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The Correlahedron

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with P. Heslop and L. Mason, arXiv:1701.00453 [hep-th]

Introduction

- Original topic in AdS/CFT: **n-point functions of stress-tensor multiplets** in $\mathcal{N} = 4$ SYM

$$\mathcal{T}|_{\rho^4} = \mathcal{L}\,, \qquad G_n = \langle \mathcal{T}(X_1) \ldots \mathcal{T}(X_n) \rangle = \int \mathcal{D}W \, e^{\frac{i}{g^2} \int d^4x_0 \, d^4\rho_0 \, \mathcal{T}(X_0)} \, \mathcal{T}(X_1) \ldots \mathcal{T}(X_n)$$

- Loop corrections by **Lagrangian insertions** (so integrating out points in higher tree correlators) $\mathcal{N} = 1, 2$ supergraphs manageable at two loop, hard beyond. **Trees not immediately rational!**
- Structure of one- and two-loop results $\Rightarrow$ **higher-loop four-point** integrands by **graph theory** [Eden, Heslop, Korchemsky, Sokatchev (2011,2012)]
- G_n yield integrands for **scattering amplitudes** in a **light-like limit**. [Eden, Korchemsky, Sokatchev (2010)]
- **Twistor Feynman rules** for amplitudes give **rational results**. Reference twistor $\mathbf{Z}^*$

$$S = \int d^4x \, D^4 \, \mathcal{T}\,, \qquad S_{\text{MHV}} = \int d^4 \, D^4 \, (D')^4 \, \mathcal{L}_{\text{MHV}} \qquad \Rightarrow \qquad \mathcal{T} = (D')^4 \, \mathcal{L}_{\text{MHV}}$$

- Put $\mathcal{L}_{\text{MHV}}$ at **outer points**. $(\mathbf{D'})^{\mathbf{4}}$ only needed at the **very end**. [Chicherin, Doobary, Eden, Heslop, Korchemsky, Mason, Sokatchev (2014)]
- **Same formalism** for correlators, amplitudes $\Rightarrow$ **correlahedron** modelled on **amplituhedron** [Arkani-Hamed, Trnka (2013)]

- **Correlahedron geometry** from twistor **graphs** in a $\mathbf{Z}^* \to \mathbf{0}$ limit. **Correlators** from $\mathbf{d}\log$?

Supersymmetic correlator/amplitude duality

Super amplitudes: variables (Z_i, χ_i, χ'_i)

$$\frac{\widehat{\mathcal{A}}_n}{\mathcal{A}^{\text{tree}}_{n\,\text{MHV}}} = \sum_{l=0}^{\infty} \sum_{k=0}^{n-4} a^l \, \widehat{\mathcal{A}}^{(l)}_{n;k} \,, \qquad \widehat{\mathcal{A}}^{(l)}_{n;k} \sim \chi^{4k} \,,$$

where $k = 0$ is the MHV part, $k = 1$ (i.e. $O(\chi^4)$) corresponds to NMHV, ...

Hypothesis for the $\bar{\rho} = \mathbf{0}$ part:

$$\lim_{x^2_{i,i+1} \to 0} \sum_{l \geq 0} a^l \frac{G^{(l)}_n}{G^{\text{tree}}_{n;0}} = \left(\sum_{l=0}^{\infty} a^l \, \widehat{\mathcal{A}}^{(l)}_n \right)^2 , \qquad G^{(l)}_n := \sum_{k=0}^{n-4} G^{(l)}_{n;k} \,, \qquad G^{(l)}_{n;k} \sim \rho^{4k}$$

in the planar limit.

Variable transformation:

$$\bar{\rho}_i = 0 \,, \qquad \chi_i = \langle i | (\rho_i - \rho_{i,i+1} \, y^{-1}_{i,i+1} \, y_i) \,, \qquad \chi'_i = \langle i | \rho_{i,i+1} \, y^{-1}_{i,i+1} \,, \qquad \langle i | = \lambda^{\alpha}_i$$

MHV diagrams for correlators

Vertices modelled on the MHV n-point amplitude: [Boels, Bullimore, Mason, Skinner]

$$\mathcal{L}_{\text{MHV}}(x,\theta) \;=\; \log \text{Det}(\bar\partial + \mathcal{A}|_X) \;=\; \text{Tr}\Big(\log\bar\partial + \sum_{n=2}^{\infty}\frac{1}{n}\,\bar\partial^{-1}\mathcal{A}_1\,\bar\partial^{-1}\mathcal{A}_2\ldots\bar\partial^{-1}\mathcal{A}_n\Big)$$

The **chiral** superfield

$$\mathcal{A}(Z,\chi) \;=\; a(Z) + \chi^A\,\gamma_A(Z) + \frac{1}{2!}\,\chi^A\chi^B\,\phi_{AB} + \frac{\epsilon_{ABCD}}{3!}\,\chi^A\chi^B\chi^C\,\tilde\gamma^D(Z) + \frac{\epsilon_{ABCD}}{4!}\,\chi^A\chi^B\chi^C\chi^D\,g(Z)$$

depends on the **twistor line**

$$(Z,\chi) \;=\; \sigma^\alpha\,(Z_\alpha,\chi_\alpha)\,, \qquad (Z^C_\alpha,\chi^I_\alpha) \;=\; \lambda^\alpha_\beta\,(\epsilon^{\beta\gamma}, x^{\beta\dot\gamma}, \theta^{\beta I})\,, \qquad (\bar\partial|_{X_i})^{-1} \;=\; \frac{1}{\sigma^\alpha_{i-1}\,\sigma_{i\alpha}} \;=\; \frac{1}{z_{ii-1}}\,.$$

Propagator: (fixed reference twistor W^*)

$$\langle\mathcal{A}(\sigma_1 W_1)\,\mathcal{A}(\sigma_2 W_2)\rangle \;=\; \delta^{4|4}(\sigma_1 W_1 + \sigma_2 W_2 + W^*)\,, \qquad W \;=\; (Z,\chi)$$

Analytic superspace:

$$\mathcal{T} \;=\; D'^4\,\mathcal{L}\,, \qquad D' \;=\; (-y,1)\,(\partial_\theta - i\,\bar\theta\partial_x)$$

Vertices:

$$\frac{1}{2}\,\frac{1}{\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_1\rangle}, \quad \frac{1}{3}\,\frac{1}{\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_3\rangle\langle\sigma_3\sigma_1\rangle}, \quad \frac{1}{4}\,\frac{1}{\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_3\rangle\langle\sigma_3\sigma_4\rangle\langle\sigma_4\sigma_1\rangle}, \quad \dots$$

Bosonic part of the propagator: Let

$$\sigma_1^1 = t_1\,, \quad \sigma_1^2 = t_2\,, \quad \sigma_2^1 = t_3\,, \quad \sigma_2^2 = t_4\,,$$

$$(Z_1)_1 \to 1\,, \quad (Z_1)_2 \to 2\,, \quad (Z_2)_1 \to 3\,, \quad (Z_2)_2 \to 4\,, \quad Z^* \to 5\,.$$

Then:

$$\delta^4(\sigma_1^\alpha\, Z_{1\alpha} + \sigma_2^\alpha\, Z_{2\alpha} + Z^*) \;=\; \delta^4(t_1\,1 + t_2\,2 + t_3\,3 + t_4\,4 + 5) \;=\; \frac{1}{\langle 1234\rangle}\,\delta\left(t_1 + \frac{\langle 5234\rangle}{\langle 1234\rangle}\right)\dots$$

Choose $\lambda_1, \lambda_2 = \mathbb{I}_2 \;\to\; \langle 1234\rangle = x_{12}^2$, so the propagator may be replaced by

$$\langle \mathcal{A}(\sigma_1 W_1)\,\mathcal{A}(\sigma_2 W_2)\rangle \;=\; \frac{\delta^4(\sigma_1^\alpha\,\chi_{1\alpha} + \sigma_2^\alpha\,\chi_{2\alpha} + \chi^*)}{x_{12}^2}$$

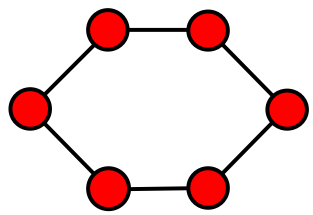
where

$$\sigma_1^\alpha \;=\; \frac{\langle Z_1^\alpha\, X_2\, Z^*\rangle}{x_{12}^2}\,, \quad \sigma_2^\alpha \;=\; \frac{\langle Z_2^\alpha\, X_1\, Z^*\rangle}{x_{12}^2}\,, \qquad X_i^{[IJ]} \;=\; Z_i^{\beta[I} Z_{i\beta}^{J]}\,.$$

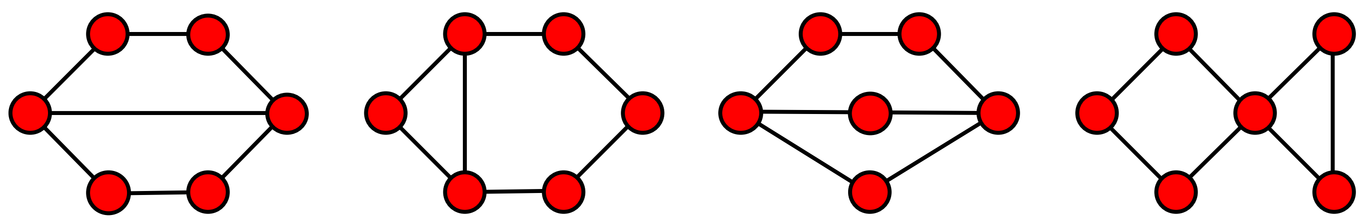
- All σ's are **frozen**!

NMHV level: $k = \#(\text{propagators}) - \#(\text{outer points})$, **Loops** by Lagrangian insertion

MHV six-point tree



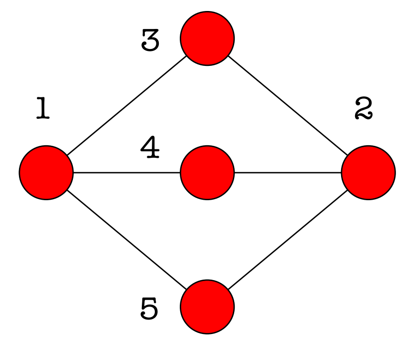
NMHV six-point tree



- **Algebraic** - with Feynman graphs $O(\rho^4)$ tree is in fact a two-loop calculation in x space!
- **Chiral** prior to $(D'_1)^4 \dots (D'_n)^4$. **On-shell limit:** $W_i^\alpha \to (W_{i-1}, W_i)$
- Closer to explaining correlator/amplitude duality, **Grassmannian** formulation of correlators **"Correlahedron"** based on $Gr(n+k, 2n)$, c.f. [Arkani-Hamed, Trnka (2013)]

Twistor graphs in special kinematics

- $\chi^* = \mathbf{0},\ \mathbf{Z}^* \to \mathbf{0} \ \Rightarrow\ \delta^4(\sigma_{ij} Z_i + \sigma_{ji} Z_j)$ implies **colinearity**



Example: **NMHV five-point tree**, select the graph Γ on the right:

Park-Taylor:

$$\mathrm{PT} = \frac{1}{(\Sigma_{134}\Sigma_{145}\Sigma_{153})\,(\Sigma_{234}\Sigma_{245}\Sigma_{253})\,\Sigma_{312}^2\,\Sigma_{412}^2\,\Sigma_{512}^2}\,, \qquad \Sigma_{ijk} = \sigma_{ij}^{\alpha}\,\sigma_{ik\,\alpha}$$

Propagators:

$$\begin{aligned}
\Pi_{13} &: \ \delta^{4|4}(\hat{W}_1 + \hat{W}_5) \\
\Pi_{14} &: \ \delta^{4|4}(\hat{W}_2 + \hat{W}_7) \\
\Pi_{23} &: \ \delta^{4|4}(\hat{W}_3 + \hat{W}_6) \\
\Pi_{24} &: \ \delta^{4|4}(\hat{W}_4 + \hat{W}_8) \\
\Pi_{15} &: \ \delta^{4|4}(\hat{W}_9 \ + \tfrac{\Sigma_{135}}{\Sigma_{134}}\hat{W}_2 - \tfrac{\Sigma_{145}}{\Sigma_{134}}\hat{W}_1) \\
\Pi_{25} &: \ \delta^{4|4}(\hat{W}_{10} + \tfrac{\Sigma_{235}}{\Sigma_{234}}\hat{W}_4 - \tfrac{\Sigma_{245}}{\Sigma_{234}}\hat{W}_3)
\end{aligned}
\qquad
\begin{aligned}
\hat{W}_1 &= \sigma_{13} W_1, & \hat{W}_2 \ &= \sigma_{14} W_1, \\
\hat{W}_3 &= \sigma_{23} W_2, & \hat{W}_4 \ &= \sigma_{24} W_2, \\
\hat{W}_5 &= \sigma_{31} W_3, & \hat{W}_6 \ &= \sigma_{32} W_3, \\
\hat{W}_7 &= \sigma_{41} W_4, & \hat{W}_8 \ &= \sigma_{42} W_4, \\
\hat{W}_9 &= \sigma_{51} W_5, & \hat{W}_{10} &= \sigma_{52} W_5.
\end{aligned}$$

- Choose $\hat{W}_1, \ldots, \hat{W}_4$ as independent, solve δ's for the others.
- Keep the Z-part of the δ's instead of solving it as $\sigma = \frac{\langle \ldots Z^* \rangle}{x^2}$. **$\delta$ instead of propagator, no $\mathbf{Z}^*$**.

The product of all $\mathbf{x_{ij}^2}$ that are **not in the graph** is

$$x_{12}^2 = \langle X_1 X_2 \rangle = \frac{\langle \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \hat{Z}_4 \rangle}{\Sigma_{134}\Sigma_{234}}, \ldots \quad \Rightarrow \quad \frac{1}{x_{12}^2 x_{34}^2 x_{35}^2 x_{45}^2} = \mathrm{PT} * \left(\frac{\Sigma_{134}\Sigma_{234}\Sigma_{312}\Sigma_{412}\Sigma_{512}}{\langle \hat{1}\,\hat{2}\,\hat{3}\,\hat{4} \rangle} \right)^4$$

Hodges bosonisation (here $n + k = 5 + 1 = 6$)

$$Z^{4+a} = \chi^I \phi_I^a \,, \qquad a \in \{1, \ldots, n+k\}$$

Postulate: (10×10 determinant)

$$G_{5;1}^{(0)} = \frac{\langle X_1 X_2 X_3 X_4 X_5 \rangle^4}{x_{12}^2 \ldots x_{45}^2}$$

$$X_1 = \frac{\sigma_{14}\hat{Z}_1 - \sigma_{13}\hat{Z}_2}{\Sigma_{134}}, \ldots \quad \Rightarrow \quad \langle X_1 X_2 X_3 X_4 X_5 \rangle = \frac{\langle \hat{1}\,\hat{2}\,\hat{3}\,\hat{4}\,\hat{5}\,\hat{6}\,\hat{7}\,\hat{8}\,\hat{9}\,\hat{10} \rangle}{\Sigma_{134}\Sigma_{234}\Sigma_{312}\Sigma_{412}\Sigma_{512}}$$

- Suppose δ's of $\mathbf{\Gamma}$, e.g. $\hat{1}^C + \hat{5}^C = 0$, $C \in \{1, \ldots, 4\}$.
- **If** also $\hat{1}^C + \hat{5}^C = 0$, $C \in \{5, \ldots, 10\}$, two columns equal $\Rightarrow$ **linear zero**. Denote as $\hat{\theta}_1 + \hat{\theta}_5$ etc.

$$\langle \hat{1} \ldots \hat{10} \rangle^4 = c(\hat{\theta}_1 + \hat{\theta}_5)^4 (\hat{\theta}_2 + \hat{\theta}_7)^4 (\hat{\theta}_3 + \hat{\theta}_6)^4 (\hat{\theta}_4 + \hat{\theta}_8)^4 (\sigma_{15}\theta_1 + \hat{\theta}_9)^4 (\sigma_{25}\theta_2 + \hat{\theta}_{10})^4 \,,$$

$$\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}_3 = \hat{\theta}_4 = 0 \Rightarrow c = \langle \hat{1}\hat{2}\hat{3}\hat{4} \rangle^4 \,.$$

Thus

$$\left.\frac{\langle X_1 X_2 X_3 X_4 X_5\rangle^4}{\prod_{ij\notin\Gamma} x_{ij}^2}\right|_{x_{ij}=0\,:\,ij\notin\Gamma} = \lim_{Z^*,\chi^*\to 0} \Gamma\big|_{\delta^{24}}$$

- Works for both non-vanishing graphs.
- Six-point NMHV tree: kinematics of every one diagram fixes the correlator, but need $(D')^4$.

Hedronising graphs

Bosonisation procedure for δ-functions:

$$\begin{aligned}
\delta^{4|4}(\hat{W}_1+\hat{W}_5) &= \delta^4(\hat{Z}_1+\hat{Z}_5)\delta^4(\hat{\chi}_1+\hat{\chi}_5)\\
&= \delta^4(\hat{Z}_1+\hat{Z}_5)\int d^4\phi^1(\hat{\chi}_1^I\phi_I^1+\hat{\chi}_5^I\phi_I^1)^4\\
&= \int d^4\phi^1\delta^4(\hat{Z}_1+\hat{Z}_5)(\hat{Z}_1^5+\hat{Z}_5^5)^4\\
&= \int d^4\phi^1\delta^4(\hat{Z}_1+\hat{Z}_5)\int d\rho_1(\rho_1)^4\delta(\hat{Z}_1^5+\hat{Z}_5^5-\rho_1)\\
&= \int d^4\phi^1\,d\rho_1\,(\rho_1)^4\,\delta^5(\hat{Z}_1+\hat{Z}_5-\rho_1\delta_5^I)
\end{aligned}$$

The graph Γ defines a $Y = \sigma.Z$:

$$Y = \begin{pmatrix} \hat{Z}_1^I + \hat{Z}_5^I & \hat{Z}_1^5 + \hat{Z}_5^5 & 0 & 0 & 0 & 0 & 0 \\ \hat{Z}_2^I + \hat{Z}_7^I & 0 & \hat{Z}_2^6 + \hat{Z}_7^6 & 0 & 0 & 0 & 0 \\ \hat{Z}_3^I + \hat{Z}_6^I & 0 & 0 & \hat{Z}_3^7 + \hat{Z}_6^7 & 0 & 0 & 0 \\ \hat{Z}_4^I + \hat{Z}_8^I & 0 & 0 & 0 & \hat{Z}_4^8 + \hat{Z}_8^8 & 0 & 0 \\ \hat{Z}_9^I + \sigma_{15} Z_1^I & 0 & 0 & 0 & 0 & \hat{Z}_9^9 + \sigma_{15} Z_1^9 & 0 \\ \hat{Z}_{10}^I + \sigma_{25} Z_2^I & 0 & 0 & 0 & 0 & 0 & \hat{Z}_{10}^{10} + \sigma_{25} Z_2^{10} \end{pmatrix}$$

Covariantly, we can write

$$G_{5;1}^{(0)} = \int d^{24}\phi \int \frac{\langle Y\, d^4 Y_1\rangle \ldots \langle Y\, d^4 Y_6\rangle\, \delta(Y; Y_0)\, \langle X_1 \ldots X_5\rangle^4}{\langle Y X_1 X_2\rangle \ldots \langle Y X_4 X_5\rangle}\,.$$

NMHV six-point tree: Y has size $(n+k) \times (n+k+4) = 7 \times 11$.
There are 11-brackets and 4-brackets, complete these by Y. Denote

$$\langle \dot{Z}_i^\alpha \rangle = \langle X_1 \ldots Z_i^\alpha \ldots X_6\rangle\,.$$

Schematically:

$$G_{6;1}^{(0)} = \int d^{28}\phi \int \frac{\langle Y\, d^4 Y_1\rangle \ldots \langle Y\, d^4 Y_7\rangle\, \delta(Y; Y_0)\, \sum \langle Y X X\rangle \langle Y X X\rangle \langle Y Z X Z\rangle \langle Y Z X Z\rangle \langle \dot{Z}\rangle \langle \dot{Z}\rangle \langle \dot{Z}\rangle \langle \dot{Z}\rangle}{\langle Y X_1 X_2\rangle \ldots \langle Y X_5 X_6\rangle}$$

Can fix the numerator from any one twistor Feynman graph in the $Z^* \to 0$ limit.

Conclusions

- The **correlahedron geometry** and its **positivity constraints** are: (**n** points, $\mathrm{N^{k}MHV}$)

$$Y \in \mathrm{Gr}(n+k, n+k+4)\,, \qquad \langle Y X_i X_j \rangle > 0 \;\forall\; i,j \in \{1,\ldots,n\}$$

- **Twistor diagrams** at $\mathbf{Z}^* \to \mathbf{0}$ point out the **geometry** and **fix the volume forms**.
- **Loops** by **Lagrangian insertion** (integrating out points), **no hiding particles**
- **Permutation symmetry** between all points $\Rightarrow$ no positivity constraints on outer data.
- Similar to the amplituhedron, **volume forms** by **cylindrical decomposition**?
- **Problems: size** of Grassmannians, **gauge invariance** without $(D')^4$, **double poles** in residues.

- **Shortcut** to the **squared amplituhedron**?
- **Freeze** the first **n rows** of Y to the span of the $X_i \wedge X_{i+1}$. **Project** to the transverse space.
- Works for loop integrands.
- The volume forms become those of the squared amplituhedron.
  We have weaker positivity $\langle Y\, i\, i+1\, j\, j+1 \rangle > 0$ (and not $\langle Y 1235 \rangle > 0$ etc.) so that various regions are found, c.f. [Arkani-Hamed, Thomas, Trnka (2017)].

Cylindrical decomposition — five-point NMHV tree

Use $GL(6)$ from the left and $GL(2)^5$ from the right to gauge $Y \in Gr(6,10)$ to the form

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e & f \end{pmatrix}$$

With $(X_1 X_2 X_3 X_4 X_5) = \mathcal{I}_{10}$ the positivity requirements $\langle Y X_i X_j \rangle > 0 \;\forall\; i,j \in \{1,\ldots,5\}$ become

$$-e + cf > 0,\; ab > 0,\; ab - bc - e - af + cf > 0,\; 1 - c - e - f + cf > 0,\; (-1+a)(-1+b) > 0$$

Cylindrical decomposition: sum over regions of the form

$$\left\{ (x_1,\ldots,x_n) : \quad \begin{array}{c} a < x_1 < b, \\ a(x_1) < x_2 < b(x_2), \\ a(x_1,x_2) < x_3 < b(x_1,x_2), \\ \ldots \end{array} \right\}, \qquad a < x_i < b \quad \to \quad d\log\left(\frac{x_i - b}{x_i - a}\right).$$

Here we find a co-ordinatised version of the earlier result from graphs:

$$\frac{(a-b)^2 da\, db\, dc\, de\, df}{(a-1)a(b-1)b(e-cf)(-cf+c+e+f-1)(ab-af-bc+cf-e)}$$