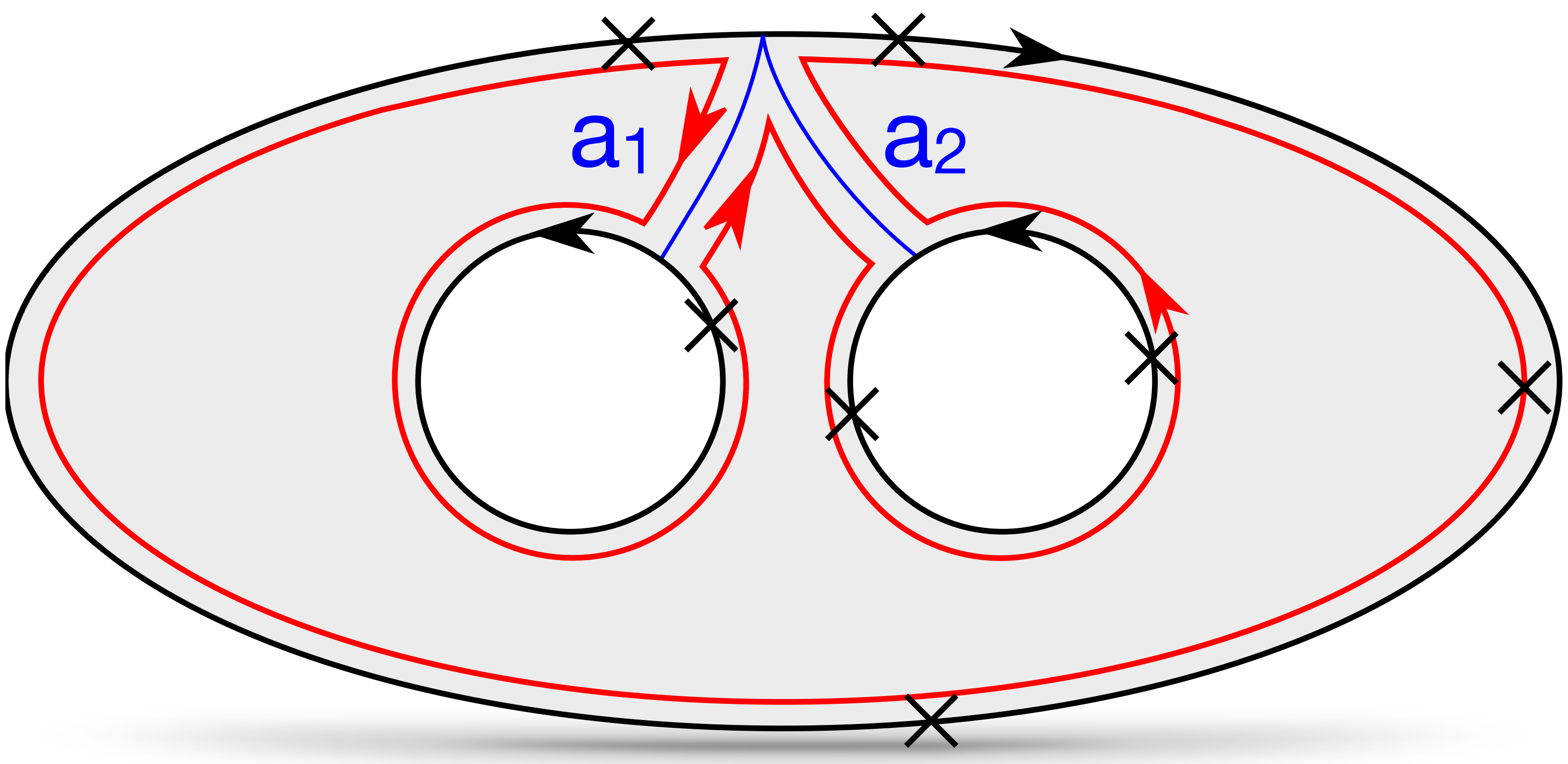
$$\sum_{r=1}^{|\alpha|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\alpha_s}}\Big)\mathcal{A}^{(2)}(\dots,\alpha_{s-1},1,\alpha_s,\dots|\beta|\gamma)+$$
$$\sum_{r=1}^{|\beta|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\beta_s}}\Big)\mathcal{A}^{(2)}(\alpha|\dots,\beta_{s-1},1,\beta_s,\dots|\gamma)[e^{-i\alpha'\pi\ell_1\cdot k_1}]+$$
$$\sum_{r=1}^{|\gamma|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\gamma_s}}\Big)\mathcal{A}^{(2)}(\alpha|\beta|\dots,\gamma_{s-1},1,\gamma_s,\dots)[e^{-i\alpha'\pi\ell_2\cdot k_1}]$$
$$=0\,. \quad (19)$$



[Tourkine, Vanhove '16]

Higher loop relations

$$\ell_i^\mu = \int_{a_i} \partial X^\mu$$



Global definition of the loop momentum

Global definition of the loop momentum

previous interpretation:

$$\int d^d\ell_1 \ldots d^d\ell_L \Big(\text{integrand monodromies}\Big) = 0$$

$$\Longleftrightarrow \quad \Big(\text{integrand monodromies}\Big) \approx 0$$

while actually we have:

$$\Big(\text{integrand monodromies}\Big) = F(\ell_i) - F(\ell_i - K_i)$$

remain to be determined explicitly

Checks on higher loops

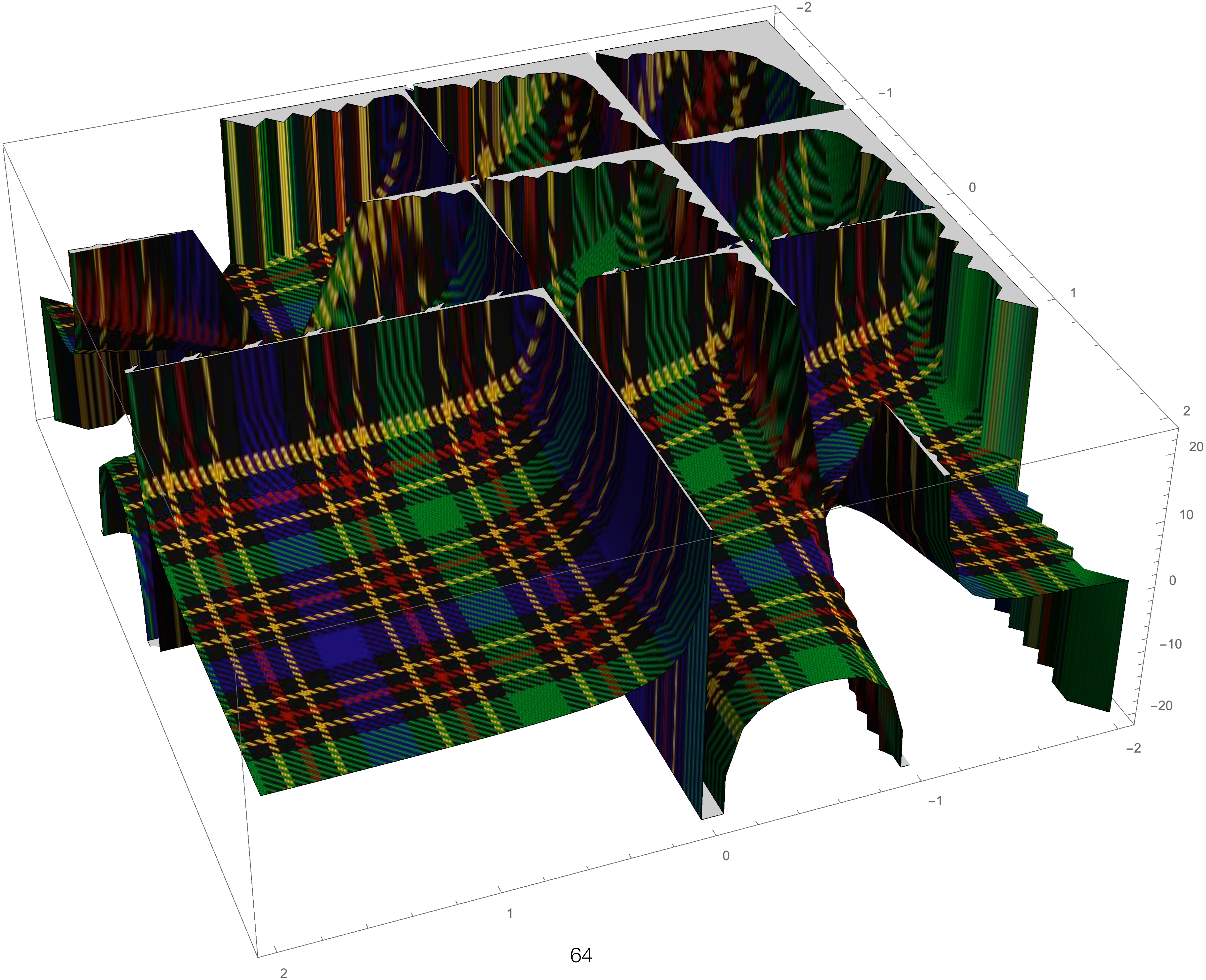
- BDDK-type relations at two loops by *[Feng, Jia, Huang 2011]*
- N=4 super-Yang-Mills 4-pt two-loop amplitude ✔
- All loop N=4 super-Yang-Mills relations of *[M. Chiodaroli, M. Gunaydin, H. Johansson, R. Roiban 1703.00421]*

Summary / results

- Verified agreement with Hohenneger & Stieberger
- Written down for the first time monodromy relations in loop-level open string theory,
- Formulae are for integrands but are exact,
    - O(1) : relate planar and non-planar
    - O(α') : integrand monodromy relations
- Valid in any theory (but may depend on the representation)

Perspectives

- Consequences of the global definition of loop momentum ?
- Control over boundary terms : connection to the labelling problem ? More generally C/K duality in loops ?
- KLT at loop-level ?
- Feynman diagrammatics consequences ? New elliptic MZVs relations ?



−2
−1
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10
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1
2