TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Inertial transfer and small-scale structures in magnetohydrodynamic turbulence

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DFG Turbulent Superstructures

What we are interested in:

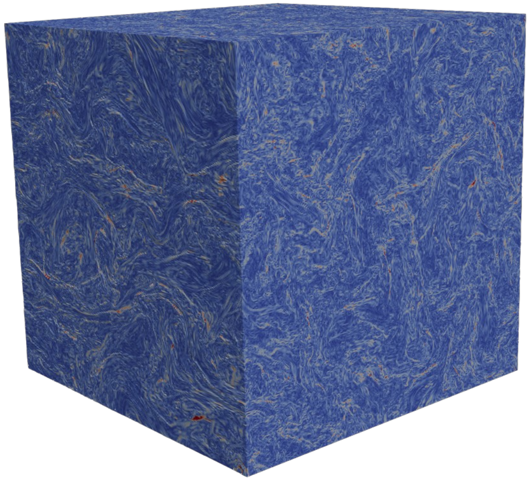
- understanding the physical processes that govern the energy cascade in MHD
- finding the length scales these processes are associated with

How we realize them:

- using a theoretical approach to construct physically interpretable observables
- disentangling the large-scale dynamics from the small-scales one

How we achieve that:

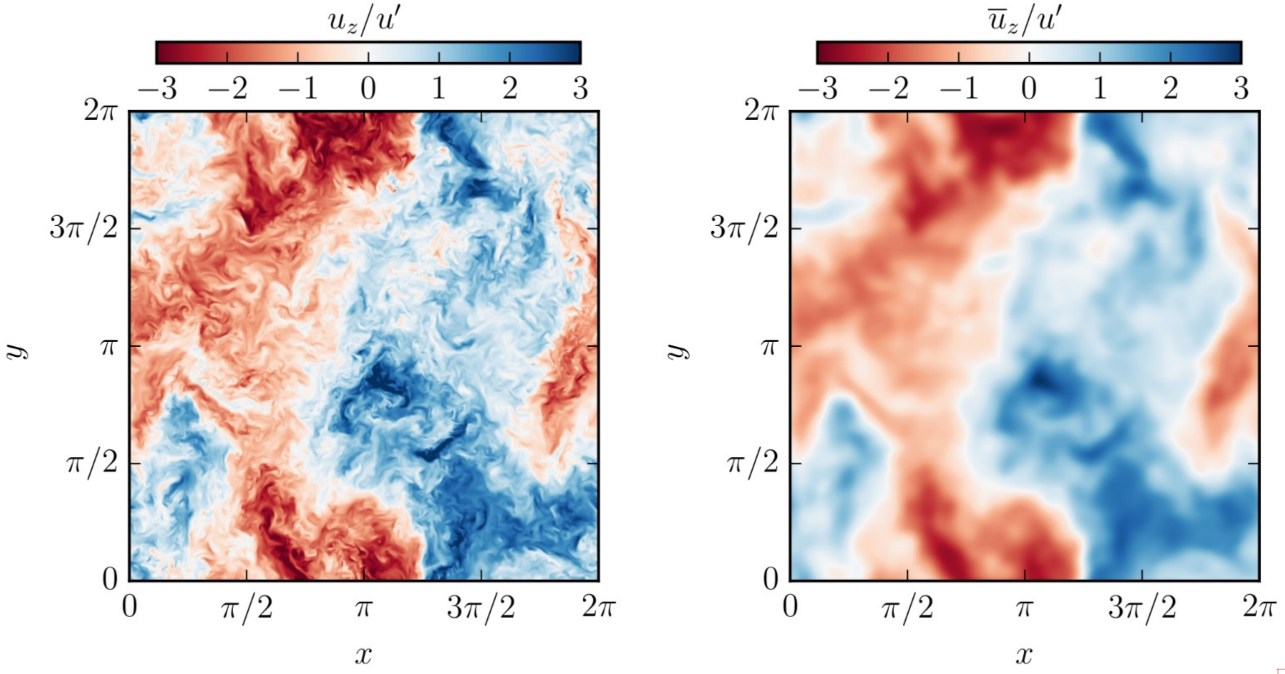
- next slides



Let us talk about hydrodynamics first

$$\begin{cases} \partial_t u_i + \partial_j \left(u_i u_j \right) = -\frac{1}{\rho} \partial_j p \delta_{ij} + \nu \, u_{i,jj} + f_i \\ \partial_i u_i = 0 \end{cases}$$

3D incompressible Navier-Stokes equations



The fluid velocity in the z direction on an xy plane in the HIT simulation with $\mathrm{Re}_\lambda = 400$:
(a) unfiltered
(b) filtered using a Gaussian filter

(P. Johnson, JFM (2021))

Separate large- and small-scale dynamics → Filtering operation

3D incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u_i + \partial_j \left(u_i u_j \right) = -\dfrac{1}{\rho} \partial_j p \delta_{ij} + \nu \, u_{i,jj} + f_i \\ \partial_i u_i = 0 \end{cases}$$

Separate large- and small-scale dynamics → Filtering operation

$$\overline{u}_i^\ell(\boldsymbol{x}) = \int_\Omega d\boldsymbol{r} \; G^\ell(\boldsymbol{r}) u_i(\boldsymbol{x}+\boldsymbol{r}). \qquad G^\ell(\boldsymbol{r}) = \frac{1}{(2\pi\ell^2)^{3/2}} \exp\left(-\frac{|\boldsymbol{r}|^2}{2\ell^2} \right)$$

$$\mathcal{F}\left\{\overline{u}_i^\ell\right\}(\boldsymbol{k}) = \mathcal{F}\left\{u_i\right\}(\boldsymbol{k}) \cdot \mathcal{F}\left\{G^\ell\right\}(\boldsymbol{k}) \equiv \mathcal{F}\left\{u_i\right\}(\boldsymbol{k}) \cdot e^{-\frac{|\boldsymbol{k}|^2 \ell^2}{2}}$$

Filtered momentum equations:

$$\partial_t \overline{u}_i^\ell + \partial_j \left(\overline{u_i u_j}^\ell\right) = -\partial_j \overline{p}^\ell \delta_{ij} + \nu\, \overline{u}^\ell_{i,jj} + \overline{f}^\ell_i$$

$$\downarrow$$

$$\partial_t \overline{u}_i^\ell + \underline{\partial_j \left(\overline{u_i}^\ell \overline{u_j}^\ell\right)} = -\partial_j \overline{p}^\ell \delta_{ij} + \nu\, \overline{u}^\ell_{i,jj} + \overline{f}^\ell_i \underline{- \partial_j \tau^\ell_{ij}}$$

$$\boxed{\tau^\ell_{ij} = \overline{u_i u_j}^\ell - \overline{u}^\ell_i \overline{u}^\ell_j}$$

Sub-Grid Scale
stress tensor

M. Germano, JFM **238** (1992)

Filtered velocity • Filtered momentum equations = Filtered field energy equation

$$\overline{u}_i^\ell \cdot \left(\partial_t \overline{u}_i^\ell + \partial_j \left(\overline{u_i}^\ell \overline{u_j}^\ell \right) \right) = \overline{u}_i^\ell \cdot \left(- \partial_j \overline{p}^\ell \delta_{ij} + \nu \, \overline{u}^\ell_{i,jj} + \overline{f}^\ell_i - \partial_j \tau^\ell_{ij} \right)$$

$$\downarrow$$

$$\partial_t \left(\frac{1}{2} \overline{u}_i^\ell \overline{u}_i^\ell \right) + \partial_j (\ldots) = \overline{u}_i^\ell \overline{f_i}^\ell + \overline{S_{ij}}^\ell \tau_{ij}^\ell - 2\nu \overline{S_{ij}}^\ell \overline{S_{ij}}^\ell$$

boundary term

vanishes in mean

input energy from forcing

energy flux across the scales

viscous dissipation

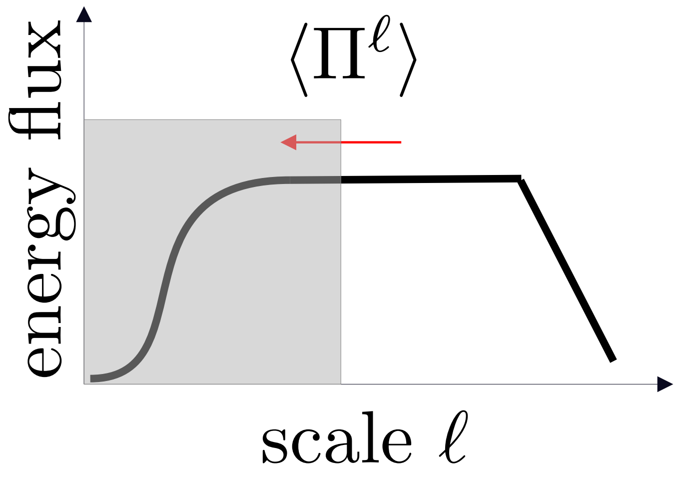
$$\tau_{ij}^\ell = \overline{u_i u_j}^\ell - \overline{u}_i^\ell \overline{u}_j^\ell$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\partial_t \overline{u}_i^\ell + \partial_j \left(\overline{u_i}^\ell \overline{u_j}^\ell\right) = -\partial_j \overline{p}^\ell \delta_{ij} + \nu\, \overline{u}_{i,jj}^\ell + \overline{f}_i^\ell - \partial_j \tau_{ij}^\ell$$

$$\tau_{ij}^\ell = \overline{u_i u_j}^\ell - \overline{u}_i^\ell \overline{u}_j^\ell$$

$$\Pi^\ell = -\overline{S}_{ji}^\ell \tau_{ij}^\ell$$



$$\overline{u}_i^\ell(\boldsymbol{x}) = \int_\Omega d\boldsymbol{r}\; G^\ell(\boldsymbol{r}) u_i(\boldsymbol{x}+\boldsymbol{r}).$$

$$G^\ell(\boldsymbol{r}) = \frac{1}{(2\pi\ell^2)^{3/2}} \exp\left(-\frac{|\boldsymbol{r}|^2}{2\ell^2}\right)$$

$$\frac{\partial \overline{u}_i^\ell}{\partial(\ell^2)} = \frac{1}{2}\Delta \overline{u}_i^\ell$$

$$\overline{u}_i^\ell(\boldsymbol{x})|_{\ell=0} = u_i(\boldsymbol{x})$$

P. Johnson, PRL **124**, 104501 (2020)

$$\partial_t \overline{u}_i^\ell + \partial_j \left(\overline{u_i}^\ell \overline{u_j}^\ell \right) = -\partial_j \overline{p}^\ell \delta_{ij} + \nu\, \overline{u}^\ell_{i,jj} + \overline{f}_i^\ell - \partial_j \tau_{ij}^\ell$$

$$\tau_{ij}^\ell = \overline{u_i u_j}^\ell - \overline{u}_i^\ell \overline{u}_j^\ell$$

$$\boxed{\begin{aligned} \frac{\partial \overline{u}_i^\ell}{\partial(\ell^2)} &= \frac{1}{2}\Delta \overline{u}_i^\ell \\ \overline{u}_i^\ell(\boldsymbol{x})|_{\ell=0} &= u_i(\boldsymbol{x}) \end{aligned}} \longrightarrow \boxed{\begin{aligned} \frac{\partial \tau_{ij}^\ell}{\partial(\ell^2)} &= \frac{1}{2}\Delta \tau_{ij}^\ell + \partial_k \overline{u}_i^\ell \partial_k \overline{u}_j^\ell \\ \tau_{ij}^\ell(\boldsymbol{x})|_{\ell=0} &= 0 \end{aligned}}$$

$$\Pi^\ell = -\ell^2 \overline{S}^\ell_{ji} \partial_k \overline{u}_i^\ell \partial_k \overline{u}_j^\ell - \overline{S}^\ell_{ji} \int_0^{\ell^2} d\theta\, \overline{\partial_k \overline{u}_i^{\sqrt{\theta}} \partial_k \overline{u}_j^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}} - \overline{\partial_k \overline{u}_i^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}} \overline{\partial_k \overline{u}_j^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}$$

↑ single-scale

↑ multi-scale

P. Johnson, PRL **124**, 104501 (2020)

$$\begin{aligned}\Pi^\ell =& -\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{u}}^\ell)^t \nabla\overline{\boldsymbol{u}}^\ell (\nabla\overline{\boldsymbol{u}}^\ell)^t\right) \\ & - \mathrm{tr}\left((\nabla\overline{\boldsymbol{u}}^\ell)^t \int_0^{\ell^2} d\theta\ \overline{\nabla\overline{\boldsymbol{u}}^{\sqrt{\theta}}(\nabla\overline{\boldsymbol{u}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla\overline{\boldsymbol{u}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}\ \overline{(\nabla\overline{\boldsymbol{u}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}}\right)\end{aligned}$$

$$\boxed{\Pi^\ell = \Pi^\ell_{s,SSS} + \Pi^\ell_{m,SSS} + \Pi^\ell_{s,S\Omega\Omega} + \Pi^\ell_{m,S\Omega\Omega} + \Pi^\ell_{m,SS\Omega}}$$

$$\overline{S}^\ell = \left(\nabla\overline{\boldsymbol{u}}^\ell + (\nabla\overline{\boldsymbol{u}}^\ell)^t\right)/2$$

$$\overline{\Omega}^\ell = \left(\nabla\overline{\boldsymbol{u}}^\ell - (\nabla\overline{\boldsymbol{u}}^\ell)^t\right)/2$$

$$\Pi^\ell_{s,SSS} = -\ell^2 \mathrm{tr}\left((\overline{S}^\ell)^t \overline{S}^\ell (\overline{S}^\ell)^t\right)$$

single-scale strain self-amplification

$$\Pi^\ell_{s,S\Omega\Omega} = -\ell^2 \mathrm{tr}\left((\overline{S}^\ell)^t \overline{\Omega}^\ell (\overline{\Omega}^\ell)^t\right)$$

single-scale vortex stretching

$$\Pi^\ell_{m,SSS} = -\int_0^{\ell^2} d\theta\ \mathrm{tr}\left((\overline{S}^\ell)^t \overline{\overline{S}^{\sqrt{\theta}}(\overline{S}^{\sqrt{\theta}})^t}^\phi - (\overline{S}^\ell)^t \overline{\overline{S}^{\sqrt{\theta}}}^\phi\ \overline{(\overline{S}^{\sqrt{\theta}})^t}^\phi\right)$$

multi-scale strain self-amplification

$$\Pi^\ell_{m,S\Omega\Omega} = -\int_0^{\ell^2} d\theta\ \mathrm{tr}\left((\overline{S}^\ell)^t \overline{\overline{\Omega}^{\sqrt{\theta}}(\overline{\Omega}^{\sqrt{\theta}})^t}^\phi - (\overline{S}^\ell)^t \overline{\overline{\Omega}^{\sqrt{\theta}}}^\phi\ \overline{(\overline{\Omega}^{\sqrt{\theta}})^t}^\phi\right)$$

multi-scale vortex stretching

$$\Pi^\ell_{m,SS\Omega} = -\int_0^{\ell^2} d\theta\ \mathrm{tr}\left((\overline{S}^\ell)^t \overline{\overline{S}^{\sqrt{\theta}}(\overline{\Omega}^{\sqrt{\theta}})^t}^\phi - (\overline{S}^\ell)^t \overline{\overline{\Omega}^{\sqrt{\theta}}}^\phi\ \overline{(\overline{S}^{\sqrt{\theta}})^t}^\phi\right)$$

couples resolved-scale strain rate w/ subfilter correlation of strain rate & vorticity

P. Johnson, PRL **124**, 104501 (2020)

G. L. Eyink, JFM **549**, 159-190 (2006) structure of expression generic for advective nonlinearity

→ can generalise to coupled advection diffusion equations → **MHD**

→ can decompose kinetic, magnetic, cross-helicity fluxes

Generalized expression

$$\Pi^\ell = -\,\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t\nabla\overline{\boldsymbol{B}}^\ell(\nabla\overline{\boldsymbol{C}}^\ell)^t\right)$$

$$-\,\mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t\int_0^{\ell^2}d\theta\;\overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}\;\overline{(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}}\right)$$

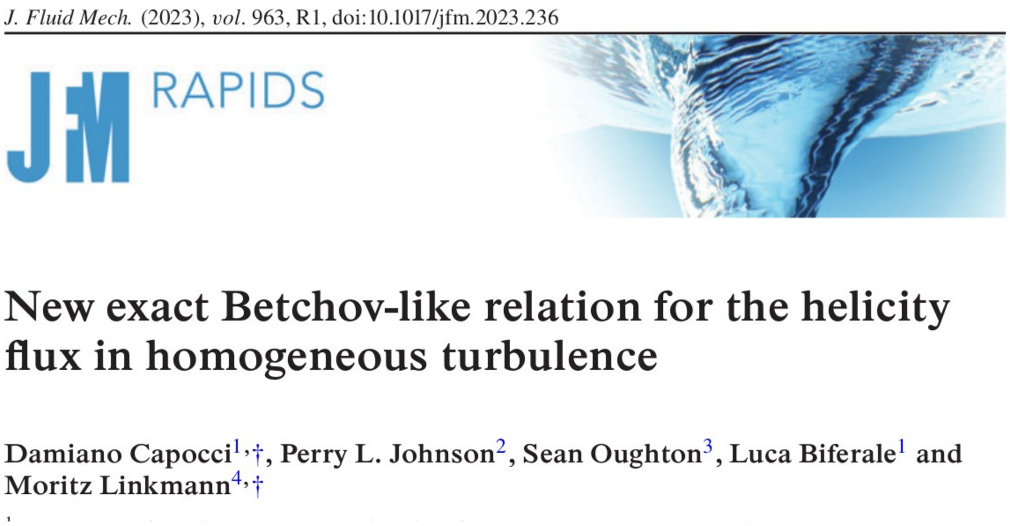
structure of expression generic for advective nonlinearity

→ can generalise to coupled advection diffusion equations → **MHD**

→ can decompose kinetic, magnetic, cross-helicity fluxes

Generalized expression

$$\Pi^\ell = -\,\ell^2 \mathrm{tr}\left((\nabla \overline{\boldsymbol{A}}^\ell)^t \nabla \overline{\boldsymbol{B}}^\ell (\nabla \overline{\boldsymbol{C}}^\ell)^t \right) \\ -\,\mathrm{tr}\left((\nabla \overline{\boldsymbol{A}}^\ell)^t \int_0^{\ell^2} d\theta \; \overline{\nabla \overline{\boldsymbol{B}}^{\sqrt{\theta}} (\nabla \overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla \overline{\boldsymbol{B}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}} \; \overline{(\nabla \overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} \right)$$



J. Fluid Mech. (2023), *vol.* 963, R1, doi:10.1017/jfm.2023.236

JFM RAPIDS

New exact Betchov-like relation for the helicity flux in homogeneous turbulence

Damiano Capocci[1],†, Perry L. Johnson[2], Sean Oughton[3], Luca Biferale[1] and Moritz Linkmann[4],†

applied to HD helicity flux

MHD equations

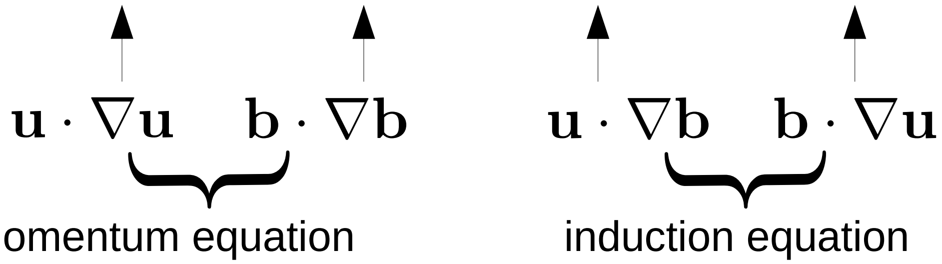
$$\begin{cases}\partial_t \mathbf{u} = -\nabla P^* - (\mathbf{u}\cdot\nabla)\,\mathbf{u} + (\mathbf{b}\cdot\nabla)\,\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \partial_t \mathbf{b} = -\,(\mathbf{u}\cdot\nabla)\,\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{b}\end{cases} \qquad P^* = P + \frac{b^2}{2}$$

Generalized expression

$$\Pi^\ell = -\,\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \nabla\overline{\boldsymbol{B}}^\ell (\nabla\overline{\boldsymbol{C}}^\ell)^t\right)$$
$$-\,\mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \int_0^{\ell^2} d\theta\; \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}\;\overline{(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}}\right)$$

MHD energy fluxes

$$\Pi^\ell = \Pi^{I,\ell} - \Pi^{M,\ell} + \Pi^{A,\ell} - \Pi^{D,\ell}$$



$$\underbrace{\mathbf{u}\cdot\nabla\mathbf{u} \quad \mathbf{b}\cdot\nabla\mathbf{b}}_{\text{momentum equation}} \qquad \underbrace{\mathbf{u}\cdot\nabla\mathbf{b} \quad \mathbf{b}\cdot\nabla\mathbf{u}}_{\text{induction equation}}$$

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Generalized expression

$$\Pi^\ell = -\,\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \nabla\overline{\boldsymbol{B}}^\ell (\nabla\overline{\boldsymbol{C}}^\ell)^t\right)$$
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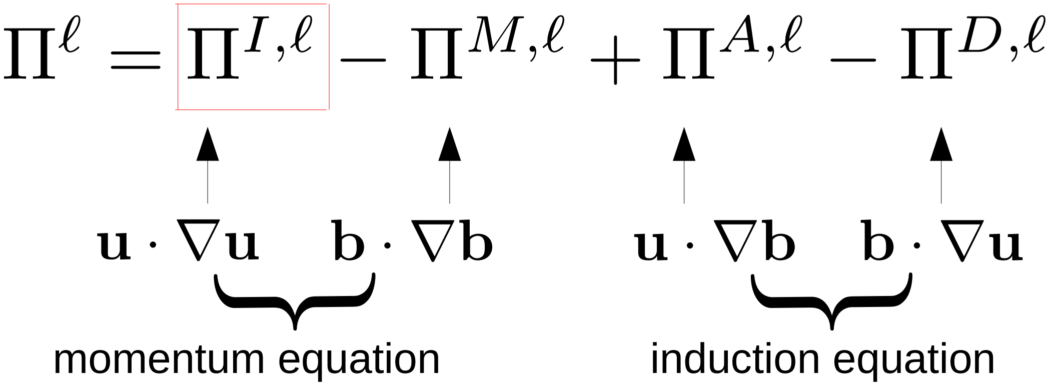
MHD energy fluxes



$$\Pi^\ell = \boxed{\Pi^{I,\ell}} - \Pi^{M,\ell} + \Pi^{A,\ell} - \Pi^{D,\ell}$$

$$\underbrace{\mathbf{u}\cdot\nabla\mathbf{u} \quad \mathbf{b}\cdot\nabla\mathbf{b}}_{\text{momentum equation}} \qquad \underbrace{\mathbf{u}\cdot\nabla\mathbf{b} \quad \mathbf{b}\cdot\nabla\mathbf{u}}_{\text{induction equation}}$$

DNS datasets

- nonlinear saturated dynamo $Pm = 1$
- statistically stationary
- background magnetic field $\begin{cases} B_0 = 0 \\ B_0 = 10 \end{cases}$
- Large-scale random forcing scheme $k_{\mathrm{f}} \in [3, 5]$ w/ minimal cross-helicity injection
- periodic BC on domain $[0, 2\pi]^3$ → pseudo-spectral method
- viscous/hyperviscous
- 1024^3 - 2048^3 collocation points

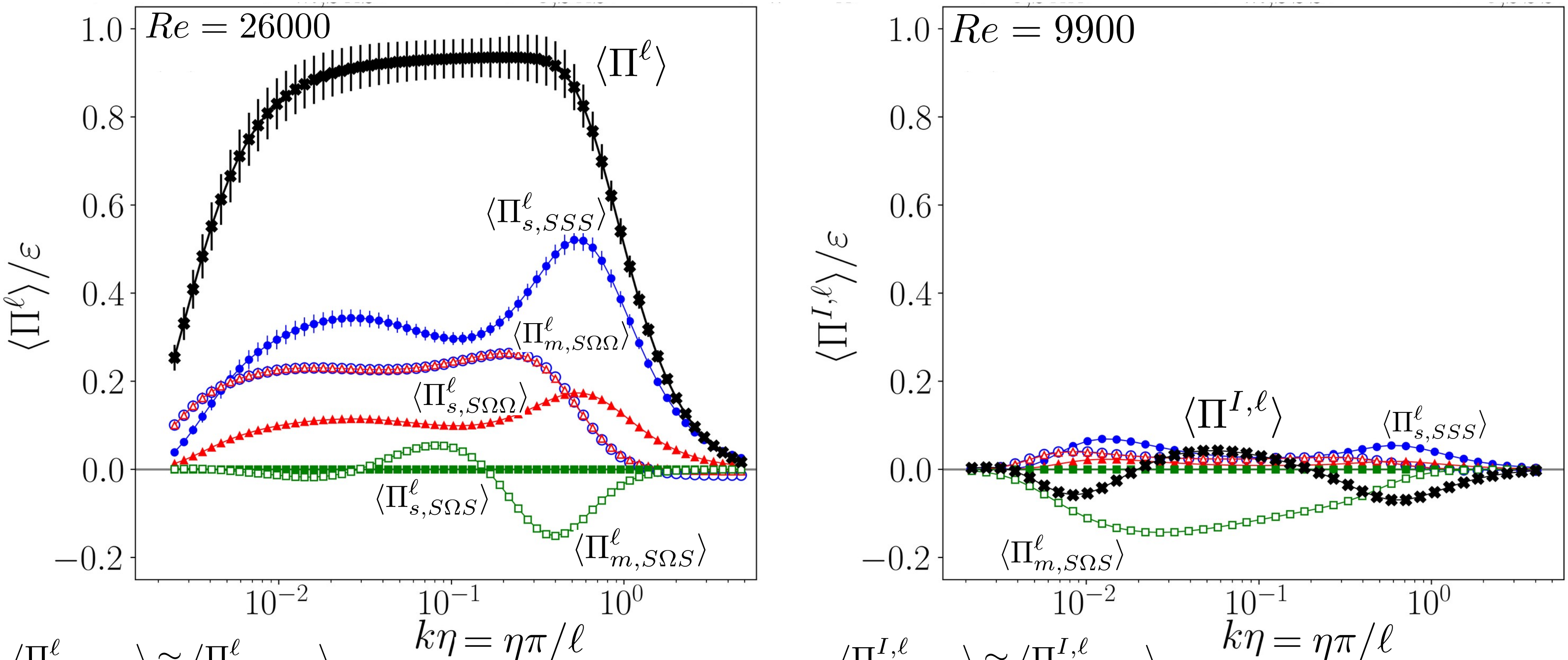
$$k_{\max}\eta_u = 1.38$$
$$k_{\max}\eta_b = 1.37$$

Decomposed inertial flux $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$

$$\Pi^{I,\ell} = \Pi^{I,\ell}_{s,SSS} + \Pi^{I,\ell}_{m,SSS} + \Pi^{I,\ell}_{s,S\Omega\Omega} + \Pi^{I,\ell}_{m,S\Omega\Omega} + \Pi^{I,\ell}_{m,S\Omega S}$$

Navier-Stokes

MHD with $B_0 = 0$



$\langle\Pi^\ell_{m,SSS}\rangle \approx \langle\Pi^\ell_{m,S\Omega\Omega}\rangle$

$\langle\Pi^{I,\ell}_{m,SSS}\rangle \approx \langle\Pi^{I,\ell}_{m,S\Omega\Omega}\rangle$

MHD equations

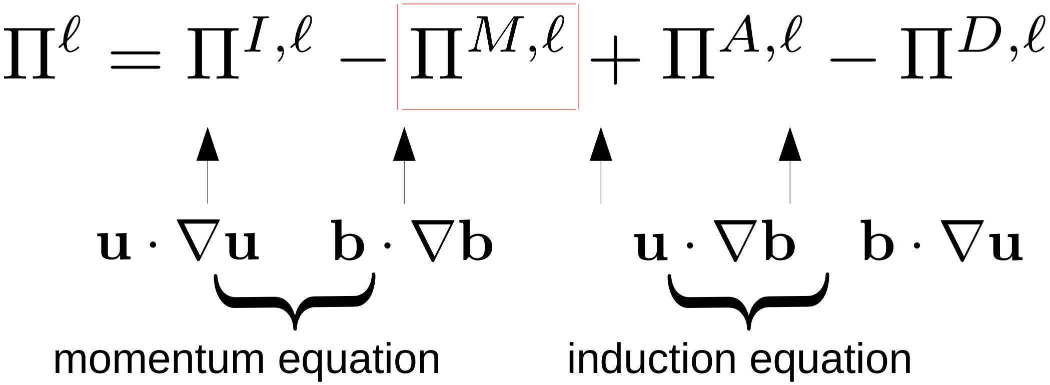
$$\begin{cases} \partial_t \mathbf{u} = -\nabla P^* - (\mathbf{u}\cdot\nabla)\,\mathbf{u} + (\mathbf{b}\cdot\nabla)\,\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \partial_t \mathbf{b} = -\,(\mathbf{u}\cdot\nabla)\,\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{b} \end{cases}$$

$$P^* = P + \frac{b^2}{2}$$

Generalized expression

$$\begin{aligned} \Pi^\ell = &-\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \nabla\overline{\boldsymbol{B}}^\ell (\nabla\overline{\boldsymbol{C}}^\ell)^t\right) \\ &- \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \int_0^{\ell^2} d\theta\; \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}\;\overline{(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}}\right) \end{aligned}$$

MHD energy fluxes

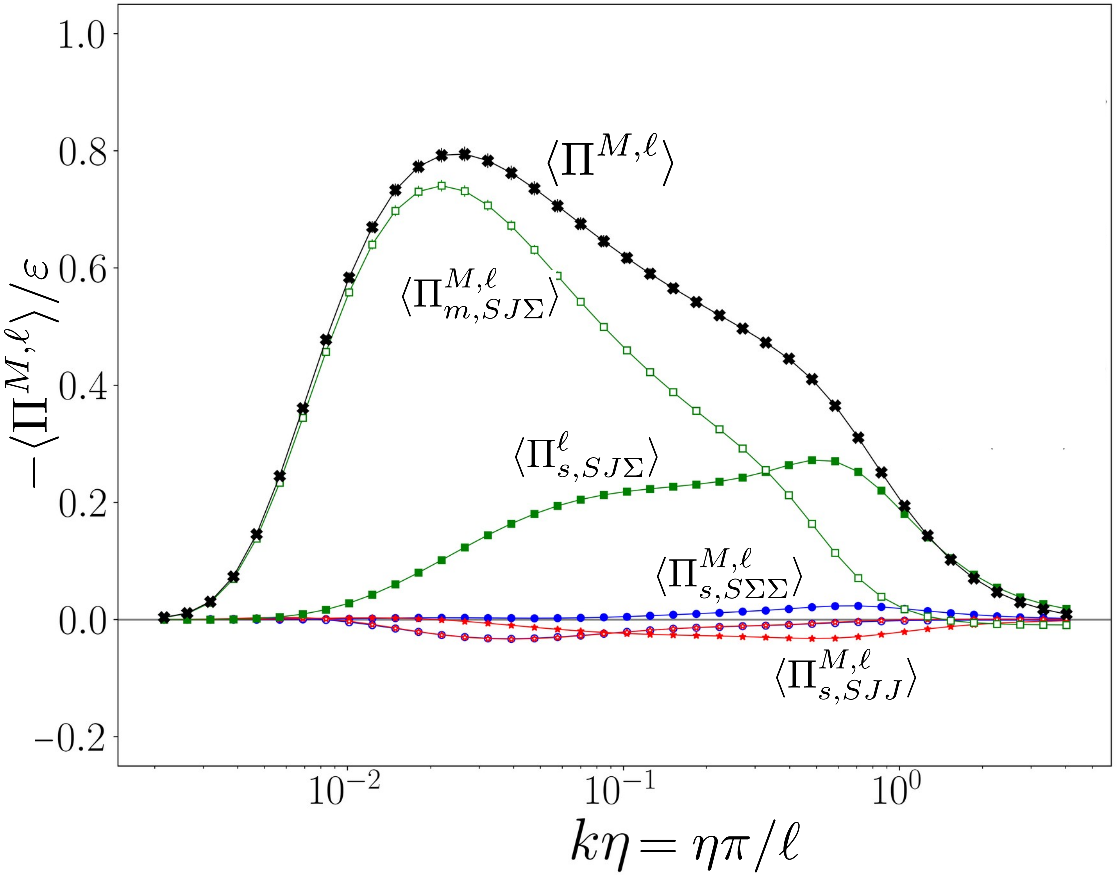


$$\Pi^\ell = \Pi^{I,\ell} - \boxed{\Pi^{M,\ell}} + \Pi^{A,\ell} - \Pi^{D,\ell}$$

$$\underbrace{\mathbf{u}\cdot\nabla\mathbf{u} \quad \mathbf{b}\cdot\nabla\mathbf{b}}_{\text{momentum equation}} \qquad \underbrace{\mathbf{u}\cdot\nabla\mathbf{b} \quad \mathbf{b}\cdot\nabla\mathbf{u}}_{\text{induction equation}}$$

Decomposed Maxwell flux $\boldsymbol{b}\cdot\nabla\boldsymbol{b}$

$$\Pi^{M,\ell} = \Pi^{M,\ell}_{s,S\Sigma\Sigma} + \Pi^{M,\ell}_{m,S\Sigma\Sigma} + \Pi^{M,\ell}_{s,SJJ} + \Pi^{M,\ell}_{m,SJJ} + \Pi^{M,\ell}_{s,SJ\Sigma} + \Pi^{M,\ell}_{m,SJ\Sigma}$$



$$\overline{\Sigma}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} + (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

$$\overline{J}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} - (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

Energy transfer almost exclusively from:

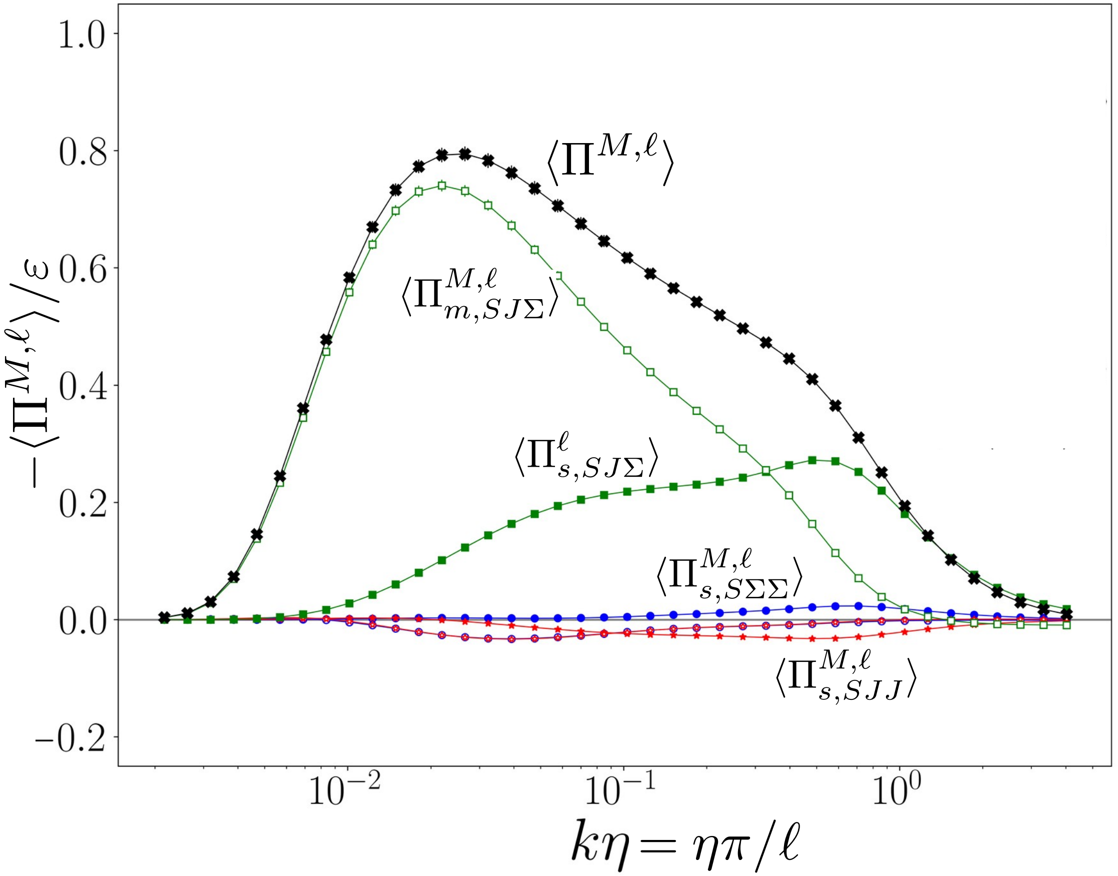
$$\Pi^{M,\ell}_{m,SJ\Sigma} =$$

$$-\int_0^{\ell^2} d\theta\ \mathrm{tr}\left(\overline{S}^{\ell}\left(\overline{\overline{J}^{\sqrt{\theta}}\overline{\Sigma}^{\sqrt{\theta}}}^{\phi} - \overline{\overline{\Sigma}^{\sqrt{\theta}}}^{\phi}\overline{\overline{J}^{\sqrt{\theta}}}^{\phi}\right)\right)$$

$$\Pi^{M,\ell}_{s,SJ\Sigma} = -\ell^2\mathrm{tr}\left(\overline{S}^{\ell}(\overline{J}^{\ell}\overline{\Sigma}^{\ell} - \overline{\Sigma}^{\ell}\overline{J}^{\ell})\right)$$

Decomposed Maxwell flux $\boldsymbol{b}\cdot\nabla\boldsymbol{b}$

$$\Pi^{M,\ell} = \underbrace{\Pi^{M,\ell}_{s,S\Sigma\Sigma} + \Pi^{M,\ell}_{m,S\Sigma\Sigma}}_{\text{extensional/restoring effect of magnetic field}} + \underbrace{\Pi^{M,\ell}_{s,SJJ} + \Pi^{M,\ell}_{m,SJJ}}_{\text{magnetic field bending}} + \underbrace{\Pi^{M,\ell}_{s,SJ\Sigma} + \Pi^{M,\ell}_{m,SJ\Sigma}}_{\text{magnetic field twisting}}$$



$$\overline{\Sigma}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} + (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

$$\overline{J}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} - (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

Energy transfer almost exclusively from:

$$\Pi^{M,\ell}_{m,SJ\Sigma} = -\int_0^{\ell^2} d\theta\ \mathrm{tr}\left(\overline{S}^{\ell}\left(\overline{\overline{J}^{\sqrt{\theta}}\,\overline{\Sigma}^{\sqrt{\theta}}}^{\phi} - \overline{\overline{\Sigma}^{\sqrt{\theta}}}^{\phi}\,\overline{\overline{J}^{\sqrt{\theta}}}^{\phi}\right)\right)$$

$$\Pi^{M,\ell}_{s,SJ\Sigma} = -\ell^2\mathrm{tr}\left(\overline{S}^{\ell}\left(\overline{J}^{\ell}\,\overline{\Sigma}^{\ell} - \overline{\Sigma}^{\ell}\,\overline{J}^{\ell}\right)\right)$$

MHD equations

$$\begin{cases} \partial_t \mathbf{u} = -\nabla P^* - (\mathbf{u}\cdot\nabla)\,\mathbf{u} + (\mathbf{b}\cdot\nabla)\,\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \partial_t \mathbf{b} = -\,(\mathbf{u}\cdot\nabla)\,\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{b} \end{cases}$$

$$P^* = P + \frac{b^2}{2}$$

Generalized expression

$$\Pi^\ell = -\,\ell^2 \mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \nabla\overline{\boldsymbol{B}}^\ell (\nabla\overline{\boldsymbol{C}}^\ell)^t \right)$$

$$-\,\mathrm{tr}\left((\nabla\overline{\boldsymbol{A}}^\ell)^t \int_0^{\ell^2} d\theta\; \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}} (\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} - \overline{\nabla\overline{\boldsymbol{B}}^{\sqrt{\theta}}}^{\sqrt{\ell^2-\theta}}\; \overline{(\nabla\overline{\boldsymbol{C}}^{\sqrt{\theta}})^t}^{\sqrt{\ell^2-\theta}} \right)$$

MHD energy fluxes

$$\Pi^\ell = \Pi^{I,\ell} - \Pi^{M,\ell} + \boxed{\Pi^{A,\ell} - \Pi^{D,\ell}}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{u}\cdot\nabla\mathbf{u} & \mathbf{b}\cdot\nabla\mathbf{b} & \mathbf{u}\cdot\nabla\mathbf{b} & \mathbf{b}\cdot\nabla\mathbf{u} \end{array}$$

$$\underbrace{\mathbf{u}\cdot\nabla\mathbf{u} \quad \mathbf{b}\cdot\nabla\mathbf{b}}_{\text{momentum equation}} \qquad \underbrace{\mathbf{u}\cdot\nabla\mathbf{b} \quad \mathbf{b}\cdot\nabla\mathbf{u}}_{\text{induction equation}}$$

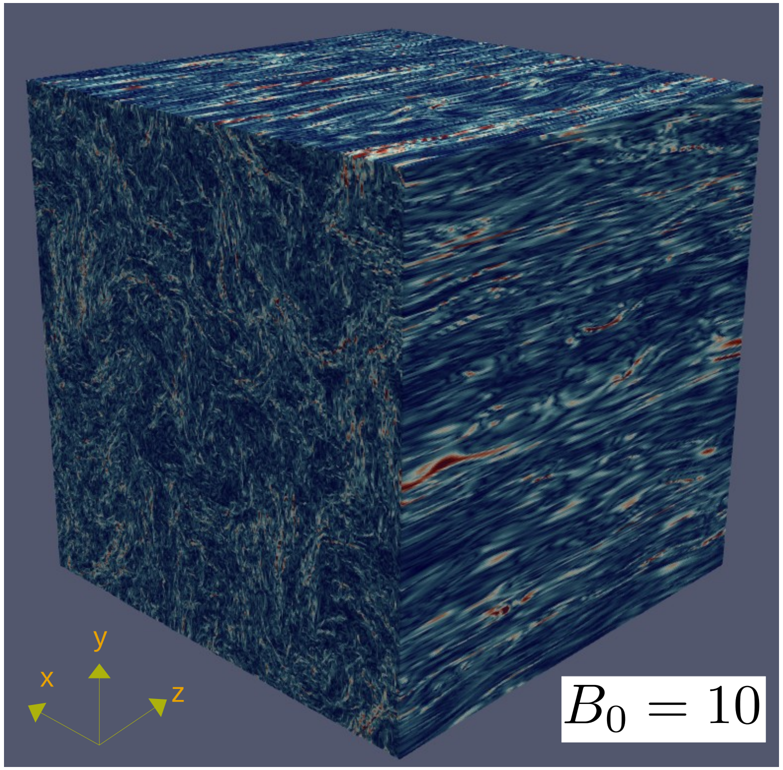
MHD equations with $B_0 \neq 0 \implies$ Anisotropy

$$\begin{cases} \partial_t \mathbf{u} = -\nabla P^* - (\mathbf{u}\cdot\nabla)\,\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{b} + (\mathbf{B_0}\cdot\nabla)\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \partial_t \mathbf{b} = -\,(\mathbf{u}\cdot\nabla)\,\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{u} + (\mathbf{B_0}\cdot\nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{b} \end{cases} \qquad P^* = P + \frac{b^2}{2}$$

$$\mathbf{B_0} = (0, 0, B_0)$$

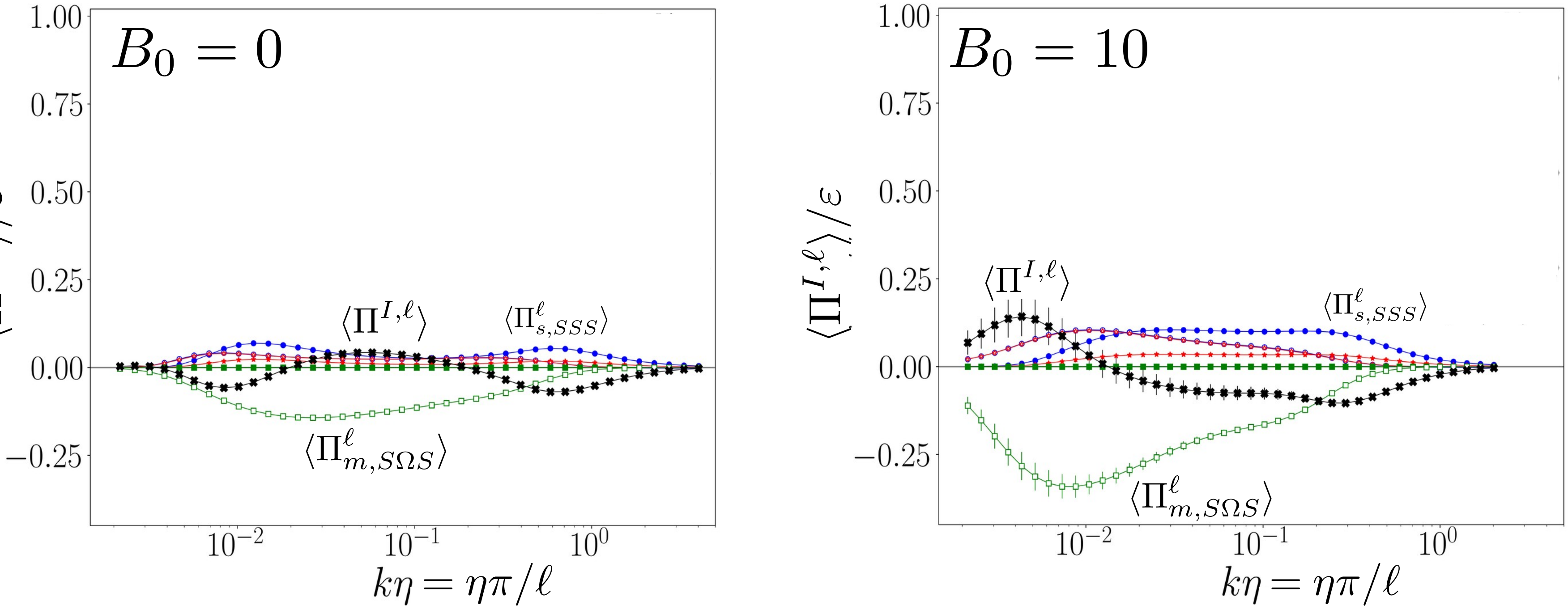
We apply the same decomposition of the SGS tensor setting:

$$\boxed{b_z + B_0 \to b_z}$$



$\mathrm{j} = \nabla \times \mathbf{b}$

magnitude of current

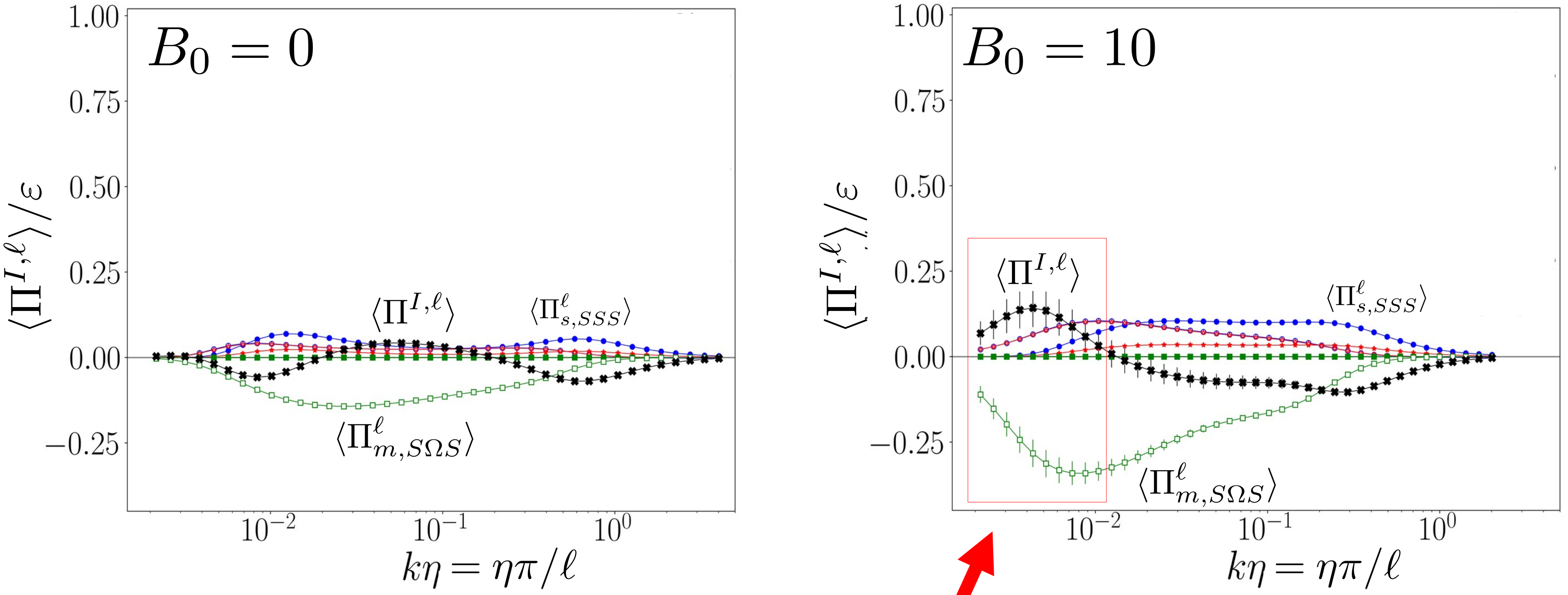


→ 2-dimensionalization: $\Pi^\ell_{m,S\Omega S}$ the only inertial component surviving in 2D

→ Enhanced inverse transfer of kinetic energy

→ $$\Pi^\ell_{m,S\Omega S} = -\int_0^{\ell^2} d\theta \ \mathrm{tr}\left(\overline{S}^\ell\left(\overline{\overline{S}^{\sqrt{\theta}}\,\overline{\Omega}^{\sqrt{\theta}}}^\phi - \overline{\overline{\Omega}^{\sqrt{\theta}}}^\phi\,\overline{\overline{S}^{\sqrt{\theta}}}^\phi\right)\right)$$

→ The system is still accumulating energy in the large scales $\Longrightarrow$ Larger errorbars for $k\eta \le 10^{-2}$



→ 2-dimensionalization: $\Pi^{\ell}_{m,S\Omega S}$ the only inertial component surviving in 2D

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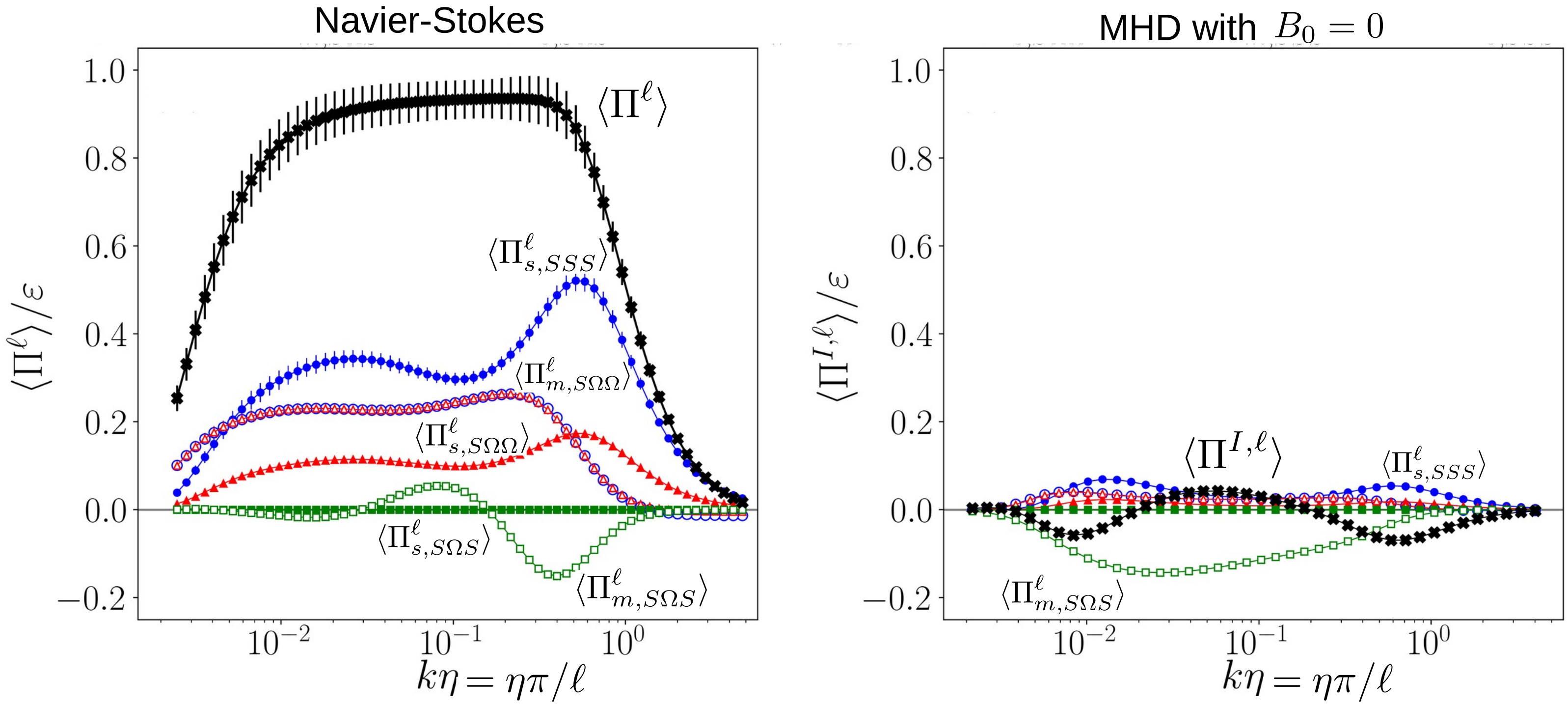
→ The system is still accumulating energy in the large scales $\Longrightarrow$ Larger errorbars for $k\eta \leq 10^{-2}$

Decomposed inertial flux $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$

$$\Pi^{I,\ell} = \Pi^{I,\ell}_{s,SSS} + \Pi^{I,\ell}_{m,SSS} + \Pi^{I,\ell}_{s,S\Omega\Omega} + \Pi^{I,\ell}_{m,S\Omega\Omega} + \Pi^{I,\ell}_{m,S\Omega S}$$



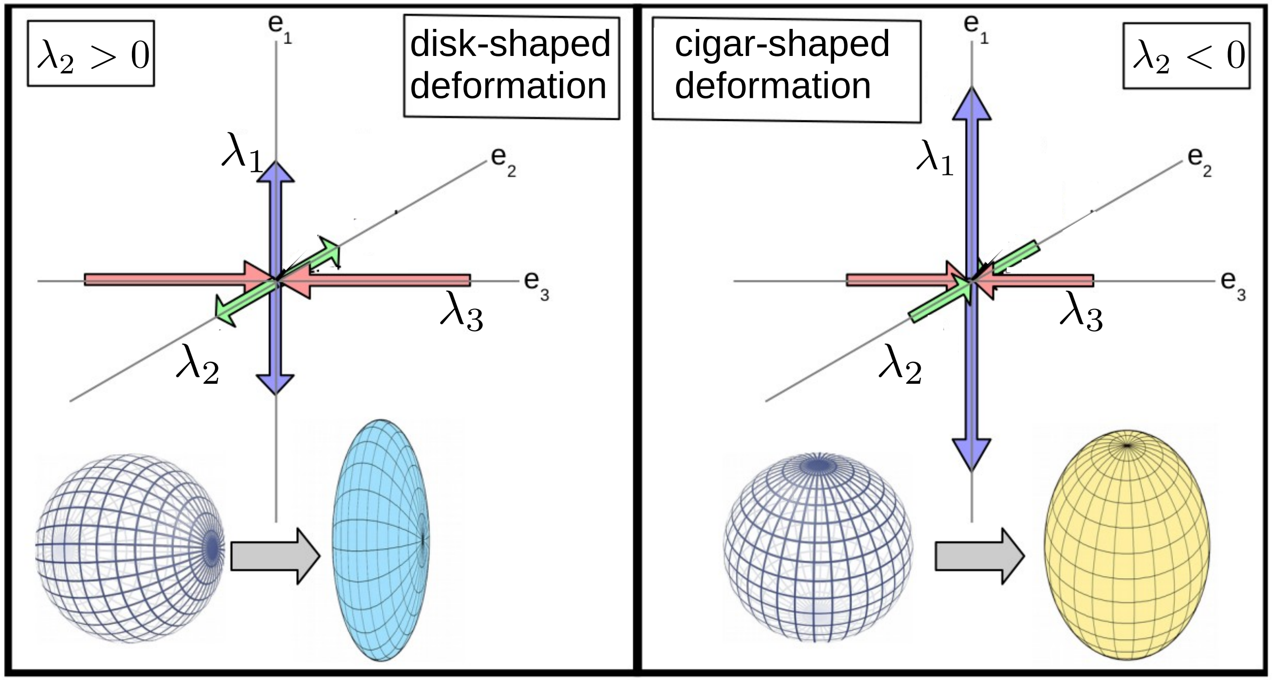
Navier-Stokes

MHD with $B_0 = 0$

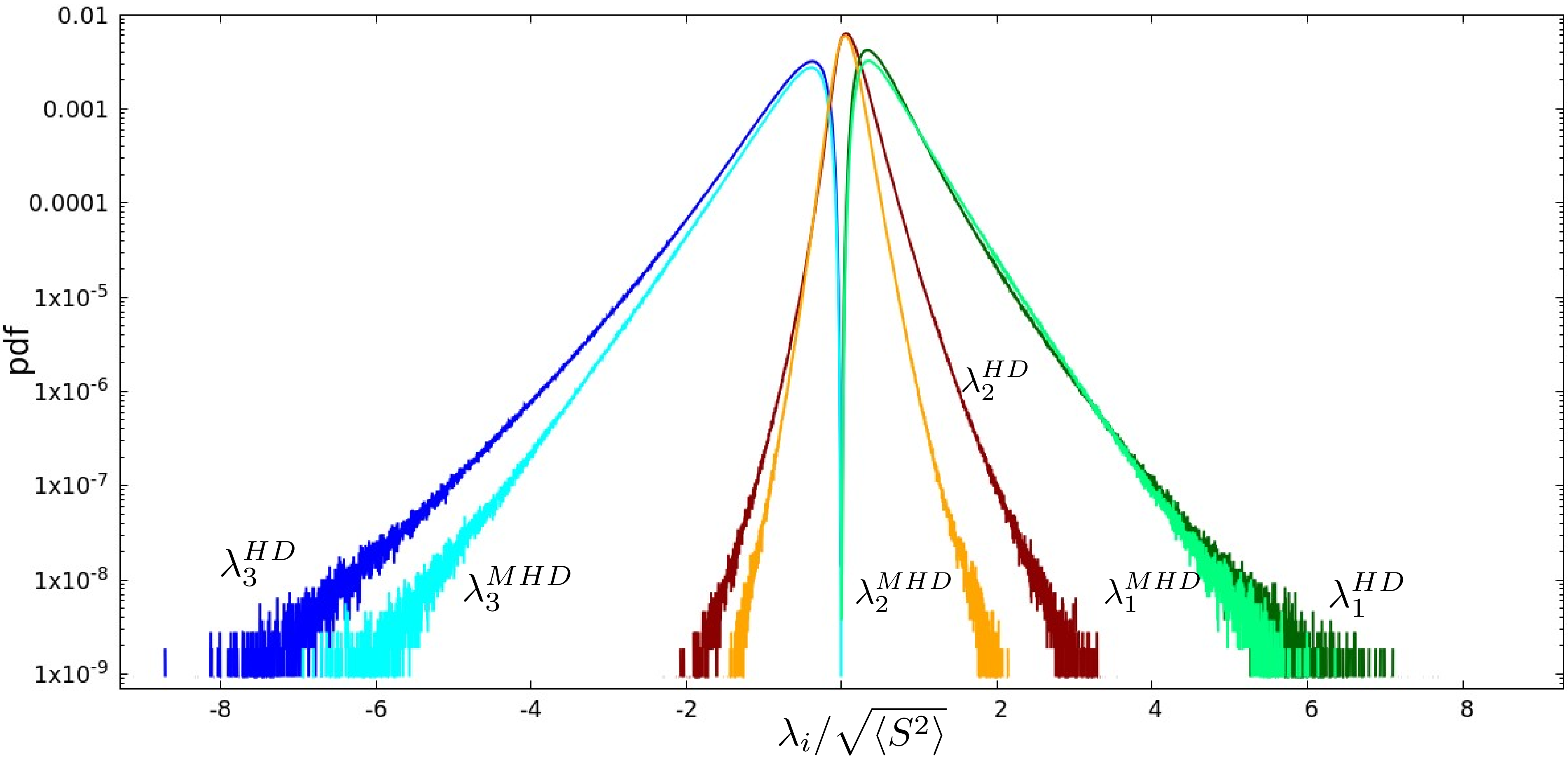
$\langle \Pi^{I,\ell}_{s,SSS} \rangle = 3 \langle \Pi^{I,\ell}_{s,S\Omega\Omega} \rangle$ because of the Betchov relation: $-\underbrace{\langle \mathrm{tr}(\overline{S}^\ell \overline{S}^\ell \overline{S}^\ell) \rangle}_{\text{strain self amplification}} = 3 \underbrace{\langle \mathrm{tr}(\overline{S}^\ell \overline{\Omega}^\ell \overline{\Omega}^\ell) \rangle}_{\text{vortex stretching}}$

- → $\mathrm{tr}\,(SSS) = 3\lambda_1\lambda_2\lambda_3$ in the reference frame where S_{ij} is diagonal
- → Fluid sphere deformation: contractile/extensional directions along the eigenvectors
- → Incompressibility i.e. $\nabla \cdot \mathbf{u} = 0$ provides $\lambda_1 + \lambda_2 + \lambda_3 = 0$
  - → $\lambda_1 \geq 0,\ \lambda_3 \leq 0$ but $\underline{\lambda_2 \lesseqgtr 0}$: 2 possible configurations

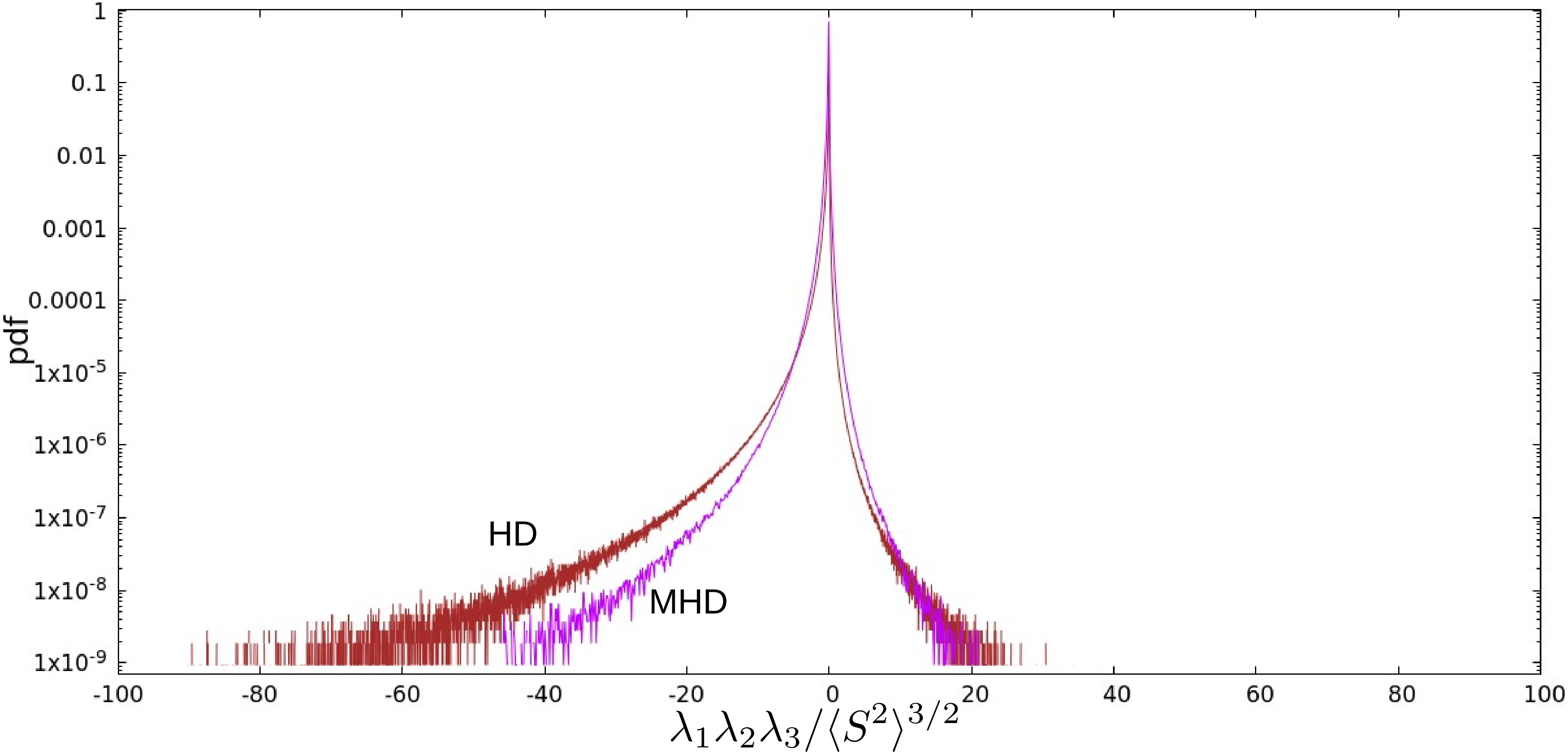
$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

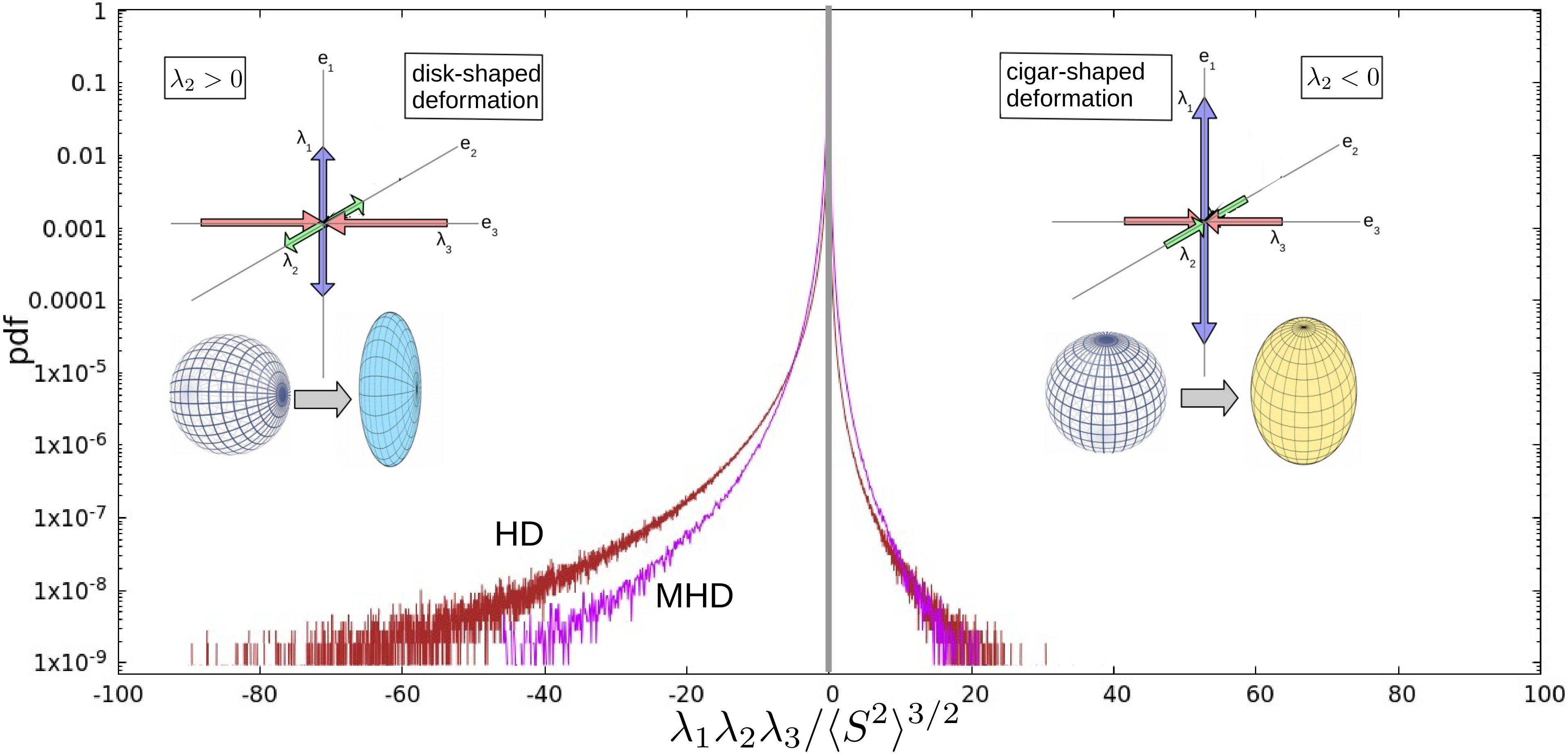


(adapted from P. Johnson, JFM (2021))



→ In MHD $P(\lambda_2^{MHD})$ is more symmetric than $P(\lambda_2^{HD})$


pdf
1
0.1
0.01
0.001
0.0001
1x10^{-5}
1x10^{-6}
1x10^{-7}
1x10^{-8}
1x10^{-9}
HD
MHD
-100
-80
-60
-40
-20
0
20
40
60
80
100
$\lambda_1\lambda_2\lambda_3/\langle S^2\rangle^{3/2}$


$\lambda_2 > 0$
disk-shaped deformation
cigar-shaped deformation
$\lambda_2 < 0$
e1
e2
e3
λ1
λ2
λ3
HD
MHD
pdf
1
0.1
0.01
0.001
0.0001
1x10-5
1x10-6
1x10-7
1x10-8
1x10-9
-100
-80
-60
-40
-20
0
20
40
60
80
100
$\lambda_1\lambda_2\lambda_3/\langle S^2\rangle^{3/2}$

Summary

- exact decomposition of flux terms in coupled advection-diffusion equations
  - kinetic, cross & magnetic helicity fluxes can be also decomposed
- application to the energy fluxes in MHD turbulence
  - depletion of vortex stretching compared to Navier-Stokes
  - The Lorentz force provides the leading contribution to the MHD kinetic energy flux
  - 2-dimensionalisation in MHD leads to increased inverse transfer of kinetic energy
- HD and MHD have different small-scale flow structures
  - the strain-rate tensor eigenvalues distributions are more symmetric in MHD than in HD
  - in HD there is a clear prevalence for disk-shaped deformation of flow structures
  - in MHD there is less difference in likelihood of disk-shaped and cigar-shaped deformation
- guidance for SGS modelling

Thank you for your attention

Decomposed advection $\boldsymbol{u}\cdot\nabla\boldsymbol{b}$ and dynamo $\boldsymbol{b}\cdot\nabla\boldsymbol{u}$ fluxes

$$\overline{\Sigma}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} + (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

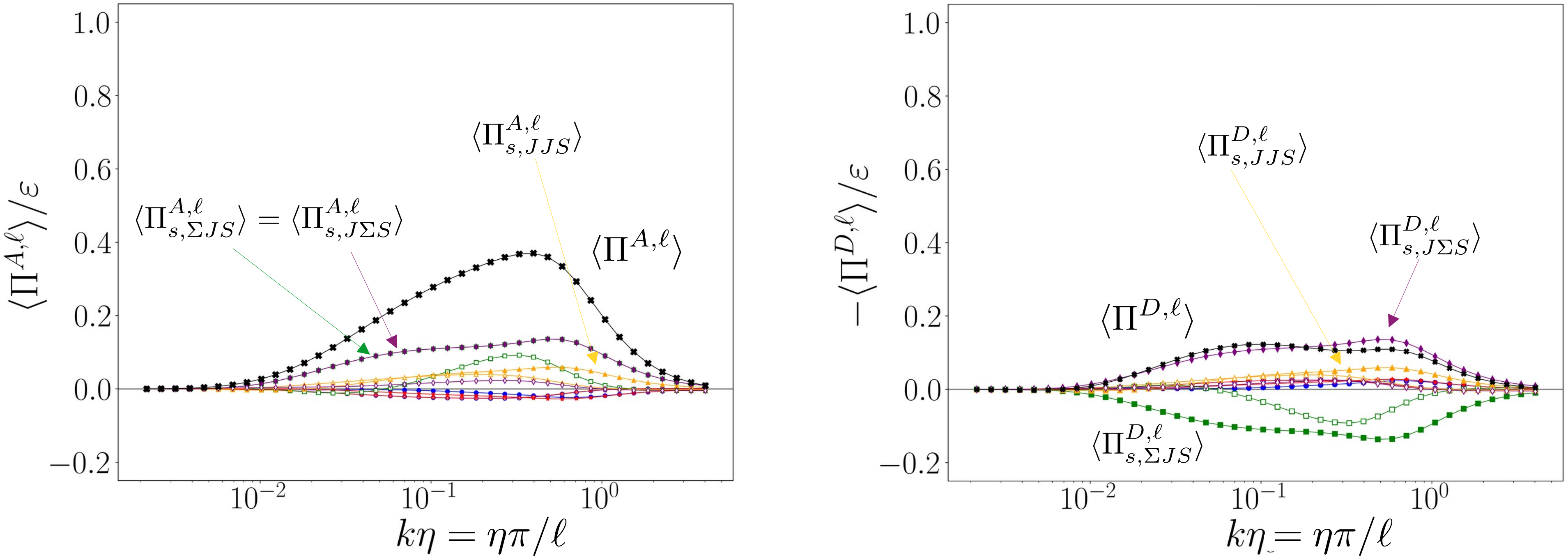
$$\overline{J}^{\ell} = \left(\nabla\overline{\boldsymbol{b}}^{\ell} - (\nabla\overline{\boldsymbol{b}}^{\ell})^{t}\right)/2$$

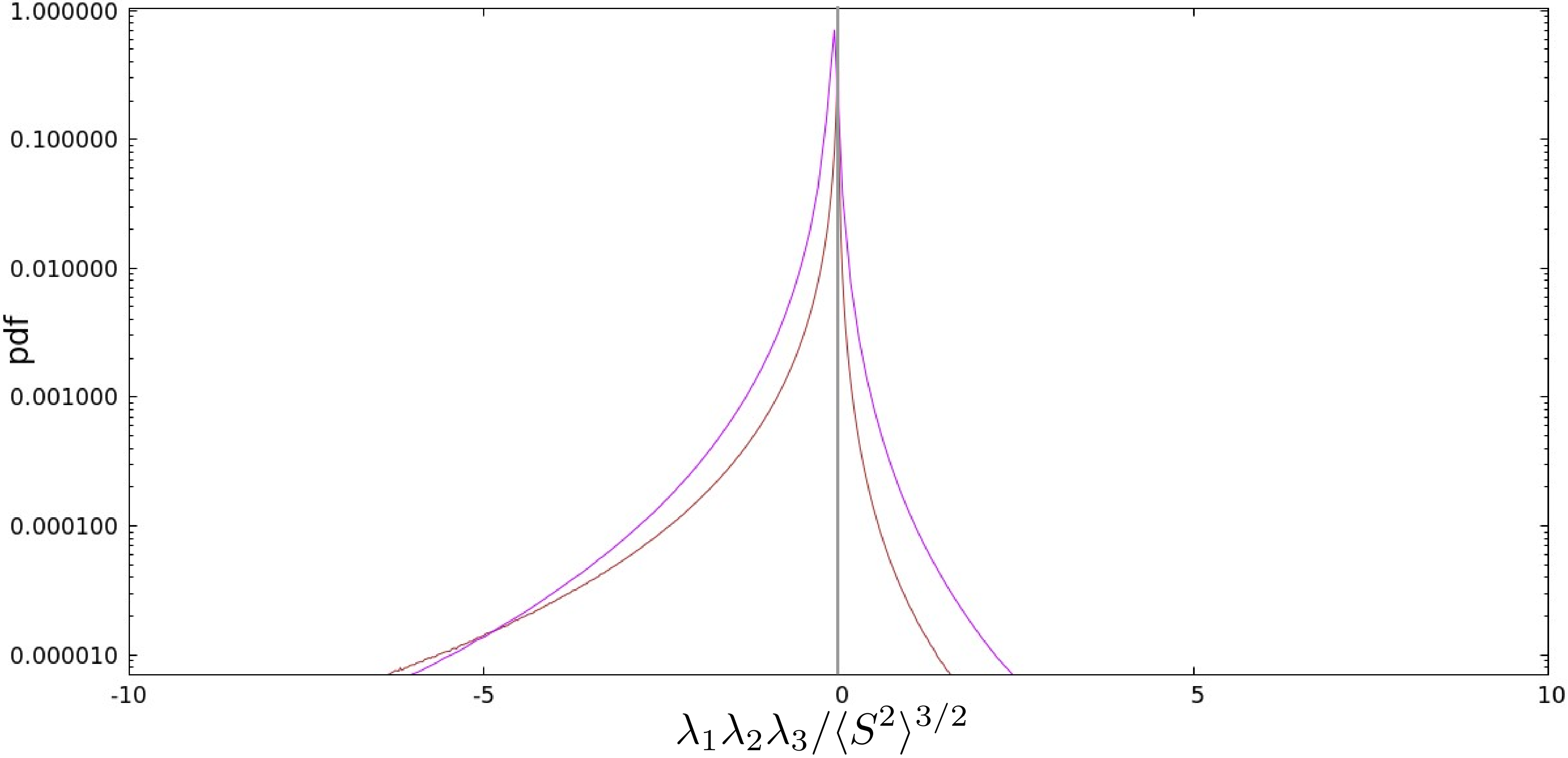
→ They are electric field energy subfluxes

→ common physical origin

$$\underline{\Pi^{A,\ell}} = \Pi^{A,\ell}_{s,\Sigma\Sigma S} + \Pi^{A,\ell}_{m,\Sigma\Sigma S} - \Pi^{A,\ell}_{s,J\Sigma S} - \Pi^{A,\ell}_{m,J\Sigma S} - \Pi^{A,\ell}_{s,JJS} - \Pi^{A,\ell}_{m,JJS} + \Pi^{A,\ell}_{m,\Sigma J\Omega} + \Pi^{A,\ell}_{m,J\Sigma\Omega}$$

$$\Pi^{D,\ell} = \underline{\Pi^{A,\ell}} + 2\,\Pi^{A,\ell}_{s,J\Sigma S} + 2\,\Pi^{A,\ell}_{m,J\Sigma S} - 2\,\Pi^{A,\ell}_{m,\Sigma J\Omega} - 2\,\Pi^{A,\ell}_{m,J\Sigma\Omega}$$




1.000000
0.100000
0.010000
0.001000
0.000100
0.000010
pdf
-10
-5
0
5
10
$\lambda_1\lambda_2\lambda_3/\langle S^2\rangle^{3/2}$

$$\langle \Pi^{I,\ell} \rangle - \langle \Pi^{M,\ell} \rangle$$

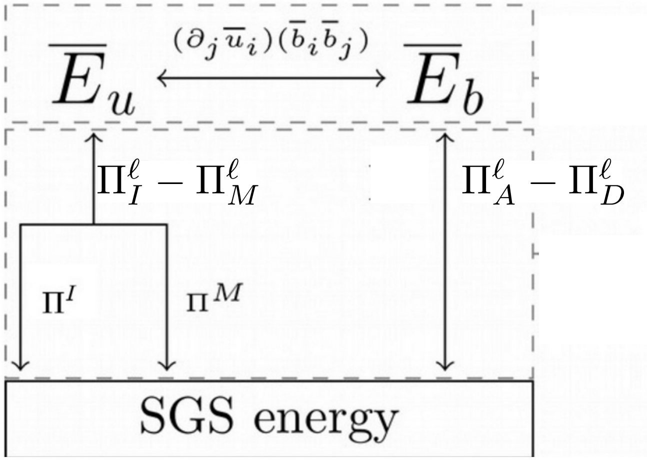
$$\mathcal{W}^{\ell}$$

$$\langle \Pi^{A,\ell} \rangle - \langle \Pi^{D,\ell} \rangle$$

$$\Pi^\ell = \Pi^\ell_I - \Pi^\ell_M + \Pi^\ell_A - \Pi^\ell_D$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{u}\cdot\nabla\mathbf{u} & \mathbf{b}\cdot\nabla\mathbf{b} & \mathbf{u}\cdot\nabla\mathbf{b} & \mathbf{b}\cdot\nabla\mathbf{u} \end{array}$$

$$\underbrace{\mathbf{u}\cdot\nabla\mathbf{u} \quad \mathbf{b}\cdot\nabla\mathbf{b}}_{\text{momentum equation}} \qquad \underbrace{\mathbf{u}\cdot\nabla\mathbf{b} \quad \mathbf{b}\cdot\nabla\mathbf{u}}_{\text{induction equation}}$$



$$\begin{cases} \partial_t \frac{1}{2} \overline{u_i}^\ell \overline{u_i}^\ell + \partial_j(\ldots) = -\Pi^\ell_I + \Pi^\ell_M - \Pi^\ell_{conv} + \epsilon^\ell_{in} - 2\nu \overline{S_{ij}}^\ell \overline{S_{ij}}^\ell \\ \partial_t \frac{1}{2} \overline{b_i}^\ell \overline{b_i}^\ell + \partial_j(\ldots) = -\Pi^\ell_A + \Pi^\ell_D + \Pi^\ell_{conv} - 2\eta \overline{\Sigma_{ij}}^\ell \overline{\Sigma_{ij}}^\ell \end{cases}$$

$$\Pi^\ell_{conv} = (\partial_j \overline{u_i}^\ell)\, \overline{b_i}^\ell \overline{b_j}^\ell$$

resolved scale
conversion term

Offermans, G. P., Biferale, L., Buzzicotti, M., & Linkmann, M. POP (2018)

H. Aluie NJP (2017)

Computational costs per configuration

1024^3 collocation points

- → Direct numerical simulation:
  - → Resources: 32 nodes
  - → Run time: 145 h
- → Data analysis:
  - → Resources: 32 nodes
  - → Run time: 60 h

Why hyperviscosity?

hyperviscous

viscous

Computational costs

2048^3 collocation points

- → Direct numerical simulation:
    - → Resources: 64 nodes
    - → Run time: 360 h
- → Data analysis:
    - → Resources: 64 nodes
    - → Run time: 230 h