Feynman Integrands and Scattering Equations

Amplitudes 2017

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Scattering Equations and the CHY Approach.

Since the remarkable work of Witten on the $\mathcal{N} = 4$ super Yang–Mills theory, the on-shell methods for the computation of scattering amplitudes have been deeply studied during the last years. In particular, the Cachazo–He–Yuan (CHY) approach. [Witten-03,Cachazo-He-Yuan-13, Mason, Skinner-13, Berkovits-13].

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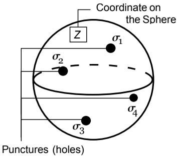
Some Properties (Tree-level).

- It is applicable in arbitrary dimension.
- It can be applied for a large family of interesting theories including scalars, gauge bosons, gravitons and mixing interactions.

Moduli Space

- Let us consider a Sphere with punctures (holes).



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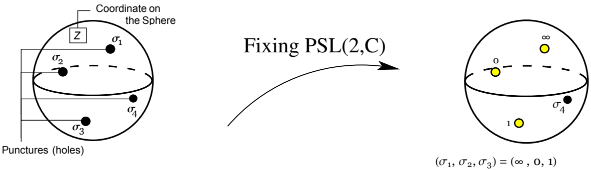

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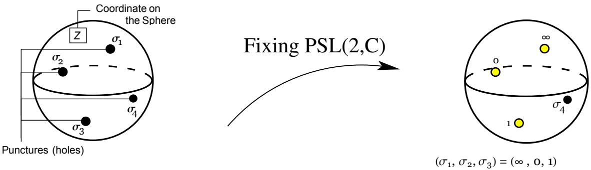

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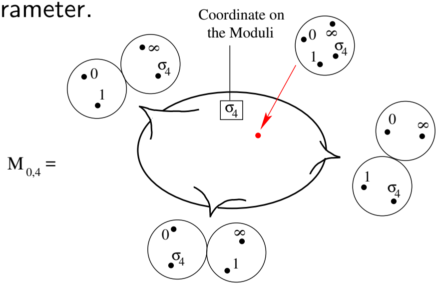
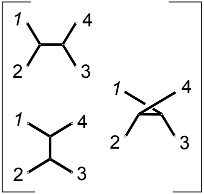
- σ_4 is a free parameter.



Scattering Equations

- One can define the scattering equations as a map from the factorization limits to the boundary of the Moduli Space. For example, by considering four massless particles, i.e. $s + t + u = 0$, in Φ^3, [Nima's talk]



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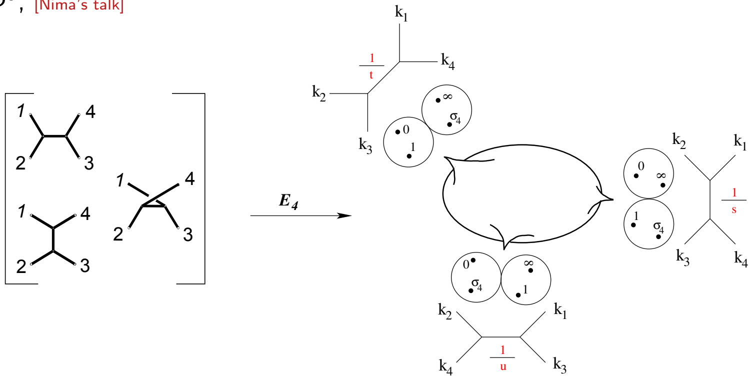

- $E_4 := \frac{t}{\sigma_4 - \infty} + \frac{u}{\sigma_4 - 0} + \frac{s}{\sigma_4 - 1} = 0 \quad \Rightarrow \quad \sigma_4 = \frac{u}{s+u} = -\frac{u}{t}$

Scattering Equations (S.E.)

- In general, the scattering equations are given by [Gross, Mende-87, O. Schlotterer's talk, Lipstein's talk, L. Mason's talk]

$$E_a = \sum_{b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0, \quad \sigma_{ab} := \sigma_a - \sigma_b, \quad a = 1, \ldots n.$$

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- $\mathrm{PSL}(2,\mathbb{C})$: $\quad \sum_{a=1}^{n} E_a = \sum_{a=1}^{n} \sigma_a \, E_a = \sum_{a=1}^{n} \sigma_a^2 \, E_a = 0.$

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- Cachazo-He-Yuan Approach, 2013 (Contour Integral over the Moduli Space and localized on $E_a = 0$ (S.E.). Mason-Skinner, Ambitwistor, 2013)

$$\mathbf{I}_n = \int_\Gamma d\mu_n^{(0)}\, \mathcal{I}_n(\sigma), \qquad d\mu_n^{(0)} := \frac{d^n\sigma}{\mathrm{Vol}(\mathrm{PSL}(2,\mathbb{C}))} \left[\frac{\sigma_{ij}\sigma_{jk}\sigma_{ki}}{\prod_{a\neq i,j,k} E_a} \right],$$

Γ is a contourn defined by $E_a = 0$ (S.E.)

- To obtain a well define $\mathbf{I}_n$ over $M_{0,n}$, the meromorphic form, $d\mu_n^{(0)}\, \mathcal{I}_n$, must be invariant under $\sigma_a = \frac{A\ \sigma_a + B}{C\ \sigma_a + D}$, with $A\,D - B\,C = 1$.

Integrands and Graphs on a Sphere

- Under $\sigma_a = \frac{A\,\sigma_a + B}{C\,\sigma_a + D}$, with $A\,D - B\,C = 1$

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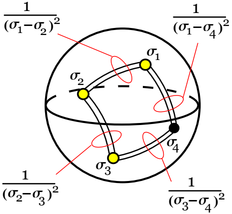

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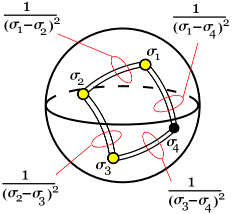

CHY graph $\rightarrow$ weight four means four lines at each vertex.

Integrands and Theories

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\cdot\mathbf{PT}^{(0)}_\alpha \times \mathrm{Pf}'\Psi := \mathrm{A}^{\mathrm{YM}}(\alpha_1,\ldots,\alpha_n) & (\text{Yang}-\text{Mills}) \\
\cdot\mathbf{PT}^{(0)}_\alpha \times \det{}'A := \mathrm{A}^{\mathrm{NLSM}}(\alpha_1,\ldots,\alpha_n) & (\text{NLSM}) \\
\cdot\mathrm{Pf}'\Psi \times \mathrm{Pf}'\Psi := \mathcal{M}(1,\ldots,n) & (\text{Gravity}) \\
\cdot\det{}'A \times \det{}'A := \mathcal{M}^{\mathrm{GAL}}(1,\ldots,n) & (\text{Galileon}) \\
\vdots &
\end{array}\right\} \begin{array}{c}\text{C.H.Y} \\ (2013)\end{array}$$

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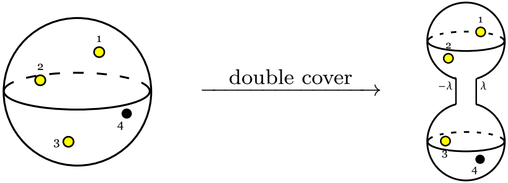

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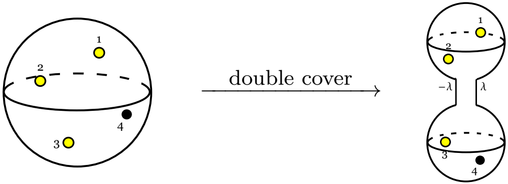

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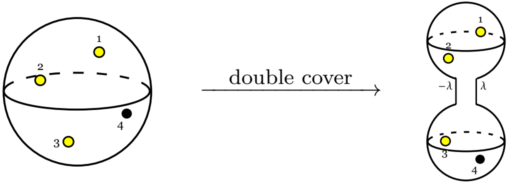

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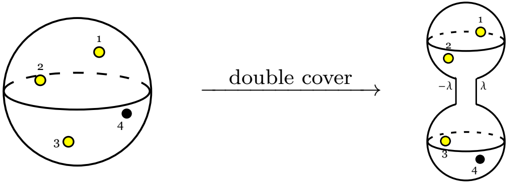

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and Γ is the contour defined by the equations

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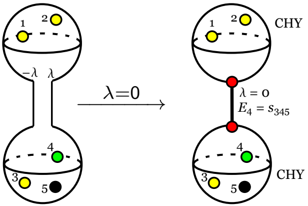

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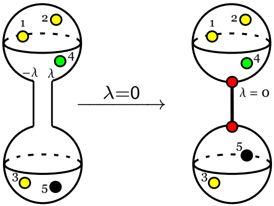

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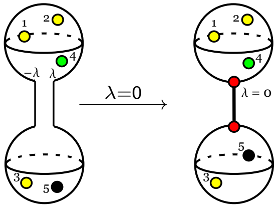
$$\lambda = 0, \quad C_a := y_a^2 - \sigma_a^2 + \lambda^2 = 0, \;\; a = 1,\ldots,n, \quad E_d^\lambda = 0, \quad b \neq p,q,r,m$$

- the DC prescription is able to gauge 4 particles, since λ has been promoted as a variable. ($\mathrm{PSL}_2 \xrightarrow{extended} \mathrm{GL}_2 = \mathrm{PSL}_2 \times \mathbb{C}^*$).



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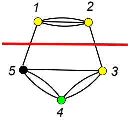
- The DC formalism sums over all possible configurations, and each configuration depends on the gauge fixing. This is a advantage to simplify the computation.



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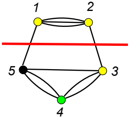

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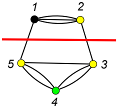

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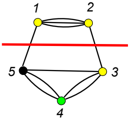

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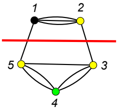

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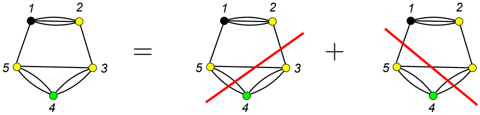

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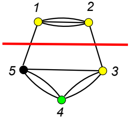

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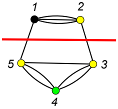

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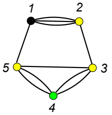

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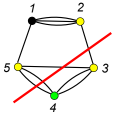
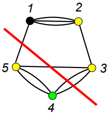


$$+$$



- The idea is to develop a $\lambda-$algorithm for the whole Pfaffian. (Work in progress with N. E. J. Bjerrum-Bohr and Poul H. Damgaard). In other words, we hope to obtain a covariant version for the CSW recurrence relation [Cachazo-Svrcek-Witten-2004].

Elliptic curve

- The DC prescription can be generalized to any algebraic curve, in particular we consider an elliptic curve (Torus) $y^2 = z(z-1)(z-\lambda) \subset \mathbb{CP}^2$, [C. Cardona and H.G. 2016].

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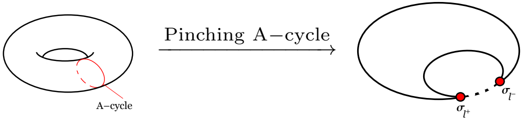

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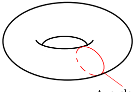
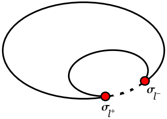
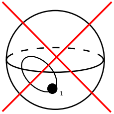


$\xrightarrow{\text{Pinching A-cycle}}$



, $k_{\ell^+} = -k_{\ell^-} = \ell$, $\ell^2 \neq 0$.

- Unlike to the sphere, on the Torus there is a global holomorphic form, $\omega = \frac{dz}{y}$. This holomorphic form has a CHY interpretation.



,

dim $(H^0(\Omega^1)) = 0$

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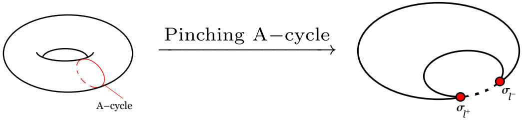

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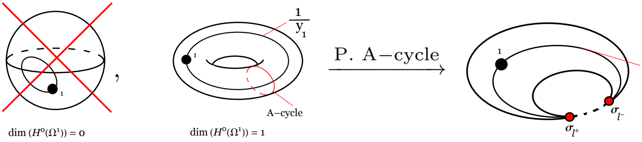

$$\omega_{1:1}^{\ell^+\ell^-} = \frac{\sigma_{\ell^+\ell^-}}{\sigma_{1\ell^+}\,\sigma_{1\ell^-}}.$$

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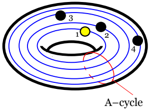
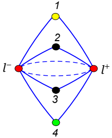


$$\xrightarrow{\text{P. A}-\text{cycle}}$$



$$\frac{(y_1^2)}{(y_1)^2(y_2)^2(y_3)^2(y_4)^2} \xrightarrow{\text{P. A}-\text{cycle}} (1,\ell^+,\ell^-)^2\frac{(\omega_{1:1}^{\ell^+\ell^-})^2\cdots(\omega_{4:4}^{\ell^+\ell^-})^2}{(\ell^+,\ell^-)^2}$$

- A dashed line means a numerator, σ_{ij}, a blue line means two black lines and $(i_1,\ldots,i_n) := \sigma_{i_1 i_2}\cdots\sigma_{i_{n-1}i_n}\sigma_{i_n i_1}$.

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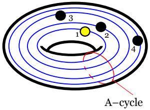

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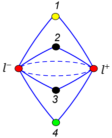

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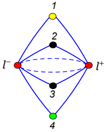
$$\int \frac{d\mu_{4+2}^{(1)}}{\ell^2}$$



$$= \frac{1}{\ell^2} \underbrace{\sum_{\alpha \in S_4} \frac{1}{(\ell \cdot k_{\alpha_1})[(\ell + k_{\alpha_1} + k_{\alpha_2})^2 - \ell^2](\ell \cdot k_{\alpha_4})}}_{\text{Linear in } \ell,\ \text{Q}-\text{cut.}}$$

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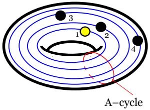

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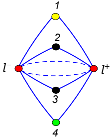

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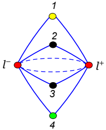

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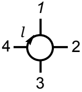

$+\ \text{per}_{1234}$

[Geyer, Mason, Monteiro, Tourkine-15, Baadsgaard, Bohr, Bourjaily, Caron, Damgaard, Feng-15, Cachazo, He, Yuan-15, Cardona, H.G.-16]

Two Loop.

- The previous program can be extended to an hyperelliptic curve, $y^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3) \subset \mathbb{CP}^2$. On this curve there are two global holomorphic forms, $\Omega_1(z) = \frac{dz}{y}$, $\Omega_2(z) = \frac{z\,dz}{y}$.

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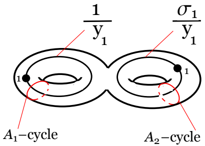

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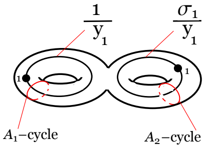

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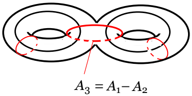
- Since A_1 does not feel to A_2, then we consider a cycle which is able to link them.



$$\xrightarrow{\text{dual form}} \quad \omega_{1:1}^{\ell_1^+ \ell_1^-} - \omega_{1:1}^{\ell_2^+ \ell_2^-}$$

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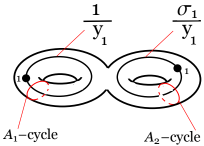

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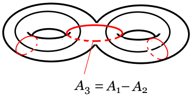

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- So, we have the following base for global quadratic differentials , $H^0(\Omega^2, \Sigma_2)$, [H.G, S. Mizera, G. Zhang-16.]

$$q_a^1 := \omega_{a:a}^{\ell_1^+\ell_1^-}\left(\omega_{a:a}^{\ell_1^+\ell_1^-} - \omega_{a:a}^{\ell_2^+\ell_2^-}\right), \ q_a^2 = \omega_{a:a}^{\ell_2^+\ell_2^-}\left(\omega_{a:a}^{\ell_2^+\ell_2^-} - \omega_{a:a}^{\ell_1^+\ell_1^-}\right), \ q_a^3 = \omega_{a:a}^{\ell_1^+\ell_1^-}\,\omega_{a:a}^{\ell_2^+\ell_2^-}$$

- We define the following CHY-integrand at two loop [H.G, S. Mizera, G. Zhang-16.]

$$\mathcal{I}_n^{(2)} = \frac{1}{(\ell_1^+, \ell_2^+, \ell_2^-, \ell_1^-)^2} \left\{ \prod_{a=1}^{i} q_a^1 \prod_{b=i+1}^{m} q_b^2 \prod_{c=m+1}^{n} q_c^3 + \mathrm{per}\,(1,2,3) \right\}.$$

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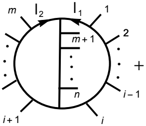

$$+\,\mathrm{per}_{1\ldots\imath} + \mathrm{per}_{\imath+1\ldots m} + \mathrm{per}_{m+1\ldots n} + \mathrm{per}_{\{\ell_1,\ell_2,(\ell_1+\ell_2)\}}$$

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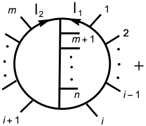


$$+\,\mathrm{per}_{1\ldots\imath} + \mathrm{per}_{\imath+1\ldots m} + \mathrm{per}_{m+1\ldots n} + \mathrm{per}_{\{\ell_1,\ell_2,(\ell_1+\ell_2)\}}$$

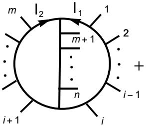
- In particular, let us consider the following CHY integrand,

- We define the following CHY-integrand at two loop [H.G, S. Mizera, G. Zhang-16.]

$$\mathcal{I}_n^{(2)} = \frac{1}{(\ell_1^+,\ell_2^+,\ell_2^-,\ell_1^-)^2}\left\{\prod_{a=1}^{i} q_a^1 \prod_{b=i+1}^{m} q_b^2 \prod_{c=m+1}^{n} q_c^3 + \mathrm{per}\,(1,2,3)\right\}.$$

- Using the scattering equations proposed by [Geyer, Mason, Monteiro, Tourkine-16], $(\ell_1^2 \neq 0, \ell_2^2 \neq 0)$, and by modifying the λ-algorithm, we obtain

$$\int \frac{d\mu_{n+4}^{(2)}\,\mathcal{I}_n^{(2)}}{\ell_1^2\ell_2^2} \xrightarrow[\text{not in }(\ell_1+\ell_2)]{\text{pfi in }\{\ell_1,\ell_2\},}$$



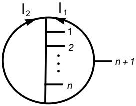
$$+\,\mathrm{per}_{1\ldots\iota} + \mathrm{per}_{\iota+1\ldots m} + \mathrm{per}_{m+1\ldots n} + \mathrm{per}_{\{\ell_1,\ell_2,(\ell_1+\ell_2)\}}$$

- In particular, let us consider the following CHY integrand,

$$\frac{1}{\ell_1^2\ell_2^2}\int d\mu_{(n+1)+4}^{(2)}\,(\ell_1^+,\ell_2^+,\ell_2^-,\ell_1^-)^{-2}\; q_{n+1}^1 \prod_{b=1}^{n} q_b^3 =$$

$$\frac{1}{\underbrace{\ell_1^2\ell_2^2\,(-\ell_1\cdot k_{n+1})}_{\text{linear in }\ell_1}}\sum_{\alpha\in S_n}\frac{1}{\underbrace{(\ell_1+\ell_2)^2(\ell_1+\ell_2+k_{\alpha_1})^2\cdots(\ell_1+\ell_2-k_{\alpha_{n+1}})^2}_{\text{quadratic in }(\ell_1+\ell_2)}} + \{\ell_{1,2}\to-\ell_{1,2}\}$$

$$\frac{1}{\ell_1^2\ell_2^2}\int d\mu^{(2)}_{(n+1)+4}\,\frac{q^1_{n+1}\prod_{b=1}^{n}q^3_b}{(\ell_1^+,\ell_2^+,\ell_2^-,\ell_1^-)^2}\xrightarrow{\text{pfi in }\ell_1}$$



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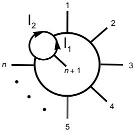

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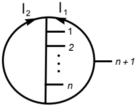

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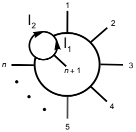

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- From the CHY approach, we have obtained an integrand which looks like a Φ^3 integrand at one loop with quadratic propagators.

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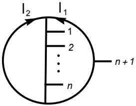

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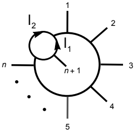

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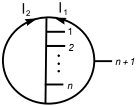

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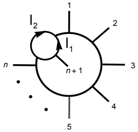

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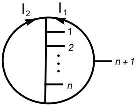
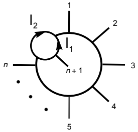
$$\frac{1}{\ell_1^2\ell_2^2}\int d\mu^{(2)}_{(n+1)+4}\frac{q^1_{n+1}\prod_{b=1}^n q_b^3}{(\ell_1^+,\ell_2^+,\ell_2^-,\ell_1^-)^2}\xrightarrow{\text{pfi in }\ell_1}$$  $+\ \text{per}_{1\ldots n} =$  $+\ \text{per}_{1\ldots n}$

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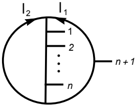

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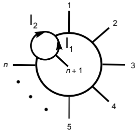

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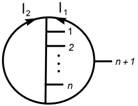

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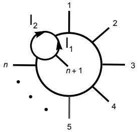

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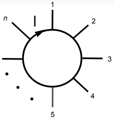
- Hence, we Identify the off-shell momentum, $k_{\tilde\ell_1^+}+k_{\tilde\ell_2^+}$, with the loop momentum at one loop, $k_{\tilde\ell_1^+}+k_{\tilde\ell_2^+}:=\ell$, namely

Quadratic Propagators

$$\delta^{(D)}(k_{\tilde{\ell}_1^+} + k_{\tilde{\ell}_2^+} - \ell)\mathcal{A} = \sum_{\alpha \in S_n} \frac{1}{\ell^2(\ell + k_{\alpha_1})^2 \cdots (\ell - k_{\alpha_n})^2} =$$



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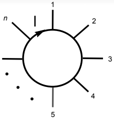

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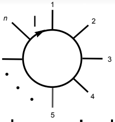

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QUADRATIC PROPAGATORS

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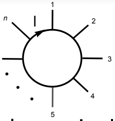

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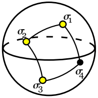

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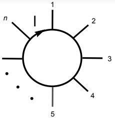

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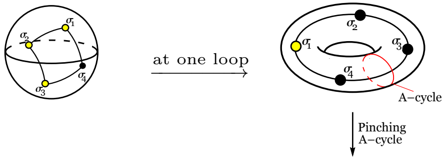

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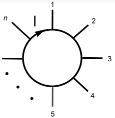
$$\delta^{(D)}(k_{\tilde{\ell}_1^+} + k_{\tilde{\ell}_2^+} - \ell)\mathcal{A} = \sum_{\alpha\in S_n} \frac{1}{\ell^2(\ell+k_{\alpha_1})^2\cdots(\ell-k_{\alpha_n})^2} =$$  $+ \text{per}_{1\ldots n}$

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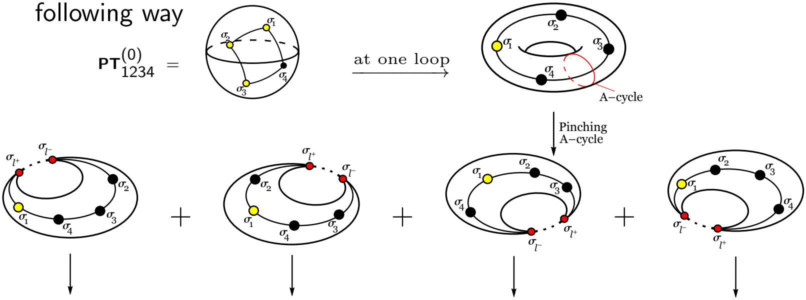
$\mathbf{PT}^{(0)}_{1234} =$ 

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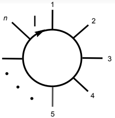

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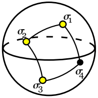
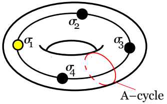
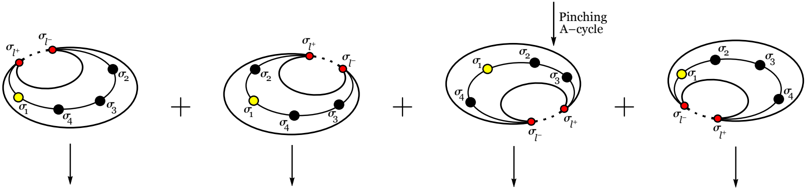


$\xrightarrow{\text{at one loop}}$





$$\mathbf{PT}^{(1)}_{1234}=\omega^{\ell^+:\ell^-}_{1:2}\frac{1}{\sigma_{23}\sigma_{34}\sigma_{41}}+\frac{1}{\sigma_{12}}\omega^{\ell^+:\ell^-}_{2:3}\frac{1}{\sigma_{34}\sigma_{41}}+\frac{1}{\sigma_{12}\sigma_{23}}\omega^{\ell^+:\ell^-}_{3:4}\frac{1}{\sigma_{41}}+\frac{1}{\sigma_{12}\sigma_{23}\sigma_{34}}\omega^{\ell^+:\ell^-}_{4:1}$$

[He-Yuan-13, Baadsgaard, Bohr, Bourjaily, Damgaard, Feng-15.]

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Linear propagators	Quadratic propagators
$d\mu^{(1)}_{n+2}$	$d\mu^{(0)}_{n+4}$

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$$\mathbf{PT}^{(1)}_{\alpha} := \sum_{\pi\in\mathrm{cyc}(\alpha)} \frac{1}{\sigma_{\pi_1\pi_2}\sigma_{\pi_2\pi_3}\cdots\sigma_{\pi_{n-1}\pi_n}}\omega^{\ell^+:\ell^-}_{\pi_n:\pi_1}\,.$$

- Nevertheless, the integrands obtained by $m^{(1)}[\alpha|\beta]$ are linear in the loop momentum. In order to get quadratic propagators one can apply the ideas given here. Let us recall the symmetrized n-gon

Linear propagators	Quadratic propagators
$d\mu^{(1)}_{n+2}$	$d\mu^{(0)}_{n+4}$
$(\ell^+,\ell^-)^{-2}$, $k_{\ell^+} = -k_{\ell^-} := \ell,\ \ell^2 \neq 0$	$(\tilde{\ell_1^+},\tilde{\ell_2^+},\tilde{\ell_2^-},\tilde{\ell_1^-})^{-2}$ $k^2_{\ell_1^+} = k^2_{\ell_1^-} = k^2_{\ell_2^+} = k^2_{\ell_2^-} = 0$

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$(\ell^+,\ell^-)^{-2}$, $k_{\ell^+} = -k_{\ell^-} := \ell,\ \ell^2 \neq 0$	$(\tilde{\ell}^+_1,\tilde{\ell}^+_2,\tilde{\ell}^-_2,\tilde{\ell}^-_1)^{-2}$ $k^2_{\ell^+_1} = k^2_{\ell^-_1} = k^2_{\ell^+_2} = k^2_{\ell^-_2} = 0$
$(\omega^{\ell^+:\ell^-}_{1:1}\cdots\omega^{\ell^+:\ell^-}_{n:n}) \times (\omega^{\ell^+:\ell^-}_{1:1}\cdots\omega^{\ell^+:\ell^-}_{n:n})$	$(\omega^{\tilde{\ell}^+_1:\tilde{\ell}^-_1}_{1:1}\cdots\omega^{\tilde{\ell}^+_1:\tilde{\ell}^-_1}_{n:n}) \times (\omega^{\tilde{\ell}^+_2:\tilde{\ell}^-_2}_{1:1}\cdots\omega^{\tilde{\ell}^+_2:\tilde{\ell}^-_2}_{n:n})$

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$(\ell^+,\ell^-)^{-2}$, $k_{\ell^+} = -k_{\ell^-} := \ell,\ \ell^2\neq 0$	$(\tilde{\ell}_1^+,\tilde{\ell}_2^+,\tilde{\ell}_2^-,\tilde{\ell}_1^-)^{-2}$ $k^2_{\ell_1^+} = k^2_{\ell_1^-} = k^2_{\ell_2^+} = k^2_{\ell_2^-} = 0$
$(\omega^{\ell^+:\ell^-}_{1:1}\cdots\omega^{\ell^+:\ell^-}_{n:n})\times(\omega^{\ell^+:\ell^-}_{1:1}\cdots\omega^{\ell^+:\ell^-}_{n:n})$	$(\omega^{\tilde{\ell}_1^+:\tilde{\ell}_1^-}_{1:1}\cdots\omega^{\tilde{\ell}_1^+:\tilde{\ell}_1^-}_{n:n})\times(\omega^{\tilde{\ell}_2^+:\tilde{\ell}_2^-}_{1:1}\cdots\omega^{\tilde{\ell}_2^+:\tilde{\ell}_2^-}_{n:n})$
$\mathbf{PT}^{(1)}_{\alpha}\times\mathbf{PT}^{(1)}_{\beta}$	$\mathbf{PT}^{\tilde{\ell}_1^+:\tilde{\ell}_1^-}_{\alpha}\times\mathbf{PT}^{\tilde{\ell}_2^+:\tilde{\ell}_2^-}_{\beta}$

Φ^3 Biadjoint at one loop with Quadratic Propagators

- The natural proposal to obtain Φ^3 biadjoint at one loop with quadratic propagators is given by [H.G., C. Lopez-Arcos, P. Talavera-1707XXX]

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- The natural proposal to obtain Φ^3 biadjoint at one loop with quadratic propagators is given by [H.G., C. Lopez-Arcos, P. Talavera-1707XXX]

$$\mathfrak{m}^{(1)}[\alpha|\beta] = \int d\Omega \int d\mu_{n+4}^{(0)} \times \frac{1}{(\tilde{\ell}_1^+, \tilde{\ell}_2^+, \tilde{\ell}_2^-, \tilde{\ell}_1^-)^2} \, \mathbf{PT}_{\alpha}^{\tilde{\ell}_1^+ : \tilde{\ell}_1^-} \times \mathbf{PT}_{\beta}^{\tilde{\ell}_2^+ : \tilde{\ell}_2^-}$$

$$d\Omega := d(k_{\tilde{\ell}_1^+} + k_{\tilde{\ell}_2^+}) d(k_{\tilde{\ell}_1^-}) d(k_{\tilde{\ell}_2^-}) \delta(k_{\tilde{\ell}_1^+} + k_{\tilde{\ell}_2^+} - \ell) \delta(k_{\tilde{\ell}_1^-} + k_{\tilde{\ell}_1^+}) \delta(k_{\tilde{\ell}_2^-} + k_{\tilde{\ell}_2^+}).$$

Conclusions and Perspectives

- The DC prescription is a powerful tool to compute CHY-graph. To extend this idea to string theory. For example, to find an alternative method to compute the coefficients in the α' expansion.
- The DC prescription is extended in a natural form to an elliptic curve. How to do that for an hyperelliptic curve? In addition, how to get the scattering equations at two loop given by Mason et al from this approach? .
- We have been able to obtain quadratic propagator in Φ^3 at one loop, by including four more massless points. We would like to extend this approach to YM. [O. Schlotterer's talk].
- By dimensional reduction, it is possible to reproduce the Feynman $i\epsilon$ in the quadratic propagators prescription. To generalize to higher loops.

Thank you very much for your attention.