Hidden Regions and Contour Deformation

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Olsson, Stone [To Appear]
Gardi, Herzog, Ma [To Appear]
Gardi, Herzog, Ma, Schlenk [2211.14845]
Heinrich, Jahn, Kerner, Langer, Magerya, Olsson, Põldaru, Schlenk, Villa [2108.10807, 2305.19768]

Outline

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Feynman & Lee-Pomeransky representation

Sector Decomposition

Method of Regions (MoR)

Hidden regions due to cancellation

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Finding and evaluating integrals with pinch singularities for *generic* kinematics

MoR and Hidden Regions due to Cancellation

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Concept

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Introduction

Parameter Space

Can exchange integrals over loop momenta for integrals over parameters

Feynman Parametrisation

$[\mathrm{d}\boldsymbol{\alpha}] = \prod_{e \in G} \frac{\mathrm{d}\alpha_e}{\alpha_e}$ $\quad$ $\boldsymbol{\alpha}^{\nu} = \prod_{e \in G} \alpha_e^{\nu_e}$

$$I(\boldsymbol{s}) = \frac{\Gamma(\nu - LD/2)}{\prod_{e \in G} \Gamma(\nu_e)} \int_0^{\infty} [d\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \delta\left(1 - H(\boldsymbol{\alpha})\right) \frac{\left[\mathscr{U}(\boldsymbol{\alpha})\right]^{\nu-(L+1)D/2}}{\left[\mathscr{F}(\boldsymbol{\alpha};\boldsymbol{s})\right]^{\nu-LD/2}}$$

$\mathscr{U}, \mathscr{F}$ homogeneous polynomials of degree L and $L+1$

Lee-Pomeransky Parametrisation

$$I(\boldsymbol{s}) = \frac{\Gamma(D/2)}{\Gamma\left((L+1)\,D/2 - \nu\right) \prod_{e \in G} \Gamma(\nu_e)} \int_0^{\infty} [d\boldsymbol{x}]\, \boldsymbol{x}^{\nu} \left(\mathscr{G}(\mathbf{x},\boldsymbol{s})\right)^{-D/2}$$

$$\mathscr{G}(\mathbf{x};\boldsymbol{s}) = \mathscr{U}(\mathbf{x}) + \mathscr{F}(\mathbf{x};\boldsymbol{s})$$

Lee, Pomeransky 13

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^{N+1}_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \frac{[\mathscr{U}(\boldsymbol{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{x},\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - H(\boldsymbol{x}))$$

Singularities

1. UV/IR singularities when some $x \to 0$ simultaneously $\Longrightarrow$ Sector Decomposition
2. Thresholds when $\mathscr{F}$ vanishes inside integration region $\Longrightarrow$ Contour Deformation

Sector decomposition

Find a local change of coordinates for each singularity that factorises it (blow-up)

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

Kaneko, Ueda 10

$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\text{Singularities}} \Bigg(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\text{Finite}} \Bigg)^t$$

Singularities **Finite**

*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

Method of Regions

Consider expanding an integral about some limit:
$p_i^2 \sim \lambda Q^2$, $p_i \cdot p_j \to \lambda Q^2$ or $m^2 \sim \lambda Q^2$ for $\lambda \to 0$

Issue: integration and series expansion do not necessarily commute

Method of Regions

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

1. Split integrand up into regions (R)
2. Series expand each region in λ
3. Integrate each expansion over the whole integration domain
4. Discard scaleless integrals (= 0 in dimensional regularisation)
5. Sum over all regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

Finding Regions

Assuming all c_i have the same sign we rescale $\boldsymbol{s} \to \lambda^{\boldsymbol{\omega}} \boldsymbol{s}$ ← $s_i \to \lambda^{\omega_i} s_i$

Newton Polytope

$$I \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t \to \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i} \lambda^{r_{i,N+1}}\right)^t \to \mathcal{N}^{N+1}$$

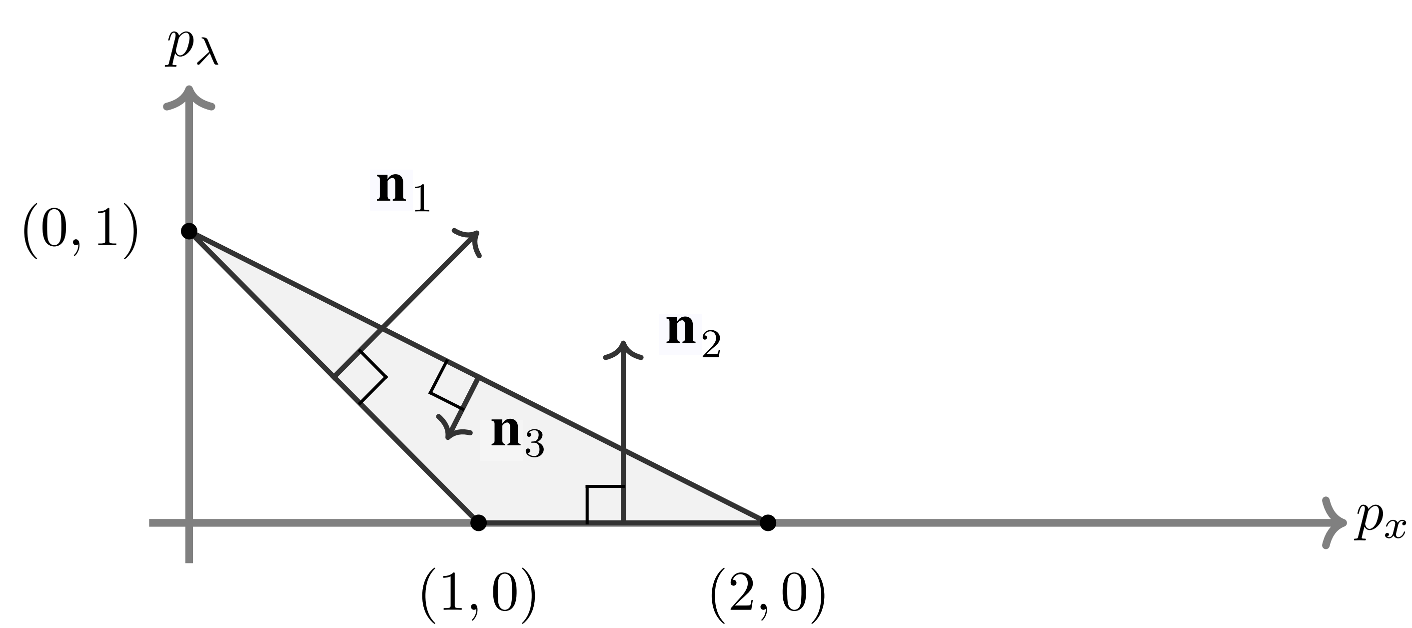
Normal vectors w/ positive λ component define change of variables $\mathbf{n}_f = (v_1, \ldots, v_N, 1)$

$$\boldsymbol{x} = \lambda^{\mathbf{n}_f} \mathbf{y}\,, \qquad \lambda \to \lambda$$

Pak, Smirnov 10; Semenova, A. Smirnov, V. Smirnov 18

Example

$p(x, \lambda) = \lambda + x + x^2$



$1,2 \in F^+$

$3 \notin F^+$

Original integral I may then be approximated as $I = \sum_{f \in F^+} I^{(f)} + \ldots$

Additional Regulators/ Rapidity Divergences

MoR subdivides $\mathcal{N}(I) \to \{\mathcal{N}(I^R)\} \Longrightarrow$ new (internal) facets $F^{\text{int.}}$

New facets can introduce spurious singularities not regulated by dim reg

Lee Pomeransky Representation:

$$\mathcal{N}(I^{(R)}) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_{\mathbb{R}^N_{\geq 0}} \left[\mathrm{d}\mathbf{y}_f \right] \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle + \frac{D}{2} a_f} \left(c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f} \right)^{-\frac{D}{2}}$$

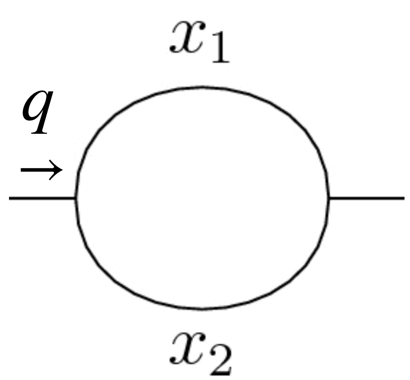
If $f \in F^{\text{int}}$ have $a_f = 0$ need analytic regulators $\boldsymbol{\nu} \to \boldsymbol{\nu} + \boldsymbol{\delta\nu}$

Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Põldaru, Schlenk, Villa 21; Schlenk 16

Regions due to Cancellation

What happens if c_i have different signs?

Consider a 1-loop massive bubble at *threshold* $y = m^2 - q^2/4 \to 0$



$$I = \Gamma(\epsilon) \int d\alpha_1 d\alpha_2 \frac{\delta(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)^{-2+2\epsilon}}{\left(\mathcal{F}_{\text{bub}}(\alpha_1,\alpha_2;q^2,y)\right)^{\epsilon}}$$

$$\mathcal{F}_{\text{bub}} = \frac{q^2}{4}(\alpha_1 - \alpha_2)^2 + y(\alpha_1+\alpha_2)^2$$

Can split integral into two subdomains $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ then remap

$$\begin{aligned} \alpha_1 &= \alpha_1'/2 \\ \alpha_2 &= \alpha_2' + \alpha_1'/2 \end{aligned} : \quad \mathcal{F}_{\text{bub},1} \to \frac{q^2}{4}\alpha'^2_2 + y(\alpha'_1 + \alpha'_2)^2 \quad \text{(for first domain)}$$

Jantzen, A. Smirnov, V. Smirnov 12

Before split: only **hard** region found $(\alpha_1 \sim y^0, \alpha_2 \sim y^0)$

After split: also **potential** region found $(\alpha_1 \sim y^0, \alpha_2 \sim y^{1/2})$

Regions due to Cancellation

Various tools attempt to find such re-mappings using **linear** changes of variables

ASY/FIESTA Jantzen, A. Smirnov, V. Smirnov 12

- Check all pairs of variables (α_1, α_2) which are part of monomials of opposite sign
- For each pair, try to build linear combination $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$ s.t negative monomial vanishes
- Repeat until all negative monomials vanish **or** warn user

ASPIRE Ananthanarayan, Pal, Ramanan, Sarkar 18; B. Ananthanarayan, Das, Sarkar 20

- Consider Gröbner basis of $\left\{\mathcal{F}, \partial\mathcal{F}/\alpha_1, \partial\mathcal{F}/\alpha_2, \ldots\right\}$ (i.e. $\mathcal{F}$ and Landau equations)
- Eliminate negative monomials with linear transformations $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$

This is not enough to straightforwardly expose all regions in parameter space

Integrals with Pinch Singularities

Landau Equations

Polynomials $\mathscr{U}, \mathscr{F}$ can vanish (gives singularities) for some $\alpha_i \to 0$ (end-point)

Additionally, due to signs in $\mathscr{F}$ it can vanish due to cancellation of terms

Avoid poles on real axis by deforming contour (roughly speaking…):

$$\alpha_k \to \alpha_k - i\varepsilon_k(\boldsymbol{\alpha})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) \to \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) - i\sum_k \varepsilon_k \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_k} + \mathcal{O}(\varepsilon^2)$$

If $\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$ and $\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})/\partial\alpha_j = 0 \ \forall j$ simultaneously, contour will vanish exactly where the deformation is required, above conditions are just the Landau equations

Landau Equations (parameter space):

$$(L+1)\mathscr{F} = \sum_{k=1}^{N} \alpha_k \frac{\partial \mathscr{F}}{\partial \alpha_k}.$$

$$1) \quad \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$$

$$2) \quad \alpha_j \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_j} = 0 \quad \forall j$$

Leading: $\alpha_j \neq 0 \forall j$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Looking for Trouble: Algorithm

Generally, solutions of the Landau equations depend on $\mathbf{s}$.
Let us restrict our search to solutions with *generic* kinematics

$$\mathcal{F} = -\sum_i s_i \left[f_i(\boldsymbol{\alpha}) - g_i(\boldsymbol{\alpha}) \right] = \sum_i \mathcal{F}_{i,-} + \mathcal{F}_{i,+}$$

$$\mathcal{F}_{i,-} = -\, s_i\, f_i(\boldsymbol{\alpha}), \quad \mathcal{F}_{i,+} = s_i\, g_i(\boldsymbol{\alpha}), \quad f_i(\boldsymbol{\alpha}), g_i(\boldsymbol{\alpha}) \geq 0$$

Algorithm (finds integrals which *potentially* have a pinch in the massless case)

For each s_i:

1) Compute $\mathcal{F}_{i,-}$, $\mathcal{F}_{i,+}$

2) If $\mathcal{F}_{i,-} = 0$ or $\mathcal{F}_{i,+} = 0 \rightarrow$ **Exit (no cancellation)**

3) If $\partial \mathcal{F}_{i,-} / \partial \alpha_j = 0$ or $\partial \mathcal{F}_{i,+} / \partial \alpha_j = 0$ set $\alpha_j = 0 \rightarrow$ Goto 1

Else $\rightarrow$ **Exit (potential cancellation)**

Much more sophisticated algorithms for solving Landau equations exist

(E.g.) Mizera, Simon Telen 21; Fevola, Mizera, Telen 23

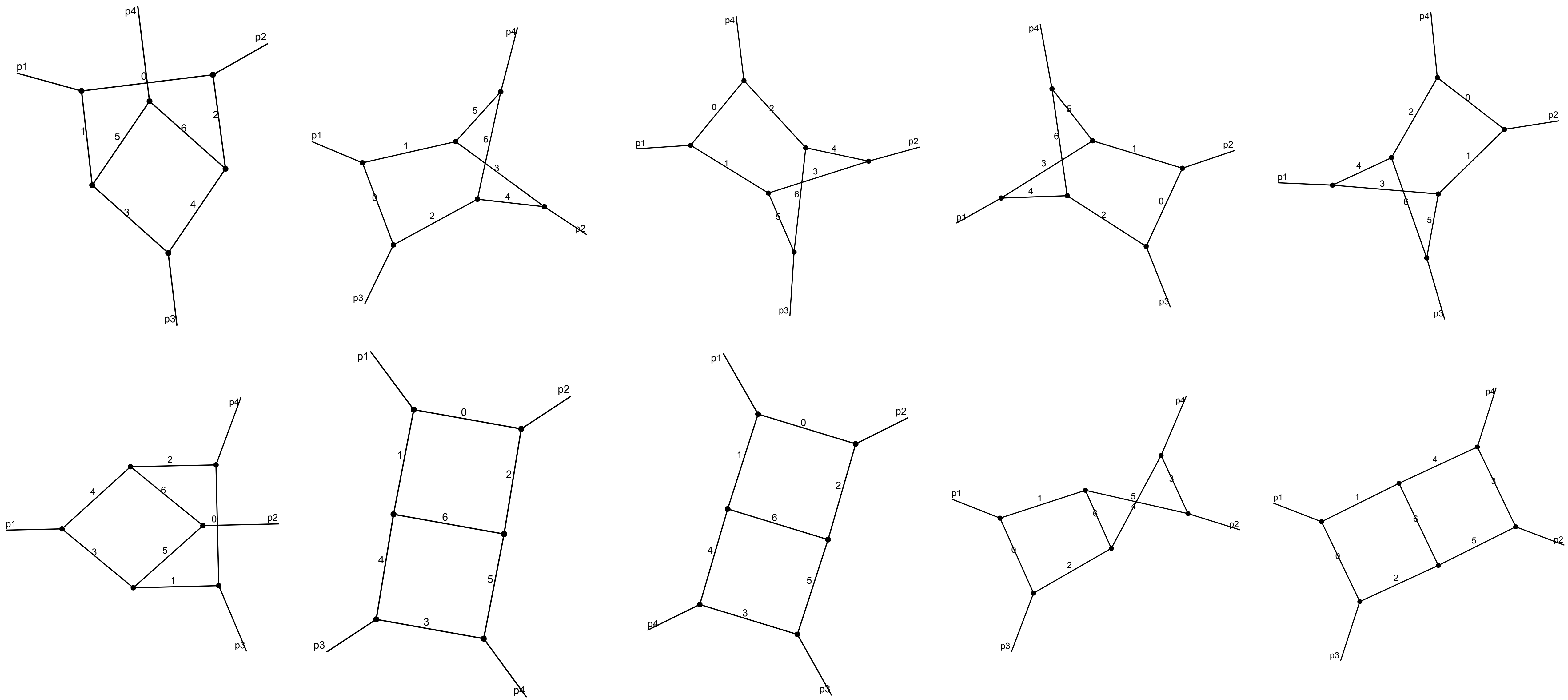
(See also) Gambuti, Kosower, Novichkov, Tancredi 23

Looking for Trouble: 1- & 2-loops

We considered massless 4-point scattering amplitudes ($s_{23} = -s_{12} - s_{13}$)

@1-loop: found no candidates (trivially)

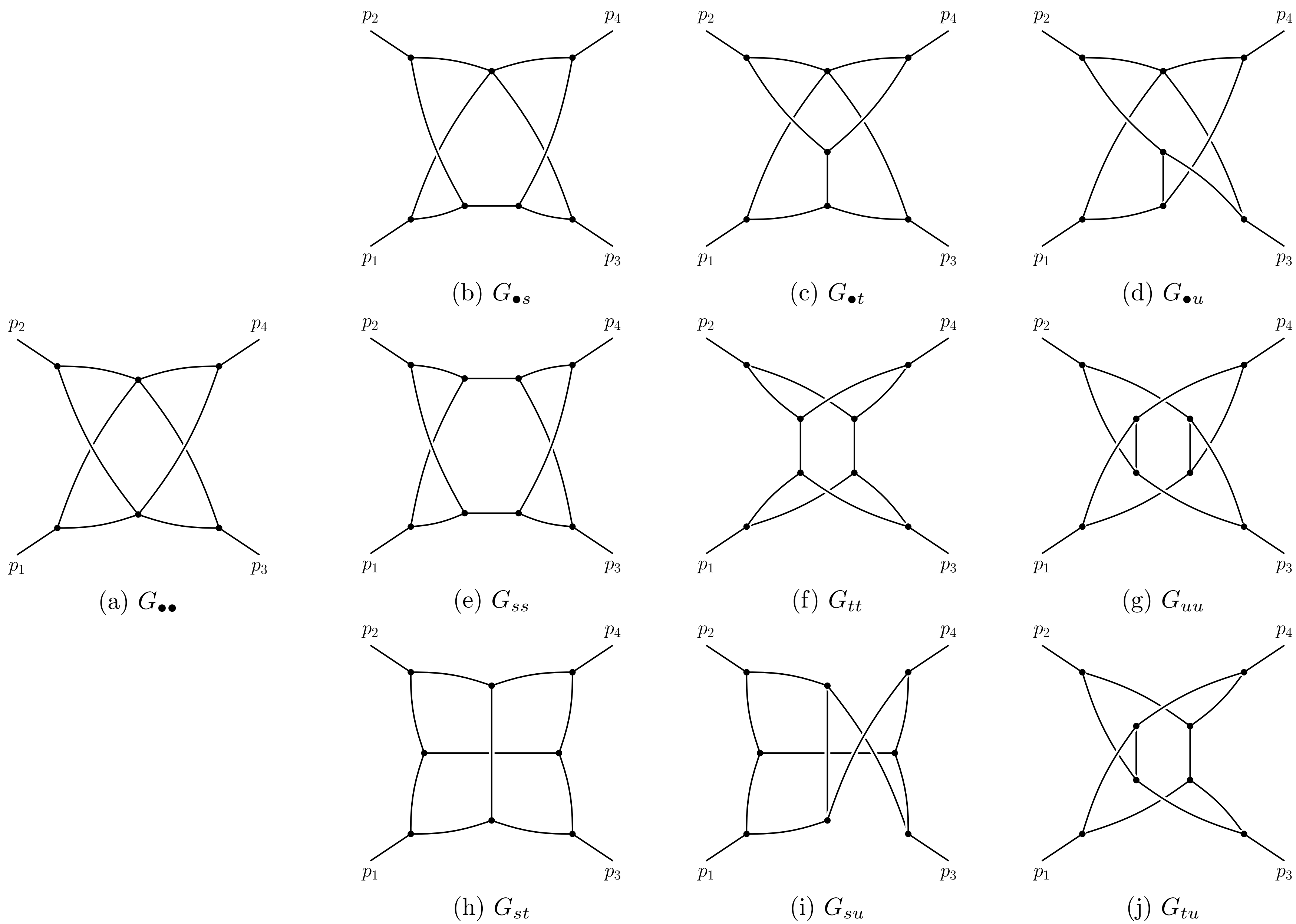
@2-loop:



+ … no candidates (!)

Looking for Trouble: 3-loops

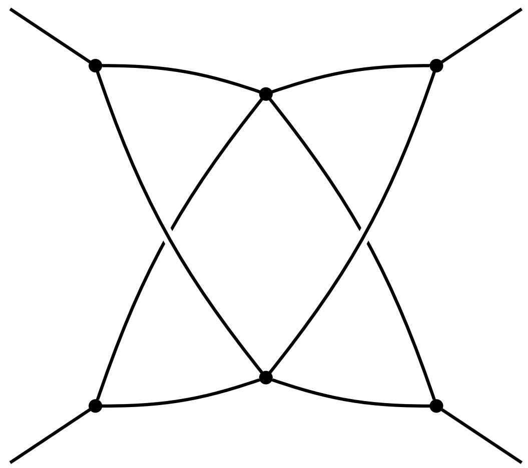
@3-loop: finally some interesting candidates



The complete set of corresponding master integrals for generic s_{12}, s_{13} are known

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21;

Interesting Example



$$= \int_0^\infty \mathrm{d}x_0 \,\ldots \mathrm{d}x_7 \,\frac{\mathscr{U}(\mathbf{x})^{4\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1-x_7)$$

$$\mathscr{U}(\alpha) = \alpha_0\alpha_2\alpha_4 + \alpha_0\alpha_2\alpha_5 + \alpha_0\alpha_2\alpha_6 + (29 \text{ terms})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} = s_{12}\,\alpha_5(\alpha_3\alpha_6 - \alpha_2\alpha_7) + s_{13}\,\alpha_3(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\vdots$$

$$\frac{\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} = s_{12}\,\alpha_2(\alpha_1\alpha_4 - \alpha_0\alpha_5) + s_{13}\,\alpha_4(\alpha_1\alpha_2 - \alpha_0\alpha_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathscr{F}$ vanishes!

Interesting Example

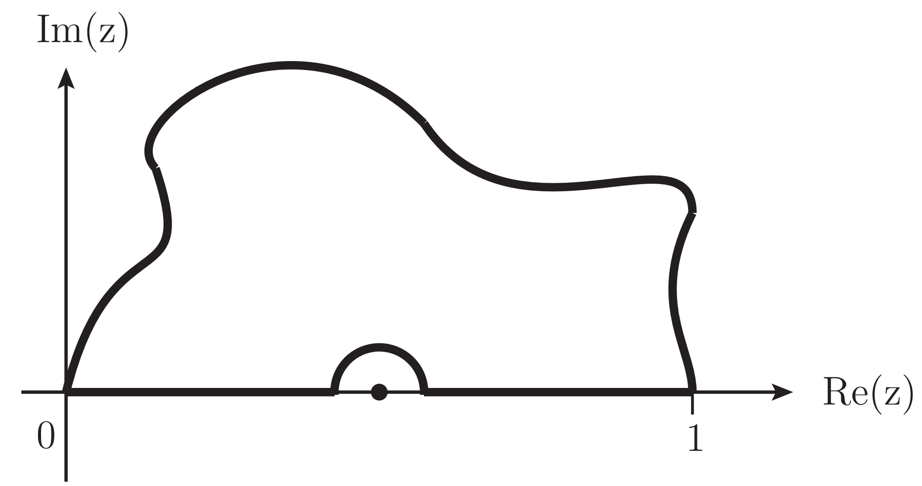
Let's try to compute this with sector decomposition (pySecDec)

Fails to find contour…

Contour Deformation

Feynman integral (after sector decomp):

$$I \sim \int_0^1 [\mathrm{d}\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \frac{[\mathcal{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})]^{N-LD/2}}$$



Deform integration contour to avoid poles on real axis

Feynman prescription $\mathcal{F} \to \mathcal{F} - i\delta$ tells us how to do this

Expand $\mathcal{F}(z = \boldsymbol{\alpha} - i\boldsymbol{\tau})$ around $\boldsymbol{\alpha}$, $\mathcal{F}(z) = \mathcal{F}(\boldsymbol{\alpha}) - i \sum_j \tau_j \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j} + \mathcal{O}(\tau^2)$

Choose $\tau_j = \lambda_j \alpha_j (1 - \alpha_j) \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j}$ with small constants $\lambda_j > 0$

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;…

Contour Deformation

But for this class of examples $\mathcal{F}(\boldsymbol{\alpha})$ and all $\partial\mathcal{F}(\boldsymbol{\alpha})/\partial\alpha_i$ vanish at the same point inside the integration domain

→ *pinch* singularity

Example

$$
\begin{aligned}
\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) &= -s_{12}\,(\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7)-s_{13}\,(\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} &= s_{12}\;\alpha_5(\alpha_3\alpha_6-\alpha_2\alpha_7)+s_{13}\;\alpha_3(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
&\vdots\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} &= s_{12}\;\alpha_2(\alpha_1\alpha_4-\alpha_0\alpha_5)+s_{13}\;\alpha_4(\alpha_1\alpha_2-\alpha_0\alpha_3)
\end{aligned}
$$

vanish for

$$
\alpha_2=\frac{\alpha_0\alpha_3}{\alpha_1},\qquad \alpha_4=\frac{\alpha_0\alpha_5}{\alpha_1},\qquad \alpha_6=\frac{\alpha_0\alpha_7}{\alpha_1}\,.
$$

Resolution

The problem is that we have monomials with different signs…

Asy2.1 PreResolve->True

```
-bash                                                                  ⌥⌘1
MACTHXJONES:fiesta sj$ cat diagram2636.m
Get["asy2.1.m"];

Print["Diagram2636"];
result = AlphaRepExpand[{k1,k2,k3}, {k1^2, k2^2,k3^2, (k1+p1)^2, (k2+p2)^2, (k3+p3)^2,
(k1+k2+k3)^2, (k1+k2+k3+p1+p2+p3)^2}, {p1^2->0, p2^2->0, p3^2->0, p1*p2->s12/2, p1*p3->
s13/2, p2*p3->-s12/2-s13/2}, {s12 -> 1, s13 -> -1/5}, PreResolve->True] (* 3-loop box *
)
Print[result];
Print["=========="];
Exit[];

MACTHXJONES:fiesta sj$ wolframscript -file diagram2636.m
Diagram2636
Asy2.1
Variables for UF: {k1, k2, k3, p1, p2, p3}
WARNING: preresolution failed
{}
==========
MACTHXJONES:fiesta sj$
```

Correctly identifies that iterated linear changes of variables are not sufficient to resolve the singularity and reports that pre-resolution has failed

Resolution

1) Rescale parameters to *linearise* singular surfaces

$$\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7)-s_{13}\,(\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7)$$

$$\alpha_0\to\alpha_0\alpha_1,\ \alpha_2\to\alpha_2\alpha_3,\ \alpha_4\to\alpha_4\alpha_5,\ \alpha_6\to\alpha_6\alpha_7$$

$$\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_4-\alpha_0)(\alpha_6-\alpha_2)-s_{13}(\alpha_2-\alpha_0)(\alpha_6-\alpha_4)\right]$$

2) Split the integral by imposing $\alpha_i\geq\alpha_j\geq\alpha_k\geq\alpha_l$

$$\begin{aligned}\alpha_0&\to\alpha_0+\alpha_2+\alpha_4+\alpha_6,\\ \alpha_2&\to\alpha_2+\alpha_4+\alpha_6,\\ \alpha_4&\to\alpha_4+\alpha_6,\\ \alpha_6&\to\alpha_6\end{aligned}\qquad +\text{perms}$$

$$\begin{aligned}\mathcal{F}_1(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_0+\alpha_2)(\alpha_2+\alpha_4)-s_{13}(\alpha_0)(\alpha_4)\right]\\ \mathcal{F}_2(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2)(\alpha_0+\alpha_2+\alpha_6)+s_{13}(\alpha_0)(\alpha_6)\right]\\ &\vdots\\ \mathcal{F}_{24}(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2+\alpha_4)(\alpha_4+\alpha_6)-s_{13}(\alpha_2)(\alpha_6)\right]\end{aligned}$$

All coefficients of s_{12}, s_{13} now have definite sign

Result

Can now obtain results numerically ($s_{12} = 1,\ s_{13} = -1/5$)

$$I_1 = \epsilon^{-4}\left[(-3.8842800687 + 5.2359902003j) \pm (4.458 \cdot 10^{-6} + 3.638 \cdot 10^{-6}j)\right] + \ldots$$
$$I_2 = \epsilon^{-4}\left[(-7.9291803033 + 20.943767810j) \pm (9.149 \cdot 10^{-5} + 1.061 \cdot 10^{-4}j)\right] + \ldots$$
$$I_3 = \epsilon^{-4}\left[(18.5195704502 - 15.707988011j) \pm (5.897 \cdot 10^{-5} + 5.897 \cdot 10^{-5}j)\right] + \ldots$$
$$I_4 = \epsilon^{-4}\left[(-13.294034089) \pm (2.068 \cdot 10^{-5})\right] + \ldots$$
$$I_5 = \epsilon^{-4}\left[(12.7432949988 - 23.561968275j) \pm (1.605 \cdot 10^{-5} + 1.415 \cdot 10^{-5}j)\right] + \ldots$$
$$I_6 = \epsilon^{-4}\left[(-4.0702330904) \pm (2.018 \cdot 10^{-6})\right] + \ldots$$

Agrees with analytic result

$$\begin{aligned} I &= 4\ (I_1 + I_2 + I_3 + I_4 + I_5 + I_6) \\ &= \epsilon^{-4}\left[8.340\mathbf{55} - 52.36\mathbf{08}j\right] + \mathcal{O}(\epsilon^{-3}) \\ I_{\text{analytic}} &= \epsilon^{-4}\left[8.3400403922 - 52.3598775598j\right] + \mathcal{O}(\epsilon^{-3}) \end{aligned}$$

Note: even after resolution this integral is slow to compute numerically, possible to vastly improve performance by avoiding contour deformation entirely

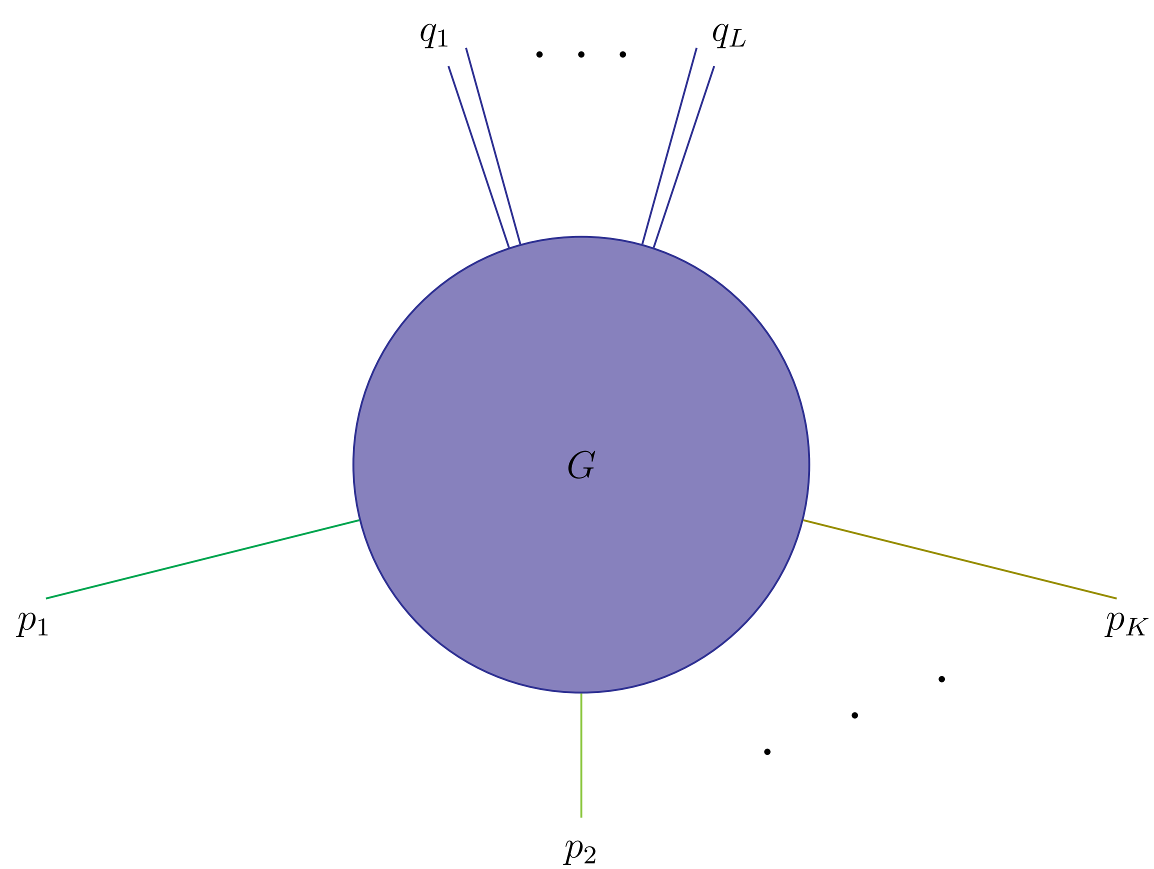
→ **We will return to this point shortly**

MoR and Hidden Regions

On-Shell Expansion

On-shell expansion provides a way to explore emergence of IR singularities starting from an object free of IR singularities (off-shell Green's function)

Consider an arbitrary loop, $(K+L)$-leg wide-angle scattering graph



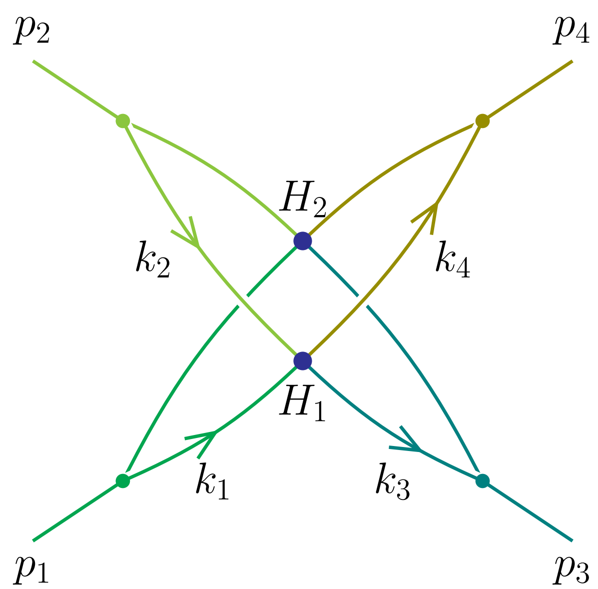
on-shell: $p_i^2 \sim \lambda Q^2 \quad (i = 1,\ldots,K),$

off-shell: $q_j^2 \sim Q^2 \quad (j = 1,\ldots,L),$

wide-angle: $p_k \cdot p_l \sim Q^2 \quad (k \neq l)\,.$

Cancellations of the type just observed lead to new regions that are *hidden* in the straightforward Newton polytope approach as they do not originate from an end-point singularity

On-Shell Expansion



Consider a collinear/jet configuration

$$p_i^2 = \lambda Q^2, \quad p_i \cdot v_i \sim \lambda Q, \quad p_i \cdot \bar{v}_i \sim Q, \quad p_i \cdot v_{i\perp} \sim \sqrt{\lambda} Q$$

Let us introduce a fourth (extra) loop momentum and consider the mode with all k_i collinear to p_i

$$k_i^\mu = Q\left(\xi_i v_i^\mu + \lambda \kappa_i \bar{v}_i^\mu + \sqrt{\lambda}\tau_i u_i^\mu + \sqrt{\lambda}\nu_i n^\mu\right)$$

Botts, Sterman 89

Momentum conservation at H_1 vertex ($k_1 + k_2 = k_3 + k_4$) implies not all ξ_i are independent:

$$\xi_2 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cos^2(\theta)\left(\tan\left(\frac{\theta}{2}\right)\Delta\tau - \cot\left(\frac{\theta}{2}\right)\Sigma\tau\right) + \lambda(\kappa_2 - \kappa_1),$$

$$\xi_3 = \xi_1 + \frac{1}{2}\sqrt{\lambda}\tan\left(\frac{\theta}{2}\right)\Delta\tau + \lambda(\kappa_2 - \kappa_4),$$

$$\xi_4 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cot\left(\frac{\theta}{2}\right)\Sigma\tau + \lambda(\kappa_2 - \kappa_3)\,.$$

$$\Delta\tau \equiv \tau_1 + \tau_2 - \tau_3 - \tau_4$$
$$\Sigma\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4$$

On-Shell Expansion

Now let us analyse the leading behaviour of this integrand for small λ,

1) Loop measure can be expressed as $\int d^Dk_1 d^Dk_2 d^Dk_3 = Q^{3D} \int \prod_{i=1}^{3} d\xi_i d\kappa_i d\tau_i d\nu_i$
2) Trade large components of k_2, k_3 for small components of k_4, $\{\xi_2, \xi_3\} \to \{\kappa_4, \tau_4\}$

   Jacobian of transformation: $\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right) = \lambda^{3/2}\cos(\theta)\cot(\theta)$

Overall obtain the following scaling:

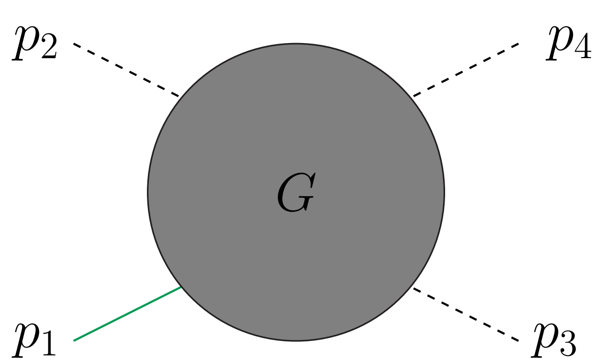
$$\int \prod_{i=1}^{3} d\xi_i \; d\kappa_i d\tau_i d\nu_i \sim \int_0^1 d\xi_1 \underbrace{\left(\int \prod_{i=1}^{3} (\lambda d\kappa_i)(\lambda^{\frac{1}{2}} d\tau_i)(\lambda^{\frac{1}{2}} d\nu_i)^{1-2\epsilon}\right)}_{\lambda^{6-3\epsilon}} \int d\kappa_4 d\tau_4 \; \underbrace{\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right)}_{\lambda^{3/2}}$$

Expect this region to scale as $\boxed{\mu = 6 - 3\epsilon + \frac{3}{2} - 8 = -\frac{1}{2} - 3\epsilon}$

Scaling of collinear propagators

On-Shell Expansion

Directly applying MoR in parameter space, we do not see this region…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	order
$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(-1,-1,-1,0,-1,0,-1,0;1)$	$1-3\epsilon$
$(-1,-1,0,-1,0,-1,0,-1;1)$	$1-3\epsilon$
$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	0

Dissecting the polytope according to our resolution procedure eliminates monomials of different sign, we now see the region in each of the 24 new polytopes

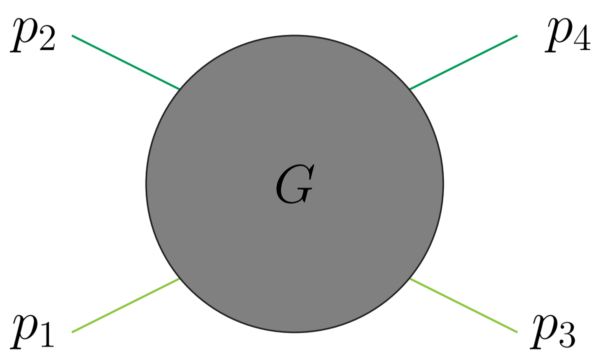
$I_1 \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	order
$(1/2,-1,1/2,-1,1/2,-1,0,-1;1)$	$(-2,-2,-2,-2,-2,-2,-2,-2;2)$	$-1/2-3\epsilon$
$(0,-1,1,-1,1,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(1,-1,1,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(1,-2,1,-2,1,-2,1,-2;1)$	$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(0,-1,0,0,0,0,0,0;1)$	$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

$\mu = -\frac{1}{2} - 3\epsilon$

→ **See Yao's talk**

Forward Scattering



Inserting $\theta \sim \sqrt{\lambda}$ into the Botts-Sterman analysis leads to one of the loop momenta becoming Glauber:

$$k_4^\mu - k_2^\mu = k_1^\mu - k_3^\mu \sim Q(\lambda, \lambda; \sqrt{\lambda})$$

We obtain $\mu = -1 - 3\epsilon$

Alternatively, can expand known analytic result in the foward limit $x = -s_{13}/s_{12}$

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21;

$$I(s_{12}, s_{13}; \epsilon) = s_{12}^{-2-3\epsilon} \mathcal{I}(x; \epsilon), \quad \mathcal{I}(x; \epsilon) \sum_{n=-4}^{\infty} \mathcal{I}^{(n)}(x)\, \epsilon^n = \sum_{n=-4}^{\infty} \sum_{k=-1}^{\infty} \mathcal{I}^{(n,k)}(L)\, x^k\, \epsilon^n$$

$L = \log(x)$

$$\mathcal{I}(x; \epsilon) = \mathrm{LP}\left\{I_{\mathrm{XX}}\right\}(L; \epsilon) + \mathcal{O}(x^0)$$

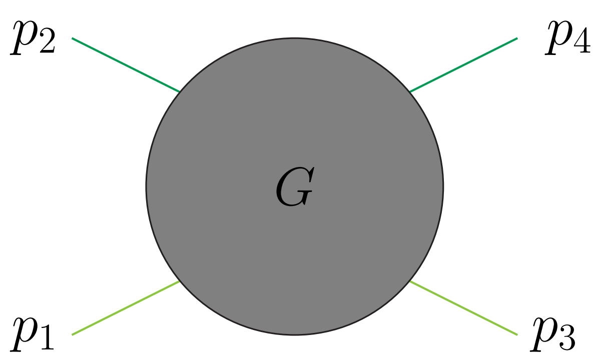
$$\mathrm{LP}\{\mathcal{I}\}(L; \epsilon) = i\pi x^{-1-3\epsilon}\Bigg(-\frac{8}{3\epsilon^4} + \frac{16}{\epsilon^3} + \frac{2\left(\pi^2 - 144\right)}{3\epsilon^2} - \frac{4\left(-58\zeta(3) + 3\pi^2 - 432\right)}{3\epsilon}$$

$$+ \frac{1}{60}\left(-27840\zeta(3) + 71\pi^4 + 1440\pi^2 - 207360\right) + \cdots \Bigg),$$

gives $\mathcal{I}(x; \epsilon) \sim x^{-1-3\epsilon}$

Forward Scattering

Directly applying MoR in parameter space, no region with correct scaling…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(-1,-1,-1,0,-1,-1,-1,0;1))$	-3ϵ
$(-1,-1,0,-1,-1,-1,0,-1;1)$	-3ϵ
$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,-1,-1,-1,0,-1,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	0

After resolution, in some polytopes we now directly see the leading region observed in the analytic result!

$I_1 \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(0,-1,0,-1,0,-1,1,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(1,-1,0,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(-1,0,0,-1,-1,0,0,-1;1)$	$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

Avoiding Contour Deformation in the Minkowski Regime

Minkowski Regime

Several conflicting definitions of the term *Minkowski regime* for Feynman Integrals

In the remainder of this talk I will use the following conventions:

(Pseudo-)Euclidean

$\mathscr{F}(\boldsymbol{\alpha}) \geq 0$ for $\boldsymbol{\alpha} \in \mathbb{R}^N_{\geq 0}$

Minkowski

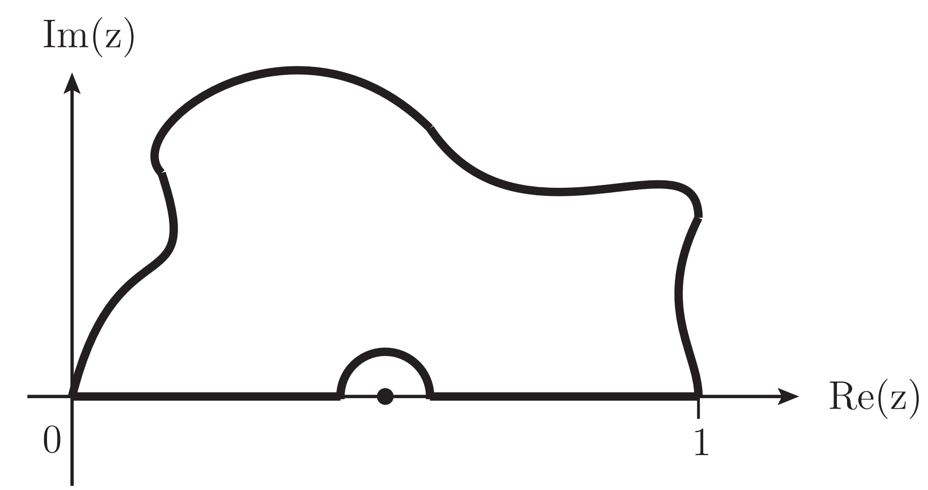
Not Euclidean/Pseudo-Euclidean

We can have $\mathscr{F}(\boldsymbol{\alpha}) < 0$ for some values of $\boldsymbol{\alpha} \in \mathbb{R}^N_{\geq 0}$

Contour Deformation

Feynman integral (after integrating δ-func.):

$$I \sim \int_0^1 [\mathrm{d}\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \frac{[\mathcal{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})]^{N-LD/2}}$$



Deform our integration contour to avoid poles on real axis

Feynman prescription $\mathcal{F} \to \mathcal{F} - i\delta$ tells us how to do this

Expand $\mathcal{F}(z = \boldsymbol{\alpha} - i\boldsymbol{\tau})$ around $\boldsymbol{\alpha}$, $\quad \mathcal{F}(z) = \mathcal{F}(\boldsymbol{\alpha}) - i \sum_j \tau_j \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j} + \mathcal{O}(\tau^2)$

Choose $\tau_j = \lambda_j\, \alpha_j (1 - \alpha_j) \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j}$ with small constants $\lambda_j > 0$

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;…

Can also generalise $\lambda_j \to \lambda_j(\boldsymbol{\alpha})$ and train the deformation with a Neural Network

Winterhalder, Magerya, Villa, SJ, Kerner, Butter, Heinrich, Plehn 22

Contour Deformation

Downsides of contour deformation:

1. Real valued integrand $\rightarrow$ complex valued integrand (slower numerics)
2. Large and complicated Jacobian from $\mathbf{x} \rightarrow \mathbf{z}$ (can be optimised)
   Borinsky, Munch, Tellander 23
3. Increases variance of function (integrand can be both > 0 and < 0)
4. Sensitive to choice of contour
5. Sometimes fails analytically and/or numerically

Summary: it is **slow, arbitrary** and can **fail**

Can we find a way to avoid contour deformation? **Yes**

Always? **I don't know***

NoCD: Avoiding Contour Deformation

Idea:

1. Construct transformations of the Feynman parameters which map the zeroes of the $\mathscr{F}$-polynomial to the boundary of integration

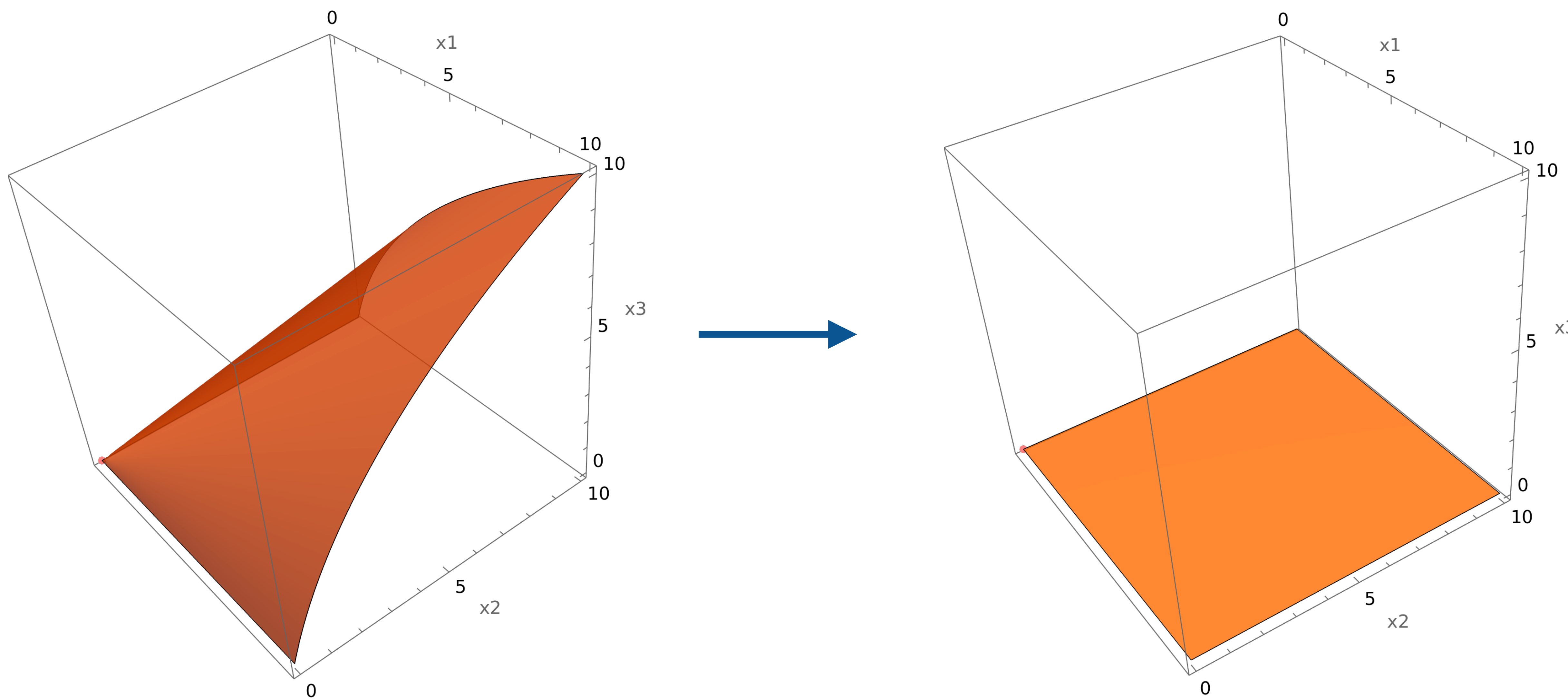


Figure: Thomas Stone

NoCD: Avoiding Contour Deformation

Idea:

2. For transformations which make $\mathscr{F}$ non-positive extract an overall minus sign (using the $i\delta$ prescription to generate the physically correct imaginary part)

3. Stitch together the resulting integrals

$$I = \sum_{n_+=1}^{N_+} I^+_{n_+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} I^-_{n_-}$$

The individual integrals $\{I^+_{n_+}, I^-_{n_-}\}$ have *manifestly* non-negative integrands

$\Longrightarrow$ no contour deformation, trivial analytic continuation, faster to integrate

NoCD: Avoiding Contour Deformation

Rules of the Game:

1. Transformations must not spoil the δ-func. constraint

   Cheng-Wu Theorem:

$$\forall S \subseteq \{1,\ldots,N\} \wedge S \neq \emptyset : \qquad \delta\left(1 - \sum_{j=1}^{N} \alpha_j\right) \to \delta\left(1 - \sum_{j \in S} \alpha_j\right)$$

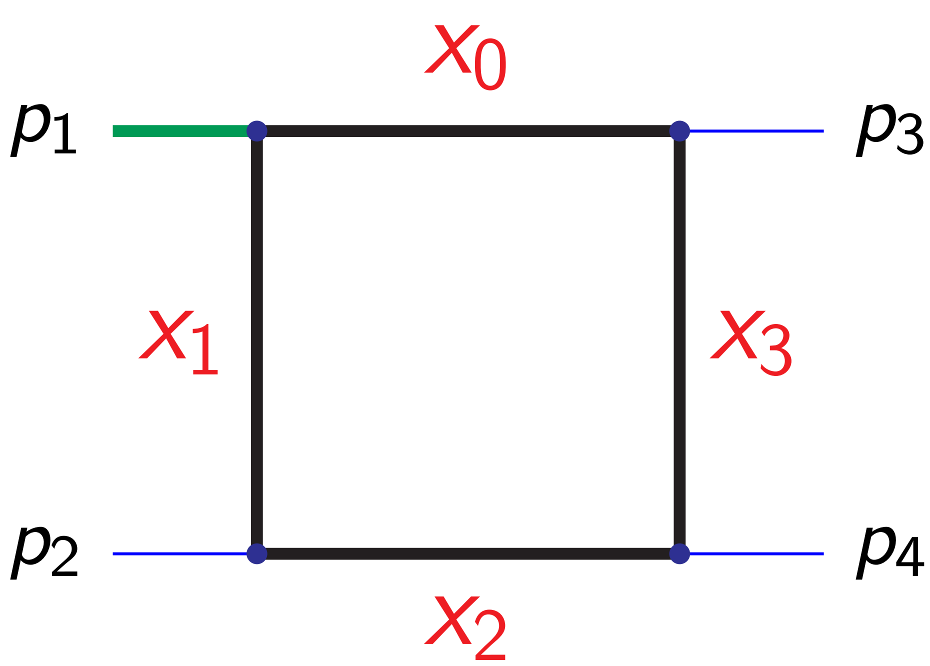
2. Transformations must preserve the sign of $\mathcal{U} \geq 0$
3. Jacobian $\mathcal{J}$ of the transformation must have a definite sign

We found the following transformations useful:

1. Rescaling: $\alpha_j \to c\alpha_j$ with $c > 0$
2. Blow-up: $\alpha_j \to \alpha_i \alpha_j$
3. Decomposition: $x_j \to x_i + x_j$

$1 = \theta(\alpha_a - \alpha_b) + \theta(\alpha_b - \alpha_a)$

NoCD: Example 1



- $\mathcal{U} = x_0 + x_1 + x_2 + x_3$
- $\mathcal{F} = -s x_0 x_2 - t x_1 x_3 - p_1^2 x_0 x_1$

Let's consider the regime: $s > 0$, $p_1^2 > 0$ & $t < 0$ $\Rightarrow$ zeroes of $\mathcal{F}$ *within* the integration volume for $\{x_0, x_1, x_2, x_3\} \in \mathbb{R}^4_{>0}$

Slide: Thomas Stone (Loops & Legs 2024)

Convention: x is now a Feynman parameter

NoCD: Example 1

$$\mathscr{F} = -\,s x_0 x_2 + |t| x_1 x_3 - p_1^2 x_0 x_1$$

$x_0 \to \frac{x_0 x_1}{s},\ x_3 \to \frac{x_2 x_3}{|t|}$

$$\mathscr{F} \to x_1 \left(x_2 \left(x_3 - x_0 \right) - \frac{p_1^2}{s} x_0 x_1 \right)$$

$x_0 > x_3:\ \ x_0 \to x_0 + x_3$

$$\mathscr{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\,\mathscr{F}_1^-$$

$x_3 > x_0:\ \ x_3 \to x_3 + x_0$

$$\mathscr{F} \to x_1 \left(-\frac{p_1^2}{s} x_0 x_1 + x_2 x_3 \right)$$

$x_2 \to \frac{p_1^2 x_0 x_2}{s},\ x_1 \to x_1 x_3$

$$\mathscr{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 \left(x_2 - x_1 \right)$$

$x_2 > x_1:\ \ x_2 \to x_2 + x_1$

$$\mathscr{F} \to \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathscr{F}_1^+$$

$x_1 > x_2:\ \ x_1 \to x_1 + x_2$

$$\mathscr{F} \to -\frac{p_1^2}{s} x_0 x_1 \left(x_1 + x_2 \right) x_3^2 =: -\,\mathscr{F}_2^-$$

NoCD: Example 1

Generate $\mathscr{U}_1^+, \mathscr{U}_1^-, \mathscr{U}_2^-$ by applying the same transformations to $\mathscr{U}$

Compute the Jacobian determinants of the transformations $\mathscr{J}_1^+, \mathscr{J}_1^-, \mathscr{J}_2^-$

Each new integral is of the form:

$$I_{n_\pm}^\pm \sim \mathscr{J}_{n_\pm}^\pm \left(\mathscr{U}_{n_\pm}^\pm\right)^{2\varepsilon} \left(\mathscr{F}_{n_\pm}^\pm\right)^{-2-\varepsilon}$$

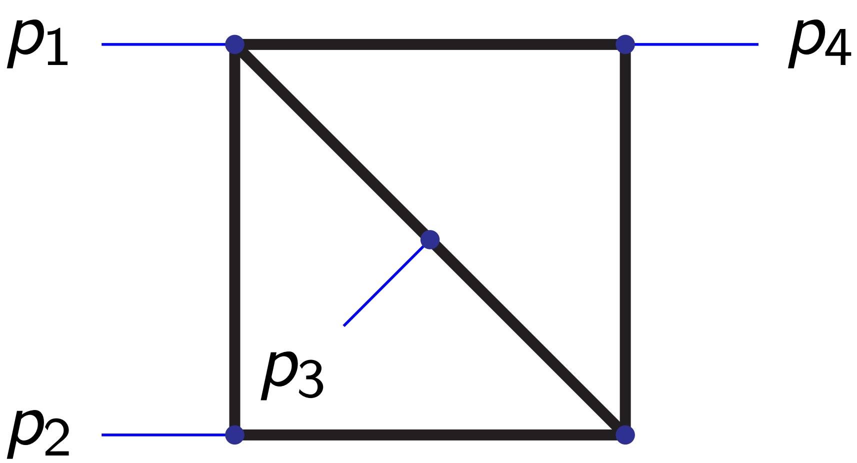
with *manifestly non-negative integrand*

We have converted the initial integral into sum of 3 integrals:

$$\boxed{I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon}\left(I_1^- + I_2^-\right)}$$

Verified result numerically against known analytic result

NoCD: Example 2



- $\mathcal{U} = x_0x_1 + x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_3 + x_1x_5 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$
- $\mathcal{F} = -sx_1x_2x_5 - tx_0x_1x_3 - ux_0x_2x_4$

Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$
Hence, $\mathcal{F}$ can be 0 *within* $\{x_i\} \in \mathbb{R}^6_{>0}$ even with $s > 0, t > 0$
Not possible to define a Euclidean region at all!
Nevertheless, the method works

Slide: Thomas Stone (Loops & Legs 2024)

NoCD: Example 2

We considered the cases:

1. $s > -t$
2. $s < -t$

We obtain *different* resolutions for each case

Nevertheless, *i*n each case we find we need 6 integrals to cover the space:

$$I = \left(I_1^+ + I_2^+ + I_3^+\right) + (-1 - i\delta)^{-2-2\varepsilon}\left(I_1^- + I_2^- + I_3^-\right)$$

Verified result numerically against known analytic result

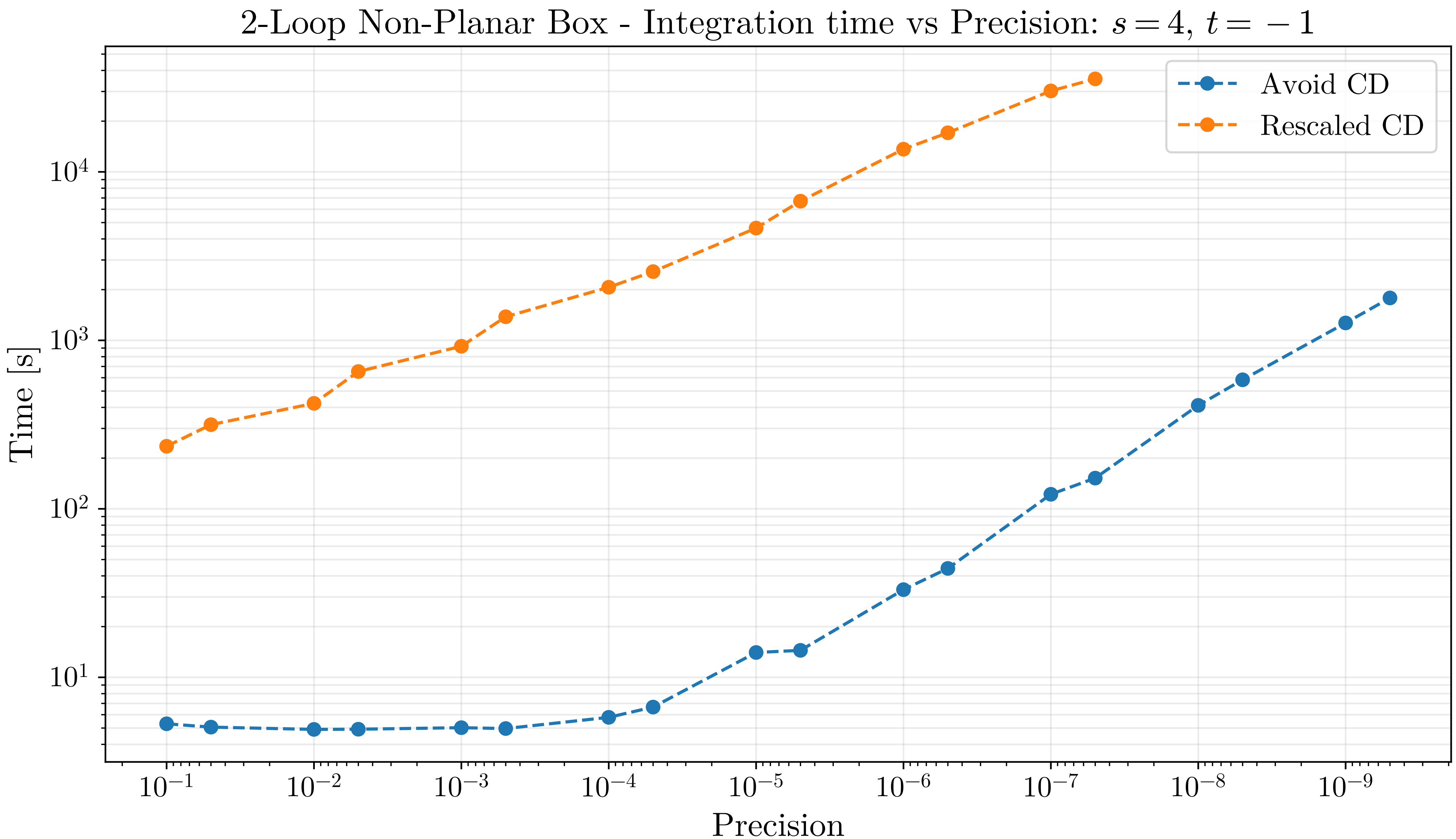
Tausk 99

Let's take a look at the time taken to numerically integrate this example…

NoCD: Example 2

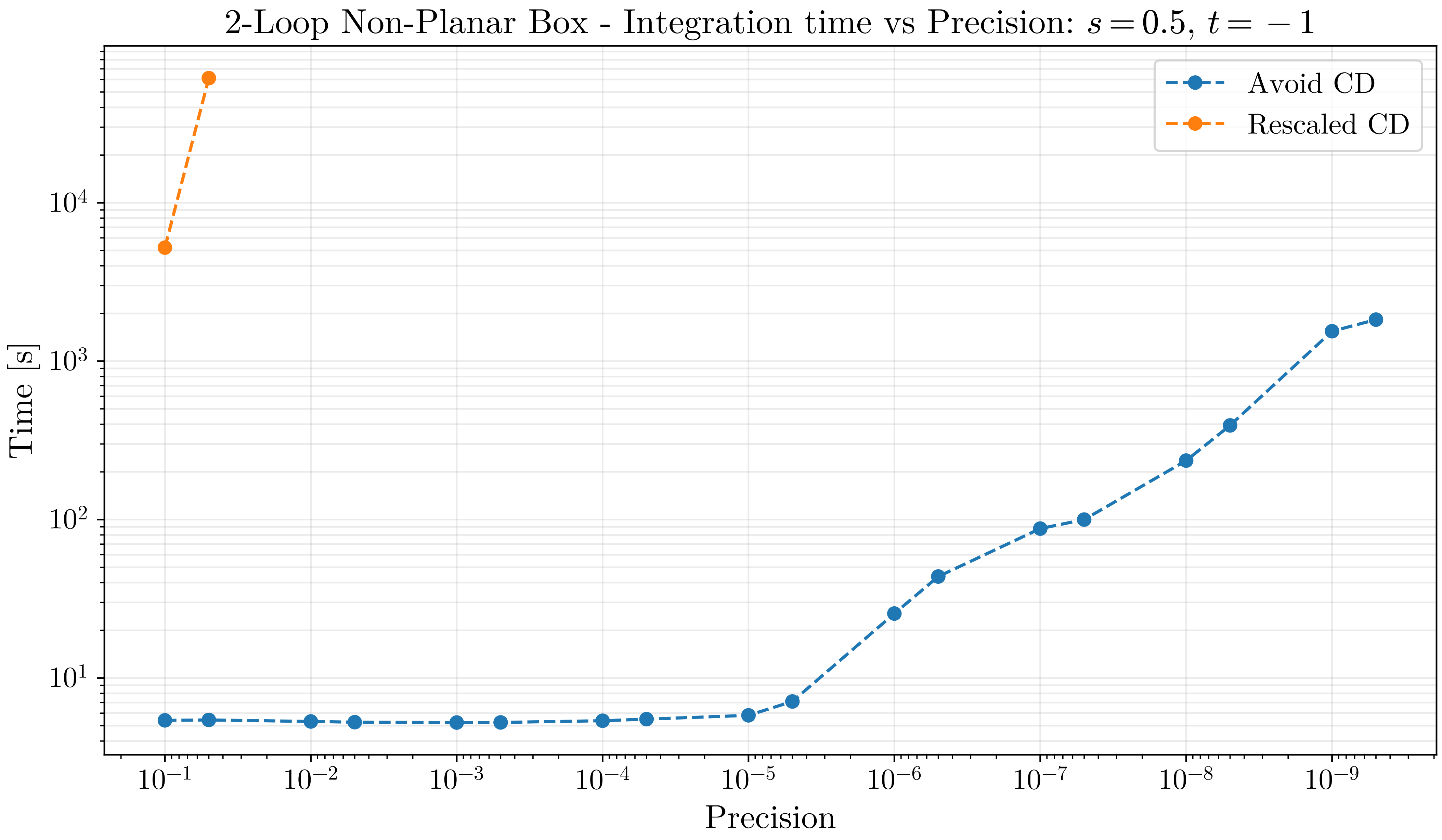
Evaluating up-to-and-including finite order with pySecDec

Heinrich, SPJ,
Kerner, Magerya,
Olsson, Schlenk 23



NoCD: Example 2

Evaluating up-to-and-including finite order with pySecDec



NoCD: Example 3

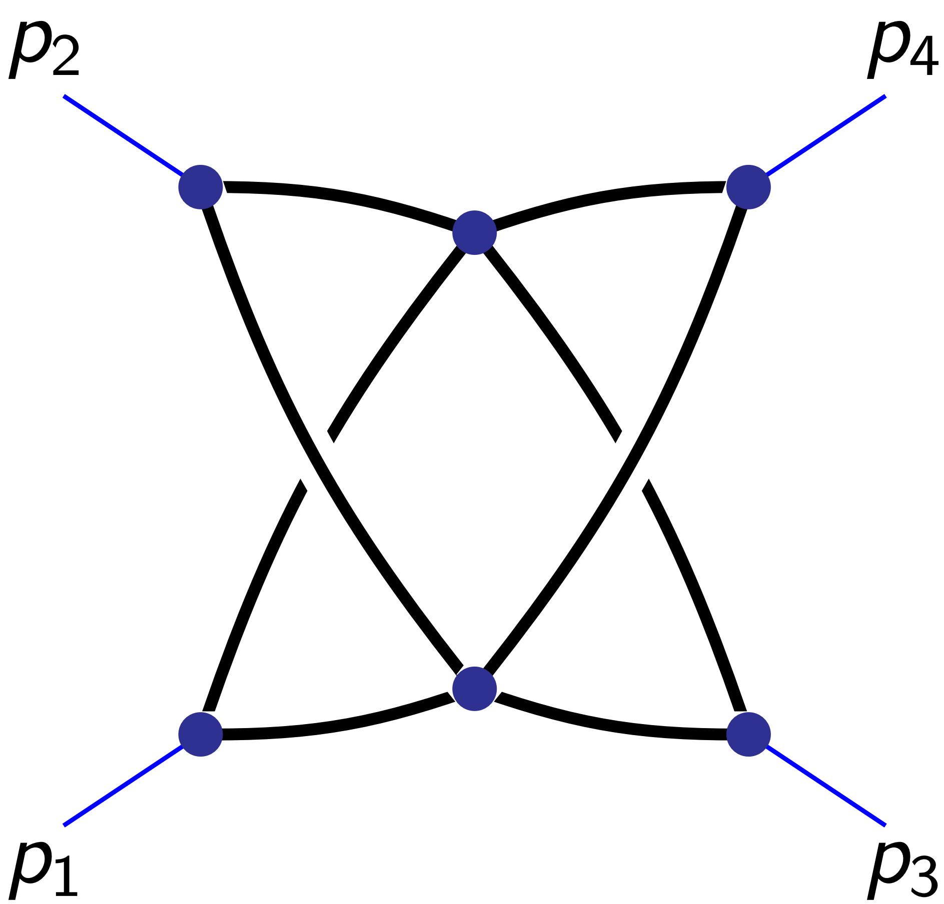


Diagram by Yao Ma

$$\mathcal{F} = -s\,(x_1x_4 - x_0x_5)\,(x_3x_6 - x_2x_7) - t\,(x_1x_2 - x_0x_3)\,(x_5x_6 - x_4x_7)$$

$$\Downarrow$$

$$\mathcal{F} \to -sx_1x_3x_5x_7\,(x_4 - x_0)\,(x_6 - x_2) - tx_1x_3x_5x_7\,(x_2 - x_6)\,(x_6 - x_4)$$

Slide: Thomas Stone (Loops & Legs 2024)

NoCD: Example 3

For $s > -t > 0$, two of the 6 independent integrals require contour deformation:

$$\mathcal{F}_3 = x_1 x_3 x_5 x_7 \left[-s x_0 x_2 + |t| \left(x_0 + x_4 \right) \left(x_2 + x_4 \right) \right]$$

$$\mathcal{F}_5 = x_1 x_3 x_5 x_7 \left[s x_6 \left(x_0 + x_2 + x_6 \right) - |t| \left(x_0 + x_6 \right) \left(x_2 + x_6 \right) \right]$$

Can express each of these in terms of 4 manifestly non-negative integrands

Putting the pieces together for the full integral:

$$I = \sum_{n_+=1}^{8} I^+_{n_+} + (-1 - i\delta)^{-2-3\varepsilon} \sum_{n_-=1}^{4} I^-_{n_-}$$

Verified result numerically against known analytic result

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21

NoCD: Example 3

Can now obtain results numerically ($s_{12} = 1,\ s_{13} = -1/5$)

$$I_3 = \epsilon^{-4}\left[(18.5195704502 - 15.707988011i) \pm (5.897 \cdot 10^{-5} + 5.897 \cdot 10^{-5} i)\right] + \ldots$$

$$I_3^{\text{NoCD}} = \epsilon^{-4}\left[(18.51948920208488 - 15.70796326794897i) \pm (4.032 \cdot 10^{-11} + 4.592 \cdot 10^{-11} i)\right] + \ldots$$

$$I_5 = \varepsilon^{-4}\left[(12.7432949988 - 23.561968275i) \pm (1.605 \cdot 10^{-5} + 1.415 \cdot 10^{-5} i)\right] + \ldots$$

$$I_5^{\text{NoCD}} = \varepsilon^{-4}\left[(12.74326269721394 - 23.5619449018131i) \pm (4.125 \cdot 10^{-11} + 6.919 \cdot 10^{-11} i)\right] + \ldots$$

Full result after a few minutes integration with pySecDec:

$$I = \epsilon^{-4}\left[8.340\mathbf{55} - 52.36\mathbf{08}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

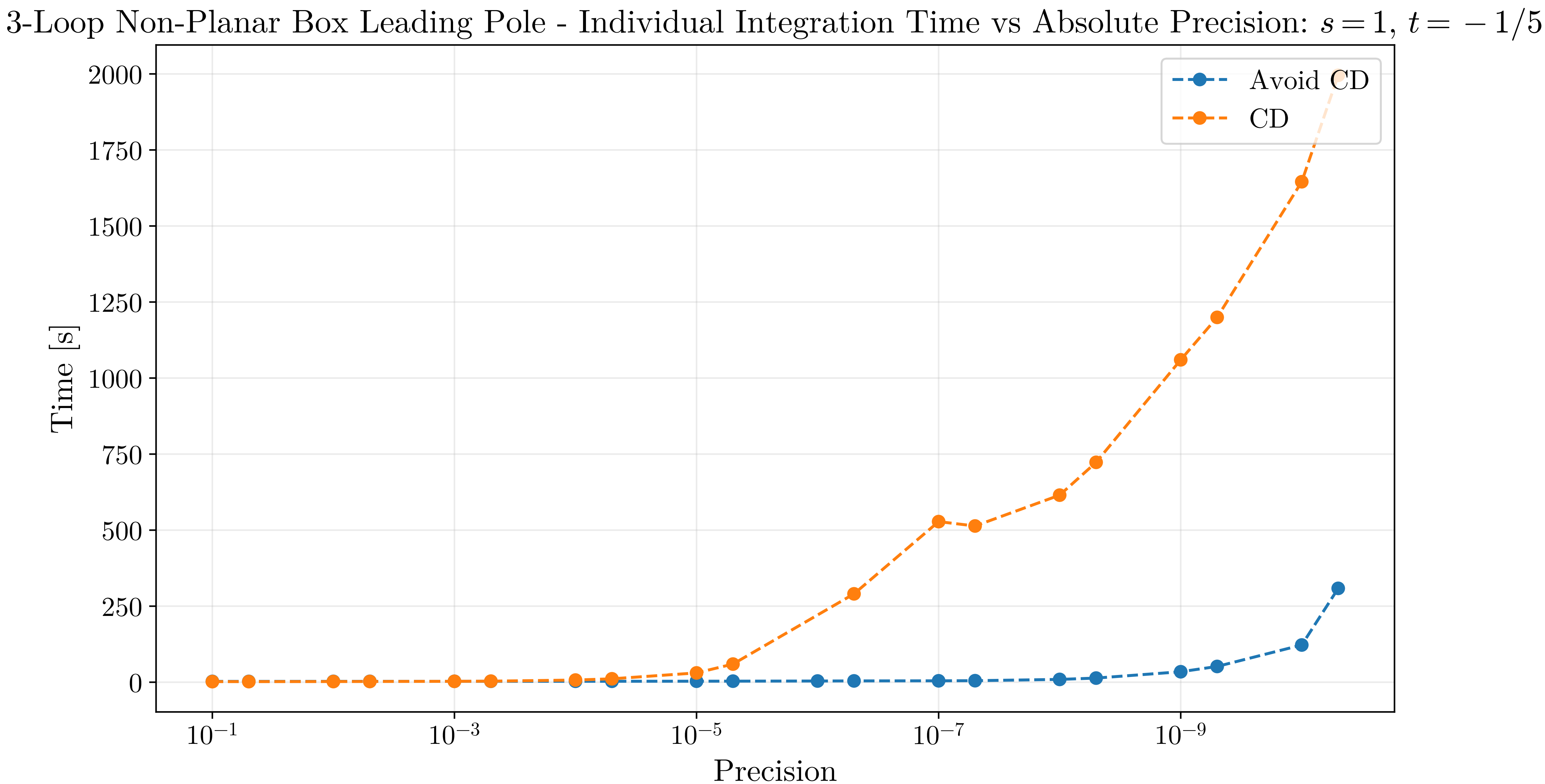
$$I^{\text{NoCD}} = \epsilon^{-4}\left[8.3400403920\mathbf{28} - 52.3598775598\mathbf{347}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

$$I^{\text{analytic}} = \epsilon^{-4}\left[8.34004039223768 - 52.35987755984493i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

Numerics are much, much faster and more stable

NoCD: Example 3

Evaluating leading pole with pySecDec



NoCD: Massive Integrals

Can this work also for massive integrals?

$$\mathscr{F}(\boldsymbol{x};\boldsymbol{s}) = \mathscr{F}_0(\boldsymbol{x};\boldsymbol{s}) + \mathscr{U}_0(\boldsymbol{x}) \sum_{j=1}^{N} m_j^2 x_j$$

Now x_j appears quadratically in $\mathscr{F}$

Transformations harder to find, even for trivial integrals

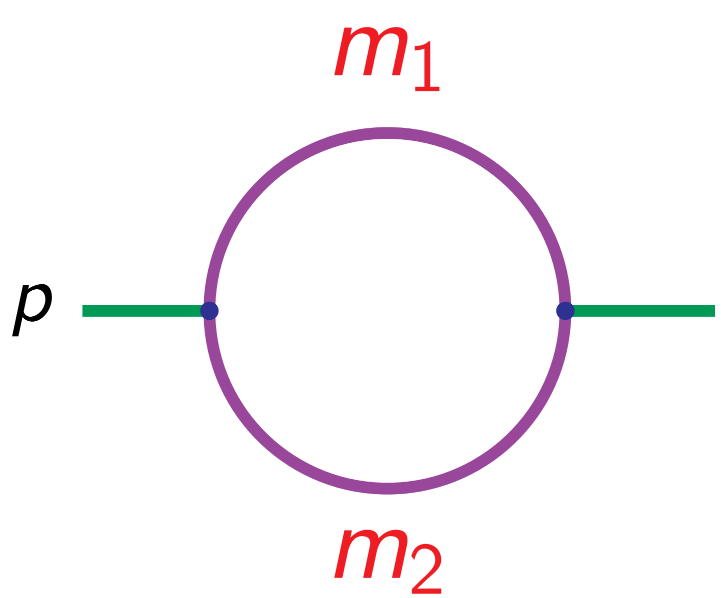
Ideas:

1. Can geometry guide us in the right direction?

2. Is this just singularity resolution? If so, how can we use existing technology?

Hironaka e.g. desing

NoCD: Massive Example 1

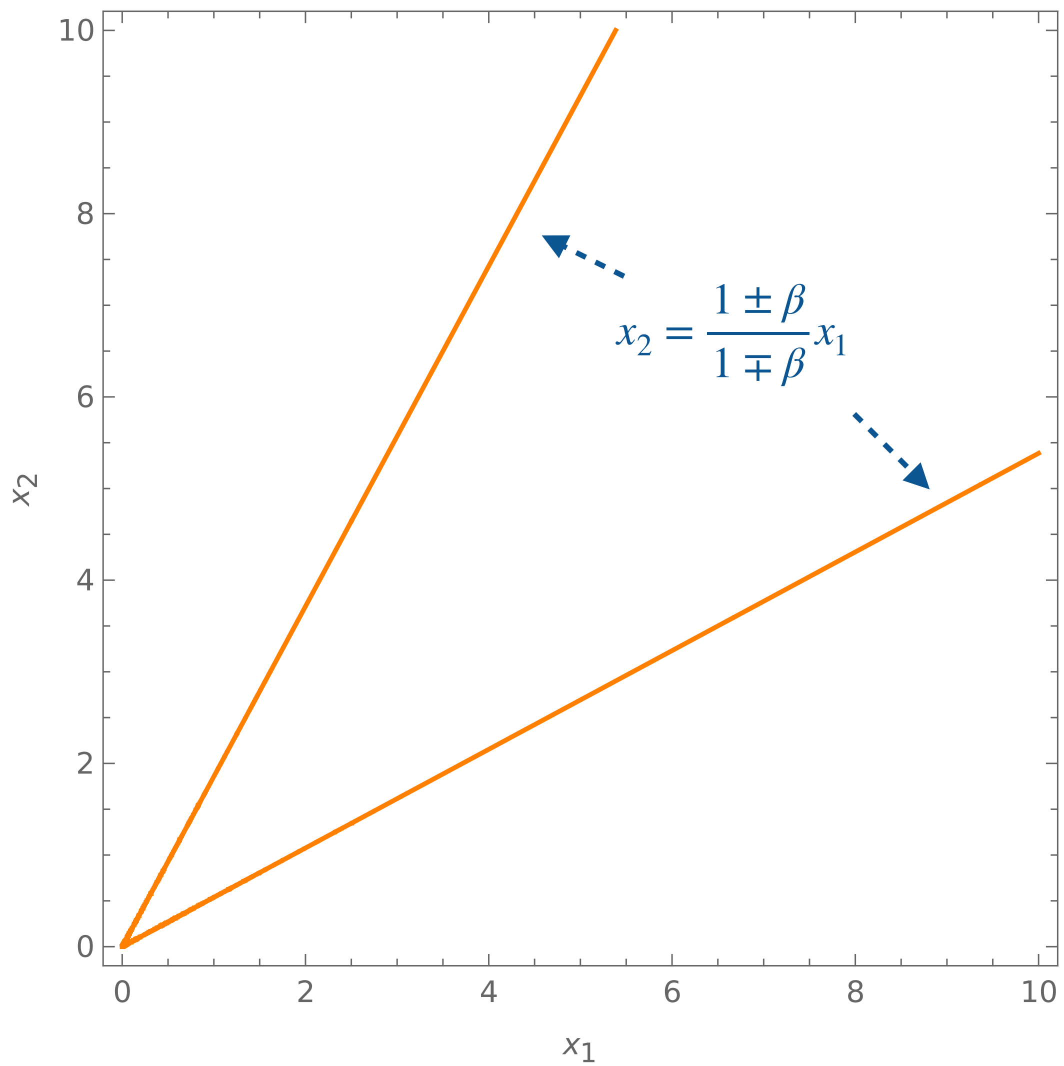


- $\mathcal{F} = -p^2 x_1 x_2 + (x_1 + x_2)\left(m_1^2 x_1 + m_2^2 x_2\right)$
- Define $\beta^2 := \frac{p^2-(m_1+m_2)^2}{p^2-(m_1-m_2)^2} \in [0,1)$
- Scale out dimension of $\mathcal{F}$ via $x_i \to \frac{x_i}{m_i}$

$$\mathcal{F} \to \widetilde{\mathcal{F}} = x_1^2 + x_2^2 - 2\frac{1+\beta^2}{1-\beta^2} x_1 x_2$$

Slide: Thomas Stone (Loops & Legs 2024)

NoCD: Massive Example 1



- Let's consider the variety of $\widetilde{\mathcal{F}}$
- 3 regions $\Rightarrow$ 3 integrals
- 2 positive regions, 1 negative region

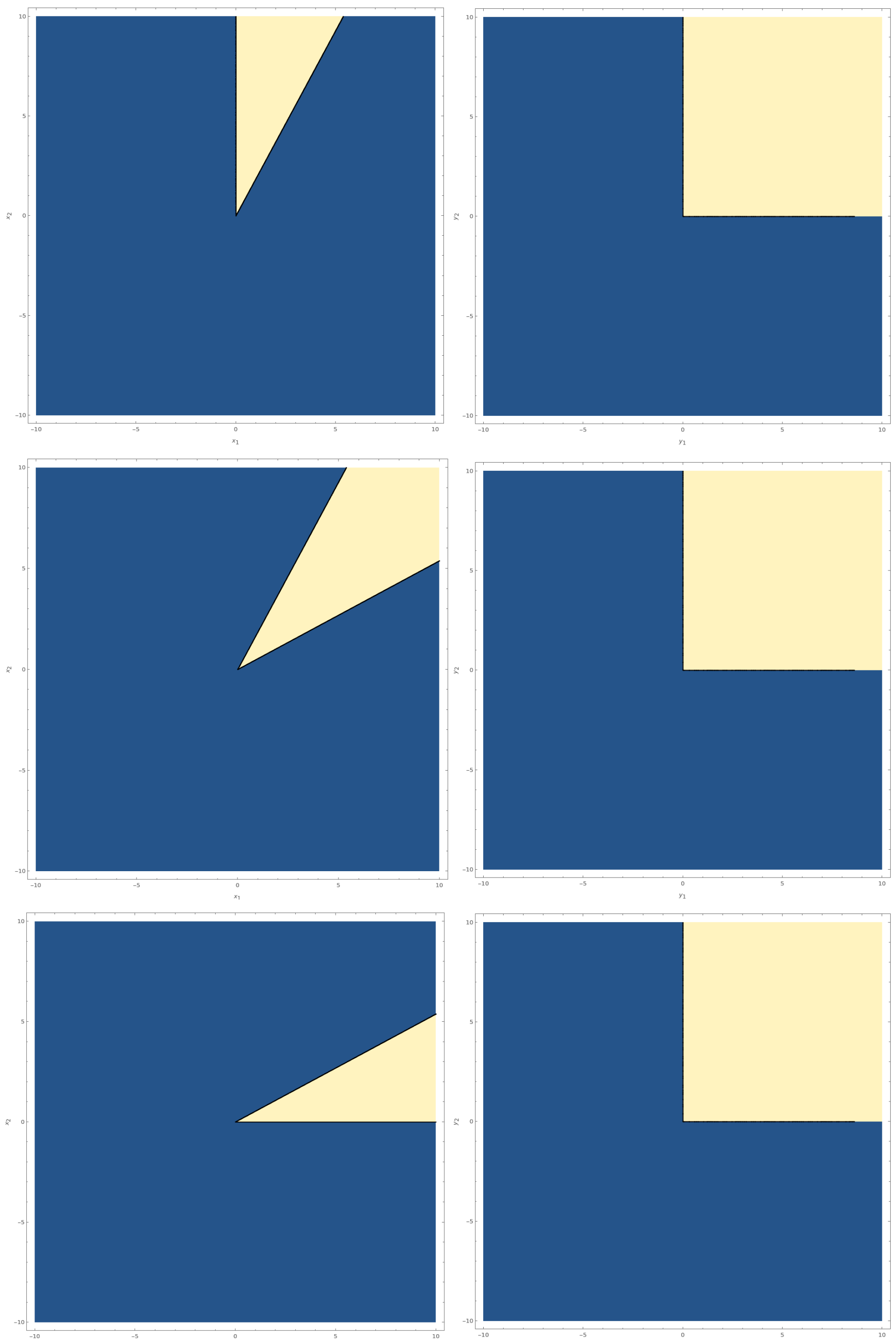
Massive Bubble

$$I = I_1^+ + I_2^+ + (-1 - i\delta)^{-\varepsilon} I_1^-$$

- Construct transformations which directly send the variety to the integration boundary

Slide: Thomas Stone (Loops & Legs 2024)

NoCD: Massive Example 1



$$\widetilde{\mathcal{F}}_1^{+} = y_2 \left(y_2 + \frac{4\beta}{1-\beta^2} y_1 \right)$$

$$\widetilde{\mathcal{F}}_1^{-} = \frac{4\beta}{1-\beta^2} y_1 y_2$$

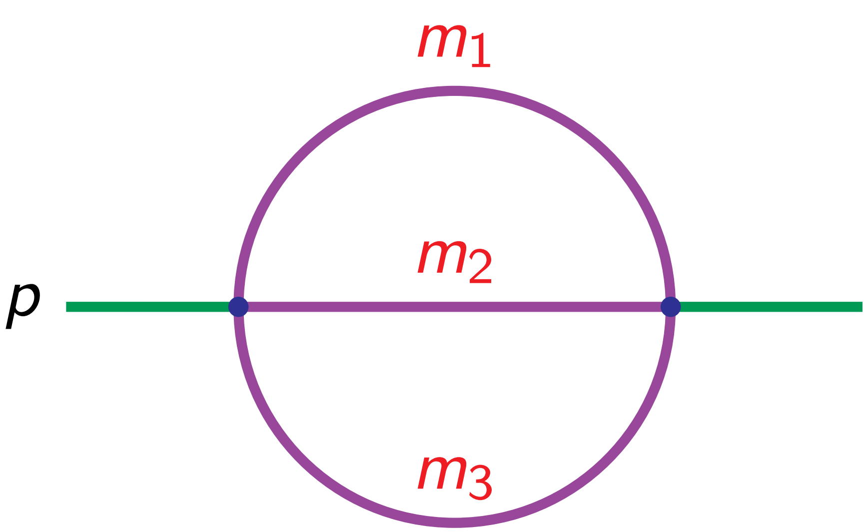
$$\widetilde{\mathcal{F}}_2^{+} = \frac{y_1 \left(4\beta y_2 + (1+\beta)^2 y_1 \right)}{1-\beta^2}$$

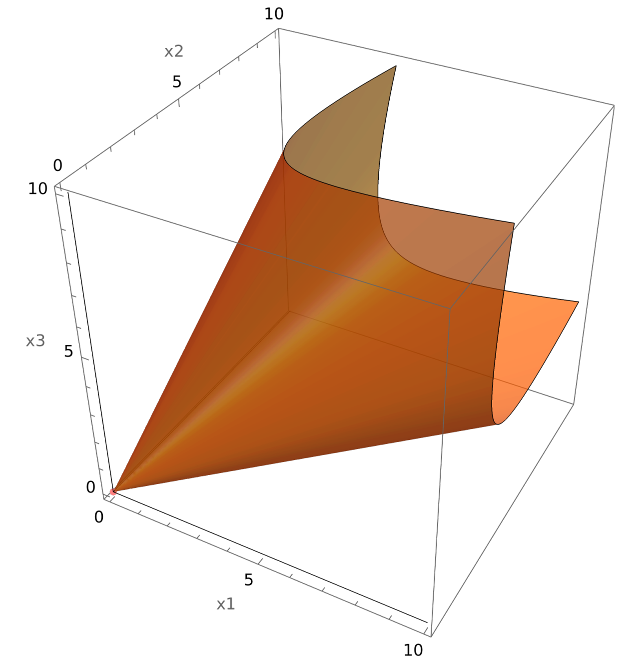
Verified result numerically **&** analytically ✓

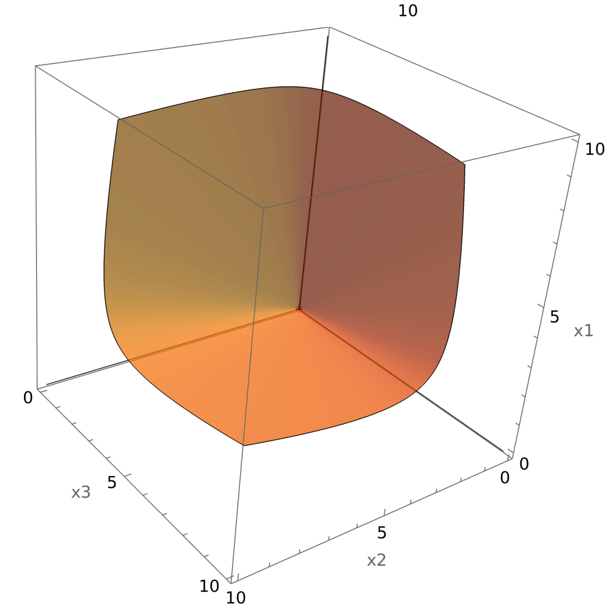
Slide: Thomas Stone (Loops & Legs 2024)

NoCD: Massive Example 2

Less clear how to proceed in more involved cases







Very happy to try smart ideas you have… or see arguments why this wont work

Conclusion

Pinched Feynman Integrals

- Studied an integral with a *pinched* contour independent of kinematics
- Found a resolution procedure to remove the pinch
- Can obtain stable numerical results only after removing pinch

MoR

- Expect regions can appear due to cancelling monomials either generically or at particular kinematic points

NoCD

- Presented method for evaluating integrals in the Minkowski regime without contour deformation
- Demonstrated procedure for some 1,2,3-loop massless & 1-loop massive integrals

Outlook

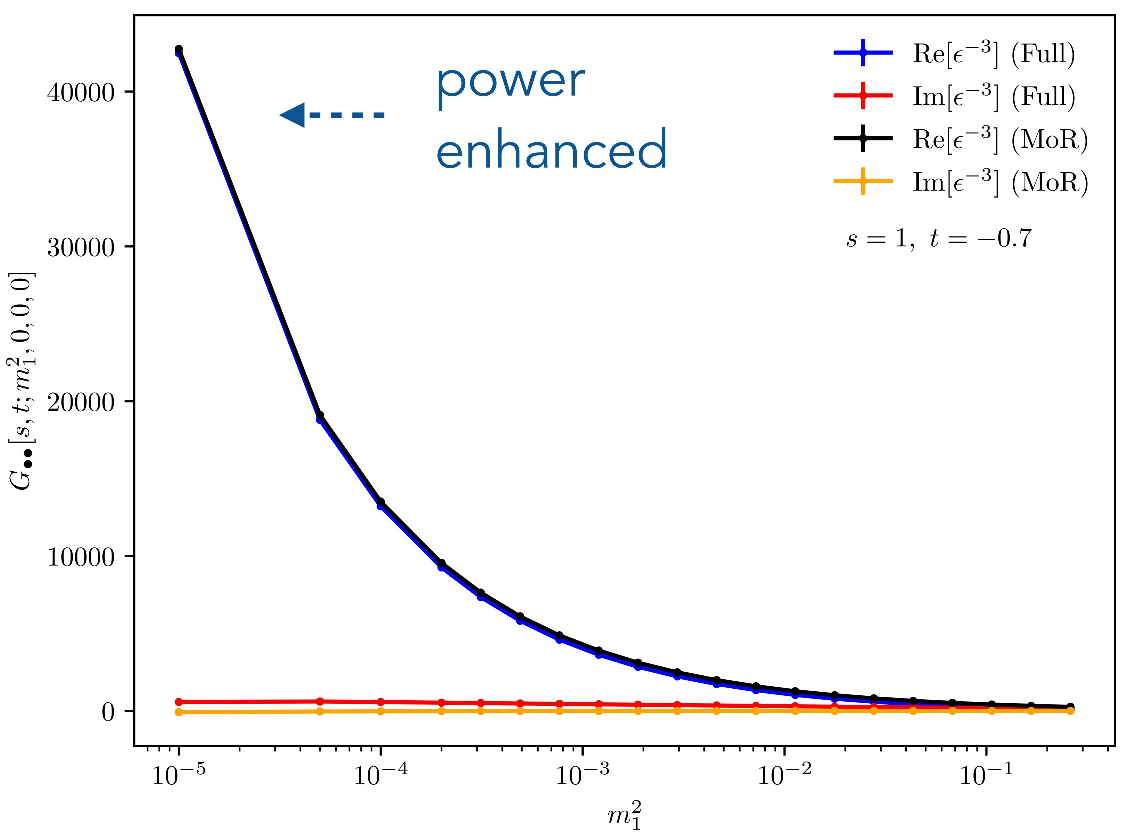
- General/automated procedure to resolve pinches and/or zeros of $\mathscr{F}$?

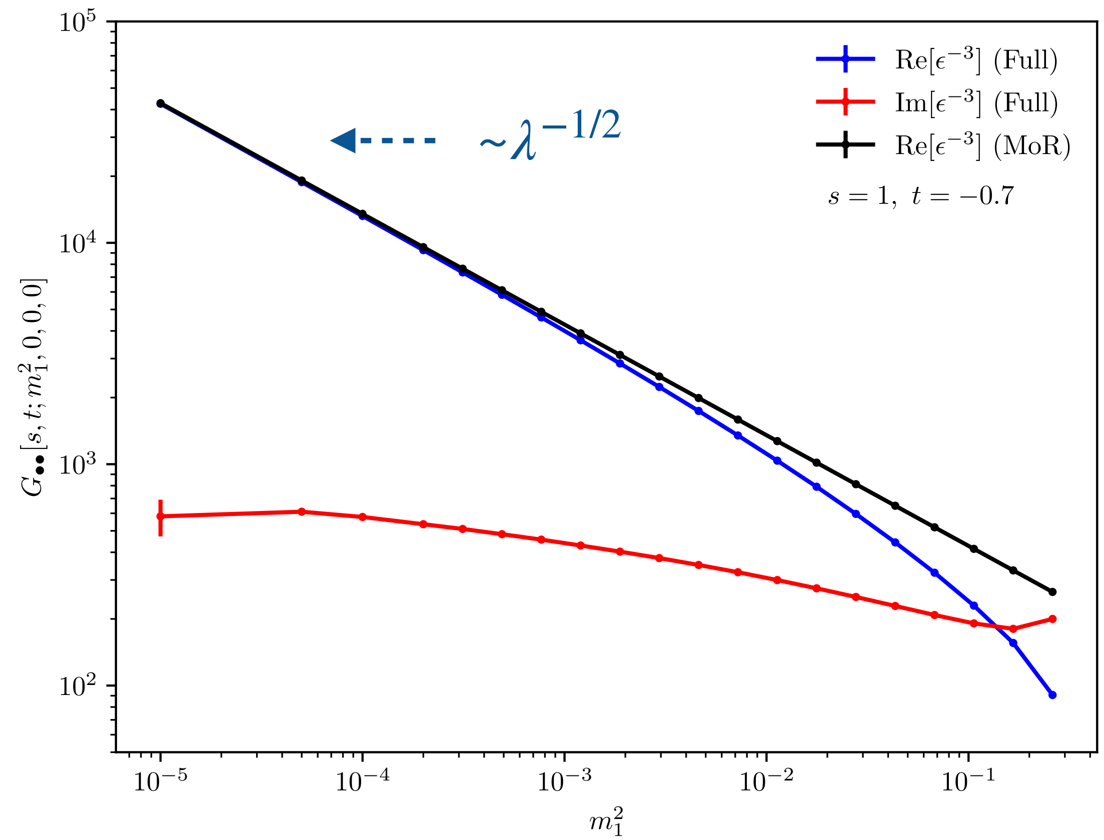
Thank you for listening!

Backup

On-Shell Expansion

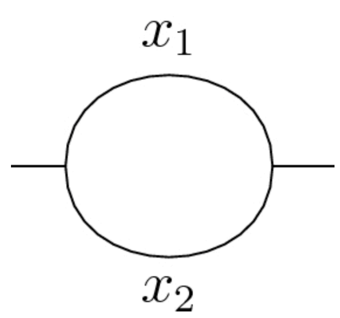
Use MoR on each of the split integrals $I_1, \ldots, I_{24}$ and summing only the leading region for each split (with $\mu = -1/2 - 3\epsilon$)





See strong numerical evidence that the split integrals (MoR) reproduce the leading behaviour of the full integral in the limit $p_1^2 \to 0$

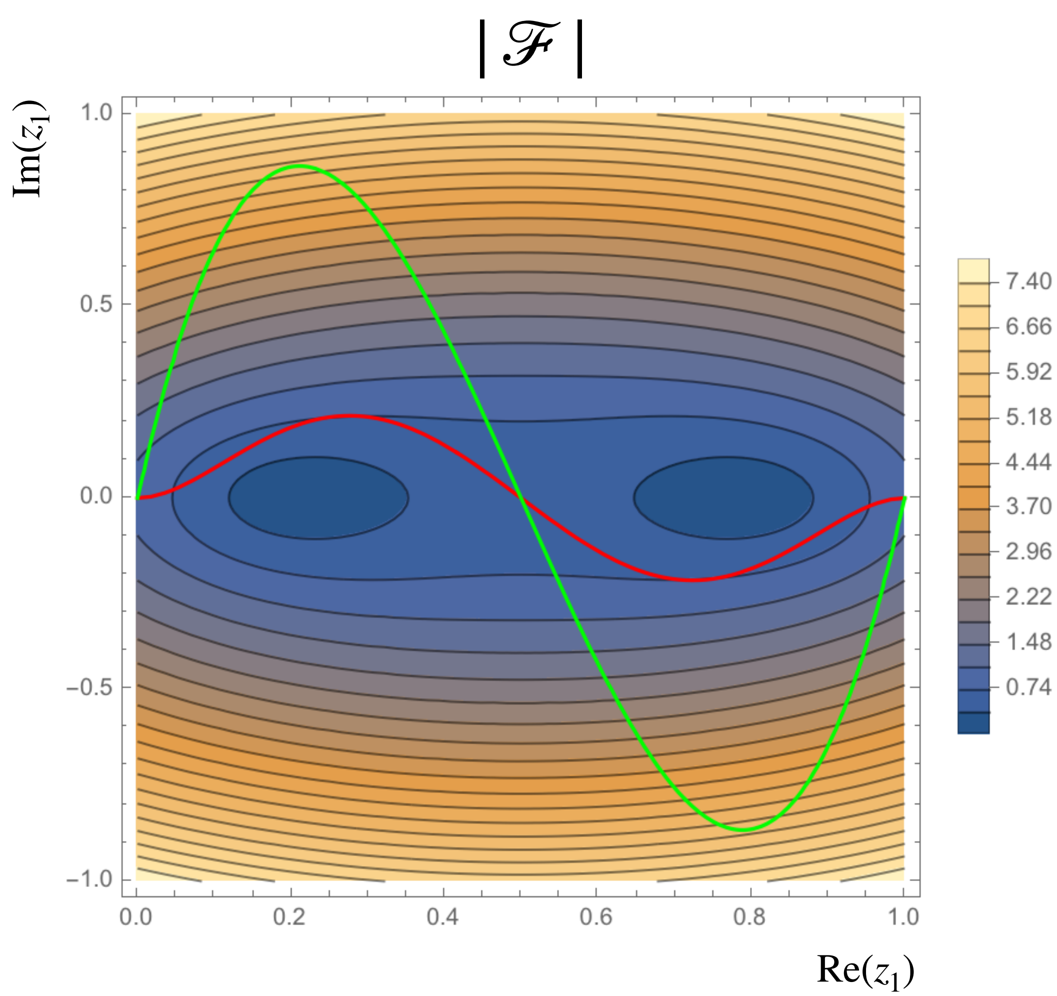
Contour Deformation

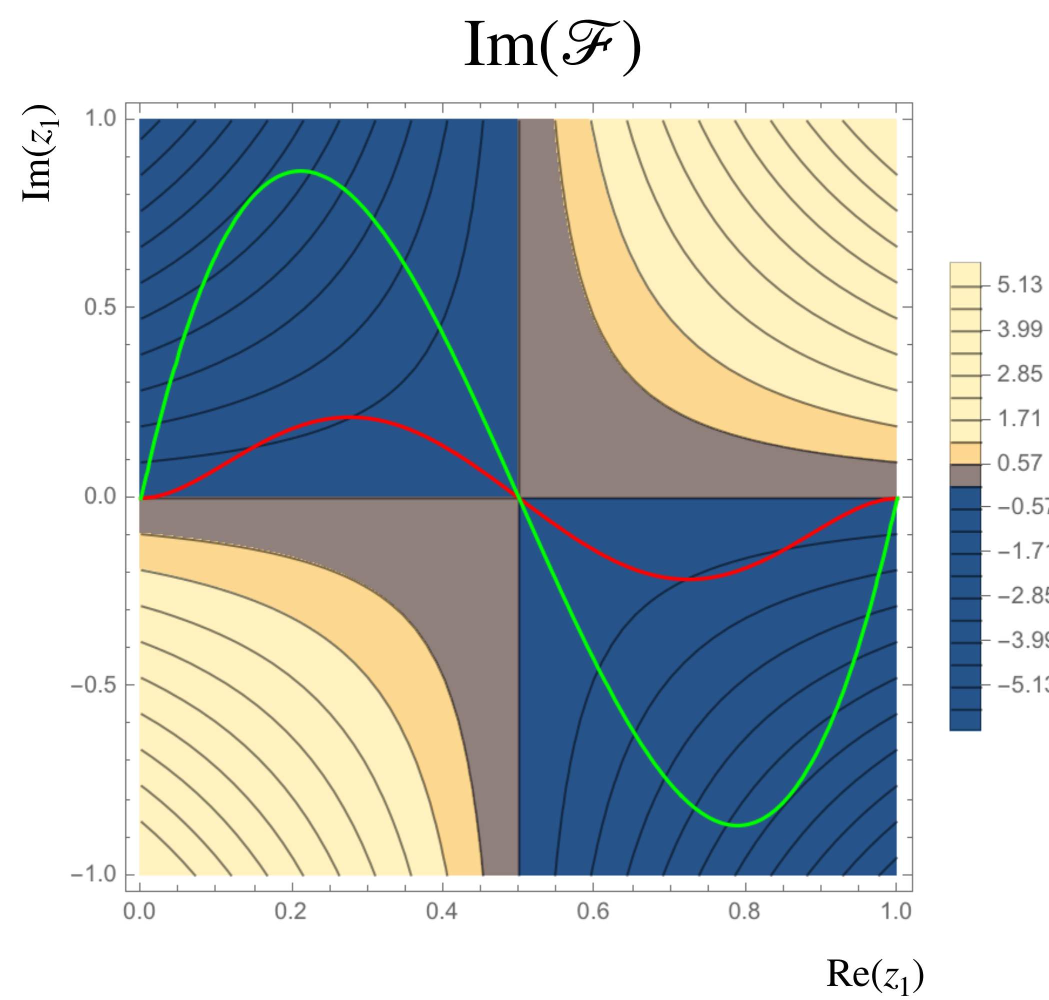


$$= \int_0^\infty \mathrm{d}x_1\, \mathrm{d}x_2\, \frac{\mathscr{U}(\mathbf{x})^{-2+2\epsilon}}{\mathscr{F}(\mathbf{x},\mathbf{s})^{\epsilon}} \delta(1 - x_1 - x_2)$$

$$\mathscr{U}(\mathbf{x}) = x_1 + x_2$$

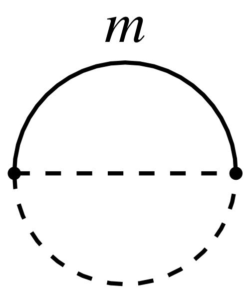
$$\mathscr{F}(\mathbf{x},\mathbf{s}) = -\, s x_1 x_2 + (m_1^2 x_1 + m_2^2 x_2)(x_1 + x_2)$$





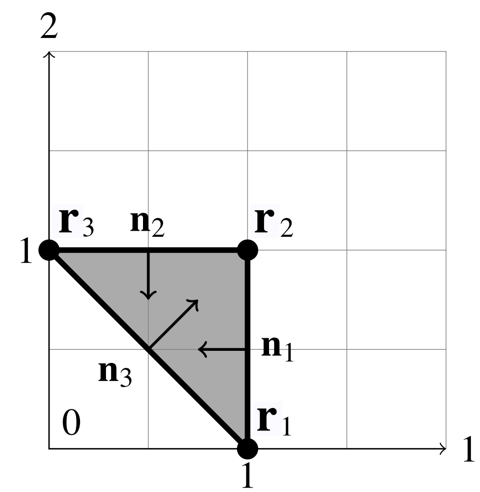
Sector Decomposition

Sector Decomposition in a Nutshell

$I =$ 

$= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\left(x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1\right)^{2-\varepsilon}}.$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathcal{N}(I) =$ 

$=$

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \mathbf{n}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \mathbf{n}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$a_1 = 1 \quad a_2 = 1 \quad a_3 = -1$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1} y_3^1,\ x_2 = y_1^0 y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1} y_2^0,\ x_2 = y_1^0 y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0 y_3^1, x_2 = y_2^{-1} y_3^1$

$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^1 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \frac{y_1^{-\varepsilon} y_2^{-\varepsilon} y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}} \left[\delta(1-y_2)+\delta(1-y_3)+\delta(1-y_1)\right]$$

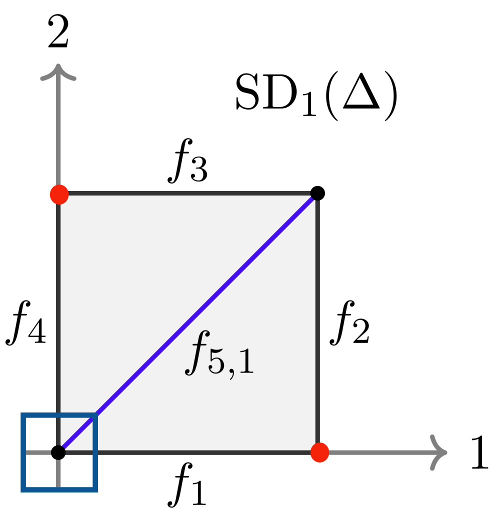
Schlenk 2016

Applications

Additional Regulators (II)

Toy Example:

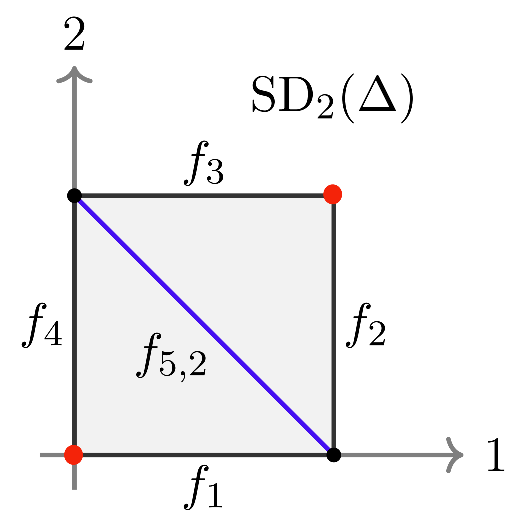
$$P_1(x,\lambda) = 1 + \lambda x_1 + x_1 x_2 + \lambda x_2$$



need analytic regulator

$$P_2(x,\lambda) = \lambda + x_1 + \lambda x_1 x_2 + x_2$$

ok!



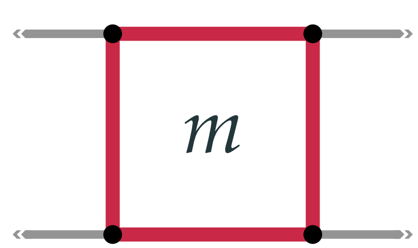
pySecDec can find the constraints on the analytic regulators for you

`extra_regulator_constraints()`:

$$v_2 - v_4 \neq 0, \quad v_1 - v_3 \neq 0$$

`suggested_extra_regulator_exponent()`:

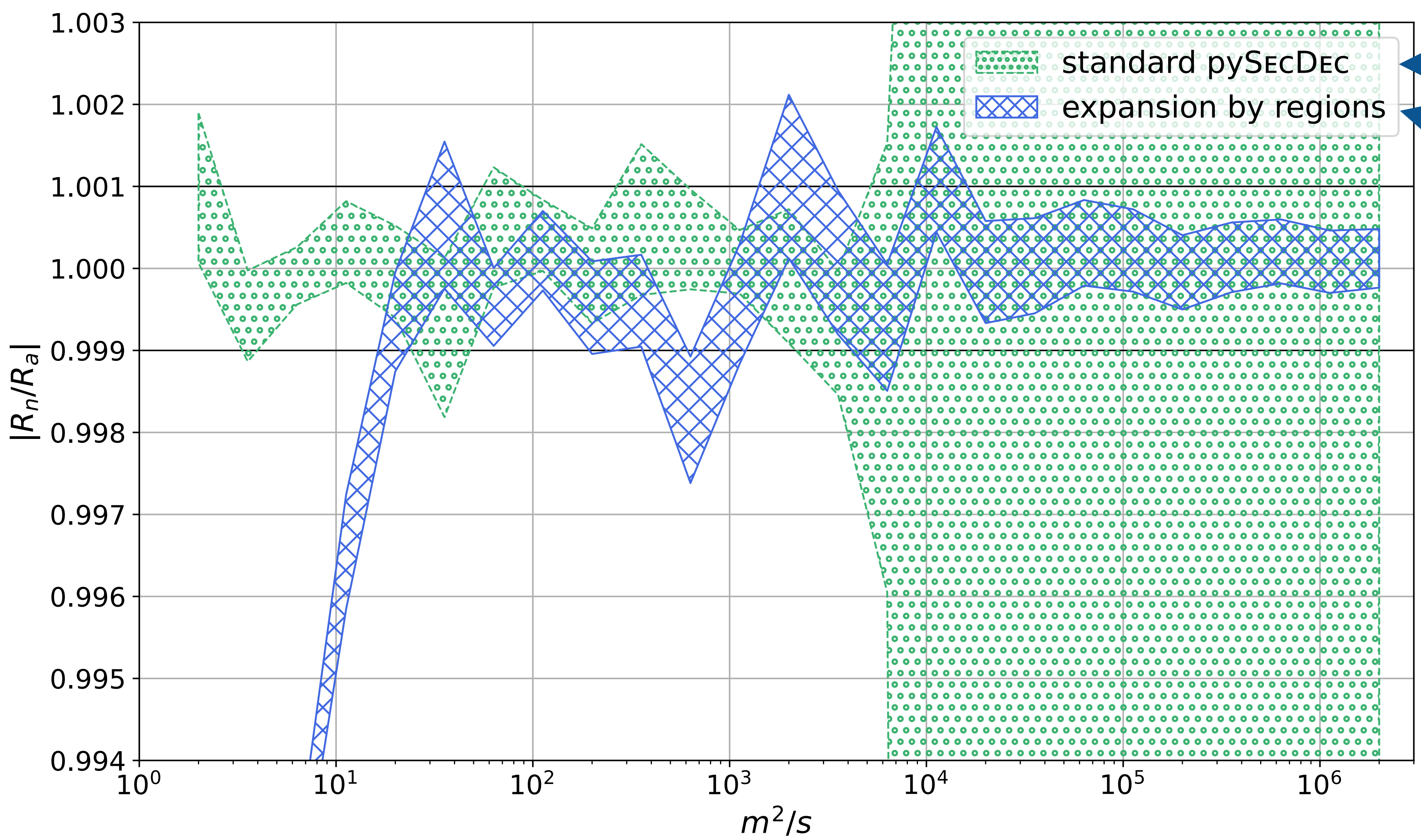
$$\{\delta\nu_1, \delta\nu_2, \delta\nu_3, \delta\nu_4\} = \{0,0,\eta,-\eta\}$$



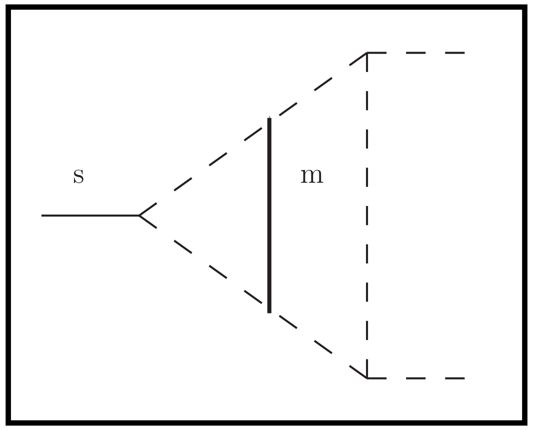
Small m expansion

Applying Expansion by Regions

Ratio of the finite $\mathcal{O}(\epsilon^0)$ piece of numerical result R_n to the analytic result R_a



(at large m^2/s)
~1 day
~2 min



For large ratio of scales (m^2/s) the EBR result is **faster** & **easier** to integrate

Lee-Pomeransky and MoR

Building Bridges: LP ↔ Propagator Scaling

Region vectors in momentum space and Lee-Pomeransky space are related, we can see this using Schwinger parameters $\tilde{x}_e$

$$\frac{1}{D_n^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \frac{d\tilde{x}_e}{\tilde{x}_e} \, \tilde{x}_e^{\nu_e} \, e^{-\tilde{x}_e D_e} \text{ , with } x_e \propto \tilde{x}_e$$

$$(D_1^{-1}, \ldots, D_N^{-1}) \sim (\tilde{x}_1, \ldots, \tilde{x}_N) \sim (x_1, \ldots, x_N)$$

Example: 1-loop form factor

$$\begin{aligned}
\text{Hard}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^0, \lambda^0), && (x_1, x_2, x_3) \sim (\lambda^0, \lambda^0, \lambda^0) \\
\text{Collinear to } p_1: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}), && (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}) \\
\text{Collinear to } p_2: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}), && (x_1, x_2, x_3) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}) \\
\text{Soft}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2}), && (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2})
\end{aligned}$$

Can connect the regions in mom. space with those we determine geometrically

Next step: automatically find (Sudakov decomposed) loop momentum scalings compatible with region vectors WIP w/ Yannick Ulrich

Building Bridges: Landau ↔ Regions

The **Landau equations** give the necessary conditions for an integral to diverge

$$1)\quad \alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) = \frac{\partial}{\partial k_a^\mu}\sum_{e\in G}\alpha_e\left(-l_e^2(k,p,q) - i\varepsilon\right) = 0 \qquad \forall a \in \{1,\ldots,L\}$$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Idea is to explore the *neighbourhood of a pinched surface*, defined by

$$1)\quad \alpha_e l_e^2(k,p,q) \sim \lambda^p \qquad \forall e \in G, \quad \text{with} \quad p \in \{1,2\}$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) \lesssim \lambda^{1/2} \qquad \forall a \in \{1,\ldots,L\}$$

with the goal of further understanding the connection between

Solutions of the Landau equations ↔ Regions

Gardi, Herzog, Ma, Schlenk 22

Method of Regions (Details/Examples)

Geometric Method

In Feynman parameter space, there is a **geometric method** for finding regions

Pak, Smirnov 10

Each region will be defined by a **region vector** $\mathbf{v} = (v_1, \ldots, v_N; 1)$, in each region we will perform a change of variables $x_i \to \lambda^{v_i} x_i$ and series expand about $\lambda = 0$

Let us start by considering some polynomial

$$P(\mathbf{x}, \lambda) = \sum_{i=1}^{m} c_i \, x_1^{r_{i,1}} \cdots x_N^{r_{i,N}} \lambda^{r_{i,N+1}}$$

c_i - non-negative coefficients

x_i - integration variables

λ - small parameter

$\mathbf{r}_i = (r_{i,1}, \ldots, r_{i,N+1}) \in \mathbb{N}^{N+1}$ - exponent vectors

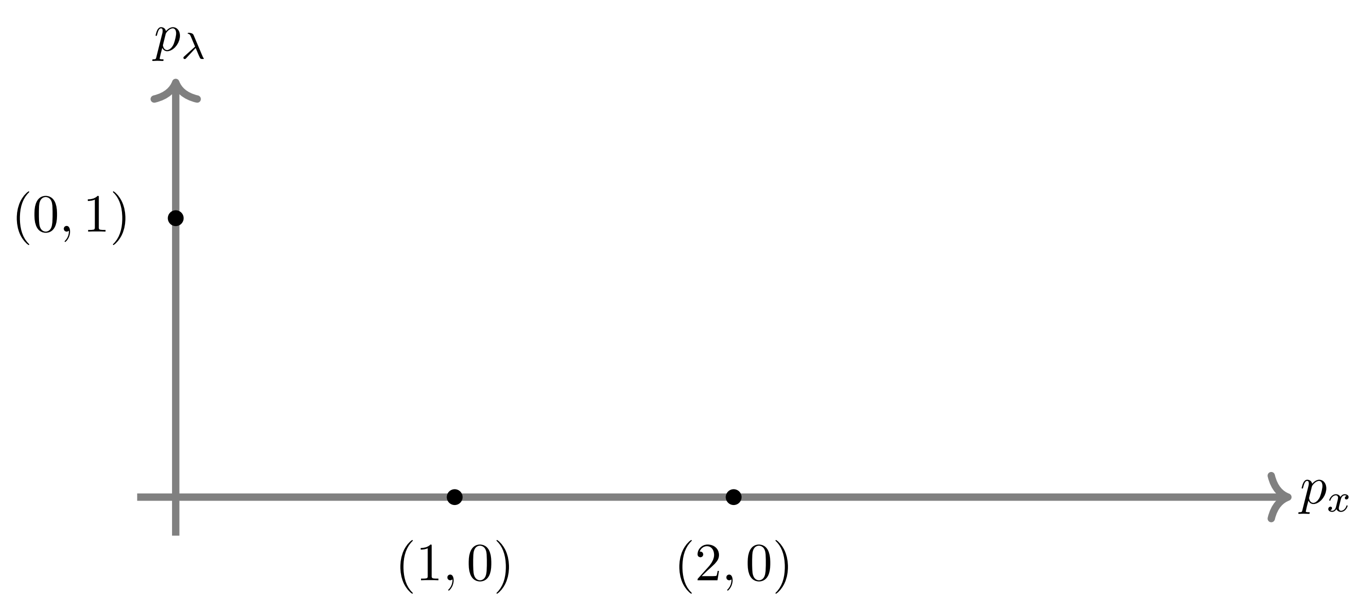
Geometric Method

Ignoring, for now, the coefficients c_i we can introduce a simple but useful picture for such polynomials:

- For each variable x_i or λ draw an orthogonal axis
- For each monomial, draw a dot at position $\mathbf{r}_i$

Example: $P(x, \lambda) = \lambda + x + x^2$ has exponent vectors

$\mathbf{r}_1 = (0,1), \mathbf{r}_2 = (1,0), \mathbf{r}_3 = (2,0)$



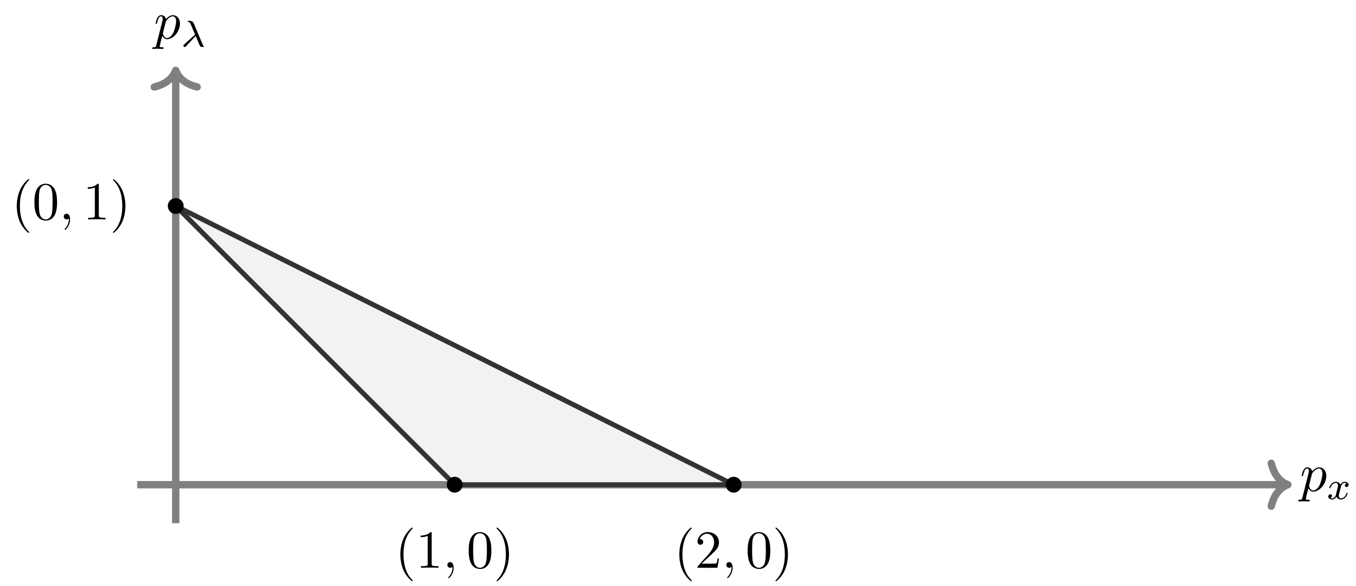
Geometric Method

We may define a **Newton polytope** of the polynomial, this is the convex hull of the exponent vectors:

$$\Delta = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \left\{ \sum_j \alpha_j \mathbf{r}_j \,|\, \alpha_j \geq 0 \wedge \sum_j \alpha_j = 1 \right\}$$

Example: $P(x, \lambda) = \lambda + x + x^2$ has exponent vectors

$\mathbf{r}_1 = (0,1), \mathbf{r}_2 = (1,0), \mathbf{r}_3 = (2,0)$



Geometric Method

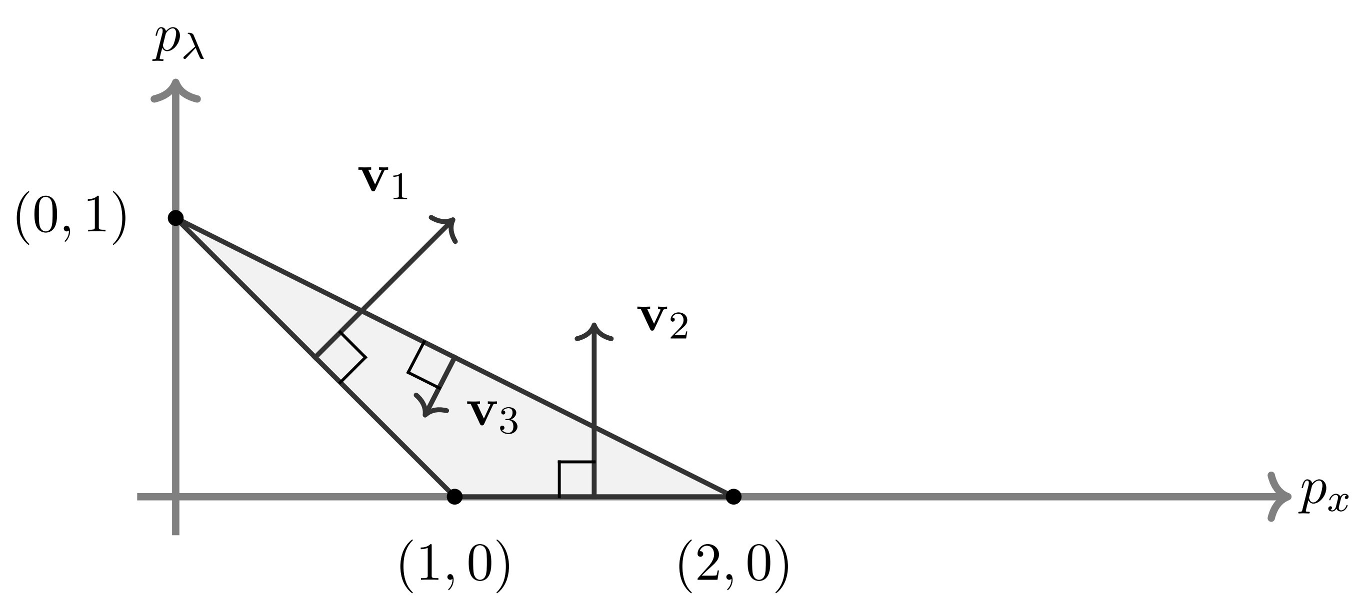
Alternatively, this polytope can also be described as the intersection of half spaces:

$$\Delta = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^{N+1} \mid \langle \mathbf{m}, \mathbf{v}_f \rangle + a_f \geq 0 \right\}$$

F - set of polytope facets, $a_f \in \mathbb{Z}$

$\mathbf{v}_f$ - inward-pointing normal vectors for each facet (co-dimension 1 face)

Several public tools exist for computing Newton polytopes/convex hulls and their representation in terms of facets exist, e.g. **Normaliz** and **Qhull**



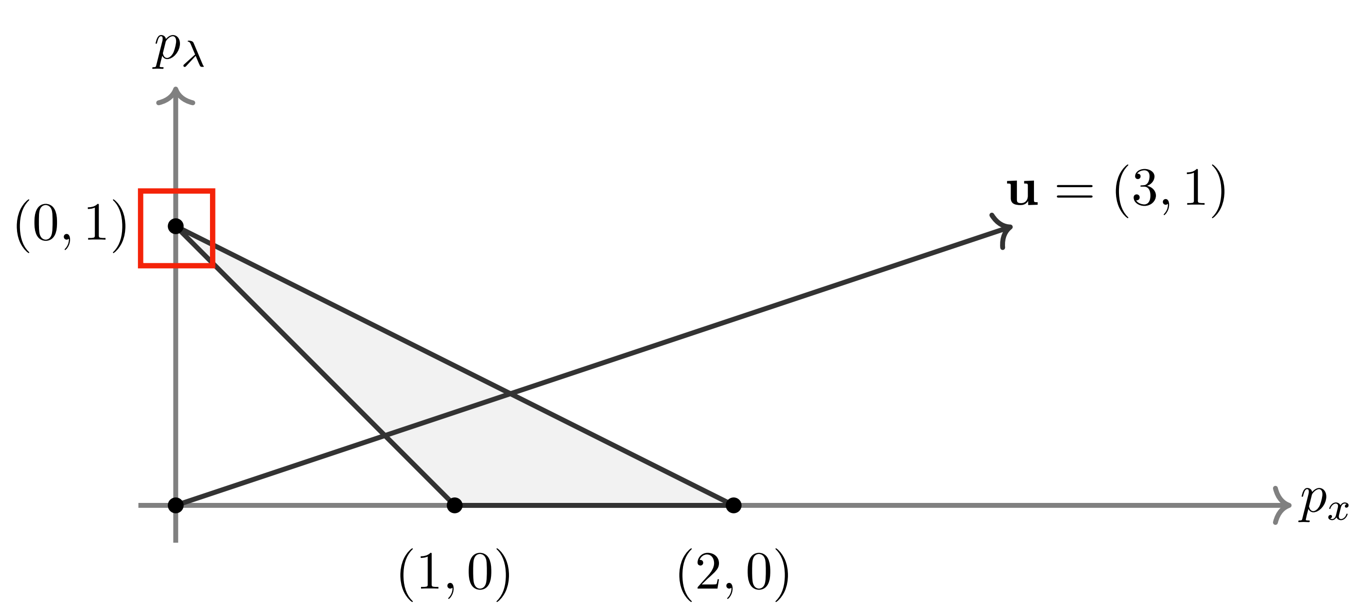
Geometric Method

Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$
Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (3,1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda} + \lambda^3 + \lambda^6$



Geometric Method

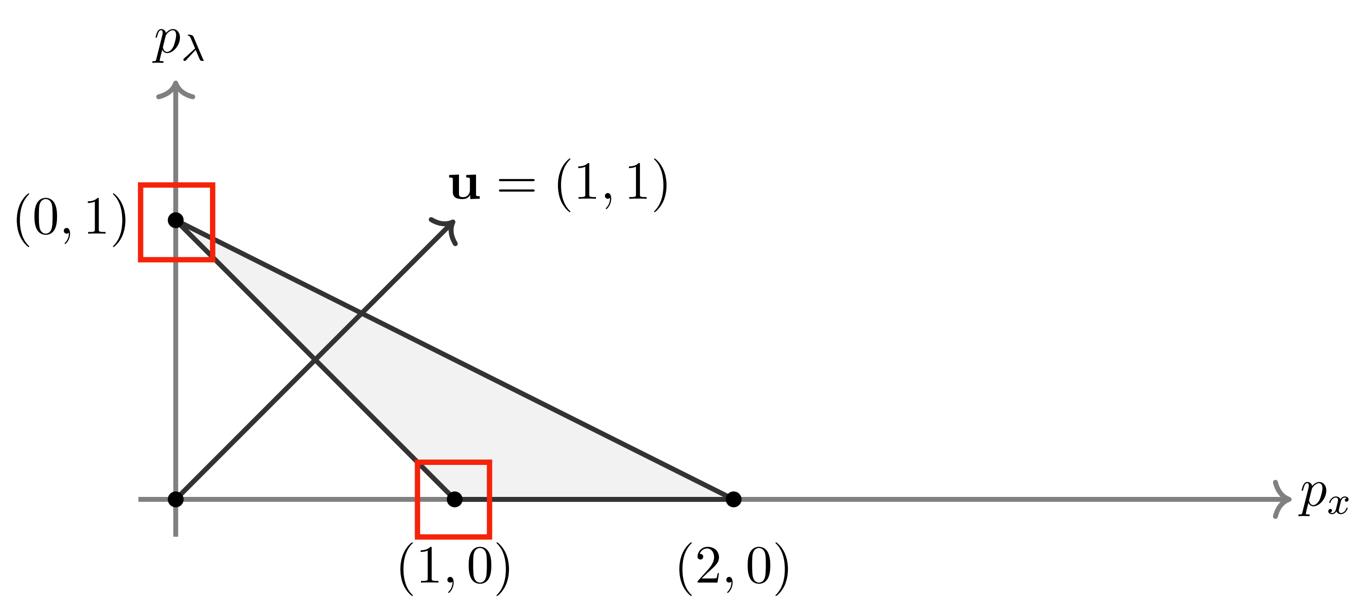
Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \, \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$

Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (1{,}1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda + \lambda} + \lambda^2$



Expanding Regions

Rewrite our polynomial as: $P(\mathbf{x}) = Q(\mathbf{x}) + R(\mathbf{x})$

With $Q(\mathbf{x})$ defined such that it contains all of the lowest order terms in λ

The binomial expansion of

$P(\mathbf{x})^m = Q(\mathbf{x})^m \left(1 + \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right)^m$ converges for $\mathbf{x} = \lambda^{\mathbf{u}}$ if $R(\mathbf{x})/Q(\mathbf{x}) < 1$

Some observations:

- An expansion with region vector $\mathbf{v}$ converges at a point $\mathbf{u}$ if the terms with minimum $< \mathbf{r}_i, \mathbf{u} >$ are contained in the terms with minimum $< \mathbf{r}_i, \mathbf{v} >$
- For any $\mathbf{u}$ the vertices with the smallest $< \mathbf{r}_i, \mathbf{u} >$ must be part of some facet F
- Since $u_{N+1} > 0$, the lowest order terms for any $\mathbf{u}$ must lie on a facet whose inwards pointing normal vector has a positive $(N+1)$-th component, let us call the set of such facets F^+ or lower facets

Claim: regions are defined by vectors normal to the facets in F^+, the integrand in each region consists of the monomials lying on the facet

Scaleless Integrals

Scaleless integrals seem to play quite an interesting role

Momentum space

In dimensional regularisation, **scaleless integrals are 0**

$$I(\{k_i\}_a, \{ck_i\}_b) = c^q\, I(\{k_i\}) \implies I(\{k_i\}) = 0, \quad \{k_i\} = \{k_i\}_a \cup \{k_i\}_b$$

Where $c \neq 1$ and $q \neq 0$ is some scaling dimension

Feynman parameter space

$$(\mathscr{U}\mathscr{F})\big(c^{\mathbf{u}}\mathbf{x}\big) = c^q(\mathscr{U}\mathscr{F})(\mathbf{x}), \quad \mathbf{u} \neq n\mathbf{1}, \quad n \in \mathbb{R}$$

Geometrical view

For Δ built from $\mathscr{U} + \mathscr{F}$

$$\dim(\Delta) = \dim(\mathbf{x}) \iff I \text{ scaleful}$$
$$\dim(\Delta) < \dim(\mathbf{x}) \iff I \text{ scaleless}$$

Important consequences:

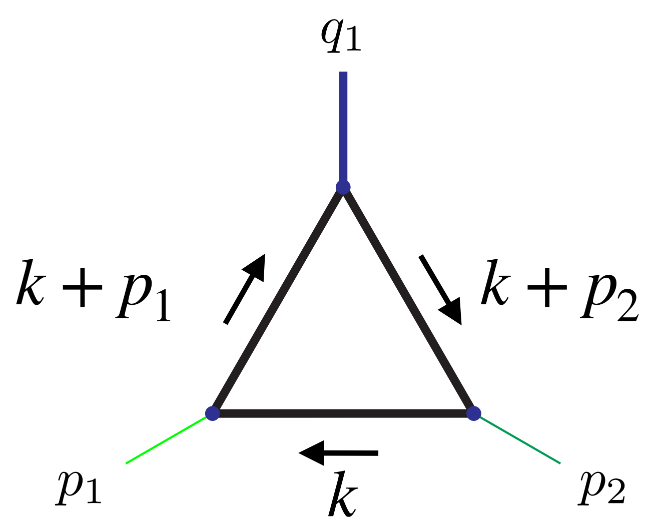
Faces of co-dimension > 1 are scaleless

``Region'' vectors not normal to a facet give scaleless integrals

Overlap contributions i.e. rescaling by two region vectors, are scaleless

Triangle Example

Consider the on-shell limit $p_1^2 \sim p_2^2 \sim \lambda q_1^2$ for $\lambda \to 0$



$$I = i\pi^{D/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k+p_1)^2(k+p_2)^2(k^2)}$$

$$p_1 = (p_1^+, p_1^-, p_1^\perp) \sim Q(\lambda, 1, \lambda^{\frac{1}{2}})$$

$$p_2 \sim Q(1, \lambda, \lambda^{\frac{1}{2}})$$

1) Split integrand up into regions

$$\text{Hard}: k_H^\mu \sim (1,1,1)\,Q$$

$$\text{Collinear to } p_1 : k_{J_1}^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})\,Q$$

$$\text{Collinear to } p_2 : k_{J_2}^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})\,Q$$

$$\text{Soft}: k_S^\mu \sim (\lambda, \lambda, \lambda)\,Q$$

2) Series expand each region in λ

$$I_H = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k^2 + 2k^+ \cdot p_1^-)(k^2 + 2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_1} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k+p_1)^2(2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_2} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^- \cdot p_1^+)(k+p_2)^2(k^2)}$$

$$I_S = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^+ \cdot p_1^- + p_1^2)(2k^- \cdot p_2^+ + p_2^2)(k^2)}$$

Analysis follows:
Becher, Broggio, Ferroglia 14

Triangle Example

3-5) Integrate each expansion over the whole integration domain, discard scaleless, sum

$$I_H = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_S = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I = I_H + I_{C_1} + I_{C_2} + I_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{P_2^2}\ln\frac{Q^2}{P_1^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

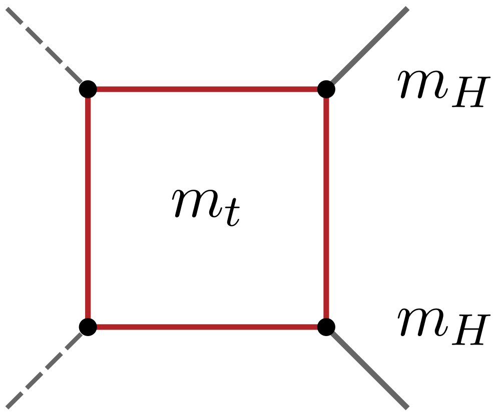
This reproduces the expected result, but why does this work (and does it always)?

1) How did we **find all the regions**?

2) Did we not **double-count** when integrating over the whole domain ?

pySecDec: EBR Box Example

Example: 1-loop massive box expanded for small $m_t^2 \ll s, |t|$



Requires the use of analytic regulators

Can regulate spurious singularities by adjusting propagators powers

$$G_4 = \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2 - m_t^2]^{\delta_1}[(k+p_1)^2 - m_t^2]^{\delta_2}[(k+p_1+p_2)^2 - m_t^2]^{\delta_3}[(k-p_4)^2 - m_t^2]^{\delta_4}}$$

Can keep $\delta_1, \ldots, \delta_4$ symbolic or $\delta_1 = 1 + n_1/2, \delta_2 = 1 + n_1/3, \ldots$ and take $n_1 \to 0^+$

Output region vectors:

$\mathbf{v}_1 = (0,0,0,0,1)$

$\mathbf{v}_2 = (-1,-1,0,0,1)$

$\mathbf{v}_3 = (0,0,-1,-1,1)$

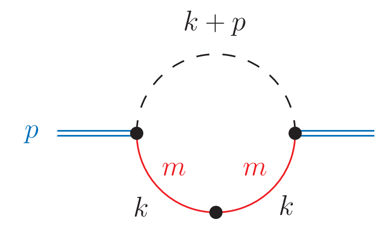
$\mathbf{v}_4 = (-1,0,0,-1,1)$

$\mathbf{v}_5 = (0,-1,-1,0,1)$

Result: s = 4.0, t = -2.82843, $m_t^2 = 0.1, m_h^2 = 0$)

$$I = -1.30718 \pm 2.7 \cdot 10^{-6} + (1.85618 \pm 3.0 \cdot 10^{-6})\, i + \mathcal{O}\left(\epsilon, n_1, \frac{m_t^2}{s}, \frac{m_t^2}{t}\right)$$

Transform the expression for the full integral:



$$F = \int\limits_{k\in D_h} \mathrm{D}k\, I + \int\limits_{k\in D_s} \mathrm{D}k\, I = \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} I$$

$$= \sum_i \Bigg(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_i^{(h)} I - \sum_j \int\limits_{k\in D_s} \mathrm{D}k\, T_j^{(s)} T_i^{(h)} I \Bigg) + \sum_j \Bigg(\int\limits_{k\in\mathbb{R}^d} \mathrm{D}k\, T_j^{(s)} I - \sum_i \int\limits_{k\in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I \Bigg)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$

All terms are integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expansion by regions $\Rightarrow$ location of boundary Λ between D_h, D_s is irrelevant.

Slide from: Bernd Jantzen, High Precision for Hard Processes (HP2) 2012

The general formalism (details)

Identities as in the examples are generally valid, under some conditions.

Consider

- a (multiple) integral $F = \int \mathrm{D}k\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \ldots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions

- $\bigcup_{x \in R} D_x = D \qquad [D_x \cap D_{x'} = \emptyset\ \forall x \neq x']\,.$
- Some of the expansions commute with each other. Let $R_\mathsf{c} = \{x_1, \ldots, x_{N_\mathsf{c}}\}$ and $R_\mathsf{nc} = \{x_{N_\mathsf{c}+1}, \ldots, x_N\}$ with $1 \leq N_\mathsf{c} \leq N$. Then: $T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \equiv T^{(x,x')}\ \forall x \in R_\mathsf{c}\,,\ x' \in R\,.$
- Every pair of non-commuting expansions is invariant under some expansion from R_c: $\forall x'_1, x'_2 \in R_\mathsf{nc},\ x'_1 \neq x'_2,\ \exists\, x \in R_\mathsf{c} :\ T^{(x)} T^{(x'_2)} T^{(x'_1)} = T^{(x'_2)} T^{(x'_1)}\,.$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) regularization. $\hookrightarrow$ All expanded integrals and series expansions in the formalism are well-defined.

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The general formalism (2)

Under these conditions, the following **identity** holds: $\left[F^{(x,\dots)} \equiv \sum_{j,\dots} \int \mathrm{D}k\, T^{(x,\dots)}_{j,\dots} I\right]$

$$F = \sum_{x\,\in\,R} F^{(x)} - \sum_{\{x_1',x_2'\}\subset R}^{\langle R_\mathrm{c}+1\rangle} F^{(x_1',x_2')} + \ldots - (-1)^n \sum_{\{x_1',\ldots,x_n'\}\subset R}^{\langle R_\mathrm{c}+1\rangle} F^{(x_1',\ldots,x_n')} + \ldots + (-1)^{N_\mathrm{c}} \sum_{x'\,\in\,R_\mathrm{nc}} F^{(x',x_1,\ldots,x_{N_\mathrm{c}})}$$

where the sums run over subsets $\{x_1',\ldots\}$ containing at most one region from R_nc.

Comments

- This identity is exact when the expansions are summed to all orders. ✓
  Leading-order approximation for $F \rightsquigarrow$ dropping higher-order terms.
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, …) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x_1',\ldots,x_n')}$ $(n \geq 2)$ are scaleless and vanish.
  [✓ if each $F_0^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x_1',x_2',\ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET. e.g. Manohar, Stewart '06; Chiu, Fuhrer, Hoang, Kelley, Manohar '09; …

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